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Jan Sperling
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Part I.

Habilitation thesis

Abstract

Quantum physics itself is a fascinating theory. When multiple degrees of freedom are considered, a manifold of quantum correlations between the individual subsystem can be observed. The aim of the present cumulative habilitation thesis is a state of the art report on the characterization techniques and measurement strategies to verify such quantum phenomena. I will mainly focus on my research which has been performed in the surrounding of the theoretical quantum optics group at the University of Rostock (head: Prof. W. Vogel) during the last couple of years.

The presented results include theoretical and – in collaboration with our experimental partners – experimental studies of complexly structured radiation fields. In this context, we study the verification of quantum properties, such as entanglement in multipartite systems and nonclassical correlations of multiple harmonic oscillators – both being crucial for the classification of quantum light fields. Beside the discussion of quantumness criteria, we also perform a quantification of these quantum effects. Furthermore, we describe a method to characterize novel quantum optical detector systems.

Zusammenfassung

Die Quantenphysik ist bereits für sich eine faszinierende Theorie. Wenn mehrere Freiheitsgrade betrachtet werden, kann eine Vielzahl von Quantenkorrelationen zwischen den Teilsystemen beobachtet werden. Das Ziel der vorliegenden, kumulativen Habitilationsschrift ist es eine Übersicht zum aktuellen Stand der Charakterisierungsmethoden und Messstrategien zu geben, um solche Quantenphänomene nachzuweisen. Ich werde mich dabei im Wesentlichen auf meine Forschung beschränken, die im Umfeld der AG Theoretische Quantenoptik an der Universität Rostock (Leiter: Prof. Dr. W. Vogel) in den letzten Jahren durchgeführt wurde.

Die präsentierten Resultate beinhalten theoretische und – in Zusammenarbeit mit unseren experimentellen Partnern – experimentelle Untersuchungen von komplex strukturierten Strahlungsfeldern. In diesem Zusammenhang studieren wir den Nachweis von Quanteneigenschaften, beispielsweise der Verschränkung in Vielparteiensystemen und nichtklassische Korrelationen einer Vielzahl von harmonischen Oszillatoren – beides ist entscheidend für die Klassifizierung von Quantenlichtfeldern. Neben der Diskussion von Nachweiskriterien des Quantencharakters, führen wir auch eine Quantifizierung dieser Effekte durch. Darüber hinaus beschreiben wir eine Methode die es erlaubt neuartige, quantenoptische Detektorsyteme zu charakterisieren.

1. Introduction and methods

$$\Delta\psi + \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\psi = 0$$

Equation (5) in *Quantisierung als Eigenwertproblem*,
E. Schrödinger, Ann. Phys. **384**, 273 (1926).

One aim of theoretical physics is the description of nature in terms of mathematical relations. An interplay between experimental and theoretical physics, mathematics, and philosophy helps us to construct models as a way towards understanding the principles of nature. At the beginning of the last century a new debate on the true paradigms of physics started that was due to the upcoming of quantum theory; see, e.g., [1–3]. Nowadays, physicist are used to this theory. Still, there is a need for a rigorous framework to identify aspects of nature which are incompatible with some classical models. Whenever an observations of quantum systems is beyond a classical expectation, our curiosity for understanding these effects awakes.

The evolution of a quantum physical system is formulated in terms of the Schrödinger equation. A separation of the time dependent part yields the time-independent Schrödinger equation, which is an eigenvalue equation of the Hamiltonian that describes the free dynamics of individual degrees of freedom together with interactions between them. The mathematical background of such eigenvalue problems is well understood employing methods from linear functional analysis and linear algebra. This foundation allows physicist to study properties of quantum systems or to predict experimental observations.

If nonlinear equations are involved, then the mathematical toolbox of linear methods is, in parts, no longer applicable. However, a lot of physical systems are characterized by nonlinear equations. Prominent example are the Navier–Stokes equations for the dynamic of fluids, Einstein's field equations for gravitation, or the propagation of light in nonlinear optical fibers being described by the so-called nonlinear Schrödinger equation. In the first part of this thesis we show that, in addition, certain quantum correlations can be described in terms of nonlinear eigenvalue equations. A number of solutions and propositions for the nonlinear eigenvalue problems have been elaborated which subsequently lead to new applicable criteria for characterizing the quantumness of complex physical systems. Those are further generalized in the second part of this document. There the topic of quantifying quantum correlations, i.e. the strength of a quantum effect, is explored. The presented approach of quantification is based on the idea of counting quantum superposition that represent one origin of quantum interferences.

The Schrödinger equation itself, has the form of a wave equation. Beside the thereby described evolution, the measurement principle in quantum mechanics is quite unique which is typically emphasized by the phrase of a spontaneous collapse of the wave function. Closely related is the well-known wave-particle duality. Hence, a rigorous description of a particular detection system is indispensable for a correct interpretation of the results of a measurement process. In the third part of this thesis we will formulate the theory of quantum optical measurement schemes which are based on very simple detectors – so-called click detectors. Despite the plain structure of such devices, it is demonstrated that they are still able to successfully detect quantum properties of radiation fields.

In summary, the distinction "classical/quantum" shall motivate the present cumulative habilitation thesis. In this first chapter we briefly review the state of the art and describe the utilized methods. The remainder of the thesis summarizes the results of my recent research in this direction.[1] As outlined above, this includes the three main items: the verification of quantum properties, the quantification of quantum correlations, and the click detection theory.

1.1. State of the art

Maxwell's theory of radiation describes optical phenomena in terms of a propagating electromagnetic waves. Hence, the electromagnetic field can exhibit interference effects. In quantum physics, the Schrödinger equation has the form of a wave equation. Thus, interferences may be observed as well. As a consequence for quantum optics, different types of interferences occur which can originate from the classical and/or quantum description of the quantized electromagnetic field. Hence, the question about the quantum character of a given state of light is a cumbersome problem to be addressed. An introduction to the theory of quantum optics can be found in the book [4].

The emerging particle of the quantized field is the photon. For quantum communications, photons play an important role as one carrier of quantum information. Hence, devices using single photons serve as one approach to implement quantum information protocols. Since quantum physics allows operations which cannot be achieved with a classical computer, the unambiguous identification of quantum correlated systems is indispensable. We refer to the book [5] for an introduction to quantum information and communications.

Two approaches to distinguish between classical correlations and quantum correlations have been formulated in the 1960s. These are the seminal works by Bell [6] and by Kochen and Specker [7]. Both address hidden variable models which are a key element of the so-called Einstein-Podolsky-Rosen paradox [3]. The two interpretations – given by Bell or Kochen and Specker – led to the notions non-locality and contextuality, respectively; see, e.g., [8–11] for recent results. In simple words both approaches are related to the question whether or not there exists a model in classical probability theory which can describe the outcome of a performed correlation measurement [12, 13].

While the approaches of Bell, Kochen and Specker start with considerations of classical statistics and show the violation in quantum physics, other methods define classical references in quantum systems. For example coherent states are the quantum analogue to the classical pendulum which yields the Glauber-Sudarshan representation [14, 15] and the notion of nonclassicality of harmonic oscillators [16, 17]. Another example involves correlations between different quantum systems [18]. Whenever the compound state of at least two degrees of freedom cannot be interpreted as a classical statistical mixture of quantum states of the individual subsystems, then the joint state is referred to as an entangled one; see [19] for a review on entanglement.

The traditional way to identify nonclassical features – e.g. nonclassicality of the har-

[1] The references in this cumulative thesis are numbered as follows:

- References (general): Arabic numbering, [1]–[228].
- Author's references in PhD: small Roman numbering, [i]–[vi].
- Author's references as post-doc: capital Roman numbering, [I]–[XXV].

monic oscillator or multipartite entanglement – is done via inequalities [20, 21]. For a correlation G under study, a lower bound for classical systems is derived: $G \geq G_{\mathrm{min\,cl.}}$. A measured correlation of the form $G < G_{\mathrm{min\,cl.}}$ is, therefore, a clear signature for the quantum correlations in the realized system.[2] In the following we will briefly review the identification of oscillator's nonclassicality and multipartite entanglement in terms of inequalities and discuss standard measurement scenarios to infer the needed correlations.

1.1.1. Nonclassicality of harmonic oscillators

The starting point of quantum optics itself may be dated back to the year 1905 when Einstein proposed an explanation for the photoelectric effect [22]. During the following decades the development in this field led to a better understanding of optical phenomena in the quantum physical framework; see [23, 24] for recent reviews and [4, 20, 25] for introductions. Coherent states have been identified as the only pure states which follow the classical motion of a harmonic oscillator and which have a classical Glauber-Sudarshan representation [26–28]. Thus, coherent states define the classical representatives in this quantum system. A mixture of coherent states can be fully interpreted in a semi-classical picture utilizing classical representatives and classical statistics. However, a photon is one example which demonstrates that such a semi-classical interpretation of the Glauber-Sudarshan representation is not always possible.

One of the early approaches to characterize the wave-particle duality of an electromagnetic field is given by the Hong-Ou-Mandel interference experiment [29] or the photon-antibunching [30–32]. The photon antibunching is related, but not identical to the notion of sub-Poisson light [34]. The latter is characterized in terms of fluctuations of the photoelectric counting statistics being below the classical shot-noise limit of this statistics [33]. Another frequently considered example of a nonclassical radiation field is squeezed light [35–38]. In case of pure states, squeezed light exhibits a minimal uncertainty between the quadrature and the corresponding momentum (a recent study can be found in [40]). The notion squeezing itself means that – similarly to sub-Poisson light – field (or quadrature) fluctuations are below the vacuum fluctuation which limits the measurement precision of classical states. Thus, squeezed light is a versatile resource for, e.g., gravitational wave detection [41, 42], which has been recently implemented by the LIGO collaboration [43, 44].

Directly accessible nonclassicality probes are based on second- or higher-order correlation functions which may be given in terms of moments. This includes moments of the field intensity, of the quadrature distribution, and of powers of annihilation and creation operators [45–50]. An experiment summarizing and applying a number of those nonclassicality criteria can be found in [51]. Remarkably, a complete moment based test of nonclassical correlations, including multi-time correlations and entanglement verification, has been discussed in a unified form in Ref. [52].

The definition of the notion nonclassicality is based on the Glauber-Sudarshan representation, which is a phase-space representation in terms of a quasiprobability distribution referred to as P function. The P function depends on the complex coherent amplitudes of a single or multiple field components. The reconstruction of a nonclassical P function is performed in [53] for the single-photon-added thermal state [47]. However, the P function turns out to exhibit, in general, singularities in terms of non-regular distributions, e.g., infinite orders of derivatives of the Dirac delta distribution.

Beside the P function, other phase-space distribution have been known, such as the

[2] Note that one could similarly study the upper bound.

Wigner function [54] or the Husimi function [55], which are regular. These distributions can be described as a P function convolved with a certain amount of Gaussian noise, which subsequently led to the notion of s-parametrized quasiprobabilities [56–58]. The parameter s addresses the amount of Gaussian noise convolution or – in terms of observables – a particular ordering of annihilation and creation operators. Most of these notions have in common that they violate the non-negativity constraint, $P \geq 0$, of classical probability theory [59, 60]. Typical reconstruction schemes for measuring these phase-space distributions are based on tomographic principles [61, 62].

If temporal correlations are taken into account then on has to extend the P function significantly [63] for including effects such as antibunching. Other approaches to identify quantumness are given by the Fourier transform of the P function which is experimentally accessible and nonclassical correlations may be unrevealed by the Bochner criterion [64–67]. Despite the fact that regular phase-space functions exists [68], the nonclassical features of highly singular P functions remained an open problem for some time. For example the previously discussed squeezed state is described by either a singular or a purely classical distribution for different s parametrized quasiprobabilities. A way for a consistent and universal regularization of the P function was introduced in [69] and the practical application was, for example, demonstrated for a squeezed state of light [70] (see also [71–75] for further reading).

1.1.2. Entanglement characterization

Let us now discuss the detection of entanglement; see [19, 21] for an introduction. Entanglement in many-body systems can be almost arbitrarily complex [76]. This is due to the manifold of individual degrees of freedom which might be quantum correlated. However, a systematic classification is feasible [77]. For quantum information processing, graph-representations of quantum states turn out be very efficient way to characterize entanglement [78–81]. A first approach to identify entanglement is based on violation of a criteria addressing local hidden-variable models, which was proposed by Clauser, Horne, Shimony, and Holt in 1969 [82]. The experimental demonstration was realized in 1981 [83]. Such criteria have been further developed to study the fundamental relation between non-locality and entanglement, cf. [84, 85].

Nowadays, the most popular technique to determine entanglement is based on so-called entanglement witnesses, which are capable to certify bipartite and multipartite entanglement [86–88]. Different methods have been proposed to enhance the range of application of this approach, such as higher mode extensions [89,90] or device independent witnesses [91]. Moreover, the notion of optimal witnesses has been introduced to further characterize such entanglement probes [92–94]. The success of the method of entanglement witnesses can be directly seen from its vast number of experimental application; see, e.g., [95–103].

In [86] it has been also shown that the entanglement witness approach is equivalent to the characterization via positive but not completely positive (PNCP) maps. It is worth mentioning that a technique has been formulated recently which can also identify multipartite entanglement with positive maps [104]. The most prominent example of a PNCP map based entanglement probe is the partial transposition criterion [105]. It is a necessary and sufficient criterion for low dimensional bipartite Hilbert spaces [86] and bipartite Gaussian states [106, 107]. Closely related are moment based approaches [108–110]. Such higher-oder moment criteria have a non-linear structure and include correlations of local observables or local uncertainty relations as studied in [111–115].

Moments of second order are particularly suited to identify entanglement of multimode Gaussian states [116–118]. One proper classification of multimode Gaussian entangled states is given in [119] and the influence of attenuation to this class of entangled states has been studied, e.g., in [120]. An particularly interesting example of a Gaussian state is a four-mode one [121], which has been experimentally realized [122]. This state cannot be identified to be entangled using the partial transposition criterion and, therefore, belongs to the class of so-called bound entangled states; see also [123] for a bipartite, non-Gaussian continuous variable example which has been additionally studied in [124]. Another four-mode state in discrete variables, which shares such a bound type of entanglement between its subsystems, is the so-called Smolin state [125–127].

Beyond the briefly mentioned types of entanglement there exists other forms and detection methods of entanglement in physical systems. For example, systems of indistinguishable particles require a careful study regarding their quantum correlations, because the spin statistic theorem requires a symmetrization of quantum states [128]. The antisymmetry or symmetry under the exchange of the subsystems has an influence on the entanglement properties in compound Fermion or Boson systems [129–131]. Moreover, if the underlying algebra of complex numbers of the Hilbert spaces is modified then some interesting forms of entanglement can arise [132–134].

For standard complex tensor product Hilbert spaces, the Schmidt decomposition is an elegant way to characterize pure states in bipartite systems, cf. [5]. Namely, the rank of this decomposition is one if and only if the state is separable. This notion of the Schmidt rank is simply the minimal number of pure product states which have to be superimposed to describe the full state. An extension of this notion to mixed states led to the notion of Schmidt number states [135]. The Schmidt number additionally quantifies the amount of entanglement and may be extended to multipartite systems [136]. An axiomatic way to quantify entanglement in general has been introduced in [137–139]. In particular an entanglement measure of pure states can always be extended to the set of mixed states using a convex roof construction [140]. Note that it has been also shown, that different measures can yield contradicting orderings of the amount of entanglement [141, 142].

1.1.3. Measurements in quantum optics

The quantum measurement process has been rigorously studied by von Neuman for the first time [143]. Let us briefly describe the concluded implications. Say a quantum state yields a certain measurement outcome $m(0)$ with a certain probability. If we repeat the same measure after some time τ, we would get an outcome $m(\tau)$ with some probability. In the limit of a vanishing delay time τ, we expect a unit probability to get the same outcome for both measurements, i.e. $\lim_{\tau\to 0} m(\tau) = m(0)$, since the system has no time to evolve into another state. Hence, this continuity requirement yields a collapse of the quantum state onto the eigenspace of the measured value $m(0)$ and, therefore, other measurement outcomes have a zero probability for $\tau \to 0$.

In quantum optics the description of the photoelectric detection process is well established, cf. [144–146]. The resulting photoelectric counting statistics, including non-unit efficiencies, can be interpreted as a quantum version of a Poisson statistics for coherent light; cf. also [147, 148]. Other states, for example photons, might exhibit a photoelectric counting statistics which has a variance below the classical shot-noise limit of the Poisson distribution [33]. Hence, nonclassical intensity correlation of a quantum light sources can be studied. However, a full characterization of the quantum state of light also requires phase-sensitive measurement.

For the characterization of field correlations, a number of phase-sensitive detection schemes have been exploited [4]. They are typically performed in terms of interferometric setups, such as homodyne detection (HD). These measurement configurations are based on unitary transformations of radiation fields [149] followed by photoelectric detection. In the limit of a strong reference signal (local oscillator), the discrete photoelectric counting within balanced HD converges to the continuous quadrature distribution of the field [150, 151]. Other detection schemes are based on unbalanced HD and weak local oscillators [152, 153], or they employ methods being suited to directly infer higher-order correlations [154, 155].

Beside a study of the measured quadrature distribution itself [156], tomographic methods allow the reconstruction of the prepared quantum state [157]. Similarly, other features of the radiation field, such as the phase-space representations or correlation functions, may be retrieved via sampling methods [158–163]. State reconstruction techniques in combination with HD setups are standard tools for inferring classical and quantum properties of light; see [164, 165] for introductions and [61, 166] for pioneering experimental realizations.

As an alternative to conventional detectors, which are described via photoelectric detection theory, novel photon counters have been realized which are designed to operate in the single photon regime; cf. the reviews [167, 168]. Some of those single-photon-resolving detectors are based on superconducting materials and, thus, require cryogenic cooling [169–171]. Charge-coupled devices – being an array of pixels – present another approach for single photon detection, which additionally allow a spatial resolution to infer correlations between individual pixels [172–180].

A third example of single-photon detectors consists of a number of avalanche photodiodes, which operate in Geiger mode. This yields a binary information denoted as “no click” or “click” event stating that no photons or at least one photon has been absorbed, respectively. If a light field is split into parts of equal intensities and each of the resulting beams is measured with such a diode, we get a certain probability distribution for the number of coincident clicks. Realizations of such detector configurations are, for example, so-called fiber-loop (or time-bin) detectors, spatial-multiplexing detectors, or equally illuminated array detectors. Since those detectors operate in the few photon regime, they are nowadays frequently used to study properties of quantum light or for the implementation of quantum information protocols [181–194].

Finally, it is worth mentioning that methods from single-photon detection on the one hand and homodyne detection on the other have been jointly applied to experimentally verify fundamental commutation relations [195].

1.2. Methods

In this section we address some of the applied methods which led to the results which are presented in these thesis. This includes the general definition of classical (mixed) reference states, the construction of quantumness witnesses, a brief discussion on statistical significances (applied in joint works with experimental collaborators), and the quantification in terms of quantum superpositions. References to my publications, where these techniques have been elaborated, are given.

1.2.1. Classical references in quantum systems

Before we start with some prominent examples of notions of quantumness, let us consider a quite general and abstract approach. Let us assume we have a particular set of pure states, $|c\rangle \in \mathcal{C}_{\rm pure}$, which are classical with respect to a given property. Then, classical statistical mixing would yield a mixed classical state,

$$\hat{\rho}_{\text{cl.}} = \int_{\mathcal{C}_{\text{pure}}} \mathrm{d}P_{\text{cl.}}(c)\, |c\rangle\langle c|, \tag{1.1}$$

with $P_{\text{cl.}}$ being a classical probability distribution. This means that the correlations with respect to the property under study are described by classical statistics only. Whenever a quantum state $\hat{\varrho}$ cannot be written in the form (1.1), we refer to it as an quantum correlated state. All classical states of the structure in Eq. (1.1) define the convex set $\mathcal{C}$ of pure – for point-like classical distributions $P_{\text{cl.}}(c) = \delta(c - c_0)$ – and mixed classical states. Quantum states which cannot be represented in this way ($\hat{\varrho} \notin \mathcal{C}$) are quantum correlated.

A first example is defined by N-mode coherent states of harmonic oscillators, $|\boldsymbol{\alpha}\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_N\rangle$, which gives rise to the Glauber-Sudarshan representation [14, 15],

$$\hat{\rho} = \int \mathrm{d}^{2N}\boldsymbol{\alpha}\, P(\boldsymbol{\alpha})|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|, \tag{1.2}$$

together with the definition of a classical state as $P = P_{\text{cl.}}$ [16, 17]. A second example is given in terms of fully separable states, $|a_1, \ldots, a_N\rangle = |a_1\rangle \otimes \cdots \otimes |a_N\rangle$, which yields mixed separable states [18] as[3]

$$\hat{\rho} = \int \mathrm{d}P_{\text{cl.}}(a_1, \ldots, a_N)\, |a_1, \ldots, a_N\rangle\langle a_1, \ldots, a_N|. \tag{1.3}$$

If a state cannot be written in this form, this refers to as an (at least partially) entangled state.

Note that in both examples it is possible to formally write any state in terms of classical pure states, if we allow P to be a pseudo-distribution. For entanglement this has been demonstrated in [196, 197] and further optimized in [ii]. This means that P might be a signed and normalized measure including some negativities in the sense of distributions. The advantage is that a distribution which cannot be considered as a classical one, $P \neq P_{\text{cl.}}$, directly identifies nonclassical states in terms of negativities. However, such a quasiprobability distribution not necessarily exists for other quantum properties which could be studied.

The application of quasiprobabilities in terms of nonclassicality filtering (being introduced in [69]) for quantum correlated light field can be found in [VII, XXI]. Optimized entanglement quasiprobabilities of dephased two-mode squeezed state are studied in [II].

1.2.2. Witnessing and nonlinear eigenvalue problems

A typical approach to certify the quantumness of a certain state is given in terms of measurable witness operators $\hat{W}$. These kinds of observables have a non-negative expectation value for classical states, $\langle\hat{W}\rangle = \mathrm{tr}(\hat{W}\hat{\rho}_{\text{cl.}}) \geq 0$ for all $\hat{\rho}_{\text{cl.}} \in \mathcal{C}$. Additionally, they may exhibit a negative mean value for nonclassical states, i.e. there exists $\hat{\varrho} \notin \mathcal{C}$ with $\langle\hat{W}\rangle = \mathrm{tr}(\hat{W}\hat{\varrho}) < 0$. The existence of such witnesses is mathematically guaranteed, which is mainly due to the convexity of $\mathcal{C}$ and the Hahn-Banach separation theorem; cf. [198] (6th ed., pp. 102). Moreover, the construction of such observables from general Hermitian operators $\hat{L}$ can be always expressed as

$$\hat{W} = \lambda_{\max}\hat{1} - \hat{L} \text{ or } \hat{W} = \hat{L} - \lambda_{\min}\hat{1}, \tag{1.4}$$

[3] Note that in finite dimensional systems a sum, instead of the integral, in Eq. (1.3) is sufficient due to Carathéodory's theorem, see also [iii] in this context.

using the least upper bound, λ_{max}, or the largest lower bound, λ_{min}, of classical expectation values,

$$\lambda_{\text{max}} = \sup_{0\neq|c\rangle\in\mathcal{C}_{\text{pure}}} \frac{\langle c|\hat{L}|c\rangle}{\langle c|c\rangle} \quad \text{or} \quad \lambda_{\text{min}} = \inf_{0\neq|c\rangle\in\mathcal{C}_{\text{pure}}} \frac{\langle c|\hat{L}|c\rangle}{\langle c|c\rangle}, \tag{1.5}$$

respectively.[4] It is also worth mentioning that the definition of $\mathcal{C}$ as the convex span of elements $|c\rangle\langle c|$ with $|c\rangle \in \mathcal{C}_{\text{pure}}$ ensures that the defined bounds $\lambda_{\text{min}\,/\,\text{max}}$ are also the bounds to mixed classical states.

The bounds (1.5) can be obtained from the optimization of the Rayleigh quotient,

$$g(c) = \tfrac{\langle c|\hat{L}|c\rangle}{\langle c|c\rangle} \to \text{optimum}. \tag{1.6}$$

If the set $\mathcal{C}_{\text{pure}}$ is continuous in some sense – e.g., it has a smooth parametrization – this optimization problem may be rewritten as $\nabla_c g(c) = 0$. After some straight forward algebra, this optimality condition may be further remodeled as a generalized, nonlinear eigenvalue problem:

$$L(c)x(c) = g(c)1(c)x(c), \tag{1.7}$$

using the notion $\nabla_c\langle c|\hat{M}|c\rangle = M(c)x(c)$. In order to understand this very formal processing, let us study one example.

One application of this technique was performed in [i] to construct bipartite entanglement witnesses. In this scenario, the classical pure states are $|c\rangle = |a\rangle \otimes |b\rangle$. Since this is already a proper parametrization the derivative has been decomposed as $\nabla_c = (\nabla_a, \nabla_b)$. Finally, Eq. (1.7) in this scenario is given by the two components

$$\hat{L}_b|a\rangle = g|a\rangle \quad \text{and} \quad \hat{L}_a|b\rangle = g|b\rangle, \tag{1.8}$$

with $\hat{L}_a = \mathrm{tr}_A(\hat{L}[|a\rangle\langle a| \otimes \hat{1}_B])$ and $\hat{L}_b = \mathrm{tr}_B(\hat{L}[\hat{1}_A \otimes |b\rangle\langle b|])$ for normalized vectors $\langle a|a\rangle = \langle b|b\rangle = 1$. The Eqs. (1.8) refer to as *(bipartite) separability eigenvalue equations*. The bounds $\lambda_{\text{min}\,/\,\text{max}}$ are consequently identical to the minimal/maximal separability eigenvalue g.

Example. *Let us consider an operator that is related to the Clauser-Horne-Shimony-Holt inequality [82] being the quantum interpretation of the Bell inequality [6]. Say* $\hat{\sigma}_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* $\hat{\sigma}_y = \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$. *We define*

$$\hat{L} = \frac{1}{2}\left(\hat{\sigma}_x \otimes \hat{\sigma}_x + \hat{\sigma}_y \otimes \hat{\sigma}_y\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.9}$$

The non-zero (standard) eigenvalues to this operator are ± 1, i.e. $-1 \leq \langle\hat{L}\rangle \leq 1$ for arbitrary states. The bounds are attained for $|\psi^{\pm}\rangle = 2^{-1/2}(0, 1, \pm 1, 0)$.

In order to find the bounds for separable states, we compute the reduced operator with an arbitrary state of the first subsystem $|a\rangle = (a_0, a_1)^{\mathrm{T}}$, with $|a_0|^2 + |a_1|^2 = 1$. That is

$$\hat{L}_a = \frac{1}{2}\langle a|\hat{\sigma}_x|a\rangle\hat{\sigma}_x + \frac{1}{2}\langle a|\hat{\sigma}_y|a\rangle\hat{\sigma}_y = \begin{pmatrix} 0 & a_0 a_1^* \\ a_0^* a_1 & 0 \end{pmatrix}. \tag{1.10}$$

[4] The denominator $\langle c|c\rangle$ in (1.5) is one, if all classical pure states are normalized.

A trivial separability eigenvalue $g = 0$ results from $a_0^* a_1 = 0$. If $a_0^* a_1 \neq 0$, we find the eigenvectors $|b\rangle = 2^{-1/2}(1, \pm \frac{a_0^* a_1}{|a_0^* a_1|})^{\mathrm{T}}$ to the eigenvalues $g = \pm |a_0^* a_1|$. Computing now the reduced operator for this solution,

$$\hat{L}_b = \frac{\pm 1}{2|a_0^* a_1|} \begin{pmatrix} 0 & a_0 a_1^* \\ a_0^* a_1 & 0 \end{pmatrix}, \tag{1.11}$$

we get the eigenvectors $2^{-1/2}(1, \pm' \frac{a_0^* a_1}{|a_0^* a_1|})^{\mathrm{T}}$, which is supposed to be identical to $|a\rangle$ for fulfilling the separability eigenvalue equations (1.8). *Equating coefficients yield $|a_0| = |a_1| = 2^{-1/2}$. Thus, we get the nontrivial separability eigenvalues $g = \pm |a_0^* a_1| = \pm 1/2$. Finally, we conclude that for all separable states holds $-1/2 \leq \langle \hat{L} \rangle_{\text{sep.}} \leq 1/2$, or*

$$|\langle \hat{L} \rangle_{\text{sep.}}| \leq 1/2. \tag{1.12}$$

For the operator (1.9) *we solve the coupled set of separability eigenvalue equations* (1.8) *for the operator. Whenever the absolute of the expectation value of this observable exceeds the bound $1/2$, entanglement is verified.*

The general approach in form of Eq. (1.7) allows the construction of necessary and sufficient conditions to identify the quantumness of any state. Beside this fact, there are also some deficiencies which should be addressed. First, the Eq. (1.7) is hard to solve. For non-trivial sets $\mathcal{C}_{\text{pure}}$, it has at least the same computational complexity as the standard eigenvalue problem [199]. Second, there exists and uncountable number of possible observables $\hat{L}$. Up to a certain point, it is not clear which witness is best suited for detecting the nonclassicality of a particular state. However, this approach turned out to be very fruitful to construct novel probes of different notions of quantumness in complex systems.

The systematic treatment in the discussed form is elaborated for entanglement and nonclassicality of harmonic oscillators in my references [i, v, VIII, XI, XVI, XVIII, XIX, XXIV].

1.2.3. Treatment of error bars

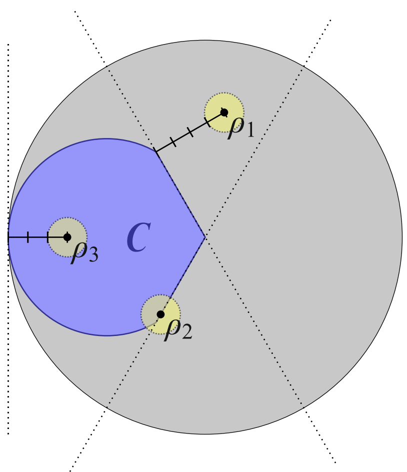


Figure 1.1.: Separation of states $\hat{\rho}_i$ ($i = 1, 2, 3$) from a convex set $\mathcal{C}$.

So far we studied the theoretical modeling of classical reference states in quantum mechanical systems in terms of a convex set $\mathcal{C}$. An analysis followed which aimed at the construction of witnesses for certifying quantum correlations using nonlinear eigenvalue equations. If such methods are applied to experimental data, then a treatment of error bars is indispensable.

In order to outline the influence of experimental errors, let us study the oversimplified example in Fig. 1.1. The blue area corresponds to the closed convex set of pure and mixed classical states $\mathcal{C}$. The gray area (excluding the blue one) is populated by the remaining class of quantum correlated states. The experimentally reconstructed states $\hat{\rho}_i$ ($i = 1, 2, 3$) are represented by the black bullets. The yellow circles around the states depicts the error – one standard deviation – which is supposed to represent the experimental uncertainty. The dashed tangential lines depict some optimal witnesses $\hat{W}_i$ which separate the classical set, $\langle\hat{W}_i\rangle \geq 0$, from a subset of the nonclassical domain, $\langle\hat{W}_i\rangle < 0$. In our case the normal vector of the separating hyperplanes $\langle\hat{W}_i\rangle = 0$ yields the minimal distance of $\hat{\rho}_i$ to the boundary of $\mathcal{C}$ in units of standard deviations. The states $\hat{\rho}_1$ and $\hat{\rho}_3$ are significantly nonclassical and classical, respectively. The state $\hat{\rho}_2$ lies on the boundary of the set of classical states $\mathcal{C}$. Therefore, it cannot be assigned with certainty to either set.

The experimentally obtained value of a witness may be given as $\langle\hat{W}\rangle = \overline{W} \pm \sigma(W)$, where $\sigma(W)$ corresponds to the standard error of the mean value $\overline{W}$. Thus the *significance*,

$$\Sigma = \frac{\overline{W}}{\sigma(W)}, \tag{1.13}$$

quantifies the experimentally verified quantumness. For a witness $\hat{W}$, the classical expectation is $\Sigma \geq 0$ for all elements of $\mathcal{C}$, i.e., the classical bound is $\Sigma = 0$. Hence, $\Sigma < 0$ is not consistent with a classical description. It implies the quantumness of the measured system which is determined with a significance $|\Sigma|$.

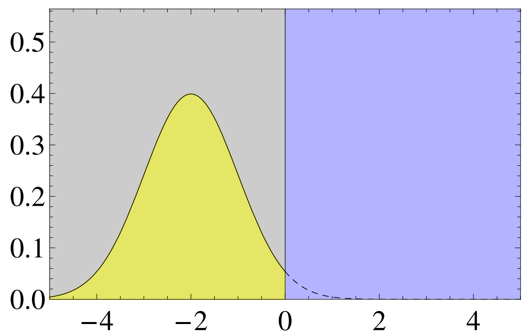


Figure 1.2.: Gaussian error distribution. Classical domain blue; quantum domain gray.

In Fig. 1.2 an example is shown for $\overline{W} = -2$ and $\sigma(W) = 1$. Here, the *confidence* (yellow area) is 97.7% to be in the quantum domain – assuming a Gaussian error distribution with the mean $\overline{W}$ and the variance $\sigma(W)$. More generally we get the confidence $C(\Sigma)$ to have a quantum correlated state, $\hat{\varrho} \notin \mathcal{C}$, for a Gaussian error distribution as

$$C(\Sigma) = \int_{-\infty}^{0} \mathrm{d}w \frac{\exp\left[-\frac{\left(w-\overline{W}\right)^2}{2\sigma(W)^2}\right]}{\sqrt{2\pi\sigma(W)^2}} = \frac{1}{2}\left(1 + \operatorname{erf}\left[\frac{-\Sigma}{\sqrt{2}}\right]\right), \tag{1.14}$$

using the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}z\, \exp[-z^2]$. For example, we have $C(-3) = 99.865\%$ for $\overline{W} = -3\sigma(W)$, and $C(-5) = 99.99997\%$ for $\overline{W} = -5\sigma(W)$. Again, a state at the classical bound, $\overline{W} = 0$, cannot be experimentally assigned to either class of states, $C(0) = 50\%$.

For an introduction to more general sampling error analysis we refer to Ref. [200]. The presented significance analysis has been applied in [XVII, XIX, XXI, XXIII].

1.2.4. Quantification in terms of superpositions

Once a quantum correlation is verified, it might occur the question: How strong is this quantum effect? For this reason entanglement measures [137–139] and nonclassicality quantifiers [67, 201–207] have been introduced. The typically considered measures are based on topological distances, having a clear geometric interpretation, or entropic quantities, such as mutual information or relative entropies; see [19] for a review.

However, we studied an algebraic approach for the quantification of quantum correlations, since the convexity of the set of classical states $\mathcal{C}$ is an algebraic notion. Therefore our approach is formulated in terms of convex ordering [XXV]. We have shown that the number r of superposition of classical pure states,

$$|\psi_r\rangle = \sum_{k=1}^{r} \lambda_k |c_k\rangle, \text{ with } |c_k\rangle \in \mathcal{C}_{\text{pure}}, \tag{1.15}$$

is a proper quantifier of the quantumness under study in this algebraic respect. Here r is supposed to be the minimal number which allows this expansion. Note that this also requires that the closure of the linear span of $\mathcal{C}_{\text{pure}}$ is the full Hilbert space, so that the representation in (1.15) is possible for any state. This is fulfilled for coherent states and separable ones. A convex roof construction allows the extension of this quantification of pure states to mixed states. This yields convex nested sets,

$$\mathcal{C}_r \subset \mathcal{C}_{r+1}, \tag{1.16}$$

which are mixtures of the pure states $|\psi_r\rangle$ and $|\psi_{r+1}\rangle$, respectively; cf. Eq. (1.15). In case of separability, the number r in Eq. (1.15) is generalized to the Schmidt number of bipartite or multipartite mixed states [5, 135, 136].

Moreover the convexity of $\mathcal{C}_r$ allows a witnessing approach as it was studied before for $\mathcal{C} = \mathcal{C}_1$. This, additionally, makes the algebraic approach a measurable one. That this further generalizes the nonlinear eigenvalue problem will be outlined in the following. Here, we use a *spinor* notion to represent superpositions of classical states.

From Dirac or Pauli equation, the notion of spinors is known. A pure quantum state $|\boldsymbol{\psi}\rangle$ is described by a complex vector of states:

$$|\boldsymbol{\phi}\rangle = \begin{pmatrix} |\phi_1\rangle \\ \vdots \\ |\phi_r\rangle \end{pmatrix} = \sum_{k=1}^{r} |\phi_k\rangle \otimes \boldsymbol{e}_k \in \underbrace{\mathcal{H} \times \cdots \times \mathcal{H}}_{r\text{-times}} = \mathcal{H}^r = \mathcal{H} \otimes \mathbb{C}^r, \tag{1.17}$$

where $\{\boldsymbol{e}_k\}_{k=1}^r$ is supposed to be the standard basis of $\mathbb{C}^r$. Say $\boldsymbol{s} = r^{-1/2} \sum_{k=1}^r \boldsymbol{e}_k$. We can define a projector, $\hat{\boldsymbol{P}} = \hat{\boldsymbol{P}}^2$, as

$$\hat{\boldsymbol{P}} = \hat{1} \otimes \boldsymbol{s}\boldsymbol{s}^\dagger : \mathcal{H} \otimes \mathbb{C}^r \to \mathcal{H} \otimes \mathbb{C}^r, \text{ with } \hat{\boldsymbol{P}}|\boldsymbol{\phi}\rangle = \frac{1}{\sqrt{r}} \left(\sum_{k=1}^{r} |\phi_k\rangle \right) \otimes \boldsymbol{s}. \tag{1.18}$$

Applying this projection to a "classical" spinor, i.e. $|\phi_k\rangle = \lambda_k |c_k\rangle$, we get $\hat{\boldsymbol{P}}|\boldsymbol{\psi}\rangle = r^{-1/2}|\psi_r\rangle \otimes \boldsymbol{s}$ with the superposition state in Eq. (1.15). Moreover the normalization of $|\psi_r\rangle$ can be rewritten as $\langle\psi_r|\psi_r\rangle = r\langle\boldsymbol{\phi}|\hat{\boldsymbol{P}}|\boldsymbol{\phi}\rangle$. Finally the bounds – leading to nonlinear eigenvalue equations – in Eq. (1.5), can be obtained from the optimization:

$$\frac{\langle\psi_r|\hat{L}|\psi_r\rangle}{\langle\psi_r|\psi_r\rangle} = \frac{\langle\boldsymbol{\phi}|\hat{\boldsymbol{L}}|\boldsymbol{\phi}\rangle}{\langle\boldsymbol{\phi}|\hat{\boldsymbol{P}}|\boldsymbol{\phi}\rangle}, \to \text{min/max}, \text{ with } \hat{\boldsymbol{L}} = \hat{L} \otimes \boldsymbol{s}\boldsymbol{s}^\dagger. \tag{1.19}$$

This rewriting of superpositions into a spinor notion allows us to derive nonlinear eigenvalue equations in the exactly same way as it has been done for $r = 1$. Consequently, the maximal or minimal nonlinear eigenvalues are the lower and upper bounds of the expectation value of an observable $\hat{L}$ for states in the set $\mathcal{C}_r$.

The algebraic quantification approach together with the construction of witnesses can be found in my publications [VI, XIV, XVI] for the nonclassicality of harmonic oscillators and, for entanglement, in [vi, XXV, XIV, XVIII].

1.2.5. Click counting and binomial distributions

It has been shown that detectors, which are based on click counting are described by a quantum version of the binomial statistics [I]. Hence, a brief reminder of some techniques employing such distributions is reasonable. A binomial statistics is given by

$$c_k = \binom{N}{k} b^k (1-b)^{N-k}, \text{ with } 0 \leq k \leq N \text{ and } 0 \leq b \leq 1. \tag{1.20}$$

The parameter b might be a random variable as well which is represented by a classical probability distribution $P_{\mathrm{cl.}}$. This yields the convolved binomial statistics,

$$c_k = \int_0^1 \mathrm{d}P_{\mathrm{cl.}}(b) \binom{N}{k} b^k (1-b)^{N-k} = \left\langle \binom{N}{k} b^k (1-b)^{N-k} \right\rangle . \tag{1.21}$$

The mean value of this statistics is $\overline{k} = \sum_{k=0}^{N} k c_k = N\langle b\rangle$. The variance may be expressed in two ways together with the variance of $P_{\mathrm{cl.}}$. That is

$$\sigma^2(k) {=} \overline{k^2} - \overline{k}^2 {=} N\langle b(1-b)\rangle + N^2\left\langle(\Delta b)^2\right\rangle {=} N\langle b\rangle\langle 1-b\rangle + N(N-1)\left\langle(\Delta b)^2\right\rangle , \tag{1.22}$$

where $\Delta b = b - \langle b\rangle$. Note that the case $\langle(\Delta b)^2\rangle = 0$ results in the well-known expression for the binomial statistics without any extra convolution. A useful tool to infer general moments is employing the generating function,

$$g(z) = \sum_{k=0}^{N} c_k z^k = \left\langle (bz + [1-b])^N \right\rangle . \tag{1.23}$$

This is due to the fact that the mth derivative of the generating function, $\partial_z^m g(z) = \sum_{k=m}^{N} c_k \frac{k!}{(k-m)!} z^{k-m} = \left\langle \frac{N!}{(N-m)!} b^m \left(bz + [1-b]\right)^{N-m} \right\rangle$ for $0 \leq m \leq N$, leads to

$$\begin{aligned} \partial_z^m g(z)\big|_{z=0} &= m! c_m = \frac{N!}{(N-m)!}\left\langle b^m (1-b)^{N-m}\right\rangle \\ \partial_z^m g(z)\big|_{z=1} &= \overline{k(k-1)\cdots(k-m+1)} = \frac{N!}{(N-m)!}\langle b^m\rangle , \end{aligned} \tag{1.24}$$

being – up to a scaling – the statistics itself and the moments of the random variable b.

The presented techniques have been applied in my references [I, III, X, XIII, XX, XXIII] to study click counting detectors in optical measurement setups.

1.3. Outline

In the remainder of this cumulative habilitation thesis, I will elucidate my research from July 2011 (PhD) until today. The so far discussed techniques represent the general framework how the individual results have been obtained. Therefore, only the conclusions of the publications will be discussed. The three main sections focus on the topics:

- characterization and verification of quantum correlations, chapter 2;
- the quantification of nonclassicality and entanglement, chapter 3;
- and measurement strategies using so-called click counters, chapter 4.

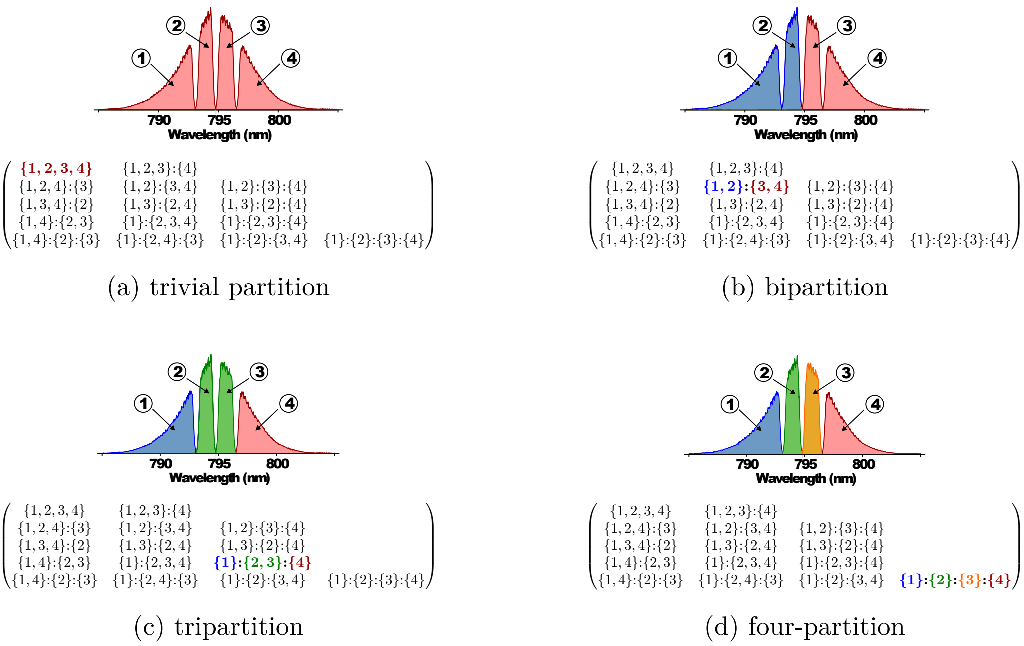
Beside specific summaries and outlooks at the end of each chapter, final concluding remarks are given in chapter 5. Copies of the published articles and available preprints – for the period under consideration – can be found in the appendix part (pp. 61).

2. Verification of correlations

In this chapter we will review several methods to certify quantum correlations in multipartite systems. This includes the detection on entanglement and nonclassicality of harmonic oscillators. The application to experiments is presented.

2.1. Witnessing multipartite entanglement

Let us discuss in more details a publication which I consider to be one of my key publications: *Witnessing multipartite entanglement* [VIII]. This will be done in combination with the experimental application of the method in Ref. [XIX], which has been performed in collaboration with the group of N. Treps and C. Fabre (École Normale Supérieure).



(a) trivial partition

(b) bipartition

(c) tripartition

(d) four-partition

Figure 2.1.: Some of the possible spectral mode decompositions of a given four-partite system. See also [XIX].

For this reason, we briefly recall the notion entanglement for a multimode system. We assume for simplicity a N-partite system given by the labels $\mathcal{I} = \{1, \cdots, N\}$. A K-partition $\mathcal{I}_1{:}\cdots{:}\mathcal{I}_K$ is a disjoint decomposition of the index set $\mathcal{I}$ into K non-empty subsets $\mathcal{I}_k$. In Fig. 2.1, an example is outlined for $N = 4$ spectral modes.[1] A pure state is separable with respect to a given K-partition, if it can be written as a tensor product of K states, each being defined in the subsystem $\mathcal{I}_k$ with $k = 1, \cdots, K$, i.e.:

$$|a_1, \cdots, a_K\rangle \text{ is separable with respect to the partition } \mathcal{I}_1{:}\cdots{:}\mathcal{I}_K. \tag{2.1}$$

[1] Special thanks to S. Gerke (Universität Rostock) and J. Roslund (Université Pierre & Marie Curie) for the figures.

If a state cannot be written as a classical mixture of these product states, then this state is referred to as entangled with respect to the given partition $\mathcal{I}_1{:}\cdots{:}\mathcal{I}_K$. The quite frequently discussed notion of a K-entangled system means that a state cannot be written as a statistical mixture of product states in *all* possible K partitions; see [18, 19]. Let us additionally mention that the trivial partition, $\mathcal{I}_1 = \mathcal{I}$, implies that all states could be formally considered as "separable" ones regarding this trivial partitioning.

As we outlined earlier, see also [88], an optimal witness for a given partition $\mathcal{I}_1{:}\cdots{:}\mathcal{I}_K$ can be written as

$$\hat{W} = \underbrace{\left[\sup_{|a_1,\cdots,a_K\rangle\neq 0} \frac{\langle a_1,\cdots,a_K|\hat{L}|a_1,\cdots,a_K\rangle}{\langle a_1,\cdots,a_K|a_1,\cdots,a_K\rangle}\right]}_{=g^{\max}_{\mathcal{I}_1:\cdots:\mathcal{I}_K}} \hat{1} - \hat{L}, \tag{2.2}$$

or, similarly, as

$$\hat{W} = \hat{L} - \underbrace{\left[\inf_{|a_1,\cdots,a_K\rangle\neq 0} \frac{\langle a_1,\cdots,a_K|\hat{L}|a_1,\cdots,a_K\rangle}{\langle a_1,\cdots,a_K|a_1,\cdots,a_K\rangle}\right]}_{=g^{\min}_{\mathcal{I}_1:\cdots:\mathcal{I}_K}} \hat{1}, \tag{2.3}$$

using a general Hermitian operator $\hat{L}$. As derived in [VIII], the optimization procedure for obtaining the bounds $g^{\max/\min}_{\mathcal{I}_1:\cdots:\mathcal{I}_K}$ yields the separability eigenvalue equations

$$\boxed{\begin{aligned} \hat{L}_{a_2,\cdots,a_K}|a_1\rangle &= g|a_1\rangle \\ &\vdots \\ \hat{L}_{a_1,\cdots,a_{K-1}}|a_K\rangle &= g|a_K\rangle, \end{aligned}} \tag{2.4}$$

with the reduced operators $\hat{L}_{a_1,\cdots,a_{k-1},a_{k+1},\cdots,a_K}$ being defined through

$$\begin{aligned} &\langle u_k|\hat{L}_{a_1,\cdots,a_{k-1},a_{k+1},\cdots,a_K}|v_k\rangle \\ =&\langle a_1,\cdots,a_{k-1},u_k,a_{k+1},\cdots,a_K|\hat{L}|a_1,\cdots,a_{k-1},v_k,a_{k+1},\cdots,a_K\rangle, \end{aligned} \tag{2.5}$$

for any $|u_k\rangle, |v_k\rangle$ in the subsystem which is represented by $\mathcal{I}_k$. The solution of the coupled system of eigenvalue equations (2.4) is given by product vectors $|a_1,\cdots,a_K\rangle$ and the real value g being the separability eigenvector and the separability eigenvalue, respectively. The bounds $g^{\max/\min}_{\mathcal{I}_1:\cdots:\mathcal{I}_K}$ for the witness construction are finally described as the maximal/minimal separability eigenvalue g. It is also worth mentioning that the standard eigenvalue problem is retrieved for the trivial partition $\mathcal{I}_1 = \mathcal{I}$. Interesting features of these equations are studied in [VIII].

The application of this approach to witness multimode Gaussian states has been demonstrated in [XIX]. In particular, multimode frequency comb states have been experimentally realized. The analytical solution $g^{\min}_{\mathcal{I}_1:\cdots:\mathcal{I}_K}$ of the separability eigenvalue equations for covariance based operators,

$$\hat{L} = \sum_{i,j=1}^{N} \left(M^{ji}_{xx}\hat{x}_i\hat{x}_j + M^{ji}_{px}\hat{p}_i\hat{x}_j + M^{ji}_{xp}\hat{x}_i\hat{p}_j + M^{ji}_{pp}\hat{p}_i\hat{p}_j\right), \tag{2.6}$$

has been derived. A numerical optimization over all the resulting witnesses, $\hat{W} = \hat{L} - g^{\min}_{\mathcal{I}_1:\cdots:\mathcal{I}_K}\hat{1}$, has been performed to get the most significant signature of entanglement.

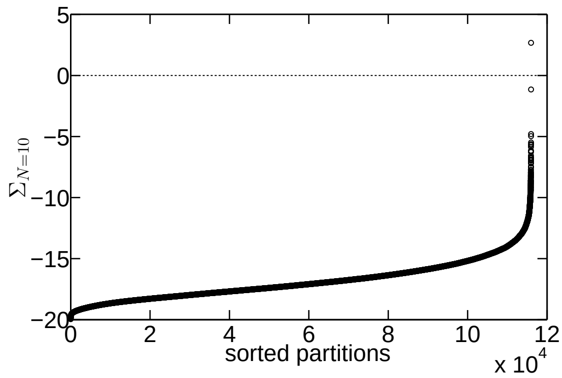


Figure 2.2.: Entanglement verification of a ten-mode frequency comb Gaussian state [XIX].

For example, the highest number of modes was $N = 10$. For this case we have 115,975 partitions. The previously described method has been applied to probe entanglement for each – except the trivial – partition. It can be seen in figure 2.2 that entanglement is certified for all non-trivial partitions, $\Sigma < 0$. To my best knowledge, such an analysis of a full multipartite entanglement has never been performed before. For the particular example only entanglement of all 511 bipartitions has been demonstrated before [208].

The method of separability eigenvalues rendered it possible to investigate the entanglement correlation properties in such a highly complex system. Although this method itself is a necessary and sufficient one, the solutions to the coupled system of separability eigenvalue equations are in general unknown. Presently we are studying a numerical iteration to find the separability eigenvalues for the construction of general – i.e. non-Gaussian – entanglement witnesses.

2.2. Multipartite Quasiprobabilities

Earlier, we discussed the regularized P function approach [69], which has been formulated for a single radiation mode and applied to experimentally realized quantum states of light, e.g., [70,71]. This method allows to apply certain filters to remove all kinds of singularities in the Glauber-Sudarshan P function without affecting the (non)classical character of the state. In [VII], we showed that this approach is also possible for multimode radiation fields. Surprisingly this can be done with a product of regularizing single-mode filters, and we can still verify quantum correlations between subsystems.

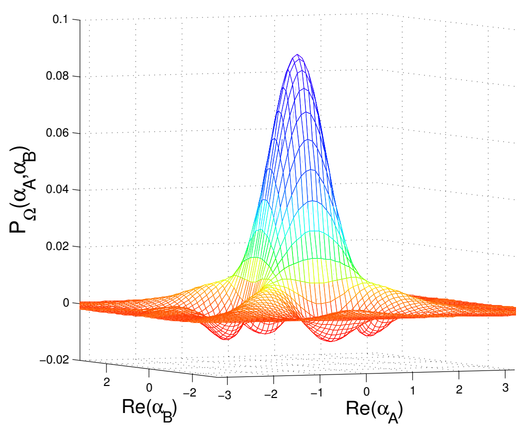


Figure 2.3.: Regularized, quantum correlated $P_\Omega(\alpha_A, \alpha_B)$ function [VII].

In order to prove this, we considered a state of the class introduced in [209],

$$\hat{\rho} = \sum_{n=0}^{\infty} (1-p)p^n |n,n\rangle\langle n,n|, \qquad (2.7)$$

with $0 < p < 1$ which can be prepared in labs. Nonclassical correlations of this fully phase-randomized two-mode squeezed-vacuum state can be directly inferred from the two-mode regularized phase-space distribution P_Ω, see Fig. 2.3. This is remarkable, because the state is classical with respect to the following notions: $\hat{\rho}$ is separable; $\hat{\rho}$ has a non-negative (classical) Wigner function; $\hat{\rho}$ has classical marginal states, i.e. $\mathrm{tr}_A(\hat{\rho})$ and $\mathrm{tr}_B(\hat{\rho})$ are classical, thermal states; and $\hat{\rho}$ is a zero-discord state – see [210,211] for the definition of this notion of correlation. Hence the multimode regularization is helpful to identify quantum correlations which would remain undetected otherwise.

In collaboration with the experimental group of B. Hage (Universität Rostock), we developed a proper multimode sampling method together with a continuous phase measurement [XXI]. Such a continuous phase measurement allows extrapolations of phase-space representation beyond the discrete phase-lock configuration, cf. 2.4. As a proof of principle we demonstrated the application to a single-mode squeezed-vacuum state, which already proves the application in multimode systems due to the product regularization approach mentioned above. The singularities of the squeezed state's P function disappeared and regular negative contributions of this phase-space representation certify the nonclassical character of such a prepared light field; see Fig. 2.4.

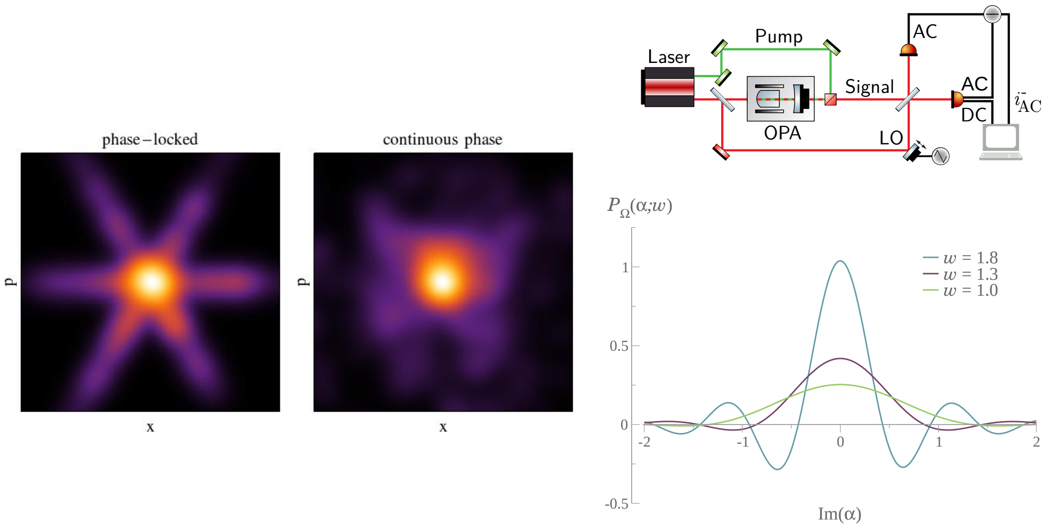


Figure 2.4.: The extrapolation of phase-space points (assuming Gaussian error model) for phase-locked (left) and random phase (center) measurements – both for 300 simulated data points. The lighter the color the better the quality of the performed extrapolation. It can be seen that the black areas of a poor extrapolation are directed in case of a phase-lock setup. Nonclassical features in this region cannot be properly reconstructed. Right: The realized continuous-phase measurement (top) and the reconstructed filtered quasiprobability (bottom) are shown for different filter parameters w, cf. [XXI].

2.3. Applications and other correlations

Beyond the multimode characterizations of entanglement and nonclassicality of radiation fields, we also applied our methods to other systems and notions of nonclassicality. Let us briefly discuss the work in this direction. A summarizing proceeding can be also found in [XV].

In collaboration with H. Fehske (Ernst-Moritz-Arndt-Universität Greifswald), the construction of entanglement witnesses has been also used to identify multipartite entangled light emitted from microcavities [IX]. The entanglement within this semiconductor structure is in a multipartite W-type configuration [213]. The emitted photons translate the information about the internal entangled polaritons into entanglement of a multimode radiation field. Hence, the characterization of the outgoing light field can be used as a probe for internal quantum correlations.

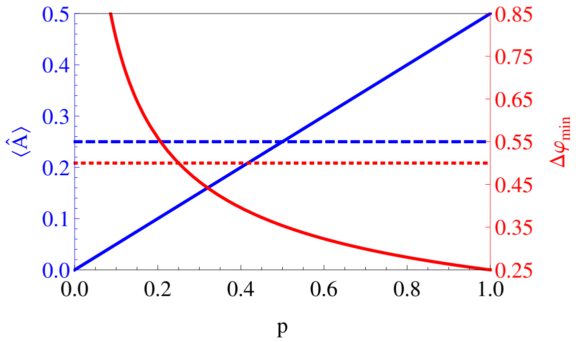


Figure 2.5.: Quantum properties of a N00N state mixed with vacuum [XIX].

Other entanglement aspects of propagating light fields were demonstrated in [XXII]. Here, the influence of atmospheric turbulences [214–216] to the entanglement of so-called N00N states [217]

$$|N00N\rangle = \frac{|0,N\rangle + |N,0\rangle}{\sqrt{2}}, \tag{2.8}$$

has been investigated. This is done in connection with super phase-resolution [218, 219], which is a quantum feature that allows the estimation of a parameter (here, the phase φ) beyond classical noise limitations. In Fig. 2.5, the phase resolution (red) is plotted together with the derived entanglement criteria (blue) for a mixture of a N00N state with vacuum (p is the probability to be in the vacuum state). As long as the expectation value of an observable $\hat{A}$ is above the blue dashed line entanglement is certified. Similarly, a phase estimate $\Delta\varphi_{\min}$ below the red dashed line implies super phase resolution.

Combining the quasiprobability approach with the notion of entanglement, optimized entanglement quasiprobabilities have been introduced in [ii]. Such a quasiprobability distribution is strictly non-negative for separable states and has negative contributions for entangled ones. In the article [II], we studied the entanglement properties in this form for the example of a two-mode squeezed state undergoing a dephasing process; cf. [51,212] for related experiments. Our optimized entanglement quasiprobabilities could identify entanglement of this state even for a significant dephasing. Note that a full dephasing yields the separable state in Eq. (2.7).

In a recent work, we also studied entanglement probes for systems of indistinguishable particles [129, 130]. This is of a fundamental interest, because the symmetrization requirements for systems of Boson and Fermion has formally the same structure as implied by entanglement of distinguishable degrees of freedom [128]. For this reason separability eigenvalue equations for indistinguishable particles have been derived and spin-statistics independent entanglement witnesses have been formulated. In Fig 2.6, we compare the entanglement of distinguishable particles and indistinguishable ones.[2]

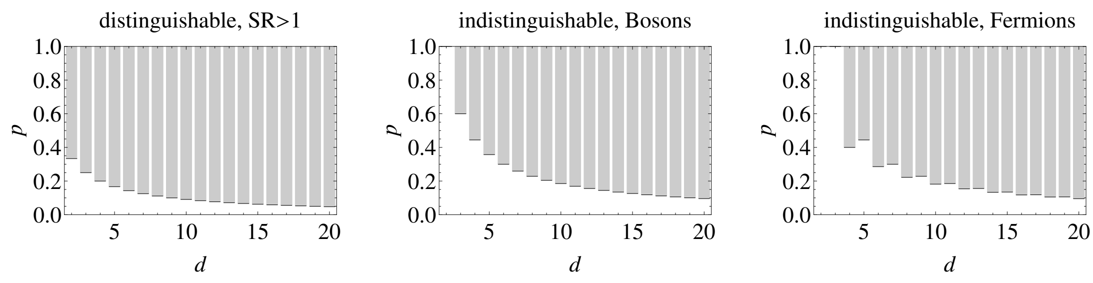


Figure 2.6.: Entanglement of mixed states, $\hat{\rho} = p|\psi\rangle\langle\psi| + (1-p)\hat{\mathbb{I}}/\mathrm{tr}\hat{\mathbb{I}}$ is verified (gray areas) for different Hilbert space dimension d. Here, $\hat{\mathbb{I}}$ differentiates between two distinguishable particles ($|\psi\rangle = d^{-1/2}\sum_{n=1}^{d}|n\rangle\otimes|n\rangle$; plot "SR>1"), two Bosons, ($|\psi\rangle = d^{-1/2}\sum_{n=1}^{d}|n\rangle\vee|n\rangle$), and two Fermions ($|\psi\rangle = (\lfloor d/2\rfloor)^{-1/2}\sum_{n=1}^{\lfloor d/2\rfloor}|2n\rangle\wedge|2n+1\rangle$).

[2] We us in Fig. 2.6 the standard notions for the symmetric tensor product, $|a\rangle\vee|b\rangle \cong |a\rangle\otimes|b\rangle + |b\rangle\otimes|a\rangle$, and the skew-symmetric tensor product, $|a\rangle\wedge|b\rangle \cong |a\rangle\otimes|b\rangle - |b\rangle\otimes|a\rangle$.

The last two items in this section are devoted to the identification of bound entanglement [XII] and quantum discord [IV]. The latter one is a collaboration with A. Miranowicz (Adam Mickiewicz University) and P. Horodecki (Technical University of Gdańsk; National Quantum Information Centre of Gdańsk). We could derive new methods to infer bounds to the quantum discord [210, 211] in two qubit systems. In the joint theoretical work [XII] with M. C. de Oliveira (Universidade Estadual de Campinas), we constructed a new type of bipartite bound entangled states which can be generated with state of the art optical devices and processes. The realization of the earlier known states was an unsolved problem for the bipartite case [123], or it required multimode scenarios [121, 122, 125, 127].

2.4. Summary and Outlook

In summary, we studied a number of methods to infer quantum correlations within multipartite systems. In particular the novel technique of separability eigenvalue equations has been studied including its application to experiments. We demonstrated that complex entanglement correlations can be identified in many physical systems, such as semiconductor structures, atmospheric channels, multiple Bosons or Fermions, and frequency comb laser systems. We also observed that multimode nonclassicality of harmonic oscillators are visualized in terms of negative probabilities being regularized phase-space distributions. Even types of correlations which cannot be observed with some other methods had been successfully identified. Continuous phase sampling allow the reconstruction of such quasiprobabilities without additional extrapolation methods as required for phase-locked measurements.

As we mentioned earlier, we are currently implementing an algorithm which can provide numerical solutions of the separability eigenvalue problem. This might yield a way to infer non-Gaussian types of multipartite entanglement. Moreover, a regularization of generalized P distributions for space-time-dependent correlations is planned; cf. [63]. In this context, we also aim at regularizing a joint system of one radiation mode coupled to a discrete variable system. Such an approach corresponds the Wigner function matrix representation in [220]. Currently this work – in collaboration with M. Bellini (Istituto Nazionale di Ottica, Firenze) – is in preparation.

This chapter has been devoted to the identification of quantum correlations. In the next chapter, we will study the quantification of such correlations.

3. Unified and universal quantification

The generation and application of quantum correlated systems also requires the determination of the strength of a quantum feature. From the information science point of view, information based quantifiers are useful, e.g., distance based measures are preferable. It was shown that such approaches can yield an ambiguous quantification [XXV, 141, 142]. Therefore we studied algebraic measures which are based on the quantum superposition principle. In this chapter a summary of this direction of research is given.

3.1. Degrees of nonclassicality and entanglement

The two notions of quantumness, entanglement and nonclassicality of harmonic oscillators, will be quantified in terms of Schmidt number and the degree of nonclassicality, respectively. The algebraic quantification also leads to a method relating both quantum aspects on an unified basis [XIV].

3.1.1. Entanglement quantification

One quantifier of bipartite entanglement is the Schmidt number (SN) [5, 135, 221]. This quantifier is shown to be universal, i.e., it has some more involved properties than other entanglement metrics [vi]. The construction of SN witnesses using generalized eigenvalue equations has been introduced in [v]. In a spinor representation, they read as

$$\begin{pmatrix} \hat{L}_{b_1;b_1} & \cdots & \hat{L}_{b_1;b_r} \\ \vdots & \ddots & \vdots \\ \hat{L}_{b_r;b_1} & \cdots & \hat{L}_{b_r;b_r} \end{pmatrix} \begin{pmatrix} |a_1\rangle \\ \vdots \\ |a_r\rangle \end{pmatrix} = g \begin{pmatrix} \hat{1}_{b_1;b_1} & \cdots & \hat{1}_{b_1;b_r} \\ \vdots & \ddots & \vdots \\ \hat{1}_{b_r;b_1} & \cdots & \hat{1}_{b_r;b_r} \end{pmatrix} \begin{pmatrix} |a_1\rangle \\ \vdots \\ |a_r\rangle \end{pmatrix}$$
$$\text{and} \quad \begin{pmatrix} \hat{L}_{a_1;a_1} & \cdots & \hat{L}_{a_1;a_r} \\ \vdots & \ddots & \vdots \\ \hat{L}_{a_r;a_1} & \cdots & \hat{L}_{a_r;a_r} \end{pmatrix} \begin{pmatrix} |b_1\rangle \\ \vdots \\ |b_r\rangle \end{pmatrix} = g \begin{pmatrix} \hat{1}_{a_1;a_1} & \cdots & \hat{1}_{a_1;a_r} \\ \vdots & \ddots & \vdots \\ \hat{1}_{a_r;a_1} & \cdots & \hat{1}_{a_r;a_r} \end{pmatrix} \begin{pmatrix} |b_1\rangle \\ \vdots \\ |b_r\rangle \end{pmatrix}, \tag{3.1}$$

with $\hat{X}_{a_k;a_l} = \mathrm{tr}_A(\hat{L}[|a_l\rangle\langle a_k| \otimes \hat{1}_B])$ and $\hat{X}_{b_k;b_l} = \mathrm{tr}_B(\hat{L}[\hat{1}_A \otimes |b_l\rangle\langle b_k|])$ for $\hat{X} = \hat{L}, \hat{1}$.

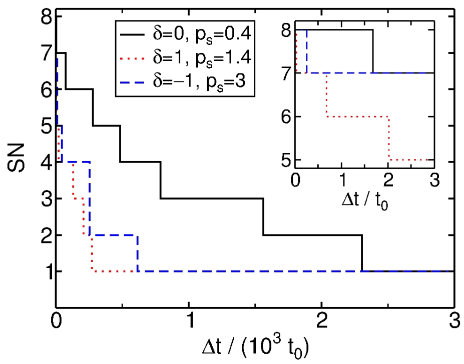


Figure 3.1.: Light from semiconductor systems [V].

In order to study the amount of entanglement from planar microcavities, this approach has been applied in collaboration with the theory group of H. Fehske (Ernst-Moritz-Arndt-Universität), see [V] and Fig. 3.1. Here, this semiconductor structure is driven with 3 pump beams resulting in emitted light which is entangled. The light may propagates in different media yielding a delay time Δt due to different optical path lengths. This dephasing induces a decay of the strength of quantum correlations, that can be directly observed in Fig. 3.1 for different system parameters.

SN witnesses allow an identification of the number of nonlocal superpositions, which has been done in collaboration with the group of S. Pádua (Universidade Federal de Minas Gerais) [XVII]. See also Fig. 3.2 for the experimental implementation. Here correlated photon pairs are produced by a spontaneous parametric down conversion (SPDC) and coincidences of such path entangled light fields are recorded. The path can be inferred using a slit aperture and spatial light modulator (SLM). The characterization of the generated states' SN has been done by using witnesses.

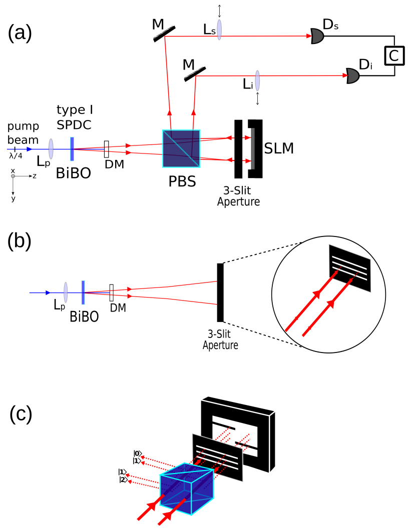


Figure 3.2.: Measurement of path entangled light [XVII].

Parallel to the determination of the SN for discrete variable systems, entanglement in continuous variables has been quantified. For this reason covariance based SN criteria have been established to quantify entanglement of Gaussian states [XI]. Other successfully applied Gaussian entanglement probes [106, 107] could only identify the presence of entanglement itself. We also considered the influence of attenuations to the amount of Gaussian entanglement.

Beyond the bipartite case, the multipartite SN is the natural extension of the bipartite one [136]. Together with the partitioning of modes this yields the notion of a *structural quantifier of entanglement* [XVIII]. Let us briefly outline the meaning of this notion. We may consider an example of an N-partite quantum system for $N = 4$. Two possible partitionings are

$$\boldsymbol{P}_2 = \{1\} : \{2,3,4\} \quad \text{and} \quad \boldsymbol{P}_3 = \{1\} : \{2,3\} : \{4\}, \tag{3.2}$$

where $\boldsymbol{P}_n$ is a n-partition. Since $\boldsymbol{P}_3$ can be considered as further splitting of $\boldsymbol{P}_2$, this 3-partition refers to as a *refinement* of the given 2-partition. With respect to each partition, one can characterize the multipartite SN r. This yields convex, nested sets $\boldsymbol{S}_{\boldsymbol{P}_n,r}$. The pure states in this set are quantum states which are a superposition of not more than r separable states with respect to the n-partition $\boldsymbol{P}_n$.

Mixed states may be obtained by a convex roof construction of those pure ones. In Fig. 3.3 the inclusion of such sets for $r > r'$ and refinements $\boldsymbol{P}'_{n'}$ of $\boldsymbol{P}_n$ are shown. The formulation of witnesses, which are capable of detecting whether a studied state is within $\boldsymbol{S}_{\boldsymbol{P}_n,r}$ or not, has been derived in [XVIII]. In this case the construction yields a quite involved set of separability eigenvalue equations; cf. Eq. (3.1) for the bipartite case. However, it has been demonstrated for some examples how they can be solved. With this method, we have been able to formulate novel entanglement criteria which allow the simultaneous determination of the entanglement structure – given by the partitioning – together with the quantification of the identified entanglement – in terms of superpositions (multipartite SN).

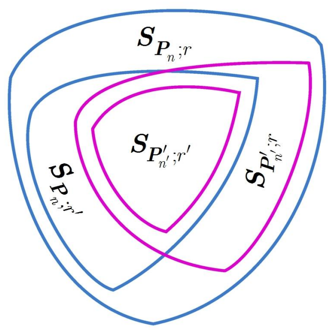


Figure 3.3.: Inclusion of structures of entangled states [XVIII].

3.1.2. Nonclassicality quantification

In the article [VI], we introduced in a first step operational measures for the nonclassicality of harmonic oscillators. These quantifiers are related to certain measurement setups. In particular, the role of noise-free measurements has been outlined. Consequently, this quantification approach characterizes a subset of nonclassical states which are useful in a specific experimental configuration.

In a second step we studied the algebraic quantification of nonclassicality. For this purpose we introduced an axiomatic approach for defining such measures. This is mainly adapted from the entanglement approach [137–139] to systems of harmonic oscillators whose classical reference are coherent states rather than separable ones. The axiomatic approach also requires an analysis of classical operations. Those maps – e.g. beam splitters, phase shifts, and displacement operations – cannot generate nonclassical light from any coherent input field. Examples of a nonclassical processes are squeezing transformations – e.g. in an optical parametric oscillator – or the photon addition protocol. The latter one renders it possible to generate nonclassical light from a classical thermal input state [74].

One example of such an algebraic measure, which fulfills the established axioms is the degree of nonclassicality, which counts the number r of superposition of coherent states. The witnessing approach has been investigated together with the group of B. Hage (Universität Rostock). The construction of such witnesses from Hermitian operators $\hat{K}$ has been done in the same way as it was performed in the case of the SN for entanglement quantification. The solution of the corresponding nonlinear eigenvalue equations led to upper bounds b_r. They limit the expectation value of $\hat{K}$ for r-classical states $\mathcal{C}_r$, cf. Fig. 3.4. Whenever this bound is exceeded, the state under study has a degree of nonclassicality larger than r – i.e. more than r superpositions of coherent states are required. The right part of Fig. 3.4 shows the example of a squeezed state projector,

$$\hat{K} = |\xi\rangle\langle\xi|, \text{ with } |\xi\rangle = (\cosh[\xi])^{-1/2} e^{-\frac{1}{2}\tanh[\xi]\hat{a}^{\dagger 2}}|0\rangle. \tag{3.3}$$

If the fidelity $\langle\xi|\hat{\rho}|\xi\rangle$ is larger than b_r, then the state $\hat{\rho}$ has a nonclassicality larger than r.

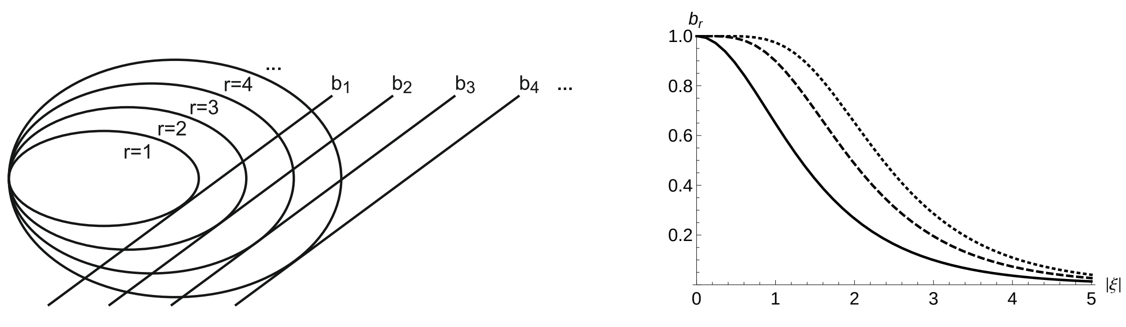


Figure 3.4.: Left: Embedded sets $\mathcal{C}_r$ of states with a degree of nonclassicality of r are shown. A separating hyperplane (witness) are depicted by the individual bounds b_r. Right: The dependence of these bound b_r for a squeezed state projection test, $\hat{K} = |\xi\rangle\langle\xi|$, is plotted in dependence of the squeezing parameter $\xi \in [0, \infty[$. The solid(dashed,dotted) curves correspond to bounds for the degree of nonclassicality $r = 1(2, 3)$. See also [XVI].

3.2. Relation between the quantification of nonclassicality and entanglement

As we discussed above, both approaches – the quantification of nonclassicality and the quantification of entanglement – can be done via counting quantum superpositions. Naturally the question arises whether there is a connection of both. An early attempt in this direction has been done in [222]. There, it has been shown that a nonclassical input state is required as the input of a beam splitter, cf. Fig. 3.5, in order to obtain entanglement at the output.

In our work [XIV], we showed that there is a deeper connection between the nonclassicality of the signal (SI) input state and the two-mode entangled output. Namely, the degree of nonclassicality of the input state is identical to the SN of the output state. As an example, let us consider an input state in the form of an even coherent state [223],

$$|\psi\rangle_{\mathrm{in}} = \frac{|\alpha\rangle + |-\alpha\rangle}{\sqrt{2(1+\exp[-2|\alpha|^2])}} \otimes |0\rangle. \tag{3.4}$$

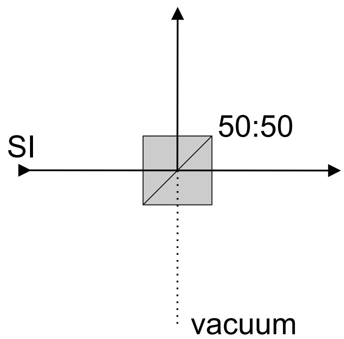


Figure 3.5.: Setup in [XIV].

Hence the output state of the configuration in Fig. 3.5 is

$$|\psi\rangle_{\mathrm{out}} = \frac{|\alpha'\rangle \otimes |\alpha'\rangle + |-\alpha'\rangle \otimes |-\alpha'\rangle}{\sqrt{2(1+\exp[-4|\alpha'|^2])}}, \text{ for } \alpha' = \frac{\alpha}{\sqrt{2}}, \tag{3.5}$$

which has a SN of two since $|\alpha'\rangle$ and $|-\alpha'\rangle$ are linearly independent. A similar relation is shown to be valid for the multipartite case [XIV].

Hence, a nonclassical single mode light source can be used to generate entanglement of the same amount as the degree of nonclassicality of the input field. This shows the connection between nonclassical light and entangled radiation fields through the quantum superposition principle. Here, it is also worth mentioning that localized emitters have similar properties [224]. Whenever the emitted light is nonclassical in one direction, then there always exists entanglement between emitted light modes propagating in different directions.

3.3. Summary and Outlook

In this chapter we studied the universal quantification of quantum correlations in terms of superpositions of classical states. For the notion of nonclassicality of harmonic oscillators, we introduced the degree of nonclassicality which corresponds to the number of superimposed coherent states. For multipartite entanglement this gave rise to the structural quantification of entanglement in terms of mode decompositions and the multipartite Schmidt number. A relation between these two notions has been derived using only linear optical elements – such as beam splitters. Witnesses to detect certain number of quantum superpositions have been constructed for both cases – nonclassicality and entanglement. Applications to path-entangled light and entangled light, which is emitted from semiconductor systems, outlined the general applicability of our technique.

For future applications, it would be also interesting to investigate the quantification of quantum processes. This means a determination of the amount of nonclassicality, that can be obtained in a given setup, has to be quantified. For example, the nonclassicality of protocols such as photon-addition should be inferred; cf. [74] for a related implementation. This information about the strength of quantumness of such an operation would be crucial for application of quantum effects and studies of noisy environments.

Moreover, there is a fundamental mathematical relation between the SN of multipartite pure states and the rank of multi-linear maps. From the fundamental point of view, an analysis of this relations has to be performed which might be done by using nonlinear eigenvalue problems. Any insight in this direction – e.g. a spectral theorem for multi-linear maps – would automatically increase the knowledge about entanglement.

4. Click measurement of radiation fields

Measurement theory is a cornerstone of quantum optics. A comprehensive understanding of a performed measurement is indispensable for uncovering quantum properties of generated light fields. Typical setups employ detectors which are described by the photoelectric detection theory. In the single photon domain, however, avalanche photodiodes (APDs) in Geiger mode play a crucial role. Such diodes produce a "click" whenever light is detected and remain silent otherwise, i.e. "no-click". If an incident light field is split into multiple fields with equal intensities then coincidences of multiple APDs may be recorded, see figure 4.1. Examples of such detection schemes are array detectors being equally illuminated or (time-bin) multiplexing detectors; see, e.g., [181, 182, 184–190].

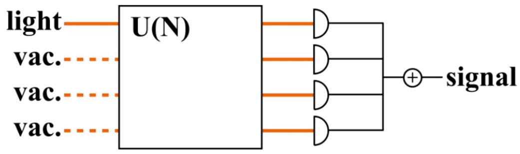


Figure 4.1.: An incident light field is split into $N = 4$ output fields with equal intensities. The same number of APDs detect each of these fields yielding a joint number of coincidence clicks k. The probability to measure k clicks is described by the click counting statistics c_k [I].

In close collaboration with G. S. Agarwal (Oklahoma State University), we have shown that the photoelectric counting theory and measurements with multiple APDs are incompatible. Therefore, we derived the click counting theory which renders it possible to describe the correct statistics of click counting devices. Based on this method, we have been able to establish new criteria to verify quantum correlations [III, X, XIII, XX]. In this chapter we aim at summarizing our research of click counting devices.

4.1. Photoelectric counting versus click counting

Firstly let us recapitulate the photoelectric detection theory, cf. [4]. The probability to measure n photoelectric counts, likewise n photons for a perfect quantum efficiency $\eta = 1$, is described by

$$p_n = \left\langle :\frac{(\eta\hat{n})^n}{n!}\mathrm{e}^{-\eta\hat{n}}: \right\rangle, \tag{4.1}$$

with $\hat{n}$ being the photon number operator and $:\cdot:$ being the normal ordering prescription; cf., e.g., [145]. As an example let us consider a coherent state $|\alpha\rangle$. In this case we get a Poisson (shot noise) statistics,

$$p_n = \frac{\lambda^n}{n!}\mathrm{e}^{-\lambda}, \text{ with } \lambda = \eta|\alpha|^2. \tag{4.2}$$

To probe the Poissonian character of this statistics, the Mandel parameter [33] has been introduced

$$Q_{\mathrm{M}} = \frac{(\Delta n)^2}{\bar{n}} - 1, \tag{4.3}$$

with $\bar{n}$ and $(\Delta n)^2$ being the mean value and the variance of the p_n statistics, respectively. If p_n describes a Poisson statistics, we have $Q_{\mathrm{M}} = 0$ (e.g, for a coherent state). If classical correlations broaden the shot noise statistics, we have a super-Poissonian radiation field, $Q_{\mathrm{M}} > 0$. Most interestingly, sub-Poissonian light, $Q_{\mathrm{M}} < 0$, is a clear signature of the quantum character of the measured system – in the notion of nonclassicality of harmonic oscillators.

As mentioned earlier, Eq. (4.1) is not the kind of statistics which describes the measurement in Fig. 4.1. In Ref. [I] the actual click counting statistics has been derived. We get the probability c_k for k clicks as

$$c_k = \left\langle : \binom{N}{k} \left(\mathrm{e}^{-\frac{\eta}{N}\hat{n}}\right)^{N-k} \left(\hat{1} - \mathrm{e}^{-\frac{\eta}{N}\hat{n}}\right)^{k} : \right\rangle, \tag{4.4}$$

where k is an integer between 0 and N and η is the efficiency of each APD. Again, the example of a coherent state reveals that this is the quantum version of a binomial statistics,

$$c_k = \binom{N}{k} (1-p)^{N-k} p^k, \text{ with } p = 1 - \mathrm{e}^{-\eta|\alpha|^2/N}. \tag{4.5}$$

Analogously to the Mandel parameter, we introduced the binomial parameter [III]

$$Q_{\mathrm{B}} = N \frac{(\Delta k)^2}{\overline{k}\left(N - \overline{k}\right)} - 1, \tag{4.6}$$

with $\overline{k}$ and $(\Delta k)^2$ being the mean value and the variance of the c_k statistics, respectively. Here a binomial statistics yields $Q_{\mathrm{B}} = 0$. A general classical state fulfills $Q_{\mathrm{B}} \geq 0$; whereas sub-binomial light, $Q_{\mathrm{B}} < 0$, certifies the quantum nature of the measured field. Shortly after introducing the binomial parameter, Bartley *et al.* directly observed sub-binomial light [225]; see also [226] for further studies.

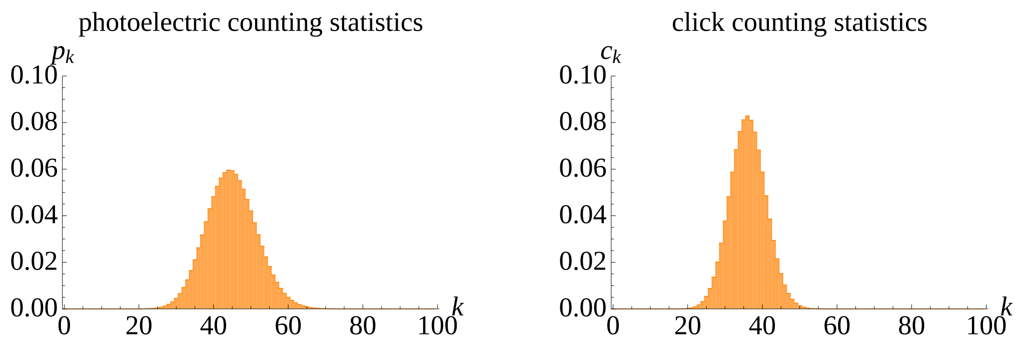


Figure 4.2.: The photoelectric and the click statistics of a coherent state with a mean photon number $|\alpha|^2 = 64$ is shown for a quantum efficiency $\eta = 60\%$. We assume $N = 100$ APDs for the click statistics.

In Fig. 4.2, we compare the properties of photoelectric counting and click counting measurements. Since a coherent field is depicted, we have $Q_{\mathrm{M}} = Q_{\mathrm{B}} = 0$. It can be seen that the click counting distribution is narrower than the Poisson one. If the Mandel parameter (4.3) is applied to the click statistics (4.5), we would get a fake negativity $Q_\mathrm{M} < 0$. Therefore, it is very important to employ the correct measurement description for determining quantum properties of states.

Moreover, let us comment that in the case of a diverging number of APDs, $N \to \infty$, the click counting distribution converges to the Poissonian one. However this convergence is quite slow, i.e., of the order $\sim 1/N$ [I]. On the other hand, for any finite number of APDs, the number of clicks k is also finite, whereas the photoelectric detection is described by $n \in \mathbb{N}$. Thus, an inversion of the click counting statistics to the photon number distribution is impossible. Therefore we derived a number of nonclassicality probes to directly employ the click counting theory, which is additionally much more convenient for experimentalists.

4.2. Correlation measurements

So far we discussed the click counting statistics c_k and the binomial parameter Q_B, cf. Eqs. (4.4) and (4.6), respectively. The latter parameter is based on the mean value and the variance of the click distribution, i.e., up to second order moments. In Ref. [X], we considered more general nonclassicality criteria. We consequently studied higher order correlations for the verification of quantum effects. Additionally, we formulated criteria for identifying quantum correlations between multiple click detector systems and described non-linear detection models of APDs.

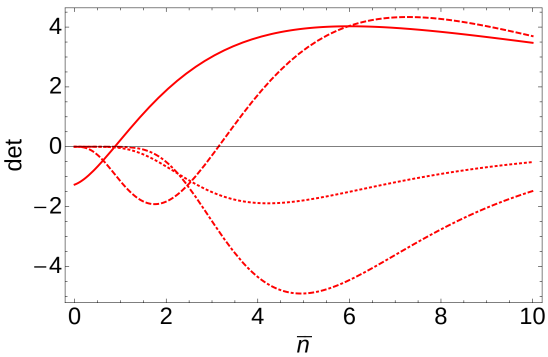


Figure 4.3.: Higher order click correlations of a SPATS, cf. [X].

In Fig. 4.3, the application of higher order moments is shown. A single photon added thermal state (SPATS) [47], being characterized by its mean thermal photon number $\bar{n}$, is measured with a click counting device consisting of $N = 8$ diodes and a quantum efficiency $\eta = 90\%$. The matrix of moments is defined as

$$\begin{aligned} M &= (\langle{:}\hat{\pi}^{m+m'}{:}\rangle)_{m,m'}, \\ \text{with } \hat{\pi} &= \hat{1} - {:}\mathrm{e}^{-\eta\hat{n}/N}{:}. \end{aligned} \tag{4.7}$$

Whenever a minor is negative, $\det M < 0$, we have successfully identified the nonclassicality of the state. The higher correlations in Fig. 4.3 are addressed through the determinant of the matrix of moments up to second (solid), forth (dashed), sixth (dotted), and eighth (dot-dashed) order.

It is equally interesting to study correlations between different click counters. In Fig. 4.4 a correlation measurement setup with two multiplexing detectors is outlined – for each detection system holds $N = 4$ and $\eta = 80\%$. The source is assumed to produce correlated photon pairs, e.g., through a down-conversion of a (not shown) pump beam. The resulting state is a two-mode squeezed-vacuum state, which is characterized by the squeezing parameter $|\xi| \in [0, 1[$. The matrix of bimodal moments is an adequate method to identify correlations between them. For example, the second order moment condition for classical light reads as

$$0 \leq \det \begin{pmatrix} \langle{:}\hat{\pi}_1^0\hat{\pi}_2^0{:}\rangle & \langle{:}\hat{\pi}_1^1\hat{\pi}_2^0{:}\rangle & \langle{:}\hat{\pi}_1^0\hat{\pi}_2^1{:}\rangle \\ \langle{:}\hat{\pi}_1^1\hat{\pi}_2^0{:}\rangle & \langle{:}\hat{\pi}_1^2\hat{\pi}_2^0{:}\rangle & \langle{:}\hat{\pi}_1^1\hat{\pi}_2^1{:}\rangle \\ \langle{:}\hat{\pi}_1^0\hat{\pi}_2^1{:}\rangle & \langle{:}\hat{\pi}_1^1\hat{\pi}_2^1{:}\rangle & \langle{:}\hat{\pi}_1^0\hat{\pi}_2^2{:}\rangle \end{pmatrix} = \langle{:}(\Delta\hat{\pi}_1)^2{:}\rangle\langle{:}(\Delta\hat{\pi}_2)^2{:}\rangle - \langle{:}(\Delta\hat{\pi}_1)(\Delta\hat{\pi}_2){:}\rangle^2, \tag{4.8}$$

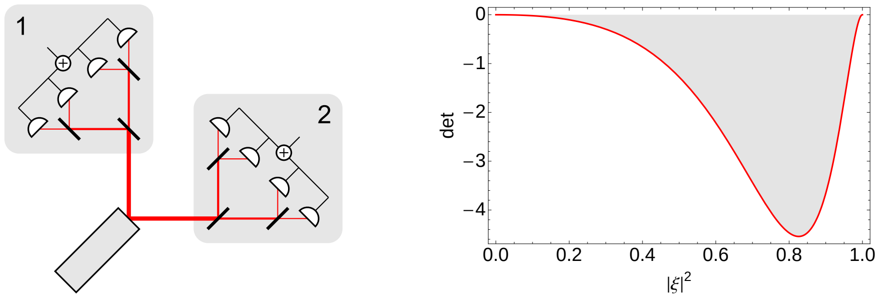


Figure 4.4.: Left plot: Correlation measurement setup using two multiplexing detectors. Right plot: Quantum correlations are shown as a function of the parameter ξ for two-mode squeezed-vacuum state, $|\xi\rangle = \sum_{n=0}^{\infty}(1-|\xi|^2)^{1/2}\xi^n|n,n\rangle$ [X].

with $\hat{\pi}_{1(2)}$ being defined for the detector 1(2) as given in Eq. (4.7). The graph in Fig. 4.4 visualizes these cross correlations for the considered state. Again a negative value corresponds to a verification of nonclassicality. Here it is important to note that the two-mode squeezed-vacuum is classical if a single mode is considered only, i.e., tracing over the other mode. Hence, if we only rely on the measurements of a single detector system, we cannot observe the quantum character of the state. Therefore, the shown negativities are authentic two-mode correlations.

Finally, let us study the detector responses beyond the linear regime. Here, this means that the "no-click" event of a single APD is no longer given by the expectation value $\langle{:}\exp[-\eta\hat{n}]{:}\rangle$; see also [227] in this context. In general the exponent could be an arbitrary function Γ of the photon number $\hat{n}$. We get for a single diode the probability $p_{\mathrm{on(off)}}$ to click (not to click) as

$$p_{\mathrm{on}} = 1 - \langle{:}\exp\left[-\Gamma(\hat{n})\right]{:}\rangle \text{ and } p_{\mathrm{off}} = \langle{:}\exp\left[-\Gamma(\hat{n})\right]{:}\rangle\,. \tag{4.9}$$

Although the particular form in (4.4) in no longer valid, we showed that the binomial character of the click statistics for multiple APDs is preserved [X]. Hence all the methods are equally applicable. In Fig. 4.5, some examples are given.

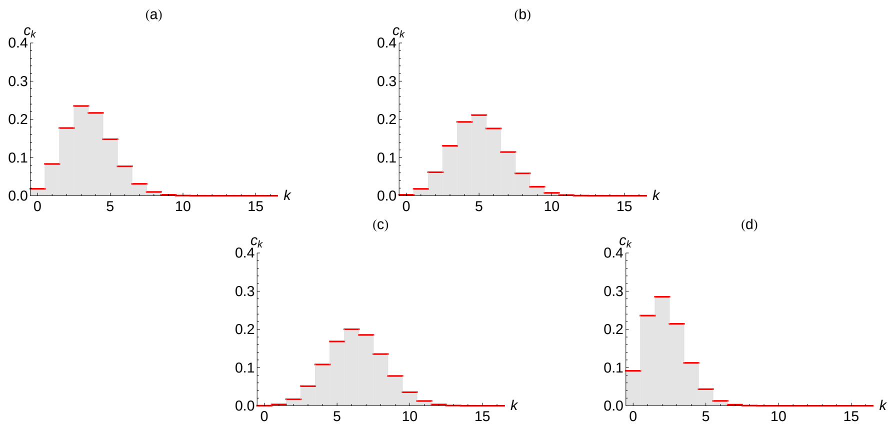


Figure 4.5.: The binomial statistics of a coherent state $|\alpha|^2 = 4$ for $N = 16$ APDs [X]. The response is (a) linear $\Gamma(x) = x$, (b) affine $\Gamma(x) = x + 2$, corresponding to a dark count rate, (c) quadratic $\Gamma(x) = x + x^2/4$, and (d) logarithmic $\Gamma(x) = x - \ln[1 + x]$, corresponding to a two-photon absorption process [228].

4.3. Phase sensitive measurements

So far the considered measurements schemes could not detect phase sensitive quantum features. In order to overcome this deficiency, we formulated the theory of balanced homodyne detection (BHD) using click counters [XX]. In Fig. 4.6, we outline this setup. The difference click statistics is shown in figure 4.7 (left) for a coherent state signal with a mean photon $|\alpha|^2 = 4$, which is also the local oscillator (LO) intensity r^2.

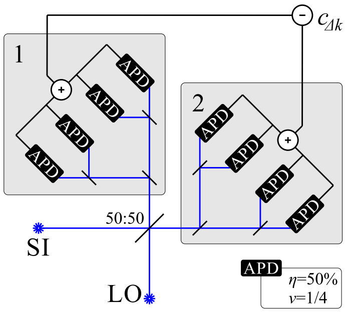


Figure 4.6.: BHD setup with click measurements [XX].

The most important modification in comparison to the BHD employing photoelectric detection theory is that the difference statistic does not yield the quadrature distribution. Here, we have to consider a nonlinear quadrature operator,

$$\hat{X}(\varphi) = 2N \mathrm{e}^{-\frac{\eta r^2}{2N}} {:}\mathrm{e}^{-\frac{\eta \hat{n}}{2N}} \sinh\left[\frac{\eta r}{2N}\hat{x}(\varphi)\right]{:}, \quad (4.10)$$

being a function of the true quadrature operator $\hat{x}(\varphi) = \hat{a}\exp[-\mathrm{i}\varphi] + \hat{a}^\dagger \exp[\mathrm{i}\varphi]$. Again, we can formulate a matrix of normally ordered moments of the click quadrature operator $\hat{X}(\varphi)$. The second order minor of this matrix yields the corresponding nonlinear squeezing condition,

$$\left\langle {:}[\Delta\hat{X}(\varphi)]^2{:}\right\rangle < \left\langle {:}[\Delta\hat{X}(\varphi)]^2{:}\right\rangle_{\mathrm{vac}} = 0. \quad (4.11)$$

The application to a squeezed signal field – given by the squeezing parameter ξ – is plotted in Fig. 4.7 (right). It can be observed that the nonclassicality of the state is certified for all squeezing parameters. Moreover, this is done for a relatively low quantum efficiency, $\eta = 50\%$, and only $N = 4$ APDs per detector.

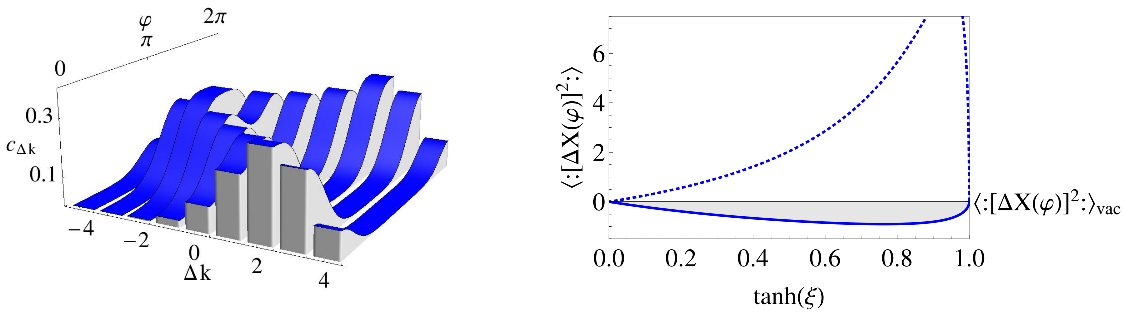


Figure 4.7.: Left: The phase sensitive difference click statistics of a coherent state is shown. Here, Δk is the difference of clicks from both detectors and φ is the LO phase, cf. Fig. 4.6. Right: The squeezed and anti-squeezed variances of the squeezed state $|\xi\rangle = (\cosh\xi)^{-1/2}\exp[-\tanh\xi\hat{a}^{\dagger 2}/2]|\mathrm{vac}\rangle$ are shown in dependence of the parameter $\xi \in [0,\infty[$. See also [XX].

Recently, we also studied an unbalanced homodyne detection scheme, cf. [152, 153], using click counters. This theoretical analysis was done in collaboration with A. Luis (University Complutense) [XXIII]. In such a case, generalized phase-space functions can be established, which are regular and can become negative for non-classical states.

4.4. Summary and Outlook

We have seen that click counters are a versatile detection system to infer quantum properties of radiation fields. The collaboration with G. S. Agarwal (Oklahoma State University) on this topic started in 2012. So far five papers have been published or submitted on different topics where click counters can be applied. Beside the above discussed approaches, we considered more applications, such as: verification of time dependent correlations [X] and state engineering protocols employing click counters [XIII].

Additionally we started a number of collaborations with experimental groups for applying our theory. This includes the groups

- H. Stolz and B. Hage (Universität Rostock, Rostock, Germany) using array detectors;
- I. A. Walmsley (University of Oxford, Oxford, UK) using multiplexing detectors;
- A. Szameit (Friedrich Schiller Universität, Jena, Germany) using waveguide based detectors;
- and Ch. Silberhorn (Universität Paderborn, Paderborn, Germany) using fiber loop detectors.

Here, the Oxford group already applied the $Q_{\rm B}$ parameter, for the first experimental demonstration of sub-binomial light [225]. Our click counting method is quite appealing to experimentalist, because directly accessible sampling formulas have been provided and a manifold of experimental imperfection have been taken into account.

Beyond a number of other straight forward extensions of the so-far conducted theoretical analysis, a main aim we have in mind is the detection of entanglement. This should be possible by employing our method of separability eigenvalue equations to phase sensitive click counting. Applications of such an approach would verify that our techniques are useful tools for quantum information technology.

5. Conclusions

5.1. Summary

We have shown how to infer quantum correlations, how they can be quantified, and how novel measurement schemes can be used to observe them. In the introduction, we reported on the state of the art and discussed the elaborated method in a general framework. The following chapters have been mainly concentrated on two aspects of quantumness: the nonclassicality of harmonic oscillators and entanglement between subsystems of compound quantum systems. Let us summarize the results we presented in this thesis.[1]

The definition of classical pure reference states led to the notion of classically correlated states via convex mixtures of those pure ones. For entanglement and nonclassicality of harmonic oscillators, this approach guided us to generalized quasiprobabilities which cannot be interpreted in terms of classical probability theory. These distributions allow the identification of quantum correlated states from negative values within these pseudo-probabilities; cf. [ii, II] for optimized quasiprobabilities of entanglement in finite dimensional subspaces [iii], and cf. E. Agudelo *et al.* [VII, XXI] for multimode regularized nonclassicality quasiprobabilities. It has been also shown in A. Luis *et al.* [XXIII] that nonclassical phase-space functions can be inferred from click counting measurements.

Witnesses – representing one form of linear correlation functions – are useful approach to identify quantum correlations. In this document, the construction of such witnesses has been formulated in terms of nonlinear eigenvalue equations. For entanglement, this led to the so-called *separability eigenvalue equations* [i, VIII]. Similar approaches have been discussed for witnessing the amount of bipartite entanglement [v], the amount of multipartite partitions F. Shahandeh *et al.* [XVIII], or the amount of nonclassicality in M. Mraz *et al.* [XVI]. In collaboration with G. S. Agarwal (Oklahoma State University) we considered correlations in terms of moments of the click counting statistics to infer nonclassical light [III, X, XX], which can be mapped to higher-order moment witnesses. Other correlations can be used to formulate bounds to the property quantum discord, cf. A. Miranowicz *et al.* [IV].

We have used the witnessing approaches to uncover quantum features in different physical systems or applications. For example, the theoretical study of entangled light, which is emitted from a semiconductor system, has been characterized by D. Pagel *et al.* [V, IX]. Moreover, the entanglement transfer in turbulent atmospheric channels can be studied on such a basis, cf. M. Bohmann *et al.* [XXII]. Covariance-based witnesses for detecting the amount of entanglement in Gaussian states have been introduced, F. Shahandeh *et al.* [XI]. In A. Reusch *et al.* [XXIV], entanglement within Boson and Fermion systems has been studied through the construction of witnesses for particular exchange symmetries.

The quantification of quantum correlations has been connected to the quantum superposition principle [XXV]. In this sense, the degree of nonclassicality has been introduced for quantifying the nonclassicality of harmonic oscillators in C. Gehrke *et al.* [VI]. Such a superposition based technique yields some universal properties of the corresponding en-

[1] The given references solely represent the author's contribution to the field. In chapter 1, the state of the art is presented in more detail. See also the proceedings [iv, XV].

tanglement measure [vi] – the Schmidt number. It also allows a unified quantification of entanglement and nonclassicality [XIV]. Additionally, the usefulness of operational measures for certain quantum applications, such as noise-free measurements or distillation protocols, has been outlined in [VI, vi].

The generation and measurement of quantum correlated states is another aspect which has been addressed. For example in the theoretical work F. E. S. Steinhoff *et al.* [XII] it has been shown how to generate bound entangled states. Path entangled states have been experimentally realized and characterized in A. J. Gutiérrez-Esparza *et al.* [XVII]. Frequency comb lasers have been demonstrated to be a versatile source of multimode entangled light, which has been analyzed in S. Gerke *et al.* [XIX] regarding all possible mode partitions. Using the true click counting statistics [I], the possibility to engineer new classes of nonclassical states has been exploited [XIII].

In summary, novel approaches have been presented which determine and classify entanglement in quantum optics and beyond. Multipartite systems of harmonic oscillators – such as multimode radiation fields – have been treated in terms of regular phase-space representations and superposition based measures of quantumness. A theoretical model for state of the art detector systems has been described and its capability to uncover quantum features has been elaborated.

The two examples of nonclassicality – in the notion of nonclassical Glauber-Sudarshan phase-space representations – and entanglement in multimode radiations fields already include a vast variety of quantum phenomena. The inspiring collaborations with experimental and theoretical partners led to a deeper understanding of these effects. Of course the results would not have been possible without the enduring hard work of all members of theoretical quantum optics groups in Rostock.

5.2. Concluding remarks

In this cumulative habilitation, I summarized my recent research in the field of theoretical quantum optics. Measuring correlations is of particular interest for demonstrating the general quantum character of nature. Multimode continuous variable systems being subjected to attenuations represent cumbersome scenarios for uncovering quantum features. I consider such realistic systems in my studies, because they serve as an optimal test bed to estimate the usefulness of novel methods that characterize quantum states.

We aimed at formulating unified concepts for achieving a progress in physics. For example, many aspects of the theory of entanglement have been unified in terms of separability eigenvalue equations. This includes quasiprobability representations, the structural quantification, and systems with different exchange symmetries. We were able to increased the knowledge of a system by surpassing limitations. For instance, the click detection theory describes an information incomplete measurement scenario. Still, a number of unknown nonclassicality probes have been formulated using diodes that can only discriminate between the presence or absence of absorbed photons.

Let us outline some of the research which I want to pursue in the future. Of course the mathematical framework of the nonlinear eigenvalue problems has to be increased in general. Further, we know that there is a relation between the equations which identify a property (e.g. entanglement through separability eigenvalue equations) and the equations that quantify the amount of quantumness (e.g. the multipartite Schmidt number). Putting this relation onto a firm foundation would lead to a deeper understanding of the quantum superpositions principle as a measure of the quantumness of physical systems. Also the question “which witness is the best for a given state” has to be addressed in the future.

The click counting theory follows the idea "what you see is what you get". This means that our methods can be directly applied to the experimentally obtained click-counting distribution without complex data processing. As mentioned earlier, we have some collaborations in progress applying the theory to different experimental setups. Additionally, It would be important to have a direct accessible entanglement probe based on the click statistics. This could be directly used by our experimental partners to observe entanglement within the existing detection settings.

Beside the open issues mentioned so far, let us comment on other aspects that require more attention in the future. The question of multi-time quantum effects is of great importance from the fundamental point of view. Since the dynamical behavior of a physical system characterizes or even defines many quantum effects, e.g. photon antibunching, a more profound theoretical description is required. For example, the question of "temporal" entanglement is, in parts, an unsolved problem. Although I am currently not able to present a solution to this issue, it surprises me that entanglement between remote particles is much better understood then entanglement of two nearby points in time. A second aspect to work on is the application of quantum light in quantum metrology. Here the classical limit for estimating a physical quantity from a measurement is the Cramér-Rao bound. The application of click detectors would clearly lead to a modified limit.

There are the huge number of open problems which are interesting to me from the fundamental or application point of view. I'm looking forward to work on at least a few of them.

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Applicant's publications (during PhD studies)

[i] J. Sperling and W. Vogel
Necessary and sufficient conditions for bipartite entanglement
Phys. Rev. A **79**, 022318 (2009). DOI: 10.1103/PhysRevA.79.022318

[ii] J. Sperling and W. Vogel
Representation of entanglement by negative quasiprobabilities
Phys. Rev. A **79**, 042337 (2009). DOI: 10.1103/PhysRevA.79.042337

[iii] J. Sperling and W. Vogel
Verifying continuous-variable entanglement in finite spaces
Phys. Rev. A **79**, 052313 (2009). DOI: 10.1103/PhysRevA.79.052313

[iv] W. Vogel, T. Kiesel, and J. Sperling
Characterizing nonclassicality and entanglement
Optics and Spectroscopy **108**, 197-205 (2010). DOI: 10.1134/S0030400X10020074

[v] J. Sperling and W. Vogel
Determination of the Schmidt number
Phys. Rev. A **83**, 042315 (2011). DOI: 10.1103/PhysRevA.83.042315

[vi] J. Sperling and W. Vogel
The Schmidt number as a universal entanglement measure
Phys. Scr. **83**, 045002 (2011). DOI: 10.1088/0031-8949/83/04/045002

Part II.

Appendix

List of publications

[note] The chronologically ordered publications have been prepared/published after my PhD (July 2011). The author's contribution are either explicitly given or it corresponds to the position in the list of authors. The page number of a copy of the manuscript is given.

Published

[I] J. Sperling, W. Vogel, and G. S. Agarwal
True photocounting statistics of multiple on-off detectors
Phys. Rev. A **85**, 023820 (2012). DOI: 10.1103/PhysRevA.85.023820

[II] J. Sperling and W. Vogel
Entanglement quasiprobabilities of squeezed light
New J. Phys. **14**, 055026 (2012). DOI: 10.1088/1367-2630/14/5/055026

[III] J. Sperling, W. Vogel, and G. S. Agarwal
Sub-Binomial Light
Phys. Rev. Lett. **109**, 093601 (2012). DOI: 10.1103/PhysRevLett.109.093601

[IV] A. Miranowicz, P. Horodecki, R. W. Chhajlany, J. Tuziemski, and J. Sperling
Analytical progress on symmetric geometric discord: Measurement-based upper bounds
Phys. Rev. A **86**, 042123 (2012). DOI: 10.1103/PhysRevA.86.042123
JS mainly contributed to Sec. IV, Theorem 1, and in parts to the writing of the paper.

[V] D. Pagel, H. Fehske, J. Sperling, and W. Vogel
Strongly entangled light from planar microcavities
Phys. Rev. A **86**, 052313 (2012). DOI: 10.1103/PhysRevA.86.052313
JS contributed substantially to the results on entanglement and writing of the paper.

[VI] C. Gehrke, J. Sperling, and W. Vogel
Quantification of nonclassicality
Phys. Rev. A **86**, 052118 (2012). DOI: 10.1103/PhysRevA.86.052118
JS contributed substantially to the results on algebraic quantification and the writing of the paper.

[VII] E. Agudelo, J. Sperling, and W. Vogel
Quasiprobabilities for multipartite quantum correlations of light
Phys. Rev. A **87**, 033811 (2013). DOI: 10.1103/PhysRevA.87.033811

JS contributed substantially to the analytic results on the filter and writing of the paper.

[VIII] J. Sperling and W. Vogel
Multipartite Entanglement Witnesses
Phys. Rev. Lett. **111**, 110503 (2013). DOI: 10.1103/PhysRevLett.111.110503

[IX] D. Pagel, H. Fehske, J. Sperling, and W. Vogel
Multipartite entangled light from driven microcavities
Phys. Rev. A **88**, 042310 (2013). DOI: 10.1103/PhysRevA.88.042310
JS contributed substantially to the results on entanglement and writing of the paper.

[X] J. Sperling, W. Vogel, and G. S. Agarwal
Correlation measurements with on-off detectors
Phys. Rev. A **88**, 043821 (2013). DOI: 10.1103/PhysRevA.88.043821

[XI] F. Shahandeh, J. Sperling, and W. Vogel
Operational Gaussian Schmidt-number witnesses
Phys. Rev. A **88**, 062323 (2013). DOI: 10.1103/PhysRevA.88.062323
JS contributed substantially to the analytical results and writing of the paper.

[XII] F. E. S. Steinhoff, M. C. de Oliveira, J. Sperling, and W. Vogel
Bipartite bound entanglement in continuous variables through degaussification
Phys. Rev. A **89**, 032313 (2014). DOI: 10.1103/PhysRevA.89.032313
JS contributed substantially to the results and writing of the paper.

[XIII] J. Sperling, W. Vogel, and G. S. Agarwal
Quantum state engineering by click counting
Phys. Rev. A **89**, 043829 (2014). DOI: 10.1103/PhysRevA.89.043829

[XIV] W. Vogel and J. Sperling
Unified quantification of nonclassicality and entanglement
Phys. Rev. A **89**, 052302 (2014). DOI: 10.1103/PhysRevA.89.052302

[XV] W. Vogel and J. Sperling
Nonclassicality, entanglement, and nonclassical correlations
in *Coherence and Quantum Optics X*, edited by N. P. Bigelow, J. H. Eberly, and C. R. Stroud (Optical Society of America, 2014), pp. 217-224.

[XVI] M. Mraz, J. Sperling, W. Vogel, and B. Hage
Witnessing the degree of nonclassicality of light
Phys. Rev. A **90**, 033812 (2014). DOI: 10.1103/PhysRevA.90.033812
JS contributed substantially to the results and writing of the paper.

[XVII] A. J. Gutiérrez-Esparza, W. M. Pimenta, B. Marques, A. A. Matoso, J. Sperling, W. Vogel, and S. Pádua
Detection of nonlocal superpositions
Phys. Rev. A **90**, 032328 (2014). DOI: 10.1103/PhysRevA.90.032328
JS contributed substantially to the theoretical results, data analysis, and writing of the paper.

[XVIII] F. Shahandeh, J. Sperling, and W. Vogel
Structural Quantification of Entanglement
Phys. Rev. Lett. **113**, 260502 (2014). DOI: 10.1103/PhysRevLett.113.260502
JS contributed substantially to the results and writing of the paper.

Preprints

[XIX] S. Gerke, J. Sperling, W. Vogel, Y. Cai, J. Roslund, N. Treps, C. Fabre
Full multipartite entanglement of frequency comb Gaussian states
arXiv:1409.5692 [quant-ph].
JS contributed substantially to the theoretical results and writing of the paper.
Phys. Rev. Lett. in press; selected as editors' suggestion.
Referece added: Phys. Rev. Lett. **114**, 050501 (2015).

[XX] J. Sperling, W. Vogel, and G. S. Agarwal
Balanced homodyne detection with on-off detector systems: observable nonclassicality criteria
arXiv:1410.8012 [quant-ph].
Europhys. Lett. in press.
Referece added: Europhys. Lett. **109**, 34001 (2015).

[XXI] E. Agudelo, J. Sperling, W. Vogel, S. Köhnke, M. Mraz, and B. Hage
Continuous sampling of the squeezed state nonclassicality
arXiv:1411.6869 [quant-ph].
JS contributed substantially to the theoretical results and writing of the paper.
Submitted to Physical Review A.

[XXII] M. Bohmann, J. Sperling, and W. Vogel
Entanglement and phase properties of noisy N00N states
arXiv:1412.3321 [quant-ph].
JS contributed substantially to the results and writing of the paper.
Submitted to Physical Review A.
Referece added: Phys. Rev. A **91**, 042332 (2015).

[XXIII] A. Luis, J. Sperling, and W. Vogel
Nonclassicality phase-space functions: more insight with less detectors
arXiv:1412.3826 [quant-ph].
JS contributed substantially to the results and writing of the paper.
Submitted to Physical Review Letters.
Referece added: Phys. Rev. Lett. **114**, 103602 (2015).

[XXIV] A. Reusch, J. Sperling, and W. Vogel
Entanglement Witnesses for Indistinguishable Particles

arXiv:1501.02595 [quant-ph].
JS contributed substantially to the results and writing of the paper.
Submitted to Physical Review Letters.

Referece added: Phys. Rev. A **91**, 042324 (2015).

[XXV] J. Sperling and W. Vogel
Convex ordering and quantification of quantumness
Overlap with arXiv:1004.1944 [quant-ph].
Phys. Scr. in press.

Referece added: Phys. Scr. **90**, 074024 (2015).

B. Personal data and scientific records

B.1. Curriculum vitae

Key facts in the applicant's CV (date: January 28, 2015) are:

- PhD in theoretical physics, Universität Rostock, 2011
- 24 refereed publication (11 as first author)
- 11 international conference contributions
- supervised 13 students
- conducted 21 courses in theoretical physics
- additional degree in mathematics (Dipl.-Math.)
- participated in writing 6 project proposals
  (DFG: SFB 652-2 B12, SFB 652-3 B12, VO 501/21-1, VO 501/22-1;
  EU FP7: *QuICTA*, *Q-CUMbER*)

B.2. Collaborations

A number of national and international collaborations has been established:
(chronological order, only those are listed with a published/submitted publication)

- Prof. G. S. Agarwal
  (Oklahoma State University, Stillwater, Oklahoma, USA)

  topic: development of click counting theory

- D. Pagel and Prof. H. Fehske
  (Ernst-Moritz-Arndt-Universität, Greifswald, Germany)

  topic: theory of entangled light from microcavity systems

- Prof. A. Miranowicz and Prof. P. Horodecki
  (Adam Mickiewicz University, Poznań, Poland; Technical University of Gdańsk, Gdańsk, Poland; National Quantum Information Centre of Gdańsk, Sopot, Poland)

  topic: theory of quantum discord

- F. E. S. Steinhoff and Prof. M. C. de Oliveira
  (Universidade Estadual de Campinas, Campinas, São Paulo, Brazil)

  topic: theory of bound entangled states

- A. J. Gutiérrez-Esparza and Prof. S. Pádua
  (Universidade Federal de Minas Gerais, Belo Horizonte, Minas Gerais, Brazil)

  topic: experiment on entanglement quantification

- M. Mraz, S. Köhnke, and Prof. B. Hage
  (Universität Rostock, Rostock, Germany)

  topic: theory of quantification of nonclassicality; experiment on verification of nonclassicality

- Y. Cai, J. Roslund, Prof. N. Treps, and Prof. C. Fabre
  (Laboratoire Kastler Brossel; Sorbonne Universités - Université Pierre & Marie Curie; École Normale Supérieure; Collège de France; CNRS, Paris, France)

  topic: experiment on entanglement in complex structured light fields

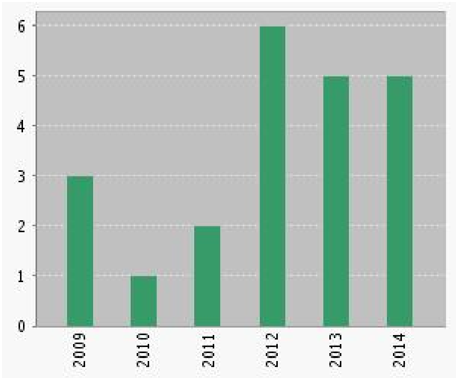
- Prof. A. Luis
  (Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, Madrid, Spain)

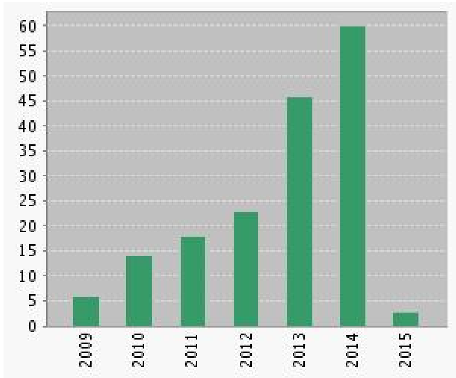
  topic: development of unbalanced homodyne click counting detection

B.3. Metrics

Records in WEB OF SCIENCE™ (core collection, date: January 17, 2015):

- 22 listed publication
- 170 citations
- h-index: 8





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