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Traditio et Innovatio

Local formulas for Ehrhart coefficients from lattice tiles

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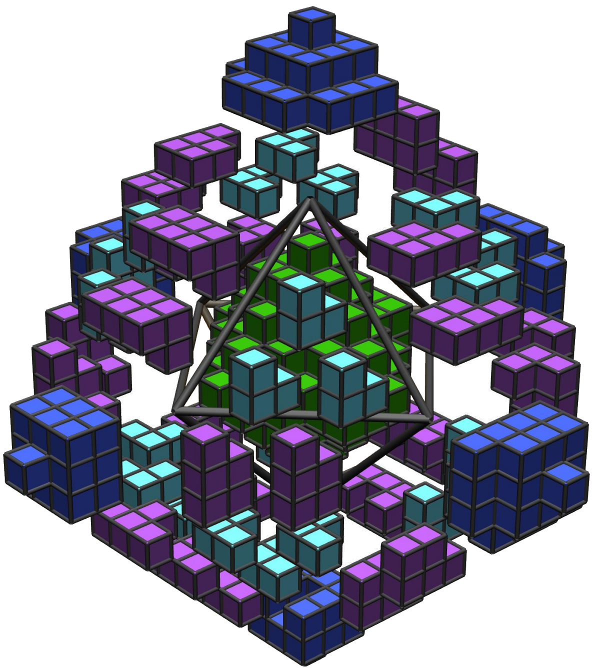

Dissertation

von

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Abstract

As shown by Peter McMullen in 1983, the coefficients of the Ehrhart polynomial of a lattice polytope can be written as a weighted sum of facial volumes. The weights in such a local formula depend only on the outer normal cones of faces or, equivalently, on their cones of feasible directions, but are far from being unique. In this thesis, we present the local formulas μ as established by Achill Schürmann and the author in [RS19]. The construction is based on choices of fundamental domains whose lattice translates tile the space and thus allows a geometric interpretation of the values of μ. Additionally, we expand μ to a function on rational polytopes that determines the coefficients of Ehrhart quasipolynomials, we prove new results about the symmetric behavior of μ and introduce a variation of μ that is well-suited for implementations.

Introduction

Solutions of systems of linear inequalities and therefore polyhedra are important concepts in a great variety of mathematical areas and applications. They are central objects in optimization, and in many cases it is necessary to consider integer solutions and thus to determine and count lattice points in polyhedra. The number of lattice points in a polytope is also called its *discrete volume* and is the subject of research of Ehrhart theory. This theory has strong connections to various fields of mathematical studies such as toric varieties in algebraic geometry (see [Ful93], [CLS11], [Dan78]) as well as areas of application such as social choice theory (see [Sch13]).

Let V be an n-dimensional Euclidean space and $\Lambda \subseteq V$ a *lattice* of rank n, e.g. $\Lambda = \mathbb{Z}^n$. Let P be a *lattice polytope*, i.e. a polytope whose vertices are in Λ with $d := \dim(P)$. In 1962 Ehrhart [Ehr62] showed that the function $\mathrm{E}_P : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ that is sending a non-negative integer to the number of lattice points in the t-th dilate tP of P, has a particularly nice form: It coincides with a polynomial in t of degree d,

$$\mathrm{E}_P(t) := \#(\Lambda \cap tP) = e_d t^d + e_{d-1} t^{d-e} + \ldots + e_1 t + e_0. \tag{1}$$

He further showed that if P is not a lattice polytope but a *rational polytope*, meaning that the vertices of P are rational with respect to the lattice, then the function E_P is still a so-called *quasipolynomial*, a generalization of a polynomial where the coefficients are allowed to be periodic functions with integer period (cf. Chapter 2).

The coefficients of Ehrhart (quasi-)polynomials are called *Ehrhart coefficients*. Despite being a vivid area of research, the knowledge of Ehrhart coefficients is still very limited. For lattice polytopes, it is known that e_d equals the relative volume of P, e_{d-1} is one half times the sum of the relative volumes of the facets of P and the constant term e_0 equals 1. In dimension 2 this provides a full description of the Ehrhart polynomial that had already been proved by Pick in 1899 [Pic99] and is known as Pick's Theorem:

$$\#(\Lambda \cap tP) = \mathrm{vol}(P)t^2 + \frac{B_P}{2}t + 1, \tag{2}$$

where $\mathrm{vol}(P)$ is the area of P and B_P is the number of lattice points on the boundary of P, which equals the sum of the relative volumes of the edges. Pick's Theorem therefore establishes a connection between discrete entities (the number of lattice points) and continuous ones (the relative volumes). It is desirable to gain such an elegant and geometrically motivated description for Ehrhart coefficients in general. Such a description is still missing, though. A simple interpretation of the coefficients as volumes of faces is not possible, if only because starting from dimension three the coefficients can be negative, as for example in the case of the Reeve tetrahedron [Ree57]. In this work we present a geometric interpretation in form of a more complex combination of sums and differences of volumes.

Our starting point is a connection between the i-th Ehrhart coefficient e_i and the volumes of the i-dimensional faces of P. This connection was suggested by Danilov in 1978 [Dan78] and positively answered by proofs of McMullen [McM83] and later also by Morelli [Mor93]. They showed the existence of what we now call a *local formula*, which is a function Φ from polyhedral cones to the real numbers that satisfies

$$e_i = \sum_{\substack{f \leq P \\ \dim(f)=i}} \Phi(\mathrm{normal}(f, P))\mathrm{vol}(f), \tag{3}$$

where $f \leq P$ are the faces of P, $\mathrm{normal}(P, f)$ the *normal cone* of P in f and $\mathrm{vol}(f)$ is the *relative (i-dimensional) volume* of f (see Section 1.1). Since the normal cone of an i-dimensional face does not change under dilation and the volume changes with exponent i, we equivalently have that a local formula is a function satisfying

$$\mathrm{E}_P(t) = \sum_{f \leq P} \Phi\left(\mathrm{normal}(P, f)\right) \mathrm{vol}(tf), \qquad \forall t \in \mathbb{Z}_{>0}.$$

Comparing the definition to Equation (2), we see that a local formula is a natural generalization of Pick's Theorem. Due to the first proof of existence, local formulas are also referred to as *McMullen's formulas.* We here stick to the more descriptive name *local formulas* as the fact that such a formula only depends on the normal cone means that global information about its boundary, neighboring faces and volume are not available. The normal cone only stores the information about the structure of the face in the vicinity of a point in the relative interior of the face, which is what makes it local.

Local formulas are an example for the close connection between counting lattice points in polyhedra and algebraic geometry. Danilov [Dan78] as well as Morelli [Mor93] both approached the topic in the context of toric varieties, using that a formula for the Todd class of a toric variety yields a formula for the number of lattice points in the corresponding polytope (see [Ful93]).

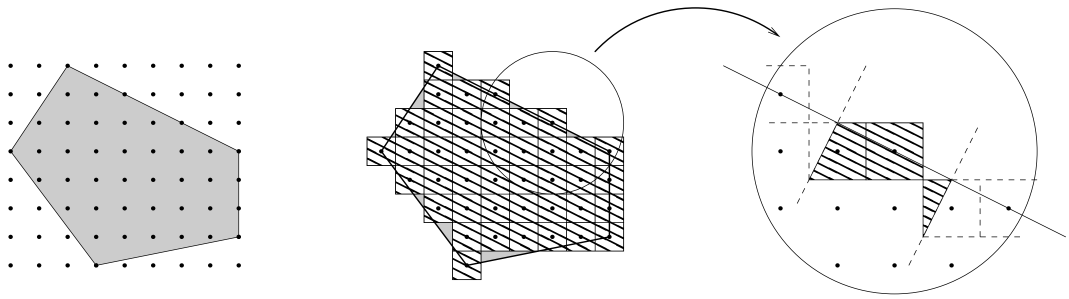


Figure 1: The domain complex of a polytope and the fundamental domains in the vicinity of a generic point on an edge with $v_f = 2$.

The use of the plural 'local formulas' is due to the fact that they are far from being unique, thus enabling mathematicians to find different ways of determining and interpreting Ehrhart coefficients. Though the proofs by McMullen and Morelli are not constructive, Morelli also gave a construction, where the weights are rational functions on certain Grassmanians. The first construction with rational values was given by Pommersheim and Thomas in 2004, [PT04], in the context of algebraic geometry. In 2007, Berline and Vergne [BV07] gave a construction of local formulas as Euler-Maclaurin formulas for polytopes.

In this work we present a different approach, based on a tiling of space by so-called *fundamental domains*. The idea goes back to an idea of Schürmann in 2004 as presented in [Sch04], with some changes, since the construction suggested there is not well-defined. At this point, we want to give an idea of our construction without going into detail. To this end, we only give an illustrative description of the necessary terminology, thorough definitions and details are given in Chapter 1. A *fundamental domain* is a set $T \subseteq V$ whose main characteristic is that lattice point translates $x + T$ with $x \in \Lambda$ form a tiling of V. Since the relative volume of a fundamental domain is 1, we can use it to interpret the number of lattice points as a volume by the equality

$$\#(\Lambda \cap tP) = \mathrm{vol}\left((\Lambda \cap tP) + T\right). \tag{4}$$

The set $(\Lambda \cap tP) + T$ is called the *domain complex* of tP. An example is given in Figure 1 in the middle. Given a face $f \leq P$, this step now allows us to determine a ratio called the *relative domain volume* v_f of how much (not necessarily an integer amount) of the domain complex appears around a *generic lattice point* of a face. *Generic* means a lattice point that is not too close to the boundary of the face. See Figure 1 for an example of the relative domain volume. Throughout this work, we denote by μ the functions that we construct as local formulas. The construction of μ is inductive, descending in the dimension of the considered face, just as the constructions that are known in the literature so far. Starting with the polytope P as a face of itself, we observe that per inner lattice point of the polytope, there is exactly one complete fundamental domain to be counted, determining $\mu(\text{normal}(P,P))$ as 1. Figure 1 also shows that the structure of the fundamental domains around a face of the polytope is periodic with respect to the lattice points in the face. That means that the ratio around one lattice point can be applied to all other generic lattice points in that face.

This reasoning is only true around generic lattice points of a face f and not for the points close to and on the relative boundary of f. However, taking the value $\mu(\text{normal}(P,f)) \cdot \text{vol}(f)$ in the computation of the Ehrhart coefficient assumes that the computed ratio is evenly spread across the whole face f. The μ-value accounts for that 'mistake' in the subsequent steps of the induction by subtracting a *correction volume* for f in the computation of all lower-dimensional faces that bound f.

That way we get the μ-value for a face f of P as

$$\mu(\text{normal}(P,f)) = v_f - \sum_{f<g\leq P} w_g^f \mu(\text{normal}(P,g)),$$

where v_f is the domain volume around a generic point in f and w_g^f the correction volume, which both only depend on the normal cones of the faces. To specify what we mean by 'around a generic point' we define *regions* that determine the relevant area in the vicinity of a lattice point for each given cone.

The main step to proving that the construction gives a local formula is to show that we get a certain tiling that covers the polytope by regions (cf. Section 1.4, Theorem 1). See Figure 2, left, for an example. This tiling ensures that when the μ values are put together according to formula (3), we exactly determine the volume in (4) and thus the number of lattice points in tP. We thereby can show our main theorem:

Theorem 2. *The function μ as defined in Section 1.2 is a local formula for Ehrhart coefficients.*

This work is about the above described construction of a local formula μ, its properties, its applications, modifications and implementation.

For a rational polytope P, let $\mathrm{E}_P(t) = c_d(t)t^d + c_{d-1}(t)t^{d-1} + \ldots + c_0(t)$ be its quasipolynomial, where the coefficients $c_0, \ldots, c_d$ are periodic functions in $t \in \mathbb{Z}_{\geq 0}$. To determine the coefficients in the way we did for lattice polytopes, we need the translation class $\text{trl}(\Lambda, f)$ of the face f with respect to the lattice Λ as an additional input value. Then it is possible to define a modified version of regions and a function μ^* on tuples of cones and translation classes in a similar way to the definition of μ. We again get a tiling by regions in Theorem 3 and can prove a generalization of Theorem 2:

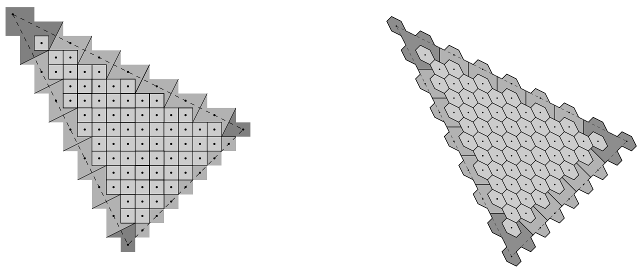


Figure 2: A simplex covered by tilings of regions.

Theorem 4. *For a rational polytope P with Erhart quasipolynomial*

$$\mathrm{E}_P(t) = c_d(t)t^d + c_{d-1}(t)t^{d-1} + \ldots + c_0(t),$$

the i-th coefficient is given by

$$c_i(t) = \sum_{\substack{f \leq P \\ \dim(f)=i}} \mu^*(\mathrm{fcone}(P, f), \mathrm{trl}(tf, \Lambda))\mathrm{vol}(tf).$$

One very important property of the functions μ and μ^* is that they behave very nicely under symmetries. The functions depend on a choice of fundamental domains and an inner product of V. This fact can be used to systematically exploit symmetries. Given a lattice polytope P and a symmetry group $\mathcal{G}$ of P, it is possible to compute so-called *Dirichlet–Voronoi cells* (see Section 3.1) that are invariant under $\mathcal{G}$. Dirichlet–Voronoi cells are closed and can be turned into fundamental domains by choosing the boundary to be half-open. While this step breaks some of the symmetry of the cell, Theorem 5 in Section 3.2 shows that this change of the boundary is negligible in the way that μ (and with some adjustments also μ^*) is invariant under $\mathcal{G}$, if the closures of the chosen fundamental domains are symmetric with respect to G (cf. Figure 2, right). Another nice behavior of μ under symmetry is given in Section 3.3, Theorem 6: If the closure of the fundamental domain is centrally symmetric, the μ-value on facets always equals 1/2, which nicely confirms the fact that the second highest Ehrhart coefficient equals 1/2.

When implementing the function μ, the biggest obstacle is that the regions it is defined on are not necessarily convex. The most common computer algebra systems working with polyhedra are not primarily designed to handle non-convex sets. A way around that is presented in Section 4: A variant of the function μ called the *brick version*, denoted as μ_b. It is very similar to the original version, but ensures that all regions are unions of fundamental domains. This way, most operations are reduced to operations on discrete finite sets, which encode the translation vector of the fundamental domains. The operations that cannot be simplified that way, as for example the computation of the correction volume, can still be reduced to operations on convex polytopes withing each fundamental domain. This approach enables us to implement the local formula μ_b in a `SageMath` program [S$^+$16]. The price to pay when changing from μ to μ_b is that some of the symmetric behavior of the original version is lost due to the fact that the brick version is more sensitive to the boundary structure of the fundamental domains. Hence, for theoretical results along the lines of [CL18] for instance, the original version is preferable. For specific examples, however, the brick version is practical. Some computations of Ehrhart polynomials using the brick version are given in Section 4.3. Plots of the computed regions using `jReality` [GHS$^+$17] after a conversion into `polymake` [GJ00] are shown in the same section as well as on the titlepage.

An interesting question is how exactly our constructions relate to the few previous ones, which leads us to several open questions. In [CL18], Castillo and Liu analyse the construction by Berline and Vergne [BV07] to investigate whether generalized permutohedra, a certain well-known class of polytopes, are Ehrhart positive, i.e. whether all their Ehrhart coefficients are nonnegative. Along the way, Castillo and Liu show that all local formulas that have a certain property, namely that they are *symmetric about the coordinates* [CL18, Def. 3.17], have the same values on the normal cones of faces of general permutohedra. In the case of centrally symmetric fundamental domains, e.g. Dirichlet–Voronoi cells, our construction of μ naturally has the property of being symmetric about the coordinates (cf. Chapter 3) and thus the values are determined and equal to the ones in the construction of Berline and Vergne on normal cones of generalized permutohedra. Despite contrary conjectures, they do not agree in general, as values in easy 2-dimensional examples can differ. An interesting yet open question is, whether previous constructions can be recovered by our construction using the fact that the great variety of fundamental domains gives us many different local formulas μ (cf. Section 5.1). It is still unclear though whether the local formulas given here are *valuations* meaning that it is possible to decompose cones and to compute the value of the whole cone from the values on the parts. The mentioned previous constructions have this nice property, while to this point there has neither been found a proof nor a counterexample for μ. A different and interesting approach to the constructions in this work is a generalization via generating functions that the author has been working on with Lukas Katthän and Sebastian Manecke. This approach is promising and some appearances of the formulas seem to suggest that it can be interpreted as a discretization of the Berline–Vergne construction. This work is still fresh though and further research is required to give substantiated conjectures and results.

The thesis is structured as follows: Section 1.1 establishes the preliminaries necessary to this work as well as important definitions, including the ones mentioned in this introduction. We then give the construction of the regions and of the function μ in a precise and brief way. To give an understanding of that definition, Section 1.3 retraces the first few inductive steps on a general polytope and, concurrently, on a simplex in dimension two as a concrete example. Section 1.4 contains the proof of Theorem 2. It is divided into two subsections, the first of which shows general properties of the regions, most importantly that the regions are bounded, which ensures that the function μ is well defined. The second part then considers the construction given an actual polytope P. It is proved that the regions along the generic points of the faces of P form a tiling of V (Theorem 1) and, finally, that μ is a local formula. The results in Sections 1.2 and 1.4 have been published by Schürmann and the author in [RS19]. The difference here is that instead of the normal cone, the *cone of feasible directions*, short *fcone* is taken as input value. This equivalent construction has the advantage of being more direct and thus shortening notation. The proofs in Section 1.4 have been revised by the author and partly improved. Section 1.3 follows loosely the elaboration of the authors overview article in [Rin19].

Chapter 2 generalizes the constructions from Chapter 1 to rational polytopes and Ehrhart quasipolynomials. The results in this chapter are original to this work. In Chapter 3 we study the symmetric behavior of μ. We start by introducing the well known Dirichlet–Voronoi cells and show how they can be used to realize symmetry invariance of μ. The main result is Theorem 5, which is original to this work. In Section 3.3 we show that the values on fcones of facets are 1/2 with a revised proof of the author's one in [Rin19]. Chapter 4.1 introduces the brick version μ_b. The source code of its implementation can be found in the Appendix and a description of the program is given in Section 4.2. The chapter finishes by showing examples computed with the program. The results in this chapter and the source code of the program are written by the author and have not been published yet. Finally, Chapter 5 gives additional information on two important components of this work, fundamental domains and the duality between normal cones and fcones. Section 5.1 analyses the definition of fundamental domains with relevant examples to show what is and what is not possible. Section 5.2 elaborates on the order of fcones and on the connections to normal cones and faces of polytopes. Since this is the first work, where the approach via fcones is chosen instead of via normal cones, it provides a guideline for switching between the two equivalent approaches, closing with an overview in Table 5.1.

Contents

Chapter 1

A local formula for Ehrhart coefficients

1.1 Definitions and preliminaries

Throughout this thesis, let V be a Euclidean space and $\Lambda \subset V$ a *lattice* of full rank, i.e. Λ is a discrete additive subgroup generated by a vector space basis of V. A *polyhedron* Q in V is the solution of finitely many linear inequalities, or, respectively, the intersection of finitely many halfspaces. A polyhedron is called *rational* with respect to the lattice Λ, if it is given by inequalities $\langle n_i, x\rangle \leq \alpha_i$, with $n_i \in \Lambda$ and $\alpha_i \in \mathbb{Z}$ for i in some finite index set I. The *dimension* of a polyhedron is the dimension of its affine hull, $\dim(Q) := \dim(\operatorname{aff}(Q))$. A *face* of Q is the intersection of Q with a hyperplane that does not intersect the relative interior of Q. The faces are again polyhedra. For two polyhedra f and Q, we write $f < Q$ if f is a face of Q and $f \leq Q$ if $f = Q$ is allowed. Formally, the empty set is a face of every polyhedron, but since we never use it, we shorten notation by assuming $f \neq \varnothing$ whenever we talk about faces $f \leq g$. If the dimension of Q is d, the $(d-1)$-dimensional faces of Q are called *facets* and the zero-dimensional faces are called *vertices*. A *polytope* P is the convex hull of finitely many points, $P = \operatorname{conv}(v_1, \dots, v_m)$. Due to a nontrivial theorem (see, for instance, Beck and Robins [BR15, Appendix]), sometimes referred to as the Minkowski Farkas-Weyl Theorem, every polytope can be written equivalently as the convex hull of finitely many points as well as the bounded intersection of finitely many halfspaces .

A *cone* here always means a convex polyhedral cone, i.e. a polyhedron that is the solution of only homogeneous linear inequalities. For a polyhedron Q and a face f of Q we define the *cone of feasible directions* of Q in the face f, short *fcone* of Q in f, by picking an s in the relative interior $\operatorname{int}(f)$ of f and defining

$$\operatorname{fcone}(Q, f) := \{x \in V \mid \exists \varepsilon > 0\colon s + \varepsilon x \in Q\}.$$

The resulting cone is independent of the choice of $s \in \operatorname{int}(f)$. This cone contains local information about the polytope in the vicinity of the interior of the face f and it is the dual of the *normal cone* of Q in f as defined in Section 5.2. As such, the fcone and the normal cone can interchangeably be used as input value for local formulas. More information on this connection can be found in Section 5.2.

Let further $\operatorname{lineal}(Q)$ be the *lineality space* of Q, i.e. the biggest linear subspace contained in Q. We denote as $\operatorname{lat}(Q) := \Lambda \cap \operatorname{lineal}(Q)$ the sublattice in $\operatorname{lineal}(Q)$ induced by Λ.

A *tiling* of a set $A \subseteq V$ is a set $\mathfrak{A}$ of subsets of A such that A is the disjoint union of the elements of $\mathfrak{A}$. The construction we will give of local formulas relies on a choice of certain lattice tiles, namely *fundamental domains* as defined below. Different choices lead to different values and thus give an infinite family of constructions.

Definition 1. For a subspace $U \subseteq V$ with induced sublattice $\operatorname{lat}(U) = U \cap \Lambda$, a *fundamental domain* $T(U)$ is a bounded subset of U such that $\{x + T(U) \mid x \in \operatorname{lat}(U)\}$ is a tiling of U and that every intersection of $T(U)$ with an affine subspace of V is Lebesgue-measurable.

Examples of and more information on fundamental domains can be found in Sections 3.1 and 5.1.

For a subset $A \subseteq V$ we denote as $\operatorname{lin}(A)$ the linear subspace parallel to its affine hull. The *relative volume* of A is the volume normalized such that a fundamental domain in $\operatorname{lin}(A)$ with respect to the lattice $\Lambda \cap \operatorname{lin}(A)$ has volume 1. Note that it is a lower dimensional volume if $\operatorname{aff}(A)$ is lower dimensional. As a convention we further set $\operatorname{vol}(A \cap B)$ to be the relative volume in $\operatorname{lin}(A) \cap \operatorname{lin}(B)$, which, in particular, means that $\operatorname{vol}(A \cap B) = 0$, if $\dim(A \cap B) < \dim(\operatorname{lin}(A) \cap \operatorname{lin}(B))$.

We will use the fundamental domains to interpret the number of lattice points inside a polyhedron as the volume of a certain set called the domain complex.

Definition 2. Let Q be a polyhedron and T a fundamental domain of V. The *domain complex* $\operatorname{DC}(Q)$ of Q is the set of all fundamental domains translated by lattice points in Q:

$$\operatorname{DC}(Q) = \bigcup_{x \in \Lambda \cap Q} x + T$$

The *covering domain complex* $\operatorname{CDC}(Q)$ of Q is the union of all lattice point translates of T that intersect Q:

$$\operatorname{CDC}(Q) = \bigcup_{\substack{x \in \Lambda \\ (x+T) \cap Q \neq \varnothing}} x + T.$$

Both, the domain complex and the covering domain complex of a polyhedron Q are unions of lattice point translates of the fundamental domain T. The covering domain complex of Q naturally covers Q as well as $\mathrm{DC}(Q)$, which justifies the name.

For a polyhedron Q and a fundamental domain $T(\mathrm{lineal}(Q))$ in the induced sublattice we define the *strip* of Q as

$$\mathrm{strip}(Q) := T(\mathrm{lineal}(Q)) + \mathrm{lineal}(Q)^{\perp}.$$

Here we use the orthogonal space $\mathrm{lineal}(Q)^{\perp}$ of $\mathrm{lineal}(Q)$ that we get by taking an inner product on V and using it to identify V with its dual. This identification can also be avoided by taking any complementary subspace of $\mathrm{lineal}(Q)$, i.e. a subspace U of V with $\mathrm{lineal}(Q) \cap U = 0$ and $\mathrm{lineal}(Q) + U = V$. The version with the orthogonal complement, however, can be used to gain invariance under certain symmetries as described in Section 3.1. In any case, we have that $\{x + \mathrm{strip}(Q) \mid x \in \mathrm{lat}(Q)\}$ is a tiling of V.

Given a (lattice) polytope P, the region will be defined inductively on the fcone of P in f for the faces $f \leq P$. Inductively here means that we start with the fcone of P in itself and we will go down in the dimension of the face. This way, when constructing the region for $\mathrm{fcone}(P, f)$, we can assume to have constructed the region for $\mathrm{fcone}(P, g)$ for all faces $f < g \leq P$. Since the benefit of a local formula is that it can be described purely on cones, independently of the polytope, we introduce the partial order '$\prec$' on cones that corresponds reciprocally to the order of faces of P:

$$D \prec C \quad :\Leftrightarrow \quad D = \mathrm{fcone}(C, F) \text{ for a face } F \text{ with } \mathrm{lineal}(C) < F \leq C.$$

This way, given two faces f and g of P, we have $\mathrm{fcone}(P, g) \prec \mathrm{fcone}(P, f)$, if and only if $g > f$.

If equality is allowed, we further write $C \preceq D$ instead of $C \prec D$ or $C = D$. If $D \preceq C$, we say that *D is an fcone of C*. Two cones C and D are called *comparable* if $D \preceq C$ or $C \preceq D$. Otherwise, they are called *incomparable*. For our inductive construction we use the fact that going up in this order means going down in the dimension of the lineality space in the sense that if $D \prec C$, then $\mathrm{lineal}(C) \subsetneq \mathrm{lineal}(D)$ and $\dim(\mathrm{lineal}(C)) < \dim(\mathrm{lineal}(D))$. An inductive construction is thus possible starting with the whole space and going down in the dimension of the lineality space. More on this order and the connections to the normal fan of a polytope can be found in Section 5.2.

1.2 Construction of regions and definition of μ

Given a choice of fundamental domains we define a function μ from rational cones to $\mathbb{R}$. In Section 1.4 we will show that μ is indeed a local formula as defined in the introduction, Equation (3). We here give a formal and very compact description of μ. For a step-by-step construction with examples and pictures, see Section 1.3. The aim of this section is to give a clear, short and formally precise definition of the construction.

The values are determined inductively, descending in the dimension of the lineality space of the cone. For not full-dimensional cones we intersect V with the linear span of the cone and consider that as our ambient space. To determine μ, we will first inductively define a map R from rational cones to subsets of V, associating a *region* to each cone. From these regions, the values of μ can be computed via volume computations.

Let C be a full-dimensional rational cone in V. For each subspace $A \subseteq V$ we assume to have chosen a fundamental domain $T(A)$.

If $C = V$, we set

$$R(V) := T(V).$$

Otherwise, if $C \neq V$, we assume we have constructed the regions $R(D)$ for all cones $D \prec C$. We define the set of *generic lattice points* X_D^C of D with respect to C to be the set of all points x in $\mathrm{lat}(D)$ that fulfill the conditions:

(I) For all halfspaces $H \preceq C$ such that $H \npreceq D$:

$$x + R(D) \subseteq \mathrm{int}(H)$$

(II) For all $E \prec C$, such that E is incomparable to D and for all $x' \in \mathrm{lat}(E)$:

$$(x + R(D)) \cap (x' + R(E)) = \varnothing$$

For later use, we also set $X_C^C := \mathrm{lat}(C)$, which is consistent with the above in the way that conditions (I) and (II) are trivially satisfied if $D = C$.

Then we define

$$R(C) := (\mathrm{strip}(C) \cap \mathrm{CDC}(C)) \setminus \bigcup_{D \prec C} \left(X_D^C + R(D) \right).$$

From this we can compute the values of the *relative domain volume* v_C in the region $R(C)$ as:

$$v_C := \mathrm{vol}\big(R(C) \cap \mathrm{DC}(C)\big).$$

And further the *correction volumes* for each $D \prec C$:

$$w_D^C := \mathrm{vol}(R(C) \cap C \cap \mathrm{lineal}(D)).$$

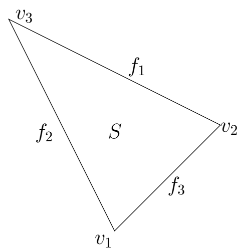
Then we get the value for C as

$$\mu(C) := v_C - \sum_{D \prec C} w_D^C \cdot \mu(D).$$

In Section 1.4 we will prove that μ is a local formula for Ehrhart coefficients (Theorem 2). In particular, we will prove in Lemma 1.3 that $R(C)$ is bounded, which means that μ and the volumes v_C and w_D^C as given above are well defined.

1.3 Application to polytopes

In addition to the definition of μ on cones in its most general way, it is useful to see how the values are determined given an actual polytope. To make this discussion as graphic as possible, we follow the inductive steps given a general full-dimensional polytope and, concurrently, given a specific simplex in the plane $\mathbb{R}^2$. In this section, we loosely follow the elaboration of the author's overview article in [Rin19]. Throughout this section, let P be a full-dimensional lattice polytope in V.



Example. Let $S = \mathrm{conv}(v_1, v_2, v_3)$ be the simplex in $\mathbb{R}^2$ with vertices $v_1 = (1, 0)$, $v_2 = (2, 1)$ and $v_3 = (0, 2)$. We consider it as a lattice polytope in $\mathbb{R}^2$ with respect to the lattice $\mathbb{Z}^2$.

We will use S as running example in this section. To avoid unreadable expressions, we expand μ, R, X, v and w to functions on the faces of S, by setting $\mu(f) := \mu(\mathrm{fcone}(S, f))$, $R(f) := R(\mathrm{fcone}(S, f))$, $X_g^f := X_{\mathrm{fcone}(S,g)}^{\mathrm{fcone}(S,f)}$, $v_f := v_{\mathrm{fcone}(S,f)}$ and $w_g := w_{\mathrm{fcone}(S,g)}^{\mathrm{fcone}(S,f)}$, whenever the simplex S is given.

μ(P) for P as a face of itself. First, we choose and fix a fundamental domain $T(A)$ for each subspace $A \subseteq V$ and denote $T := T(V)$. Since $\mathrm{fcone}(P, P) = V$, we get

$$R(V) = T$$

and

$$v_V = \mathrm{vol}\big(R(V) \cap \mathrm{DC}(V)\big) = \mathrm{vol}(T) = 1.$$

Since V is minimal in the '$\prec$'-order, there are no correction volumes to consider. Thus, the relative domain volume directly determines the value of μ as

$$\mu(V) = v_V = 1.$$

Remark. Using this result to compute the d-th Ehrhart coefficient via Equation (3) for local formulas, we get

$$e_d = \sum_{f \in \mathcal{F}_d} \mu(\mathrm{fcone}(P, f))\ \mathrm{vol}(f) = \mu(V)\mathrm{vol}(P) = 1 \cdot \mathrm{vol}(P) = \mathrm{vol}(P)$$

as desired, since the highest Ehrhart coefficient is known to be the relative volume of the polytope.

$\mu(\mathbf{S})$ **for S as a face of itself.** The region $R(S)$ is given as the fundamental domain $T = T(\mathbb{R}^2)$ of $\mathbb{Z}^2$, which here we choose to be the square with edge length 1 and the origin as barycenter. Then we have $\mu(S) = v_S = \mathrm{vol}(T) \ = \ 1$.

$\mu(\mathbf{F})$ **for facets F of P.** Let $F < P$ be a facet of P. Then $H^+ := \mathrm{fcone}(P, F)$ is a halfspace containing a hyperplane H. Following the construction from Section 1.2, we need to determine the sets of generic lattice points $X_D^{H^+}$ for cones $D \prec H^+$, which means $D = \mathrm{fcone}(P, g)$ for some $g > F$. Since F is a facet, we have $g = P$ and $D = \mathrm{fcone}(P, P) = V$. The only halfspace containing H^+ is H^+ itself and $H^+ \not\preceq V$. Thus the points $x \in \Lambda$ satisfying Property (I), are the ones with

$$x + R(V) \subseteq \mathrm{int}(H^+).$$

Regarding Property (II), there is nothing to check, since all fcones of H^+ are comparable to V. Altogether, the region $R(H^+)$ is defined as

$$R(H^+) = \big(\mathrm{strip}(H) \cap \mathrm{CDC}(H^+)\big) \setminus \left(X_V^{H^+} + T\right), \tag{1.1}$$

with the set of generic lattice points given as

$$X_V^{H^+} = \{x \in \Lambda \mid (x + R(V)) \subseteq H^+\}.$$

We can simplify (1.1) due to

$$\mathrm{CDC}(H^+) \setminus \left(X_V^{H^+} + T\right) = Y_H + T,$$

where we define $Y_H := \{x \in \Lambda \mid (x + T) \cap H \neq \varnothing \ \}$ and thus

$$R(H^+) = \mathrm{strip}(H) \cap (Y_H + T)\,,$$

That means that $R(H^+)$ equals $\mathrm{strip}(H)$ intersected with the union of all lattice translates of the fundamental domain T that intersect H.

After the construction of $R(H^+)$, the relative domain volume in $R(H^+)$ can be computed as

$$v_{H^+} = \mathrm{vol}(R(H^+) \cap \mathrm{DC}(H^+))$$

and the correction volume as

$$w_V^{H^+} = \mathrm{vol}(R(H^+) \cap H^+).$$

That yields the value of μ as

$$\mu(H^+) = v_{H^+} - w_V^{H^+} \cdot \mu(V) = v_{H^+} - w_V^{H^+}.$$

$\mu(\mathbf{f_i})$ for the edges $\mathbf{f_1, f_2}$ and $\mathbf{f_3}$ of S. In all 1-dimensional subspaces, we choose the fundamental domain to be the line segment with barycenter at the origin. For $i \in \{1,2,3\}$, let L_i be the line through the origin parallel to the affine hull of f_i, i.e. $L_i = \mathrm{lineal}(\mathrm{fcone}(S, f_i))$ and let $L_i^+ := \mathrm{fcone}(S, f_i)$. As shown above for a general polytope, the regions $R(f_i)$ look as follows:

$$\begin{aligned} R(f_i) = &\ \mathrm{strip}(L_i) \cap \mathrm{CDC}(L_1) \backslash (X_S^{f_1} + T) \\ = &\ \mathrm{strip}(L_i) \cap (Y_{L_i} + T), \end{aligned}$$

where $\mathrm{strip}(f_i) := T(L_i) + {L_i}^{\perp}$ and

$$Y_{L_i} = \{x \in \mathbb{Z}^2 \mid (x + T) \cap L_i \neq \varnothing\}.$$

An illustration of the construction of the region $R(f_1)$ is given in Figure 1.1.

The areas of v_{f_1} and of $w_S^{f_1}$ are shown in Figure 1.2. Altogether we get

$$\mu(f_1) = v_{f_1} - w_S^{f_1} = 2 - 3/2 = 1/2.$$

Analogously, we can construct $R(f_2)$ and $R(f_3)$ and compute v_{f_2}, v_{f_3}, $w_S^{f_2}$ and $w_S^{f_2}$ to get

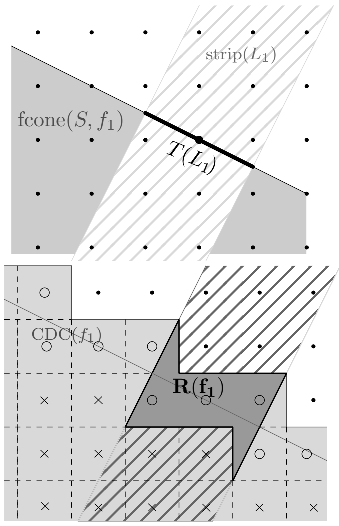


Figure 1.1: Construction of $R(f_1)$. The lattice points marked with an '$\times$' are the points in $X_S^{f_1}$, the ones in Y_{L_1} are marked '$\circ$'.

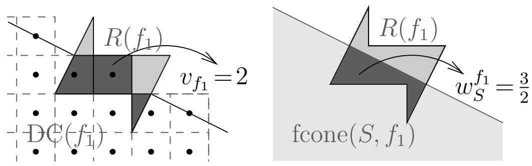


Figure 1.2: v_{f_1} and $w_S^{f_1}$.

$$\begin{aligned} \mu(f_2) &= v_{f_2} - w_S^{f_2} = 2 - \frac{3}{2} = \frac{1}{2}, \\ \mu(f_3) &= v_{f_3} - w_S^{f_3} = 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Later, in Section 3.3, we will see that the values for facets always equal 1/2, as long as the fundamental domains are chosen to be centrally symmetric.

$\mu(\mathbf{f})$ for codimension 2 faces f of P. Let f be a codimension two face of P. Then there are exactly two facets F_1 and F_2 of P that meet in f. That means that $W := \mathrm{fcone}(P, f)$ is a wedge defined by the intersection of the halfspaces $H_1^+ := \mathrm{fcone}(P, F_1)$ and $H_2^+ := \mathrm{fcone}(P, F_2)$, whose lineality spaces are the hyperplanes H_1 and H_2, respectively. W has three faces that are not equal to $H_1 \cap H_2$, namely W itself, $W \cap H_1$ and $W \cap H_2$ with $\mathrm{fcone}(W, W) = V$, $\mathrm{fcone}(W, W \cap H_1) = H_1^+$

and $\mathrm{fcone}(W, W \cap H_2) = H_2^+$. We thus have to consider the sets of generic lattice points for V, H_1^+ and H_2^+:

$$\begin{aligned}
X_V^W &= \{x \in \Lambda \qquad\quad |(x+T) \subseteq \mathrm{int}(W)\ \}.\\
X_{H_1^+}^W &= \{x \in \mathrm{lat}(H_1^+)|(x + R(H_1^+) \subseteq \mathrm{int}(H_2^+) \text{ and for all } x' \in \mathrm{lat}(H_2^+):\\
&\qquad\qquad\qquad\quad (x + R(H_1^+)) \cap (x' + R(H_2^+)) = \varnothing\} \text{ and}\\
X_{H_2^+}^W &= \{x \in \mathrm{lat}(H_2^+)|(x + R(H_2^+) \subseteq \mathrm{int}(H_1^+) \text{ and for all } x' \in \mathrm{lat}(H_1^+):\\
&\qquad\qquad\qquad\quad (x + R(H_2^+)) \cap (x' + R(H_1^+)) = \varnothing\},
\end{aligned}$$

Then the region $R(W)$ is given by

$$\begin{aligned}
R(W) =& \mathrm{strip}(W) \cap \mathrm{CDC}(W)\\
&\setminus \left((X_V^W + T) \cup (X_{H_1^+}^W + R(H_1^+)) \cup (X_{H_2^+}^W + R(H_2^+))\right).
\end{aligned}$$

The relative domain volume is

$$v_W = \mathrm{vol}(R(W) \cap \mathrm{DC}(W)).$$

We also have to consider three correction volumes:

$$\begin{aligned}
w_V^W &= \mathrm{vol}(R(W) \cap W),\\
w_{H_1^+}^W &= \mathrm{vol}(R(W) \cap H_1 \cap W) \text{ and}\\
w_{H_2^+}^W &= \mathrm{vol}(R(W) \cap H_2 \cap W).
\end{aligned}$$

The value of $\mu(W)$ then is

$$\mu(W) = v_W - w_{H_1^+}^W \cdot \mu(H_1^+) - w_{H_2^+}^W \cdot \mu(H_2^+) - w_V^W \cdot \mu(V)$$

$\mu(\mathbf{v_i})$ for the vertices $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ of $\mathbf{S}$ For $i \in \{1, 2, 3\}$, the fcone $C_i := \mathrm{fcone}(S, v_i)$ is a two-dimensional pointed cone with apex in 0. Since $\mathrm{lineal}(C_i) = \{0\}$, the only fundamental domain we can choose for C_i is $T(C_i) = \{0\}$. Since $\mathrm{lineal}(C_i)^\perp = \mathbb{R}^2$, we have $\mathrm{strip}(C_i) = \mathbb{R}^2$. We start with the construction of the region of the vertex v_2. S has three faces g with $v_2 < g \leq S$, namely S itself, f_1 and f_3. Thus, the region $R(v_2)$ is given as

$$\begin{aligned}
R(v_2) = \mathbb{R}^2 \setminus \Big(& (X_{f_1}^{v_2} + T(L_1)) \cup (X_{f_3}^{v_2} + T(L_3))\\
& \cup (X_S^{v_2} + T)\Big).
\end{aligned}$$

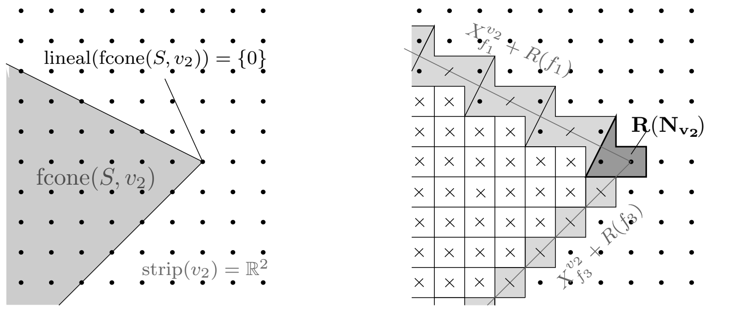


Figure 1.3: Construction of $R(v_2)$.

The construction of $R(v_2)$ is shown in Figure 1.3. One might already see that the union of all regions in the picture forms the set $\mathrm{CDC}(C_2)$, a fact that is the subject of Lemma 1.2.

Having constructed the region, we can directly compute the relative domain volume v_{v_2} and the correction volumes $w_S^{v_2}$, $w_{f_1}^{v_2}$ and $w_{f_2}^{v_2}$, as shown in Figure 1.4. Note that while v_{v_2} and $w_S^{v_2}$ are full-dimensional volumes, the correction volumes $w_{f_1}^{v_2}$ and $w_{f_2}^{v_2}$ are relative volumes taken in the one-dimensional subspaces $L_1 = \mathrm{lineal}(\mathrm{fcone}(S, f_1))$ and $L_2 = \mathrm{lineal}(\mathrm{fcone}(S, f_2))$, respectively. Altogether we get the μ-value for the fcone of S at the vertex v_2 as

$$\begin{aligned}\mu(v_2) &= v_{v_2} - w_{f_1}^{v_2} \cdot \mu(f_1) - w_{f_3}^{v_2} \cdot \mu(f_3) - w_S^{v_2} \cdot \mu(S)\\ &= \frac{7}{4} - \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} - \frac{7}{8} \cdot 1\\ &= \frac{3}{8}.\end{aligned}$$

Analogously, we can compute $\mu(v_3) = 1/4$ and $\mu(v_1) = 3/8$. For reasons of symmetry, the latter has to equal the value $\mu(v_2)$ — see Section 3.1 for details.

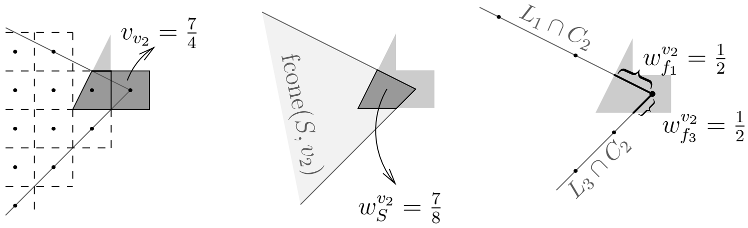


Figure 1.4: v_{v_2}, $w_S^{v_2}$, $w_{f_1}^{v_2}$ and $w_{f_3}^{v_2}$.

1.4 Proof of Theorem 2

In this section, we prove that the function μ we constructed is indeed a local formula. The proof can roughly be divided into two parts. In the first part we show general properties of the regions, the most important ones being that we get a tiling of the covering domain complex of a cone by regions and that the regions are bounded. Boundedness ultimately means that the function μ as a combination of the relative domain volumes and the correction volumes is well defined. The second part then takes the view from a polytope and we use the tiling from the first part to give a tiling of the covering domain complex of the polytope. This tiling then enables us to show that the μ-values on its fcones indeed determine the Ehrhart coefficients as required for a local formula.

1.4.1 Properties of the regions and well-definedness of μ

The most important property we want to show about the regions is that they are bounded, which means that μ is well defined. In order to show this, we need some other properties, especially that for a cone C we can get a tiling of the covering domain complex $\mathrm{CDC}(C)$ by translates of $R(C)$ and of $R(D)$ for $D \prec C$. A picture of this tiling coming arises in the construction of $R(v_2)$ in the Example of the Simplex in Section 1.3 and can be seen in Figure 1.3. This result is given in Lemma 1.2. To prove it, we first show $\mathrm{lat}(C)$-invariance of the set of generic lattice points:

Lemma 1.1. *Let C be a full-dimensional cone and $D \prec C$. Then the set of generic lattice points X_D^C of D is invariant under translation by points in* $\mathrm{lat}(C)$*.*

Proof. Let $y \in \mathrm{lat}(C) = \mathrm{lineal}(C) \cap \Lambda$. We want to show that $x + y$ is in X_D^C if and only if $x \in X_D^C$. Since $\mathrm{int}(H)$ is invariant under translation by $\mathrm{lineal}(C)$ for all halfspaces H with $H \prec C$, in particular under translation by $-y$, it immediately follows that Property (I) is satisfied for x if and only if it is satisfied for $x + y$. Also, for all $E \prec C$, we have $\mathrm{lat}(C) \subseteq \mathrm{lat}(E)$. That means

$$(y + x + R(D)) \cap (x' + R(E)) = \varnothing \qquad \text{for all } x' \in \mathrm{lat}(E)$$

is equivalent to

$$(x + R(D)) \cap (x'' + R(E)) = \varnothing \qquad \text{for all } x'' \in \mathrm{lat}(E)$$

by setting $x'' = -y + x'$. This shows that $x + y$ satisfies Property (II) if and only if x does, which finishes the proof. $\square$

Using this, we can show that we get a tiling of the covering domain complex CDC(C) by regions. This result is a basis for most of the following results and can, in particular be used to create a tiling by regions around the faces of a polytope as shown in Theorem 1 in the following subsection.

Lemma 1.2. *For any full-dimensional cone C we have a tiling*

$$\{x + R(D) : \; D \preceq C, \; x \in X_D^C\}$$

of the covering domain complex CDC(C)*, consisting of lattice point translates of regions.*

Proof. In general, for a lattice L in V (not necessarily of full rank) and subsets $A, B \subseteq V$ with the properties that $A + L = V$ and $B + L = B$ we have that

$$L + (A \cap B) = B. \tag{1.2}$$

We show both inclusions:

- $\subseteq$: $L + (A \cap B) \subseteq L + B = B$.
- $\supseteq$: Let $x \in B$. Since $V = L + A$, we can write $x = l + a$ with $l \in L$ and $a \in A$. Then $a = x - l \in B + L = B$ and hence $x \in L + (A \cap B)$.

By definition $\mathrm{strip}(C) = T(C) + \mathrm{lineal}(C)^{\perp}$ and we have $\mathrm{lat}(C) + \mathrm{strip}(C) = V$. Since $\Lambda \cap C$ is invariant under translation by $\mathrm{lat}(C)$, we have that $\mathrm{CDC}(C) = \{x + T \mid x \in \Lambda \cap C\}$ is invariant under translation by $\mathrm{lat}(C)$. Using Lemma 1.2, the set

$$B := \mathrm{CDC}(C) \setminus \bigcup_{D \prec C} X_D^C + R(D),$$

is $\mathrm{lat}(C)$-invariant. With $L := \mathrm{lat}(C)$ and $A := \mathrm{strip}(C)$ Equation 1.2 then yields

$$\mathrm{lat}(C) + R(C) = \mathrm{CDC}(C) \setminus \bigcup_{D \prec C} X_D^C + R(D). \tag{1.3}$$

Thus, we have

$$\begin{aligned}\mathrm{CDC}(C) =& (\mathrm{lat}(C) + R(C)) \cup \bigcup_{D \prec C} X_D^C + R(D)\\ =& \bigcup_{D \preceq C} X_D^C + R(D),\end{aligned}$$

using $X_C^C = \mathrm{lat}(C)$.

Since $R(C) \subseteq \mathrm{strip}(C)$, we know that the translates of $R(C)$ by points in $\mathrm{lat}(C)$ do not intersect. For each $D \prec C$, the set X_D^C is a subset of $\mathrm{lat}(D)$, so the same argument shows that the sets $\{x + R(D) : \ x \in X_D^C\}$ have pairwise empty intersections. Now we only need to show that for two faces $D, D' \prec C$ the sets of the form $x + R(D)$ and $y + R(D')$ with $x \in X_D^C$ and $y \in X_{D'}^C$ do not intersect. If D or D' is an fcone of the other one, say $D' \prec D$, we have

$$\begin{aligned}
X_D^C + R(D) &\subseteq \operatorname{lat}(D) + R(D) \\
&\overset{(1.3)}{=} \operatorname{CDC}(D) \backslash \left(\bigcup_{E \prec D} (X_E^D + R(E)) \right) \\
&\subseteq \operatorname{CDC}(D) \backslash \left(X_{D'}^D + R(D') \right) \\
&\subseteq \operatorname{CDC}(D) \backslash \left(X_{D'}^C + R(D') \right).
\end{aligned}$$

The last inclusion follows from the property that $X_{D'}^C \subseteq X_{D'}^D$ whenever $D' \prec D \prec C$, since there are just more restrictions in $X_{D'}^C$. If D and D' are incomparable, then $X_D^C + R(D)$ and $X_{D'}^C + R(D')$ do not intersect by Property (II) of X_D^C. □

We can now show that the regions are bounded. This is a highly non-trivial result. We show that there is a radius $r \in \mathbb{R}_{>0}$ such that $R(C)$ is contained in a certain bounded set determined by the radius r. Since efficient computation of the regions relies heavily upon a good estimate for this radius, particular attention is paid to the exact constraints needed for r to be a sufficiently large bound.

Lemma 1.3. *Let C be a full-dimensional cone. Then $R(C)$ is bounded.*

Proof. Since it is used frequently, we denote $L_C := \operatorname{lineal}(C)$ for full-dimensional cones $C \subseteq V$. We prove the lemma inductively going down in the dimension of L_C. We want to show that there exists a certain bounded set that contains $R(C)$. The bounded sets we want to consider are cylinders around L_C with radius $r \in \mathbb{R}_{>0}$:

$$\operatorname{Cyl}(r, C) := (L_C + B_r) \cap \operatorname{strip}(C),$$

where B_r is the open ball around the origin with radius r. Since $T(C) \subseteq L_C$, we can alternatively describe $\operatorname{Cyl}(r_C, C)$ as

$$\operatorname{Cyl}(r, C) = T(C) + (B_r \cap {L_C}^{\perp}).$$

For $C = V$ we simply notice that $R(V) = T$, which is bounded.

Now let C be a full-dimensional cone with $V \prec C$ and we assume that $R(D) \subseteq \operatorname{Cyl}(r_D, D)$ for all faces $D \prec C$ with suitable $r_D \in \mathbb{R}_{>0}$. By construction, $R(C) \subseteq \operatorname{strip}(C)$. What we now need to show is that $R(C) \subseteq (L_C + B_r)$ for some $r \in \mathbb{R}_{>0}$. We choose r defined by the following constraints:

Construction of r:

For each face $D \prec C$ we have $R(D) \subseteq \mathrm{Cyl}(r_D, D)$ by the inductive hypothesis. We further define $\alpha_D, \alpha \in \mathbb{R}_{>0}$ by

$$\alpha_D := r_D + 2 \cdot \max_{E \prec D} r_E$$

and

$$\alpha := \max_{D \prec C} \alpha_D.$$

We recall that by Lemma 1.2 we have a tiling of $\mathrm{CDC}(D)$ such that

$$\mathrm{CDC}(D) = \bigcup_{E \preceq D} \left(X_E^D + R(E) \right).$$

So there exists $\beta \in \mathbb{R}_{>0}$ such that

$$U(D) := \bigcup_{E \preceq D} \bigcup_{x \in X_E^D \cap B_\beta} x + R(E) \tag{1.4}$$

contains $\mathrm{Cyl}(\alpha_D, D) \cap \mathrm{CDC}(D)$. As a finite union of bounded sets, $U(D)$ is bounded and we can find $\gamma_D \in \mathbb{R}_{>0}$ such that $U(D) \subseteq B_{\gamma_D}$.

For $D, E \prec C$ with $C \cap L_D \cap L_E = L_C$ and for all $\varepsilon \in \mathbb{R}_{>0}$, there exists an $\delta(\varepsilon, D, E) \in \mathbb{R}_{>0}$ such that for all $x \in C \cap L_D$:

$$\mathrm{dist}(x, L_C) > \delta(\varepsilon, D, E) \quad \Rightarrow \quad \mathrm{dist}(x, L_E) > \varepsilon$$

We define δ as the maximum

$$\delta := \max\{\delta(\gamma_D + \gamma_E, D, E) \mid D, E \prec C \text{ with } C \cap L_D \cap L_E = L_C\}.$$

Next, we define t as a radius such that $T(D) \subseteq B_t$ for all $D \prec C$. And finally we define r as

$$r := \sqrt{(\delta + 2t)^2 + \alpha^2}.$$

By construction we know that $R(C) \subseteq \mathrm{strip}(C) \cap \mathrm{CDC}(C)$. Let $p \in \mathrm{CDC}(C)$ with $\mathrm{dist}(p, L_C) > r$. To prove the statement, we need to prove that $p \notin R(C)$.

Case 1: For all $V \neq D \prec C$, we have $\mathrm{dist}(p, L_D) \geq \alpha_D$

In particular we have

$$\mathrm{dist}(p, L_H) \geq \alpha_H > 2r_V \tag{1.5}$$

for all halfspaces $H \prec C$. Let $x \in \Lambda$ be the lattice point with $p \in (x + T)$. Since $T \subseteq B_{r_V}$, we have $\mathrm{dist}(p, x) < 2r_V$. Thus, Equation 1.5 shows that either $(x + T) \subseteq C$ or $(x + T) \cap C = \varnothing$. The second case is not possible, since we chose $p \in \mathrm{CDC}(C)$. Hence, $x + T \subseteq C$. Together with Equation 1.5 that means

$$x + T = x + R(V) \subseteq \mathrm{int}(H)$$

for all halfspaces $H \prec C$. Hence, $x \in X_V^C$ and $p \notin R(C)$.

Case 2: There exists a $D \prec C$, with $D \neq V$ and $\operatorname{dist}(p, L_D) < \alpha_D$.
Under all fcones of C with the above property, let D be the maximal one in the fcone order $\prec$.

Define $y := p|_{L_D}$ to be the orthogonal projection of p onto L_D and let $x \in \operatorname{lat}(D)$ with $y \in x + T(D)$. That means $p \in x + \operatorname{Cyl}(\alpha_D, D)$. We also have $p \in \operatorname{CDC}(C) \subseteq \operatorname{CDC}(D)$. Thus, $p \in (x + \operatorname{Cyl}(\alpha_D, D)) \cap \operatorname{CDC}(D)$, which is covered by $x + U(D)$. $p \in x + U(D)$ means there is a $D_1 \preceq D$ and an $a \in X^D_{D_1}$, such that $p \in (x + a + R(D_1))$.

Our goal is to show that $(x + a) \in X^C_{D_1}$, which directly yields

$$p \in (x + a + R(D_1)) \subseteq V \backslash R(C).$$

To this end, we use our carefully defined radii to show that there is a certain minimal distance between x and the linear spaces L_E for all $E \prec C$ incomparable to D.

First of all, the Pythagorean theorem yields

$$\operatorname{dist}(y, L_C)^2 + \operatorname{dist}(p, L_D)^2 = \operatorname{dist}(p, L_C)^2.$$

Using $\operatorname{dist}(p, L_D) < \alpha_D$, $\operatorname{dist}(p, L_C) > r$ and the definition of r then gives us

$$\begin{aligned}
\operatorname{dist}(y, L_C)^2 &= \operatorname{dist}(p, L_C)^2 - \operatorname{dist}(p, L_D)^2 \\
&> r^2 - \alpha_D^2 \quad = (\delta + 2t)^2 + \alpha^2 - \alpha_D^2 \\
&> (\delta + 2t)^2.
\end{aligned}$$

We have now shown that $\operatorname{dist}(y, L_C) > \delta + 2t$ and since $\operatorname{dist}(x, y) < 2t$, we have $\operatorname{dist}(x, L_C) > \delta$. We further have $x \in C \cap L_D$, since D was chosen maximal and otherwise there would have been another cone D' with $D \prec D' \prec C$ with $\operatorname{dist}(p, L_{D'}) < \alpha_{D'}$. By definition of δ, we then get

$$\operatorname{dist}(x, L_E) > \gamma_D + \gamma_E \tag{1.6}$$

for all $E \prec C$ incomparable to D and $C \cap L_D \cap L_E = L_C$.

To show that (1.6) holds for all $E \prec C$ incomparable to D, let $E \prec C$ be incomparable to D with $L_C \subsetneq C \cap L_D \cap L_E$. Then there is a cone C' with $D, E \prec C' \prec C$ and $C' \cap L_D \cap L_E = L_{C'}$. This cone C' is the join of D and E, more information on this can be found in Section 5.2.

By minimality of L_D, we have that

$$\operatorname{dist}(p, L_{C'}) > \alpha_{C'} > r_{C'}.$$

Then with exactly the same arguments as above, substituting C by C' in every step on the way, we also get

$$\operatorname{dist}(x, L_E) > \gamma_D + \gamma_E.$$

Together with (1.6), it shows that

$$\operatorname{dist}(x, L_E) > \gamma_D + \gamma_E \tag{1.7}$$

for all $E \prec C$ incomparable to D.

Since $(x + a + R(D_1)) \subseteq U(D) \subseteq B_{\gamma_D}$ and $R(E) \subseteq B_{r_E} \subseteq B_{\gamma_E}$, we have

$$(a + R(D_1)) \cap (b + R(E)) = \varnothing$$

for all $E \prec C$ incomparable to D and $b \in \operatorname{lat}(E) \subseteq L_E$.

If there is an $E \prec C$ comparable to D but not to D_1, transitivity of '$\prec$' only admits the case $E \prec D$. We have $a \in X^D_{D_1}$ and thus by Lemma 1.1 also $(x + a) \in X^D_{D_1}$. Hence,

$$(a + R(D_1)) \cap (b + R(E)) = \varnothing$$

for all $E \prec D$ incomparable to D_1 and $b \in \operatorname{lat}(E) \subseteq L_E$ by Property (II).

Let $H \prec C$ be a halfspace with $H \not\preceq D$. Then $(x + B_{\gamma_D}) \cap L_H = \varnothing$ by Equation (1.7). That means $x + B_{\gamma_D}$ is either completely inside or completely outside of H, not intersecting the boundary. Assuming it is outside means that p is also outside. But since also $p \in \operatorname{CDC}(C)$, we have $\operatorname{dist}(p, L_H) < 2r_V < \gamma_H$ for all halfspaces $H \prec C$ with $H \not\preceq D$, which is a contradiction to Inequality (1.7). Hence, $x + B_{\gamma_D} \subseteq H$ and therefore also $x + a + R(D_1)$ is in $\operatorname{int}(H)$.

Now if $H \prec C$ with $H \prec D$ but $H \not\preceq D_1$, we can use Property (I) of $X^D_{D_1}$ for $x + a$ and, again, get

$$(x + a + R(D_1)) \subseteq \operatorname{int}(H).$$

Together it shows that $x + a$ also complies with Property (I) of X^C_D and thus we have shown $(x + a) \in X^C_D$. That means $p \in (x + a + R(D_1)) \subseteq V \backslash R(C)$.

Hence, we have shown that $R(C)$ is bounded. □

The following two results are further properties of the regions that we will need later on.

Lemma 1.4. *Let C be a full-dimensional cone. Then $T(C) \subseteq R(C)$.*

Proof. For $C = V$, we have $R(V) = T(V)$ and nothing is show.

Otherwise,

$$R(C) = \operatorname{strip}(C) \cap \operatorname{CDC}(C) \backslash \bigcup_{D \prec C} (X^C_D + R(D)).$$

$T(C)$ is the fundamental domain in $\operatorname{lineal}(C)$ and $\operatorname{lineal}(C) \subseteq \operatorname{CDC}(C)$. Also, $T(C) \subseteq \operatorname{strip}(C) = T(C) + \operatorname{lineal}(C)^{\perp}$. Therefore, we only need to check that

$$(X^C_D + R(D)) \cap T(C) = \varnothing, \quad \text{for all } D \prec C.$$

Since $T(C) \subseteq \mathrm{lineal}(C)$, it suffices to show that

$$(X_D^C + R(D)) \cap \mathrm{lineal}(C) = \varnothing, \qquad \text{for all } D \prec C.$$

For each $D \prec C$, there is at least one halfspace $H \prec C$ that does not contain D and thus for all $x \in X_D^C$ we have

$$(x + R(D)) \subseteq \mathrm{int}(H)$$

by property (I) for X_D^C. Using $\mathrm{lineal}(C) \subseteq \mathrm{bd}(H)$, we conclude

$$(X_D^C + R(D)) \cap \mathrm{lineal}(C) = \varnothing,$$

as we wanted to show. □

Lemma 1.5. $X_D^C \subseteq C \cap \mathrm{lineal}(D)$.

Proof. By construction, $X_D^C \subseteq \mathrm{lat}(D) = \Lambda \cap \mathrm{lineal}(D)$. To show that $X_D^C \subseteq C$, we use the fact that

$$C = \bigcap_{\substack{H \prec C \\ H \text{halfspace}}} H,$$

from Section 5.2. We already have $X_D^C \subseteq \mathrm{lineal}(D) \subseteq H$ for all halfspaces $H \preceq D$. For a halfspace $H \prec C$ with $H \not\preceq D$ Property (I) of X_D^C yields $x+R(D) \subseteq \mathrm{int}\, H$ for all $x \in X_D^C$. Since by Lemma 1.4 $x \in x+T(D) \subseteq x+R(D)$, we have $X_D^C \subseteq C$. □

1.4.2 From fcones to Ehrhart coefficients

In this section, we consider a full-dimensional lattice polytope P and its dilates tP for $t \in \mathbb{Z}_{>0}$. To shorten notation, for faces $f \leq P$ we write $R(f)$ instead of $R(\mathrm{fcone}(P, f))$ and for $f < g \leq P$ we write X_g^f instead of $X_{\mathrm{fcone}(P,g)}^{\mathrm{fcone}(P,f)}$, but keep in mind that these sets do not depend on the faces, but only their fcones. Note that in particular $\mathrm{fcone}(P, f) = \mathrm{fcone}(tP, tf)$ for all $f \leq P$ and $t \in \mathbb{Z}_{>0}$ (cf. Section 5.2), so that we also have $R(f) = R(\mathrm{fcone}(tP, tf))$.

To gain a tiling of the covering domain complex of P, we compose the tilings from Lemma 1.2 for the fcones of the vertices using the following sets of generic lattice points in a face:

Definition 3. For a face $f \leq P$ define $\mathcal{X}(tf) \subseteq \Lambda$, the set of all *generic lattice points in tf*, as the following:

$$\mathcal{X}(tf) := \bigcap_{v \text{ vertex of } f} X_f^v + tv. \tag{1.8}$$

Since $X_v^v = \mathrm{lat}(\mathrm{fcone}(P, v)) = \{0\}$ for a vertex v of P, this definition is also valid for $\mathcal{X}(tv)$ and yields $\mathcal{X}(tv) = tv$.

Since dilated vertices are lattice points, the sets $\mathcal{X}(tf)$ are indeed subsets of Λ and they are also subsets of tf as we can quickly show:

Lemma 1.6. *For all $f \leq P$ and $t \in \mathbb{Z}_{>0}$, we have $\mathcal{X}(tf) \subseteq tf$.*

Proof. By applying Lemma 1.5, we get

$$X_f^v \subseteq \operatorname{lineal}(\operatorname{fcone}(P, f)) \cap \operatorname{fcone}(P, v) = \operatorname{fcone}(f, v).$$

Using

$$\bigcap_{v \text{ vertex of } P} \operatorname{fcone}(P, v) + v = P,$$

thus yields

$$\mathcal{X}(tf) \subseteq \bigcap_{v \text{ vertex of } f} \operatorname{fcone}(f, v) + tv = tf.$$

More information on the properties of fcones, among which are the ones we used here, can be found in Section 5.2. □

Theorem 1. *Let $P \subseteq V$ be a full-dimensional lattice polytope. Then there exists a $t_0 \in \mathbb{Z}_{>0}$ such that for each $t \geq t_0$ we have a tiling of $\operatorname{CDC}(tP)$ into translated regions of the form*

$$\{x + R(f) \mid \ f \leq P,\, x \in \mathcal{X}(tf)\}. \tag{1.9}$$

Proof. Let P be a full-dimensional lattice polytope with vertices $v_1, \ldots, v_m \in \Lambda$. We start by specifying t_0. Lemma 1.3 yields that $R(f)$ is bounded for each $f \leq P$. Thus, for any $f, g \leq P$ that do not intersect, there is a $t_{fg} \in \mathbb{Z}_{>0}$ such that $(R(f) + t_{fg} \cdot f) \cap (R(g) + t_{fg} \cdot g) = \varnothing$. We set

$$t_0 = \max\{t_{fg} \mid \ f, g \leq P \text{ and } f \cap g = \varnothing\}.$$

By Lemma 1.6 we know that $\mathcal{X}(tf) \subseteq tf$ for all $f \leq P$. Thus the choice of t_0 shows that for all $t \in \mathbb{Z}_{>0}$ with $t \geq t_0$ the sets $x + R(f)$ and $y + R(g)$ with $x \in \mathcal{X}(tf)$ and $y \in \mathcal{X}(tg)$ have empty intersections.

If $f \cap g \neq \varnothing$, we find $j \in \{1, \ldots, m\}$ such that $v_j \in f \cap g$. Then for $x \in \mathcal{X}(tf)$ and $y \in \mathcal{X}(tg)$, we have $(x - tv_j) \in X_f^{v_j}$ and $(y - tv_j) \in X_g^{v_j}$. By Lemma 1.2 for $\operatorname{fcone}(P, v_j)$, the sets $(x - tv_j) + R(f)$ and $(y - tv_j) + R(g)$ do not intersect, which then also holds for their translations $x + R(f)$ and $y + R(g)$.

It remains to show that (1.9) is indeed a covering of $\mathrm{CDC}(tP)$. To this end, let $p \in \mathrm{CDC}(tP)$ be an arbitrary point. Translating the sets in Lemma 1.2 by the respective vertex yields

$$tv_i + \mathrm{CDC}(\mathrm{fcone}(P, v_i)) = \bigcup_{\substack{f \leq P \\ v_i \in f}} \left(X_f^{v_i} + tv_i + R(f) \right)$$

for each vertex v_i of P. Hence, for each $i \in \{1, \ldots, m\}$ we find $f_i \leq P$ with $v_i \in f_i$ and $x_i \in X_{f_i}^{v_i}$ such that $p \in (x_i + tv_i + R(f_i))$.

Let v_i be a vertex, such that f_i is smallest in dimension. Without loss of generality we can assume $i = 1$. Then we have $p \in (x_1 + tv_1 + R(f_1))$. We want to show that $x_1 + tv_1 \in \mathcal{X}(tf_1)$, because then $x_1 + tv_1 + R(f_1)$ is an element of the set in (1.9) and contains p.

Let's assume this is not the case. After possibly renumbering, we can assume that $(x_1 + tv_1) \notin (X_{f_1}^{v_2} + tv_2)$. In particular, we have $v_2 \in f_1$.

But then we can find $f_2 \leq P$ with $v_2 \in f_2$ and $x_2 \in X_{f_2}^{v_2}$ such that $p \in (x_2 + tv_2 + R(f_2))$. This yields

$$p \in (x_2 + tv_2 + R(f_2)) \cap (x_1 + tv_1 + R(f_1)) \,,$$

and thus

$$(x_2 + R(f_2)) \cap (\underbrace{x_1 + tv_1 - tv_2}_{\in \mathrm{lat}(f_1)} + R(f_1)) \neq \varnothing, \tag{1.10}$$

which contradicts $x_2 \in X_{f_2}^{v_2}$ by property ((II)), unless f_1 and f_2 are comparable.

The case $f_2 \subsetneq f_1$ is not possible, since $\dim(f_2) \geq \dim(f_1)$ by assumption on the minimality of the dimension of f_1. The case $f_1 = f_2$ is not possible either, since $x_2 + tv_2 + R(f_1)$ and $x_1 + tv_1 + R(f_1)$ can only intersect if $x_2 + tv_2 = x_1 + tv_1$, in which case $x_1 + tv_1 \in (X_{f_2}^{v_2} + tv_2)$.

We are left with the case $f_1 \subseteq f_2$ to be excluded. We now can consider $X_{f_2}^{f_1}$ and have the inclusion $X_{f_2}^{v_2} \subseteq X_{f_2}^{f_1}$ (since we only add conditions when going from $X_{f_2}^{f_1}$ to $X_{f_2}^{v_2}$). Since $x_2 \in X_{f_2}^{v_2}$, we also have $x_2 \in X_{f_2}^{f_1}$. Then the sets $(x_1 + tv_1 - tv_2 + R(f_1))$ and $(x_2 + R(f_2))$ are part of the tiling that we get by applying Lemma 1.2 to $\mathrm{fcone}(P, f_1)$. But, as we see in equation (1.10), the two sets do intersect, which is a contradiction. So the assumption was wrong and we have shown that the sets in (1.9) cover $\mathrm{CDC}(tP)$.

□

Theorem 2. *The function μ on full-dimensional rational cones in V as defined in Section 1.2 is a local formula for Ehrhart coefficients. That is, for every lattice polytope P with Ehrhart polynomial $\mathrm{E}_P(t) = e_d t^d + e_{d-1}t^{d-1} + \cdots + e_1 t + e_0$, $t \in \mathbb{Z}_{\geq 0}$, we have*

$$e_i = \sum_{\substack{f \leq P \\ \dim(f)=i}} \mu(\mathrm{fcone}(P, f))\mathrm{vol}(f),$$

for all $i \in \{0, \ldots, d\}$.

We recall the definition of μ for full-dimensional cones $C \subseteq V$ as

$$\mu(C) = v_C - \sum_{D \prec C} w_D^C \cdot \mu(D),$$

where v_C is the relative domain volume given by

$$v_C = \mathrm{vol}(R(C) \cap \mathrm{DC}(C))$$

and w_D^C is the correction volume for $D \prec C$, given as

$$w_D^C = \mathrm{vol}(R(C) \cap C \cap \mathrm{lineal}(D)).$$

Analogously to $R(f)$ and X_g^f we also abbreviate $\mu(f) := \mu(\mathrm{fcone}(P, f))$, $v_f := v_{\mathrm{fcone}(P,f)}$ and $w_g^f := w_{\mathrm{fcone}(P,g)}^{\mathrm{fcone}(P,f)}$.

Proof. To make the structure of the proof easier to grasp, we delay some steps into lemmas, that are shown subsequently.

Recall that T is a fundamental domain of Λ. Since the relative volume is normalized, such that every fundamental domain has volume 1, we have the following equation for every $t \in \mathbb{Z}_{\geq 0}$:

$$|tP \cap \Lambda| = \mathrm{vol}((tP \cap \Lambda) + T) = \mathrm{vol}(\mathrm{DC}(tP)). \tag{1.11}$$

Instead of counting the (discrete) number of lattice points in tP, we thus can compute the (continuous) volume of fundamental domains around each lattice point in tP.

Let $t \in \mathbb{Z}_{>0}$ be big enough, such that we have a tiling of $\mathrm{CDC}(tP)$ by regions as in Theorem 1:

$$\{x + R(f) \mid \ f \leq P, \ x \in \mathcal{X}(tf)\}.$$

We will show in Lemma 1.7 below that we can divide the volume of the domain complex into the parts in each region, which equals the DC-volume in $R(f)$:

$$\mathrm{vol}(\mathrm{DC}(tP)) = \sum_{f \leq P} |\mathcal{X}(tf)| \cdot v_f.$$

We have now divided the number of lattice points into a part that is purely in tf, namely $\mathcal{X}(tf)$, and a part that is only depending on the fcone, namely $v_f = v_{\text{fcone}(P,f)}$, for each $f \leq P$. But $|\mathcal{X}(tf)|$ is only an (integer) approximation of $\text{vol}(tf)$. This can be corrected by transferring a part of v_f to $|\mathcal{X}(tf)|$, which is where the correction volume comes into play.

With the definition of $\mu(f)$ solved for v_f we get

$$\begin{aligned} |\Lambda \cap tP| &= \sum_{f \leq P} |\mathcal{X}(tf)| \cdot v_f \\ &= \sum_{f \leq P} \left[|\mathcal{X}(tf)| \cdot \left(\mu(f) + \sum_{h > f} w_h^f \cdot \mu(h) \right) \right]. \end{aligned}$$

Note that for faces f, h of P we have $f < h$ if and only if $\text{fcone}(P, h) \prec \text{fcone}(P, f)$. See Section 5.2 for more information.

We can now expand the product and combinatorially rearrange the sum to single out $\mu(f)$:

$$\begin{aligned} |\Lambda \cap tP| &= \sum_{f \leq P} \left[|\mathcal{X}(tf)| \cdot \mu(f) + |\mathcal{X}(tf)| \cdot \sum_{h > f} w_h^f \cdot \mu(h) \right] \\ &= \sum_{f \leq P} \underbrace{\left[\sum_{g \leq f} |\mathcal{X}(tg)| \cdot w_f^g \right]}_{=: V(tf)} \cdot \mu(f). \end{aligned}$$

In the last line the expression w_f^f for the correction volume technically has not been defined yet — we simply set $w_f^f := 1$ for faces $f \leq P$, which is a consistent extension of the definition of the correction volume.

In Lemma 1.9 below we show that indeed we have

$$V(tf) = \text{vol}(tf),$$

which yields

$$|tP \cap \Lambda| = \sum_{f \leq P} \text{vol}(tf) \cdot \mu(f) \tag{1.12}$$

for all $t \in \mathbb{Z}_{\geq 0}$ with $t > t_0$ for a certain $t_0 \in \mathbb{Z}_{\geq 0}$. By Ehrhart's Theorem [Ehr62], we know that $\text{E}_P(t) = |tP \cap \Lambda|$ is a polynomial in t, as is the right hand side of Equation (1.12). Since these polynomials agree for infinitely many t, we have equality and get

$$\text{E}_P(t) = \sum_{f \leq P} \mu(N_f)\text{vol}(tf)$$

for all $t \in \mathbb{Z}_{\geq 0}$. $\square$

Lemma 1.7. *We have*

$$\mathrm{vol}(\mathrm{DC}(tP)) = \sum_{f \leq P} |\mathcal{X}(tf)| \cdot v_f$$

for all $t \in \mathbb{Z}_{\geq 0}$ big enough in the sense of Theorem 1.

Proof. Let $t \in \mathbb{Z}_{\geq 0}$ be big enough, such that by Theorem 1 we have a tiling of the covering domain complex $\mathrm{CDC}(tP)$ into regions:

$$\{x + R(N_f) : \; f \leq P, \, x \in \mathcal{X}(tf)\}.$$

To compute the volume of the domain complex $\mathrm{DC}(tP)$ we can thus compute the volume in each region and add everything up:

$$\mathrm{vol}(\mathrm{DC}(tP)) = \sum_{f \leq P} \sum_{x \in \mathcal{X}(tf)} \mathrm{vol}\big((x + R(f)) \cap \mathrm{DC}(tP)\big). \tag{1.13}$$

We recall the definition of the relative domain volume:

$$v_f = \mathrm{vol}(R(f) + \mathrm{DC}(\mathrm{fcone}(P, f))). \tag{1.14}$$

Our aim is to show that

$$\mathrm{vol}\big((x + R(f)) \cap \mathrm{DC}(tP)\big) = v_f$$

for all $x \in \mathcal{X}(tf)$. Since volume is translation invariant and $\mathrm{DC}(\;)$ commutes with translation by lattice points, we have

$$v_f = \mathrm{vol}((x + R(f)) \cap \mathrm{DC}(x + \mathrm{fcone}(P, f))). \tag{1.15}$$

Since $x \in \mathcal{X}(tf) \subseteq tf$ by Lemma 1.6, we have $tP \subseteq x + \mathrm{fcone}(P, f)$ and thus

$$(x + R(f)) \cap \mathrm{DC}(tP) \subseteq (x + R(f)) \cap \mathrm{DC}(x + \mathrm{fcone}(P, f)) \tag{1.16}$$

To show the reverse inclusion, let $y \in ((x + \mathrm{fcone}(P, f)) \cap \Lambda)$ and assume that $y \notin tP$. We want to show that

$$(x + R(f)) \cap (y + T) = \varnothing. \tag{1.17}$$

$y \notin tP$ means there is a vertex v of P with $y \notin tv + \mathrm{fcone}(P, v)$. $v = f$ is not possible since then $x = tv$ which contradicts the assumption $y \in x + \mathrm{fcone}(P, f)$. If v is not a vertex of f, we can enlarge t, such that we can assume the boundary of $tv + \mathrm{fcone}(P, v)$ to be arbitrarily far away from tf and thus also from the bounded set $x + R(f)$, since $x \in \mathcal{X}(tf) \subseteq tf$. Thus, Equation (1.17) holds, since $x + R(f) \subseteq tv + \operatorname{fcone}(P, v)$, but $y \nsubseteq tv + \operatorname{fcone}(P, v)$.

We are left to show Equation (1.17) for $y \notin tv + \operatorname{fcone}(P, v)$ for a vertex v of f not equal to f. $v < f$ means there is a facet F of P containing v but not f, such that the translated halfspace $tv + H := tv + \operatorname{fcone}(P, F)$ does not contain y. If $y \notin \operatorname{CDC}(tv + H)$, then $(y + T) \cap (tv + H) = \varnothing$, but since $x - tv \in X_f^v$, we have $x + R(f) \subseteq tv + H$ due to Property (I) and thus Equation (1.17) holds.

Otherwise, $y \in \operatorname{CDC}(tv + H)$, but $y \notin tv + H$, means $y - tv \notin X_P^F$ and thus $y - tv + T \subseteq \operatorname{lat}(H) + R(F)$ by Lemma 1.2. Since $x - tv \in X_f^v$ and by Property (II) of X_f^v we have $(x - tv + R(f)) \cap (\operatorname{lat}(H) + R(F)) = \varnothing$. That yields $(x - tv + R(f)) \cap (y - tv + T) = \varnothing$ and thus Equation (1.17) holds.

Hence, equality holds in Equation (1.16):

$$(x + R(f)) \cap \operatorname{DC}(tP) = (x + R(f)) \cap \operatorname{DC}(x + \operatorname{fcone}(P, f)) \tag{1.18}$$

Summarized we have shown that

$$\begin{aligned}
\operatorname{vol}(\operatorname{DC}(tP)) &= \sum_{f \le P} \sum_{x \in \mathcal{X}(tf)} \operatorname{vol}\big((x + R(f)) \cap \operatorname{DC}(tP)\big) \\
&\overset{(1.18)}{=} \sum_{f \le P} \sum_{x \in \mathcal{X}(tf)} \operatorname{vol}((x + R(f)) \cap \operatorname{DC}(x + \operatorname{fcone}(P, f))) \\
&\overset{(1.15)}{=} \sum_{f \le P} \sum_{x \in \mathcal{X}(tf)} \operatorname{vol}(R(f) \cap \operatorname{DC}(\operatorname{fcone}(P, f))) \\
&\overset{(1.14)}{=} \sum_{f \le P} \sum_{x \in \mathcal{X}(tf)} v_f \\
&= \sum_{f \le P} |\mathcal{X}(tf)| \cdot v_f,
\end{aligned}$$

which finishes the proof. □

Lemma 1.8. *There exists a $t_0 \in \mathbb{Z}_{>0}$ such that for each $t \geq t_0$ the dilation of a face $f < P$ by t satisfies*

$$tf \subseteq \bigcup_{g \le f} (\mathcal{X}(tg) + R(g)) \, .$$

Proof. For t_0 big enough, Theorem 1 yields that

$$tf \subseteq \bigcup_{g \le P} (\mathcal{X}(tg) + R(g)) \tag{1.19}$$

for all $t \geq t_0$. We need to show that in (1.19) the translated regions of faces g of P with $g \not\leq f$ do not intersect tf. We consider the two cases $f \cap g = \varnothing$ and $f \cap g \neq \varnothing$. First, let $g \leq P$ such that g does not intersect f. Since $R(g)$ is bounded and $\mathcal{X}(tg) \subseteq tg$ by Lemma 1.6, we can choose t_0 big enough to ensure that $(x + R(g)) \cap tf = \varnothing$ for all $t \geq t_0$ and $x \in \mathcal{X}(tg)$.

For the second case, let $g \leq P$ such that g does intersect f. Then there exists a vertex $v \in f \cap g$. Since $g \not\leq f$, we either have $f < g$ or f and g are incomparable. Either way, we find a facet F of P that contains f but not g. Since $v \leq f$, we also have $v \leq F$. Let $t \geq t_0$ and let $x \in \mathcal{X}(tg)$. Then $(x - tv) \in X_g^v$. In particular (by property ((I))), that means $((x - tv) + R(g)) \subseteq \operatorname{int}(\operatorname{fcone}(P, F))$. But $tf - tv$ is on the boundary of $\operatorname{fcone}(P, F)$ and we get that $(x - tv + R(g)) \cap (tf - tv) = \varnothing$ and hence, $(x + R(g)) \cap tf = \varnothing$.

Hence, we have shown that for all faces g with $g \not\leq f$ the intersection $tf \cap (\mathcal{X}(tg) + R(g))$ is empty, which leaves The covering in Equation (1.19) as

$$tf \subseteq \bigcup_{g \leq f} (\mathcal{X}(tg) + R(g))$$

as we wanted to show. $\square$

Lemma 1.9. *There exists a $t_0 \in \mathbb{Z}_{>0}$ such that for each $t \geq t_0$ and every face $f < P$, we have*

$$\operatorname{vol}(tf) = \sum_{g \leq f} |\mathcal{X}(tg)| \cdot w_f^g = |\mathcal{X}(tf)| + \sum_{g < f} |\mathcal{X}(tg)| \cdot w_f^g.$$

In other words, the volume of tf is given by $|\mathcal{X}(tf)|$, the number of generic lattice points in tf, plus the correction volumes times $|\mathcal{X}(tg)|$ for each face $g < f$.

Proof. We recall that for $g < f$, the correction volume w_f^g is defined by

$$w_f^g = \operatorname{vol}\left(R(g) \cap \operatorname{fcone}(P, g) \cap \operatorname{lineal}(\operatorname{fcone}(P, f))\right)$$

and $w_f^f = 1$. From Lemma 1.8 we get a covering of tf, which by Theorem 1 is also disjoint, such that

$$\begin{aligned}\operatorname{vol}(tf) &= \operatorname{vol}\left(\bigcup_{g \leq f} ((\mathcal{X}(tg) + R(g)) \cap tf)\right)\\ &= \sum_{g \leq f} \sum_{x \in \mathcal{X}(tg)} \operatorname{vol}\left((x + R(g)) \cap tf\right).\end{aligned}$$

It thus suffices to show that

$$\operatorname{vol}\left((x + R(g)) \cap tf\right) = w_f^g \tag{1.20}$$

for all $g \leq f$ and $x \in \mathcal{X}(tg)$.

For a facet F of P, let A_F be the affine hyperplane that contains F. Let further A_F^+ be the affine halfspace bounded by A_F that contains P. Then

$$tf = \bigcap_{\substack{F \text{ facet of } P \\ f \leq F}} A_F \quad \cap \bigcap_{F \text{ facet of } P} A_F^+.$$

Let $g \leq f$ and $x \in \mathcal{X}(tg)$, then since $x \in tg \subseteq tf$ (Lemma 1.6), we have

$$(x + \mathrm{lineal}(\mathrm{fcone}(P, f))) \cap (x + \mathrm{fcone}(P, g)) = \bigcap_{\substack{F \text{ facet of } P \\ f \leq F}} A_F \quad \cap \bigcap_{\substack{F \text{ facet of } P \\ g \leq F}} A_F^+$$

By translation invariance of the volume we have

$$w_f^g = \mathrm{vol}\left((x + R(g)) \cap (x + \mathrm{lineal}(\mathrm{fcone}(P, f))) \cap (x + \mathrm{fcone}(P, g))\right).$$

To proof that Equation (1.20) holds, we thus need to show that $A_F^+ \cap (x + R(g)) = \varnothing$ for all facets F of P that do not contain g. Let F be such a facet of P. If $F \cap g \neq \varnothing$, then there is a vertex v of P contained in F as well as in g. Since $x \in \mathcal{X}(tg)$, $x - tv \in X_g^v$ and by Property (I) of X_g^v, we get that

$$x - tv + R(g) \subseteq \mathrm{int}(A_F^+ - tv). \tag{1.21}$$

If $F \cap g = \varnothing$, we can assume t_0 big enough, such that $tg + R(g) \subseteq A_F^+$ and in particular $x + R(g) \subseteq A_F^+$, for all $t \geq t_0$.

□

Chapter 2

Rational polytopes

Let P be a d-dimensional rational polytope with respect to the lattice Λ. That is, the vertices of P have rational coordinates with respect to a lattice basis. The least common multiple of the denominators of these coordinates is called the *denominator of* P. As in the lattice polytope case, we can consider the function $\mathrm{E}_P \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ that counts the lattice points in dilations of P by a nonnegative integer t,

$$\mathrm{E}_P(t) := |tP \cap \Lambda|, \qquad\qquad \text{for } t \in \mathbb{Z}_{\geq 0}.$$

In contrast to the case of lattice polytopes, $\mathrm{E}_P(t)$ can not necessarily be described as a polynomial. An easy way to see this is to consider a zero-dimensional polytope (i.e. a point) that is not a lattice polytope, for example $P_0 = \{\frac{1}{2}\} \subseteq \mathbb{R}$ in the Euclidean line $\mathbb{R}$ with lattice $\mathbb{Z}$. Then the dilation $t \cdot P_0$ by a nonnegative integer t contains one lattice point if t is even and none if t is odd. Hence, E_{P_0} is not the zero function but it has infinitely many roots, so that it cannot be a polynomial. However, if we only consider even $t \in \mathbb{Z}_{\geq 0}$ or, respectively only odd t, we see that we can split E_{P_0} into two (constant) polynomials depending on the parity of the input:

$$\mathrm{E}_{P_0} = \begin{cases} 1, & t \text{ even}, \\ 0 & t \text{ odd}. \end{cases}$$

More general, we call such a function a *qusipolynomial*:

Definition 4 ([BR15])**.** A *quasipolynomial* Q is an expression of the form

$$Q(t) = c_n(t)t^n + c_{n-1}(t)t^{n-1}, \dots, c_0(t),$$

where $c_0, \dots, c_n$ are periodic functions in t. The least common period of $c_0, \dots, c_n$ is called the *period of* Q. Alternatively, for a quasipolynomial Q, there exists a positive integer N and polynomials $p_0, p_1, \dots, p_{N-1}$ such that $Q(t) = p_i(t)$, when $t \equiv i \mod N$. The minimal such N is the period of Q and for this minimal N the polynomials $p_0, p_1, \dots, p_{N-1}$ are called the *constituents* of Q. The *degree* of Q is the maximal degree of its constituents.

In this definition, it is obvious that E_{P_0} is a quasipolynomial. This is no coincidence. It is also due to Ehrhart [Ehr62], that for rational polytopes E_P is a quasipolynomial.

2.1 Local formulas for Ehrhart quasipolynomials

As shown by McMullen [McM83] and realized by Berline and Vergne [BV07], it is still possible to attain a kind of local formulas for Ehrhart quasipolynomials. In this case, however, as further information the *translation class* $\mathrm{trl}(f, \Lambda)$ of the face f with respect to the lattice Λ, as defined below, is needed. A *local formula for Ehrhart quasipolynomials* is a real valued function μ^* on cones with translation classes modulo the lattice such that for each rational polytope P we have

$$\mathrm{E}_P(t) = \sum_{f \leq P} \mu^*(\mathrm{fcone}(P, f), \mathrm{trl}(t, f)\Lambda)\mathrm{vol}(tf).$$

To define a local formula μ^*, we give a very similar construction as the one in Section 1.2, but in a 'shifted' way, by translating it by a representative of the translation class. In order to distinguish between the construction before, we call this the *affine version*, as opposed to the version before, which we here refer to as the *linear version*. In the linear version, the objects we considered, i.e. fundamental domains, domain complex, regions and so on all contained zero. In the affine version this will not necessarily be the case, which justifies the name. Whenever we consider an object that we essentially have seen before, but that we now define with translated objects, we will denote it with a '$*$'.

To define the translation class, we recall that for a subset $A \subseteq V$, the affine span of A, $\mathrm{aff}(A)$, is the smallest affine subspace containing A. We denote by $\mathrm{lin}(A)$ the linear subspace parallel to $\mathrm{aff}(A)$. As before, $\mathrm{lineal}(A)$ is the biggest linear subspace contained in A.

Definition 5. Given a subset $B \subseteq V$, the *translation class* of B with respect to the lattice Λ is the set of all points p such that the affine hull of B translated by p contains lattice points:

$$\mathrm{trl}(B, \Lambda) = \{p \in V \mid (-p + \mathrm{aff}(B)) \cap \Lambda \neq \varnothing\}.$$

Given a linear subspace $U \subseteq V$, we call a set $\mathrm{Trl} \subseteq V$ *a translation class of* U *(w.r.t.* Λ*)* if there is an affine space $A \subseteq V$ with $\mathrm{lin}(A) = U$ and $\mathrm{trl}(A, \Lambda) = \mathrm{Trl}$.

To give a concrete construction, we choose representatives of translation classes. In order to ensure that the choices fit together, we have to assume certain representatives to be compatible in the following sense:

Definition 6. Given a set $\mathcal{U}$ of linear subspaces and for each $U \in \mathcal{U}$ translation classes Trl_U of U w.r.t. Λ, we call a choice of representatives $p(U) \in \mathrm{Trl}_U$ of the translation classes for each $U \in \mathcal{U}$ *compatible* if

$$(p(U_1) + U_1) \subseteq (p(U_2) + U_2) \cap (p(U_3) + U_3)$$

whenever $U_1, U_2, U_3 \in \mathcal{U}$ with $U_1 \subseteq U_2 \cap U_3$.

2.2 Construction of μ on cones with translation classes

For each pair (U, Trl) with U a linear subspace of V and Trl a translation class of U with respect to Λ, we choose and fix a fundamental domain $T(U)$ as well as a representative $p(U, \mathrm{Trl}) \in \mathrm{Trl}$.

Let $C \subseteq V$ be a cone and Trl a translation class of $\mathrm{lineal}(C)$ with respect to Λ. Then for all $D \preceq C$ there is a unique translation class Trl_D of $\mathrm{lineal}(D)$ with $\mathrm{Trl} \subseteq \mathrm{Trl}_D$. As long as C and Trl are clear from the context, for $D \preceq C$ we write $p(D)$ instead of $p(\mathrm{lineal}(D), \mathrm{Trl})$ and define *the affine fundamental domain in* $p(D) + \mathrm{lineal}(D)$ to be

$$T^*(D) := p(D) + T(\mathrm{lineal}(D)).$$

In the following, we further assume the set of representatives $\{p(D) \mid D \preceq C\}$ to be compatible. Also, we write D^* for the translated cone $p(D) + D$, for all $D \preceq C$.

If C is not full-dimensional in V and $\mathrm{aff}(C^*) \cap \Lambda = \varnothing$, we set $R^*(C, \mathrm{Trl}) := \varnothing$. If C is not full-dimensional, but $\mathrm{aff}(C^*)$ does contain lattice points, we intersect everything with $\mathrm{aff}(C^*)$, so that we can simplify the construction by assuming $\mathrm{aff}(C^*)$ to be our ambient space with lattice $\Lambda \cap \mathrm{aff}(C^*)$.

So let $C \subseteq V$ be full-dimensional. We denote $T^* := T^*(V)$ for the (affine) fundamental domain of $V \preceq C$. For a rational polyhedron Q, we adjust the domain complex and the covering domain complex to an affine version:

$$\mathrm{DC}^*(Q) = \bigcup_{x \in \Lambda \cap Q} x + T^*$$

$$\mathrm{CDC}^*(Q) = \bigcup_{\substack{x \in \Lambda \\ (x+T^*) \cap Q \neq \varnothing}} x + T^*.$$

And the *affine strip* of (C, Trl) is defined analogously to the strip before as:

$$\mathrm{strip}^*(C) = T^*(C) + \mathrm{lineal}(C)^\perp.$$

If $C = V$ is the whole space, we define

$$R^*(V, p(V)) = T^*.$$

If C is a full-dimensional cone with $V \neq C$, we can assume to have constructed $R^*(D, p(D))$ for all $D \prec C$ and we define the *set of affine generic lattice points* ${X^*}^C_D(p(C))$ as all points x in $\mathrm{lat}(D)$ such that

(I*) For all halfspaces $H \preceq C$ with $H \not\preceq D$:

$$x + R^*(D, p(D)) \subseteq \mathrm{int}(H^*)$$

(II*) For all $E \prec C$ with E incomparable to D and for all $x' \in \mathrm{lat}(E)$:

$$(x + R^*(D, p(D))) \cap (x' + R^*(E, p(E))) = \varnothing$$

Consistently with the above, we further set ${X^*}^C_C(p(C)) := \mathrm{lat}(C)$.

Then we can define the region

$$R^*(C, p(C)) := (\mathrm{strip}^*(C) \cap \mathrm{CDC}^*(C^*)) \setminus \bigcup_{D \prec C} {X^*}^C_D(p(C)) + R^*(D, p(D)).$$

From this we can compute the values of the *affine relative domain volume* v^*_C in the region $R^*(C, p(C))$ as:

$$v^*_C := \mathrm{vol}\big(R^*(C, p(C)) \cap \mathrm{DC}^*(C^*)\big).$$

And further the *affine correction volumes* for each $D \prec C$:

$${w^*}^C_D := \mathrm{vol}(R^*(C, p(C)) \cap C^* \cap (p(D) + \mathrm{lineal}(D))).$$

Then we get the value for (C, Trl) as

$$\mu^*(C, \mathrm{Trl}) := \mu^*(C, p(C)) := v^*_C - \sum_{D \prec C} {w^*}^C_D \cdot \mu^*(D, p(D)).$$

2.3 Dependence on the choice of representatives

Given that we make a choice of representatives of the translation classes, it is interesting to see in which way these choices affect the outcome. As it turns out, a different choice of representatives can be reduced to choosing different fundamental domains, so that no real new variation is added to the construction.

Lemma 2.1. *Let $x \in \Lambda$ be a lattice point. If we choose $\{(\text{lineal}(D), x+p(D)) \mid D \preceq C\}$ as the compatible set of representatives instead of $\{(\text{lineal}(D), p(D)) \mid D \preceq C\}$, the value of μ^* does not change.*

Proof. By taking $x + p(D)$ instead of $p(D)$ for all $D \preceq C$, the affine fundamental domain $T^*(D)$, the strip, the affine cones D^* and as a result all regions are translated by x, which, in turn does not change the affine domain volume and the affine correction volumes.

Another way to look at this is to consider the origin to be shifted to another lattice point. And since the whole construction is made with respect to the lattice, indifferent of where the origin is, nothing changes. □

Lemma 2.2. *Let $\{(\text{lineal}(D), p(D)) \mid D \preceq C\}$ be a compatible choice of representatives of some translation classes. Let $E \preceq C$ and $y \in \text{lineal}(E)$. Then the regions $R^*(D, p(D))$, where we substitute $p(E)$ by $y + p(E)$ are the same as if we constructed everything with $p(E)$, but substitute $T(\text{lineal}(E))$ by $y + T(\text{lineal}(E))$.*

Proof. In both cases, the affine fundamental domain $T^*(E)$ is $y+p(E)+T(\text{lineal}(E))$. Since $y \in \text{lineal}(E)$, we also have $p(E) + E = y + p(E) + E$. □

Given two representatives p_1 and p_2 of the same translation class of some subspace U, the difference $p_1 - p_2$ can be written as $p_1 - p_2 = l + u$ with $l \in \Lambda$ and $u \in U$. Combining Lemmas 2.1 and 2.2, we conclude that the differences in the construction resulting from the choices of representatives of the translation classes are included in the variations given by the choices of fundamental domains.

2.4 Properties of the affine regions

There are certain properties that the linear regions had that we now also want for the affine regions. More specifically, we want to show a version of Lemmas 1.1 through 1.4 applied to the context of affine regions. In these cases the proofs are surprisingly easy to adapt so that we will give the correct forms of the lemmas here with only a few remarks on how to adjust the proofs. However, not all results can effortlessly be adapted to the new situation; namely Lemma 1.5 cannot be shown in the same way. We will thus prove a slightly different result instead.

In the following, let C be a full-dimensional cone with a given translation class Trl and given choices of representatives $p(D)$ and affine fundamental domains $T^*(D)$ for all $D \preceq C$.

Lemma 2.3. *For all $D \prec C$, ${X^*}^C_D(p(C))$ is invariant under translation by points in* $\mathrm{lat}(C)$.

Proof. Since also $H^* = p(H) + H$ is invariant under translation by points in $\mathrm{lat}(C)$ for all halfspaces $H \preceq C$, the assertion can be shown analogously to Lemma 1.1. $\square$

Lemma 2.4. *We have a tiling*

$$\{x + R^*(D, p(D)) \mid D \preceq C, x \in {X^*}^C_D(p(C))\}$$

of the covering domain complex $\mathrm{CDC}^*(C^*)$*, consisting of lattice point translates of regions.*

Proof. As in the proof of Lemma 1.2, we have $\mathrm{lat}(C) + \mathrm{strip}^*(C) = V$. We also have that $\mathrm{CDC}(C^*)$ is invariant under translation by $\mathrm{lat}(C)$, since C^* is. The rest of the proof follows with exactly the same arguments, substituting X^C_D by ${X^*}^C_D(p(C))$, $R(C)$ by $R^*(C, p(C))$, $\mathrm{CDC}(C)$ by $\mathrm{CDC}^*(C^*)$ and so on. $\square$

Lemma 2.5. *$R^*(C, p(C))$ is bounded.*

Proof. Though Lemma 1.3 is rather hard to prove, adjusting it to the affine case is quite easy, since there are no fundamental arguments that have to be changed.

Instead of L_C we consider $L^*_C := \mathrm{lineal}(C) + p(C) \subseteq C^*$. Then adjusting the proof of Lemma 1.3 simply consists of adding a '$*$' whenever possible. $\square$

Lemma 2.6. *We have $T^*(C) \subseteq R^*(C, p(C))$.*

Proof. As in the linear case, we have $T^*(C) \subseteq \mathrm{strip}^*(C) \cap \mathrm{CDC}^*(C^*)$. Then in the proof of Lemma 1.4 we substitute $\mathrm{lineal}(C)$ by $p(C) + \mathrm{lineal}(C)$ and all arguments hold analogously. $\square$

The strict adaptation of Lemma 1.5 does not hold in the affine case. It is possible, however, to show a slightly different version that, as we will see, suffices. As in the proof of Lemma 2.5, we set $L^*_C := p(C) + \mathrm{lineal}(C)$.

Lemma 2.7. *We have ${X^*}^C_D(p(C)) + T^*(D) \subset C^* \cap L^*_C$.*

Proof. We use that

$$C^* = \bigcap_{\substack{H \prec C \\ H \text{ a halfspace}}} H^*.$$

Let $x \in {X^*}^C_D(p(C))$. We have $\mathrm{lat}(D)+T^*(D) = L^*_D \subseteq D^*$ and thus $x+T^*(D) \subseteq L^*_C$ and also $x + T^*(D) \subseteq H^*$ for all halfspaces $H \preceq D$. Regarding the halfspaces $H \preceq C$ with $H \not\preceq D$, we can apply Lemma 2.6 and Property (I*) for ${X^*}^C_D(p(C))$ and get $x + T^*(D) \subseteq x + R^*(D, p(D)) \subseteq \mathrm{int}(H^*)$. Altogether, we have shown that $x + T^*(D) \subseteq C^*$ for all $x \in {X^*}^C_D(p(C))$. □

2.5 Proof of Theorem 4

To prove that μ^* is indeed a local formula for Ehrhart quasipolynomials, we need to change our perspective from cones to polytopes. Let P be a rational polytope. For each face $f \leq P$ we have a translation class, which changes under dilation, and cones of feasible directions, which do not change under dilation. If $\mathrm{aff}(P)$ does not contain integer points, then all regions we construct are empty and all values of μ^* are zero. If $\mathrm{aff}(P)$ does contain integer points, we intersect everything with $\mathrm{aff}(P)$. We can thus assume that P is a full-dimensional rational polytope.

We assume affine fundamental domains to be given for every affine subspace of V. As a result of Lemmas 2.1 and 2.2, the differences in the construction can be traced back to the choice of these affine fundamental domains. To proof that μ^* is a local formula for Ehrhart quasipolynomials, it thus suffices to give one valid choice of representatives for the occurring translation classes.

Let $K \in \mathbb{Z}_{>0}$ be the denominator of P. Then for every integer $0 \leq k < K$ we can compute the translation class $\mathrm{trl}(kf, \Lambda)$ of $\mathrm{aff}(kf)$ with respect to the lattice Λ. If we choose a point $p_{kf} \in kf$ for every face $f \leq P$ and $k \leq K$, then for each vertex v of P, the set $\{(\mathrm{lineal}(\mathrm{fcone}(P, f)), p_{kf}) \mid v \leq f \leq P\}$ is a compatible set of representatives of the translation class $\mathrm{trl}(kf, \Lambda)$. We choose this as the given set of representatives and define for all $f \leq P$ and $t \in \mathbb{Z}_{\geq 0}$:

$$p(tf) := p(\mathrm{lin}(\mathrm{aff}(tf)), \mathrm{trl}(tf, \Lambda)) = p_{kf},$$

where $0 \leq k < K$ with $t \equiv k \mod K$. We further denote

$$T^*(tf) := T^*(\mathrm{lineal}(\mathrm{fcone}(P, f)), p(tf))$$

for the affine fundamental domain in $p(tf) + \mathrm{lineal}(\mathrm{fcone}(P, f))$. Since we assume P to be full-dimensional, the translation class of P with respect to Λ does not change under dilation and we denote $T^* := T^*(\mathrm{fcone}(P, P), p(P))$.

We further shorten notation by setting

$$\begin{aligned} R^*(tf) &:= R^*(\mathrm{fcone}(P, f), p(tf)), \\ {X^*}^g_f(t) &:= {X^*}^{\mathrm{fcone}(P,v)}_{\mathrm{fcone}(P,f)}(p(tv)), \\ v^*_f(t) &:= v^*_{\mathrm{fcone}(P,f)}(p(tf)) \qquad \text{and} \\ {w^*}^g_f(t) &:= {w^*}^{\mathrm{fcone}(P,g)}_{\mathrm{fcone}(P,f)}(p(tg)), \qquad \text{for } g \leq f \leq P. \end{aligned}$$

Though not obvious from the notation, we keep in mind that $p(tf)$, $T^*(tf)$, $R^*(tf)$, ${X^*}^g_f(t)$, $v^*_f(t)$ and ${w^*}^g_f(t)$ do not depend on t but only on the residue class of t modulo K.

As in the linear case, we want to give a tiling by translated regions of the covering domain complex of dilations tP of P. To this end, we define the *set of generic lattice points* $\mathcal{X}^*(tf)$ in tf for $f \leq P$ and $t \in \mathbb{Z}_{>0}$ as

$$\mathcal{X}^*(tf) := \bigcap_{\substack{v \text{ vertex of P} \\ v \leq f}} {X^*}^v_f(t) + tv - p(tv).$$

Lemma 2.8. *For all $f \leq P$ and $t \in \mathbb{Z}_{>0}$, we have $\mathcal{X}^*(tf) + T^*(tf) \subseteq tf$.*

Proof.

$$\begin{aligned} tf &= \operatorname{aff}(tf) \quad \cap \bigcap_{v \text{ vertex of } f} \operatorname{fcone}(P, v) + tv \\ &= \operatorname{aff}(tf) \quad \cap \bigcap_{v \text{ vertex of } f} p(tv) + \operatorname{fcone}(P, v) + tv - p(tv) \end{aligned}$$

By Lemma 2.7 for each vertex v of f we have

$${X^*}^v_f(t) + T^*(D) \subseteq p(tv) + \operatorname{fcone}(P, v),$$

which shows that

$$\mathcal{X}^*(tf) + T^*(tf) \subseteq \bigcap_{v \text{ vertex of } f} \operatorname{fcone}(P, v) + tv.$$

Also due to Lemma 2.7, we have

$${X^*}^v_f(t) + T^*(tf) \subseteq L^*_{\operatorname{fcone}(P,f)} = \operatorname{lineal}(\operatorname{fcone}(P, f)) + p(tf),$$

and hence,

$$\mathcal{X}^*(tf) + T^*(tf) \subseteq \operatorname{lineal}(\operatorname{fcone}(P, f)) + p(tf) + tv - p(tv) = \operatorname{aff}(tf),$$

since $p(tf) - p(tv) \in \operatorname{lineal}(\operatorname{fcone}(P, f))$ for every vertex v of f. □

Theorem 3. *There exists a $t_0 \in \mathbb{Z}_{>0}$ such that for every integer $t \geq t_0$ the set*

$$\{x + R^*(tf) \mid \ x \in \mathcal{X}^*(tf)\}.$$

is a tiling of the covering domain complex $\operatorname{CDC}^*(tP)$.

Proof. Lemma 2.5 shows that all regions are bounded. By Lemma 2.8, we have $\mathcal{X}^*(tf)+T^*(tf) \subseteq tf$ and by Lemma 1.4 we have $T^*(tf) \subseteq R^*(\mathrm{fcone}(P,f), p(tf))$. That means there is a certain $r_f > 0$ such that

$$\mathcal{X}^*(tf) + R^*(tf) \subseteq tf + B_{r_f}.$$

Since by dilating P we can ensure non-intersecting faces to have a distance bigger than a given one, we find a $t_0 \in \mathbb{Z}_{>0}$ such that

$$(x + R^*(tf)) \cap (y + R^*(tg)) = \varnothing$$

for all faces $f, g \leq P$ with $f \cap g = \varnothing$, $x \in \mathcal{X}^*(tf)$, $y \in \mathcal{X}^*(tg)$ and all $t \in \mathbb{Z}_{>0}$ with $t > t_0$.

If f and g are faces of P that do intersect, then there is a vertex v of P that is contained in both, f and g. Let $x \in \mathcal{X}^*(tf)$ and $y \in \mathcal{X}^*(tg)$. Then $x - tv + p(tv) \in {X^*}^v_f(t)$ and $y - tv + p(tv) \in {X^*}^v_g(t)$ and the sets $x - tv + p(tv) + R^*(tf)$ and $y - tv + p(tv) + R^*(tg)$ do not intersect as part of the tiling in Lemma 2.4. This implies that the also

$$(x + R^*(tf)) \cap y + (R^*(tg)) = \varnothing.$$

The remainder of the proof is analogous to the one of Theorem 1 with '$*$' in the right places with one further adjustment: As we did in the first part of this proof, every time we translate by tv for some vertex v of P in the proof of Theorem 1, here we translate by $tv - p(tv)$. Then using the fact that if v_1 and v_2 are both vertices of a face $f_1 \leq P$ we have $-p(tv_1) + p(tv_2) \in \mathrm{lat}(f_1)$, every step works exactly as in the original proof. □

Theorem 4. *For a rational polytope P with Ehrhart quasipolynomial*

$$\mathrm{E}_P(t) = c_d(t)t^d + c_{d-1}(t)t^{d-1} + \ldots + c_0(t),$$

the i-th coefficient is given by

$$c_i(t) = \sum_{\substack{f \leq P \\ \dim(f)=i}} \mu^*(\mathrm{fcone}(P,f), \mathrm{trl}(tf, \Lambda))\mathrm{vol}(tf).$$

Proof. The proof is completely analogous to the one of Theorem 2. The respective versions of Lemmas 1.7, 1.8 and 1.9 are given below in Lemmas 2.9, 2.10 and 2.11. We use the fact that as for polynomials, two quasipolynomials are equal if they agree on all but finitely many values of $t \in \mathbb{Z}_{>0}$. □

Lemma 2.9. *We have*

$$\mathrm{vol}(\mathrm{DC}^*(tP)) = \sum_{f \leq P} |\mathcal{X}^*(tf)| \cdot v_f^*(t)$$

Proof. The structure of the proof is, again, analogous to the one of Lemma 1.7, but since the adjustments are more nuanced than adding a '$*$' in the right places, we give the proof in more details.

By Theorem 3, for all $t \in \mathbb{Z}_{>0}$ big enough, we have a tiling of the affine covering domain complex, which covers $\mathrm{DC}^*(tP)$, so that we can compute the volume of the affine domain complex in each part of the tiling:

$$\mathrm{vol}(\mathrm{DC}^*(tP)) = \sum_{f \leq P} \sum_{x \in \mathcal{X}^*(tf)} \mathrm{vol}((x + R^*(tf)) \cap \mathrm{DC}^*(tP)). \tag{2.1}$$

We recall that the affine relative domain volume is defined as

$$v^*_f(t) = \mathrm{vol}(R^*(tf) \cap \mathrm{DC}^*(p(tf) + \mathrm{fcone}(P, f))).$$

We want to show that for $f \leq P$, each summand of the inner sum in (2.1) equals $v^*_f(t)$. To follow the proof of 1.7, we first need to show that $\mathcal{X}^*(tf) \subseteq \Lambda$. Let $x \in \mathcal{X}^*(tf)$. Then

$$x \in tv - p(tv) + {X^*}^v_f(t) \tag{2.2}$$

for a vertex $v \leq f$. By definition, ${X^*}^v_f(t) \subseteq \Lambda$ and since $p(tv)$ is an element of the translation class of tv with respect to Λ, we also have $tv - p(tv) \in \Lambda$. Therefore, we can use that the affine domain complex commutes with translation by a lattice point and we get

$$v^*_f(t) = \mathrm{vol}((x + R^*(tf)) \cap \mathrm{DC}^*(x + p(tf) + \mathrm{fcone}(P, f))).$$

We thus want to show that

$$(x + R^*(tf)) \cap \mathrm{DC}^*(x + p(tf) + \mathrm{fcone}(P, f)) = (x + R^*(tf)) \cap \mathrm{DC}^*(tP). \tag{2.3}$$

To show inclusion from right to left, it suffices to show that $tP \subseteq x + p(tf) + \mathrm{fcone}(P, f)$. From Equation (2.2) we can deduce that

$$x + p(tf) \in tv - p(tv) + p(tf) + {X^*}^v_f(t)$$

and since ${X^*}^v_f(t) \subseteq \mathrm{lineal}(\mathrm{fcone}(P, f)) = \mathrm{lin}(\mathrm{aff}(tf))$, $tv \in tf$ and $-p(tv) + p(tf) \in \mathrm{lineal}(\mathrm{fcone}(P, f))$, we get $x + p(tf) \subseteq \mathrm{aff}(tf)$ and thus

$$tP \subseteq x + p(tf) + \mathrm{fcone}(P, f).$$

To show the inclusion from left to right in Equation (2.3), we need to show that

$$(x + R^*(tf)) \cap (y + T^*) = \varnothing$$

for all $y \in \Lambda \cap (x + p(tf) + \mathrm{fcone}(P, f))$ with $y \notin tP$.

$y \notin tP$ means there is a vertex v of P with $y \notin v + \mathrm{fcone}(P, v)$. If $f = v$ we have $x \in \mathcal{X}^*(tv)$ with $\mathcal{X}^*(tv) = \{tv - p(tv)\}$ and thus by assumption we have

$$\begin{aligned} y &\in x + p(tf) + \mathrm{fcone}(P, f) \\ &= \ tv - p(tv) + p(tv) + \mathrm{fcone}(P, v) \\ &= tv + \mathrm{fcone}(P, v). \end{aligned}$$

If v is not a vertex of f then by assuming t big enough (as we did in Theorem 3), we can ensure that tf and also $x + R^*(tf)$ are inside of $tv + \mathrm{fcone}(P, f)$ with a large enough distance to the boundary, such that $(x + R^*(tf)) \cap (y + T^*) = \varnothing$.

We are left with the case $y \notin tv + \mathrm{fcone}(P, v)$ for a vertex $v < f$. In this case there is a facet F of P that contains v but not f and for that we have $y \notin tv + \mathrm{fcone}(P, F)$.

If $y + T^* \nsubseteq \mathrm{CDC}^*(tv + \mathrm{fcone}(F, P))$, then by definition of the covering domain complex we have $(y+T^*)\cap(tv+\mathrm{fcone}(F, P)) = \varnothing$. But since $x-tv+p(tv) \in X_f^v(t)$, we have by Property (I*) that

$$x - tv + p(tv) + R^*(tf) \subseteq p(tF) + \mathrm{fcone}(P, F).$$

Since $p(tF)-p(tv) \in \mathrm{fcone}(P, F)$, the above implies $x+R^*(tf) \subseteq tv+\mathrm{fcone}(P, F)$, such that $y + T^*$ and $x + R^*(tf)$ do not intersect.

Otherwise, $y + T^* \subseteq \mathrm{CDC}^*(tv + \mathrm{fcone}(F, P))$. But since $y \in y + T^*$ and $y \notin tv + \mathrm{fcone}(P, v)$ we have

$$y - tv + p(tv) + T^* \nsubseteq p(tv) + \mathrm{fcone}(F, P) \subseteq p(tF) + \mathrm{fcone}(F, P),$$

which means $y - tv + p(tv) \notin {X^*}_P^F(t)$. Using Lemma 2.4 we see that then

$$y - tv + p(tv) + T^* \subseteq \mathrm{lat}(\mathrm{fcone}(P, F)) + R^*(tf).$$

Since $x - tv + p(tv) \in {X^*}_f^v$, we have by Property (II*)

$$(x - tv + p(tv) + R^*(tf)) \cap (\mathrm{lat}(\mathrm{fcone}(P, F)) + R^*(tf)) = \varnothing$$

and hence $(x - tv + p(tv) + R^*(tf)) \cap (y - tv + p(tv) + T^*) = \varnothing$. That yields $(x + R^*(tf)) \cap (y + T^*) = \varnothing$ as we wanted to show. □

Lemma 2.10. *There exists a $t_0 \in \mathbb{Z}_{>0}$ such that for each $t \geq t_0$ we have*

$$tf \subseteq \bigcup_{g \leq f} (\mathcal{X}^*(tg) + R^*(tg)).$$

Proof. The proof is a straightforward adaptation of the proof of Lemma 1.8. □

Lemma 2.11. *There exists a $t_0 \in \mathbb{Z}_{>0}$ such that for each $t \geq t_0$ and every face $f < P$ we have*

$$\mathrm{vol}(tf) = \sum_{g \leq f} |\mathcal{X}^*(tf)| \cdot {w^*}_f^g(tg).$$

Proof. The proof is a straightforward adaptation of the proof of Lemma 1.9. $\square$

Chapter 3

Symmetry

Symmetry of μ can be achieved by choosing symmetric fundamental domains, for example by taking *Dirichlet–Voronoi cells*, as is shown below. Exploiting symmetries has many advantages, for theoretical results as well as computationally. Given central symmetry, the values on halfspaces and thus on fcones of facets of a polytope are always 1/2. We will show this in Section 3.3.

3.1 Dirichlet–Voronoi cells and symmetry

Possibly the most natural choice of fundamental domains are *Dirichlet–Voronoi cells*. Given a space V and an inner product $\langle\cdot,\cdot\rangle$ with induced norm $\|\cdot\|$, the Dirichlet–Voronoi cell of a sublattice $L \subseteq \Lambda$ is defined as

$$\mathrm{DV}(L, \langle\cdot,\cdot\rangle) := \{x \in \mathrm{lin}(L) : \|x\| \leq \|x - a\| \text{ for all } a \in L\}.$$

Dirichlet–Voronoi cells of a lattice are always convex polytopes [CS99]. They are naturally centrally symmetric and can be forced to have certain symmetries by choosing a suitable inner product:

Let P be a lattice polytope and $\mathcal{G}$ a subgroup of all lattice symmetries of P, i.e. $\mathcal{G}$ is a finite matrix group with $A \cdot P := \{A \cdot x : x \in P\} = P$ and $A \cdot \Lambda = \Lambda$ for all $A \in \mathcal{G}$. Then we can define a $\mathcal{G}$-invariant inner product by taking

$$\langle x, y\rangle_{\mathcal{G}} := x^t G y \qquad \text{for all } x, y \in V, \tag{3.1}$$

with the Gram matrix G given by

$$G := \frac{1}{|\mathcal{G}|} \sum_{A \in \mathcal{G}} A^t A. \tag{3.2}$$

Let $\|\cdot\|_{\mathcal{G}}$ be the induced norm and let D be the Dirichlet–Voronoi cell for Λ given by that particular inner product,

$$D := \mathrm{DV}(\Lambda, \langle\cdot,\cdot\rangle_{\mathcal{G}}) = \{x \in V : \|x\|_{\mathcal{G}} \leq \|x - p\|_{\mathcal{G}} \text{ for all } p \in \Lambda\}.$$

Then D is invariant under the action of $\mathcal{G}$: Let $x \in D$, then for $A \in \mathcal{G}$ we have

$$\|Ax\|_{\mathcal{G}} = \|x\|_{\mathcal{G}} \leq \|x - p\|_{\mathcal{G}} = \|Ax - Ap\|_{\mathcal{G}} \qquad \text{for all } p \in \Lambda.$$

Since $A\Lambda = \Lambda$, we get $AD \subseteq D$ for all $A \in \mathcal{G}$. Substituting A by A^{-1}, we get $A^{-1}D \subseteq D$ which yields $D \subseteq AD$ and hence $AD = D$.

From Dirichlet–Voronoi cells to fundamental domains. In the definition given above, a Dirichlet–Voronoi cell is not yet a fundamental domain of the lattice L, since for $l \in L\backslash\{0\}$, the sets $\mathrm{DV}(L, \langle\cdot,\cdot\rangle)$ and $l + \mathrm{DV}(L, \langle\cdot,\cdot\rangle)$ can intersect on their boundaries. To fix that, we want to consider a half open variant of the Dirichlet–Voronoi cell. To give a construction, we consider *visible* and *invisible* points from a *general direction* in the following sense. Though the proofs here are due to the author, this concept is not new—see, for instance, Köppe and Verdoolaege [KV08, Theorem 3] for a more general result of this kind.

Definition 7. Let $A \subseteq V$ be a closed convex set and v an element of V. We call a point $a \in A$ *invisible from the direction of* v, if there exists an $\varepsilon > 0$ such that $a - \varepsilon v \in A$. If $a \in A$ is not invisible, i.e. for all $\varepsilon > 0$ we have $a - \varepsilon v \notin A$, it is called *visible from the direction of* v. We denote the invisible part of A from the direction of v as $\mathrm{inv}_v(A)$.

Definition 8. Let P be a polytope in V and $v \in \mathrm{lin}(P)$ an element of the linear space parallel to the affine span of P. Then we say v is a *general direction with respect to* P, if v is not orthogonal to any of the normal vectors of the facets of P.

An example of a non-general and a general direction is shown in Figure 3.1. One can think of this concept as a parallel light rays in the direction of v and everything on the boundary of P that is lit is called visible, every other point of P is dark and thus called invisible. We want the light not to shine in a direction aligned with a facet, which would mean that this facet is on the threshold between light and dark. Hence the notion of general position is introduced, excluding exactly this case. Mathematically, a general direction yields that for a point $x \in P$, any point $x - \varepsilon v$ for $\varepsilon > 0$ cannot be in the same facet as x. Another consequence is that if we have a pair of parallel facets, exactly one of them is visible and the other one is invisible (except possibly for its boundary). The interior of a polytope P is always contained in $\mathrm{inv}_v(P)$.

Using the concept of (in-)visibility from a general direction, we can turn Dirichlet–Voronoi cells into fundamental domains:

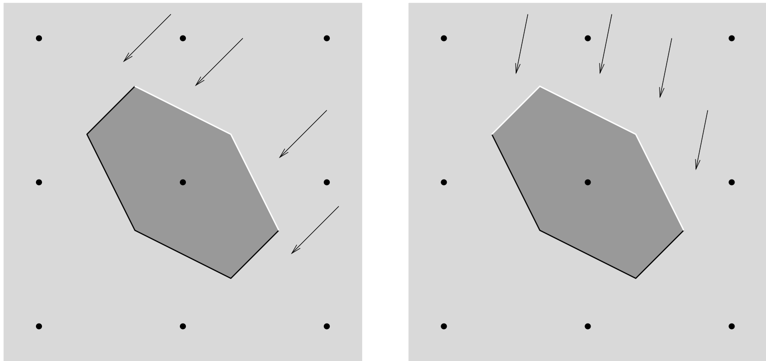


Figure 3.1: Non-general and general direction.

Lemma 3.1. *Let* $D := \mathrm{DV}(L, \langle\cdot,\cdot\rangle)$ *be a Dirichlet–Voronoi cell of a sublattice* $L \subseteq \Lambda$ *and let* $v \in V$ *be a general direction with respect to* D*. Then* $\mathrm{inv}_v(D)$ *is a fundamental domain of* L*.*

Proof. We first want to show that lattice point translates of $\mathrm{inv}_v(D)$ by two distinct points have empty intersection. Since we can assume one point to be the origin, it suffices to show that the intersection of $\mathrm{inv}_v(D)$ and $l + \mathrm{inv}_v(D)$ is empty for all $l \in L\backslash\{0\}$. We assume the contrary: Let $l \in L$ and $x \in \mathrm{inv}_v(D)$ such that $x \in l+\mathrm{inv}_v(D)$. That means $x-l \in \mathrm{inv}_v(D)$. Applying the definition of invisibility and general direction, we get scalars $\varepsilon > 0$ and $\delta > 0$ such that $x - \varepsilon v \in \mathrm{int}(D)$ and $(x - l) - \delta v \in \mathrm{int}(D)$. Since D is convex, by taking the minimum of ε and δ, we can assume $\varepsilon = \delta$. But that yields $x - \varepsilon v \in \mathrm{int}(D) \cap (l + \mathrm{int}(D))$, which is a contradiction, since $\mathrm{int}(D)$ is the set of all points strictly closer to the origin than to any other lattice point, and hence, $\mathrm{int}(D)$ and $l + \mathrm{int}(D)$ are disjoint.

We have $\mathrm{int}(D) \subseteq \mathrm{inv}_v(D) \subseteq D$. Since the union of all lattice point translates of D equals $\mathrm{lin}(L)$, in order to show that the union of lattice point translates of $\mathrm{inv}_v(D)$ equals $\mathrm{lin}(L)$, it suffices to show that for each $x \in \mathrm{bd}(D)$ there exists an $l \in L$ such that $x \in l + \mathrm{inv}_v(D)$. To this end, let $x \in \mathrm{bd}(D)$. Since v is a general direction with respect to D, for all $\varepsilon > 0$ small enough $x - \varepsilon v$ is not on the boundary of $l + D$ for any $l \in L$. Thus, there exists an $l \in L$ with $x \in l + D$ and $x - \varepsilon v \in l + \mathrm{int}(D)$. That means $x - l - \varepsilon v \in \mathrm{int}(D)$, which, together with $x - l \in D$ yields $x - l \in \mathrm{inv}_v(D)$ and hence $x \in l + \mathrm{inv}_v(D)$ as we wanted to show. $\square$

3.2 From symmetric fundamental domains to invariant local formulas

Coming back to the lattice polytope P with a given symmetry group $\mathcal{G}$, we see that for all faces f in the same $\mathcal{G}$-orbit their fcones are mapped onto each other. Assuming the same holds for the chosen fundamental domains in the construction of the regions, it is easy to see that then $A \cdot R(C) = R(A \cdot C)$ for all fcones C of P and all $A \in \mathcal{G}$. Thus $\mu(C_1) = \mu(C_2)$, whenever C_1 and C_2 are in the same $\mathcal{G}$-orbit of fcones of P.

At this point it is natural to ask about the existence of symmetric fundamental domains. We have seen that given a matrix group $\mathcal{G}$ it is easy to construct a $\mathcal{G}$-symmetric Dirichlet–Voronoi cell. The problem is that turning it into a fundamental domain by taking the invisible part of the cell can break most of its symmetries. However, the change of the boundary does not effect the value of μ as shown in the following theorem:

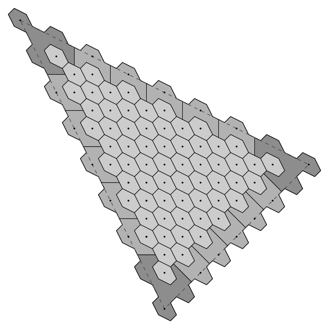


Figure 3.2: Simplex S with symmetric regions.

Theorem 5. *Let $A \in \mathcal{G}$ and $A \cdot \mathrm{cl}(T(D)) = \mathrm{cl}(T(A \cdot D))$ for all $D \preceq C$. Then*

$$\mu(C) = \mu(A \cdot C).$$

Example. We consider the simplex S from Section 1.3. The example given there is symmetric under reflection at a line in the direction $(1, -1)$. So is the square that we chose as fundamental domain in that example. It is thus not surprising that the values for the fcones of v_1 and v_2 are the same.

Taking a closer look at S, we note that it is also symmetric under the action of the matrix group $\mathcal{G}$ generated by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$ of order 3. The $\mathcal{G}$-invariant inner product that we can compute according to Section 3.1 is given by the Gram matrix $G = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. The resulting Dirichlet–Voronoi cell is the hexagon shown in Figure 3.1. With this inner product and the hexagonal fundamental domain we can construct the regions for the fcones of faces of S and get the ones shown in Figure 3.2. Then the μ-values for all fcones of vertices are $1/3$, while the values on all fcones of facets are still $1/2$.

Before we can prove Theorem 5, we need another quite useful property of the regions. The following lemma shows that adding a region that was cut out originally does not change the value, allowing is to change the regions if necessary.

Lemma 3.2. *Let C be a cone with region $R(C)$. Let $D \prec C$ and $x \in X_D^C$. Then the μ-value of C does not change if we take $R(C)' := R(C) \cup (x + R(D))$ instead of $R(C)$ to compute the value.*

Proof. If we set

$$\bar{v}_C := \mathrm{vol}(R(C)' \cap \mathrm{DC}(C)),$$
$$\bar{w}_E^C := \mathrm{vol}(R(C)' \cap C \cap \mathrm{lineal}(E))$$

for $E \prec C$ and then

$$\mu(C)' := \bar{v}_C - \sum_{E \prec C} \bar{w}_E^C \cdot \mu(E),$$

what we want to show is that

$$\mu(C) = \mu(C)'.$$

Using that $R(C)$ and $x + R(D)$ are disjoint and the same arguments as in the proof of Lemma 1.7, we get

$$\begin{aligned}
\bar{v}_C &= \mathrm{vol}(R(C)' \cap \mathrm{DC}(C)) \\
&= \mathrm{vol}((R(C) \cap \mathrm{DC}(C)) \cup ((x + R(D)) \cap \mathrm{DC}(C))) \\
&= v_C + \mathrm{vol}(R(D) \cap \mathrm{DC}(D)) \\
&= v_C + v_D.
\end{aligned}$$

Then for $\bar{w}_E^C$ we get

$$\begin{aligned}
\bar{w}_E^C &= \mathrm{vol}(R(C)' \cap C \cap \mathrm{lineal}(E)) \\
&= \mathrm{vol}(R(C) \cap C \cap \mathrm{lineal}(E)) + \mathrm{vol}((x + R(D)) \cap C \cap \mathrm{lineal}(E)) \\
&= w_E^C + \mathrm{vol}(R(D) \cap D \cap \mathrm{lineal}(E)) \\
&= \begin{cases} w_E^C + 0, \text{ if } E \not\preceq D \\ w_E^C + w_D^C, \text{ if } E \prec D \\ w_E^C + 1, \text{ if } E = D. \end{cases}
\end{aligned}$$

Line three and the first case follow from Property (I) for x, while the second case comes from applying the definition and the third is due to the fact that $R(D) \cap \mathrm{lineal}(D) = T(D)$, as we have shown in Lemma 1.4.

Altogether we get

$$\begin{aligned}
\mu(C)' &= \bar{v}_C - \sum_{E \prec C} \bar{w}_E^C \cdot \mu(E) \\
&= v_C + v_D - \sum_{E \prec C} w_E^C \cdot \mu(E) - \sum_{E \prec D} w_E^D \cdot \mu(E) - 1 \cdot \mu(D) \\
&= \mu(C) + \mu(D) - \mu(D) = \mu(C),
\end{aligned}$$

as we wanted to show. $\square$

The lemma shows that we can add a translate of a region $R(D)$ with $D \prec C$ to $R(C)$ without changing the μ-value. This extends to finitely many translates of regions and ultimately allows us to be a little less careful around the boundary as stated in Theorem 5.

Proof of Theorem 5. Here, we cannot assume that C is full-dimensional, since the linear span of C and of $A \cdot C$ might differ.

Yet, for $C = \operatorname{lin}(C)$ a linear subspace, we have $R(C) = T(C)$ and $\mu(C) = \operatorname{vol}(T(C)) = 1$ as well as $R(A \cdot C) = T(A \cdot C)$ and $\mu(A \cdot C) = \operatorname{vol}(T(A \cdot C)) = 1$.

To show that the boundary of the fundamental domains does not change the μ-values, we show that the values do not change if we change the definition of the regions in the following way:

For a cone C that is not a linear subspace of V, we define the sets $\bar{X}_D^C$ analogously to X_D^C as the set of all points $x \in \operatorname{lat}(D)$ that comply with the following constraints:

($\bar{I}$) For all halfspaces $H \preceq C$ such that $H \npreceq D$:

$$x + \operatorname{cl}(R(D)) \subseteq \operatorname{int}(H)$$

($\overline{II}$) For all $E \prec C$, such that E is incomparable to D and for all $x' \in \operatorname{lat}(E)$:

$$(x + \operatorname{cl}(R(D))) \cap (x' + \operatorname{cl}(R(E))) = \varnothing$$

In other words, we want to be a little stricter regarding the regions that we can cut out when constructing $R(C)$. We define the region $\bar{R}(C)$ given by $\bar{X}_D^C$ as follows:

$$\bar{R}(C) := (\operatorname{strip}(C) \cap CDCC) \setminus \bigcup_{D \prec C} \left(\bar{X}_D^C + \bar{R}(D)\right).$$

The pieces we this way add to $R(C)$ in each inductive step are of the form

$$\operatorname{strip}(C) \cap (\operatorname{lat}(C) + x + R(D))$$

with $x \in X_D^C$. Note that with $x \in X_D^C$ we also have $x + \operatorname{lat}(C) \subseteq X_D^C$. Thus adding $\operatorname{lat}(C)$ and intersecting with $\operatorname{strip}(C)$ is essentially the same as taking $x + R(D)$ dissected in pieces. We can thus apply Lemma 3.2 finitely many times and get that the μ-values do not change when taking $\bar{R}(C)$ instead of $R(C)$.

What we now gained is that we can make the construction of regions invariant under changes on the boundary of the fundamental domains in the way that

$\mathrm{cl}(\bar{R}(C)) = \mathrm{cl}(\bar{R}(A \cdot C))$ for $A \in \mathcal{G}$. Since v_C is a $\dim(C)$-dimensional volume, it does not change when changing the boundary of $\bar{R}(C)$. For $D \prec C$ we have

$$w_D^C = \mathrm{vol}(\bar{R}(C) \cap C \cap \mathrm{lineal}(D)),$$

which is a $\dim(\mathrm{lineal}(D))$-dimensional volume. To show that this also does not change when taking $\mathrm{cl}(R(C))$ instead of $\bar{R}(C)$, we assume that there is a part of the boundary of $\bar{R}(C)$ that has a nonempty relative volume in $\mathrm{lineal}(D)$. That cannot come from $\mathrm{strip}(C)$ or $\mathrm{CDC}(C)$, since both are $\dim(C)$-dimensional and the boundary of the former is lower dimensional in in $\mathrm{lineal}(D)$ and the boundary of the latter does not intersect C. It thus has to come from $y + \bar{R}(E)$ with $y \in \bar{X}_E^C$. If $E \neq D$, the closure of $y + \bar{R}(E)$ does not intersect $\mathrm{lineal}(D)$ by Property ($\bar{I}$). If $E = D$, its boundary intersected with $\mathrm{lineal}(D)$ is the boundary of $T(D)$ and as such also lower-dimensional in $\mathrm{lineal}(D)$.

We have thus shown that considering the closure of $\bar{R}(C)$ instead of $\bar{R}(C)$ itself does not change v_C and w_D^C and thus also not the value of μ. Together with $\mathrm{cl}(\bar{R}(C)) = \mathrm{cl}(\bar{R}(A \cdot C))$ for $A \in \mathcal{G}$, we have shown that $\mu(C) = \mu(A \cdot C)$ for all $A \in \mathcal{G}$. □

Remark. Theorem 5 can easily be adjusted to hold for the function μ^* as defined in Chapter 2 as well, with the group not only acting on the cones, but also their translation classes.

3.3 Codimension one

It is known that the second highest Ehrhart coefficient always equals 1/2 times the sum over the relative volumes of the facets of a polytope. A natural conjecture would be that all values of McMullen's formulas corresponding to facets (in this case all values on halfspaces) have the value 1/2. This is not true in general for μ, but we show here that it does hold when the closures of the chosen fundamental domains are centrally symmetric. In particular, this result always holds when taking Dirichlet–Voronoi cells as fundamental domain — regardless of the inner product. The results in this section have been published by the author in [Rin19].

Theorem 6. *Let $T(A)$ be a fundamental domain with centrally symmetric closure for each subspace $A \subseteq V$. Let P be a lattice polytope and $F < P$ a facet. Then*

$$\mu(\mathrm{fcone}(P, F)) = \frac{1}{2}.$$

Proof. In the following, let $T(A)$ be a fundamental domain with centrally symmetric closure for each $A \subseteq V$. Again, we denote $T := T(V)$.

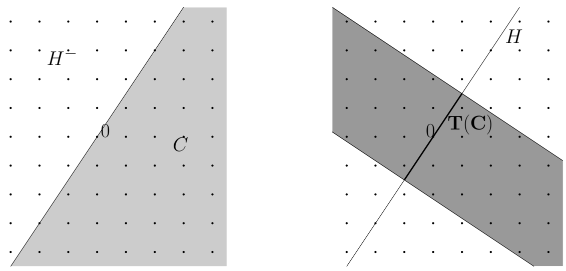


Figure 3.3: **left**: A cone C with $H = \mathrm{lineal}(C)$ and $H^- = V \backslash C$; **right**: $\mathrm{strip}(C) = T(C) + H^\perp$

Let P be a full-dimensional polytope and F a facet of P with fcone $C := \mathrm{fcone}(P, F)$. Then C is a halfspace in V. We denote $H := \mathrm{lineal}(C)$ for the hyperplane contained in C, H^+ for the open halfspace contained in C and H^- for the open halfspace on the other side of H.

Using what we have established in Section 1.3 for regions of halfspaces, we have

$$R(C) = \mathrm{strip}(C) \cap (Y_H + T),$$

where $Y_H = \{x \in \Lambda \mid (x + T) \cap H \neq \varnothing\}$. That means $Y_H + T = \mathrm{CDC}(H)$ and we can write $R(C)$ as

$$R(C) = \mathrm{CDC}(H) \cap \mathrm{strip}(C).$$

Recall that $\mu(C)$ for the halfspace C is defined as

$$\begin{aligned} \mu(C) &= v_C - \mu(V) \cdot w_V^C \\ &= \mathrm{vol}(R(C) \cap \mathrm{DC}(C)) - 1 \cdot \mathrm{vol}(R(C) \cap C). \end{aligned}$$

To show that $\mu(C) = 1/2$, we show that everything but half the fundamental domain around the origin cancels out nicely. We use the fact that everything is centrally symmetric in the following sense:

Let σ_0 be the point reflection at the origin:

$$\begin{aligned} \sigma_0 \colon V &\to V \\ v &\mapsto -v \end{aligned}$$

Then $\sigma_0(\mathrm{cl}(T)) = \mathrm{cl}(T)$ and $\sigma_0(\mathrm{cl}(T(C))) = \mathrm{cl}(T(C))$ by assumption and we further have

$$\begin{aligned}
\sigma_0(H) &= H, \\
\sigma_0(H^+) &= H^-, \\
\sigma_0(H^-) &= H^+, \\
\sigma_0(\mathrm{cl}(\mathrm{CDC}(H))) &= \mathrm{cl}(\mathrm{CDC}(H)), \\
\sigma_0(\mathrm{cl}(\mathrm{strip}(C))) &= \mathrm{cl}(\mathrm{strip}(C))
\end{aligned}$$

and as a result also

$$\sigma_0(\mathrm{cl}(R(C))) = \mathrm{cl}(R(C)).$$

Then $R(C)$ can be partitioned into three parts

$$\begin{aligned}
R(C) =&(Y_H + T) \cap \mathrm{strip}(C) \\
=&(\underbrace{(Y_H \cap H)}_{:=X_0} + T) \cap \mathrm{strip}(C) \\
&\cup (\underbrace{(Y_H \cap H^+)}_{:=X_+} + T) \cap \mathrm{strip}(C) \\
&\cup (\underbrace{(Y_H \cap H^-)}_{:=X_-} + T) \cap \mathrm{strip}(C),
\end{aligned}$$

where the unions are disjoint, since X_0, X_+, X_- are. For an illustration see Figure 3.4.

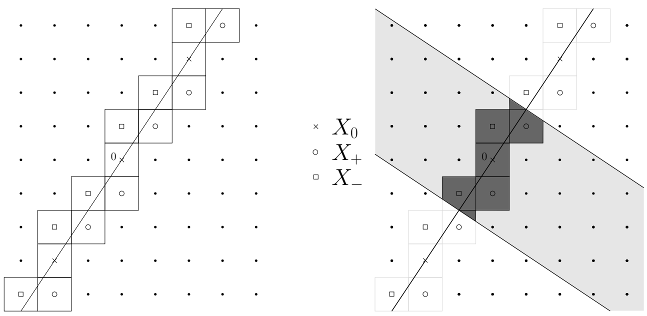


Figure 3.4: $(Y_H + T)$ and $R(C)$

By definition, $\mathrm{lat}(C) = \Lambda \cap \mathrm{lineal}(C) = \Lambda \cap H$, and therefore, $X_0 \subseteq \mathrm{lat}(C)$. Since $0 \in T$, we also have $\mathrm{lat}(C) \subseteq Y_H$ and thus $X_0 = \mathrm{lat}(C)$.

We now want to show that $\mathrm{vol}(\mathrm{strip}(C) \cap (X_0 + T) = 1$. We use the fact that two translates of T and also of $\mathrm{strip}(C)$ by different lattice points $x, y \in \mathrm{lat}(C)$

are disjoint and that $\mathrm{lat}(C) + \mathrm{strip}(C) = V$ to get

$$\begin{aligned}
\mathrm{vol}(\mathrm{strip}(C) \cap (X_0 + T)) &= \mathrm{vol}(\bigcup_{x \in \mathrm{lat}(C)} \mathrm{strip}(C) \cap (x + T)) \\
&= \sum_{x \in \mathrm{lat}(C)} \mathrm{vol}(\mathrm{strip}(C) \cap (x + T)) \\
&= \sum_{x \in \mathrm{lat}(C)} \mathrm{vol}((-x + \mathrm{strip}(C)) \cap T) \\
&= \mathrm{vol}(\bigcup_{x \in \mathrm{lat}(C)} (-x + \mathrm{strip}(C)) \cap T) \\
&= \mathrm{vol}(V \cap T) \\
&= \mathrm{vol}(T) \\
&= 1.
\end{aligned}$$

Since σ_0 does not change the volume, we can use it to gain information on the occurring volumes by using the equality:

$$\sigma_0((X_0 + \mathrm{cl}(T)) \cap \mathrm{cl}(\mathrm{strip}(C)) \cap H^+) = (X_0 + \mathrm{cl}(T)) \cap \mathrm{cl}(\mathrm{strip}(C)) \cap H^-.$$

Since $X_0 + T$, $\mathrm{strip}(C)$, H^+ and H^- are full-dimensional, a consideration of the boundaries can be neglected when taking the volume so that we get

$$\mathrm{vol}\left(((X_0 + T) \cap \mathrm{strip}(C)) \cap H^+\right) = \mathrm{vol}(((X_0 + T) \cap \mathrm{strip}(C)) \cap H^-\right).$$

The sum of both volumes equals $\mathrm{vol}(\mathrm{strip}(C) \cap (X_0 + T)) = 1$ and hence,

$$\mathrm{vol}\left(((X_0 + T) \cap \mathrm{strip}(C)) \cap H^-\right) = \frac{1}{2}.$$

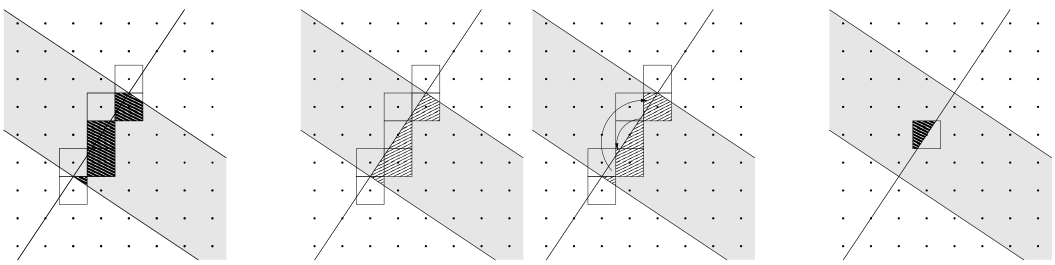


Figure 3.5: v_C, w_V^C and $w_V^C - v_C$

Then since

$$\sigma_0\left(((X_- + \mathrm{cl}(T)) \cap \mathrm{cl}(\mathrm{strip}(C))) \cap H^+\right) = (X_+ + \mathrm{cl}(T)) \cap \mathrm{cl}(\mathrm{strip}(C))) \cap H^-,$$

the two have equal volume. Now we can, again, use the fact that a full-dimensional volume can be dissected into the part inside of H^+ and the one inside H^-, while the intersection with H can be neglected. In particular, we can exploit the fact that the volume of the intersection with the closed halfspace C equals the volume of the intersection with the open halfspace H^+ and get

$$\begin{aligned}
w_V^C =&\operatorname{vol}(R(C) \cap C) \\
=&\operatorname{vol}(((X_0 + T) \cap \operatorname{strip}(C)) \cap H^+) \\
&+ \operatorname{vol}((X_+ + T) \cap \operatorname{strip}(C)) \cap H^+) \\
&+ \operatorname{vol}((X_- + T) \cap \operatorname{strip}(C)) \cap H^+) \\
=&\operatorname{vol}(((X_0 + T) \cap \operatorname{strip}(C)) \cap H^+) \\
&+ \operatorname{vol}((X_+ + T) \cap \operatorname{strip}(C)) \cap H^+) \\
&+ \operatorname{vol}((X_+ + T) \cap \operatorname{strip}(C)) \cap H^-) \\
=&\operatorname{vol}(((X_0 + T) \cap \operatorname{strip}(C)) \cap H^+) \\
&+ \operatorname{vol}((X_+ + T) \cap \operatorname{strip}(C))).
\end{aligned}$$

Together with

$$\begin{aligned}
v_C &= \operatorname{vol}(R(C) \cap ((C \cap \Lambda) + T)) \\
&= \operatorname{vol}((X_0 + T) \cap \operatorname{strip}(C)) + \operatorname{vol}((X_+ + T) \cap \operatorname{strip}(C))
\end{aligned}$$

we finally get

$$\begin{aligned}
\mu(C) &= v_C - w_V^C \\
&= \operatorname{vol}(R(C) \cap \operatorname{CDC}(C)) - \operatorname{vol}(R(C) \cap C) \\
&= \operatorname{vol}((X_0 + T) \cap \operatorname{strip}(C)) - \operatorname{vol}(((X_0 + T) \cap \operatorname{strip}(C)) \cap C^+) \\
&= \operatorname{vol}(((X_0 + T) \cap \operatorname{strip}(C)) \cap C^-) \\
&= \frac{1}{2}
\end{aligned}$$

as we wanted to show. □

Remark. To adjust this result to hold for the function μ^* from Chapter 2, the fundamental domains $T^*(A)$ for subspaces $A \subseteq V$ have to be point symmetric at the chosen representatives of the translation classes instead of centrally symmetric at the origin.

Chapter 4

Brick version and implementations

The largest obstacle in implementing the local formula μ is that the regions are not convex. The regions are inductively defined as the complement of regions of lower fcones translated by generic lattice points (see Section 1.2). As in the example of the simplex S in Section 1.3, already for halfspaces the regions can be non-convex. One way to deal with that is to use the tiling we get from Lemma 1.2 and to define the region as the union of regions that are not in their complement. To determine these regions is rather involved, but possible. The regions can then be considered as collections of (convex) polytopes that have nontrivial intersection. To compute the values for the relative domain volume and the correction volume, we can then use the inclusion-exclusion-principle. The author has a working prototype in `SageMath` [S$^+$16] implementing the construction this way. But since the cardinality of the collection of polytopes in each region and thus the complexity of all operations on the regions is very high and growing exponentially, this version is not useful already in many cases in dimension three.

Instead, we give the *brick version* μ_b as a variation of μ that elegantly circumvents the issue with non-convex polyhedral structures. Although we have to trade in some properties of symmetry, the computability outruns by far the above described implementation of the original μ.

4.1 Brick version

In this section, we want to present a version of the local formulas where the regions are unions of full translates of fundamental domains. Because of the appearing shapes we will call this variation the *brick version of* μ, notated as μ_b and construct it from *brick regions* R_b — in the hope that after some chapters of heavy terminology some figurativeness might help the mind distinguish between the original and the newly introduced version. The reason to introduce the brick version is that besides looking nice it has computational advantages: The brick regions have a natural dissection into convex polytopes, as they can be seen as a collection of translations of fundamental domains rather than non-convex polyhedral structures. So that operations and algorithms on convex polyhedra can naturally be used. Additionally, in many cases, operations can even be reduced to operations on finite point sets. See Section 4.2 for details.

As something made up of bricks, the term *wall* is imminent and thus we introduce the following definition. Again, we assume to have chosen and fixed a fundamental domain $T(U)$ that contains the origin for each subspace U of V with respect to the lattice $U \cap \Lambda$. For a polyhedron Q we set $T(Q) := T(\mathrm{lineal}(Q))$. Due to its frequent use we further set $T := T(V)$.

Definition 9. A set $A \subseteq V$ is called a *wall* if $\mathrm{DC}(A) = A$, where we recall that the domain complex of A is defined as $\mathrm{DC}(A) = (A \cap \Lambda) + T$.

Since the origin is the unique lattice point contained in T, for a set A to be a wall thus means that A is of the form $A = X + T$ for some subset $X \subseteq \Lambda$. Examples for walls are the domain complex and the covering domain complex of a set. In our analogy we deliberately ignore the fact that real-life walls do have trimmed bricks in them and that one rarely comes across a brick in the shape of a rhombic dodecahedron, to say the least.

If we recall the definition of the regions in Chapter 1, we note that the intersection with the strip is the main obstacle to being a wall. And, as it turns out, it is the only thing we need to change to achieve regions that are walls. We recall that for a polyhedron Q the strip of Q is defined as $\mathrm{strip}(Q) = T(Q) + \mathrm{lineal}(Q)^{\perp}$. Then from the lattice points of $\mathrm{strip}(Q)$ we get the *brick strip* (or *pillar*)

$$\mathrm{strip}_b(Q) := \mathrm{DC}(\mathrm{strip}(Q)).$$

Then the definition of the brick regions R_b is exactly the same as the original regions, just using the brick strip instead of the original strip:

As before, we can assume the cones to be full-dimensional as otherwise we can intersect everything with the affine span and take that as our ambient space. The definition of R_b is recursive, starting with the unique minimal element in the so-called order of fcones that we introduced in Section 1.1 and will elaborate on in Section 5.2. We define

$$R_b(V) := T.$$

Now let C be a full-dimensional cone with $V \prec C$ and we assume we have constructed the brick regions $R_b(D)$ for all cones $D \prec C$.

The set of *brick generic lattice points* of D with respect to C is the set of all points $x \in \operatorname{lat}(D)$ that fulfill the conditions:

(I_b) For all halfspaces $H \preceq C$ such that $H \npreceq D$:

$$x + R_b(D) \subseteq \operatorname{int}(H)$$

(II_b) For all $E \prec C$, such that E is incomparable to D and for all $x' \in \operatorname{lat}(E)$:

$$(x + R_b(D)) \cap (x' + R_b(E)) = \varnothing$$

We further set $X_C^C := \operatorname{lat}(C)$.

Then the brick region is given as

$$R_b(C) := (\operatorname{strip}_b(C) \cap \operatorname{CDC}(C)) \setminus \bigcup_{D \prec C} \left(X_D^C + R_b(D)\right).$$

Having constructed the regions, we can compute the values of the *(brick) relative domain volume* v_{bC} in the region $R_b(C)$ as:

$$\mathrm{v}_{bC} := \operatorname{vol}\big(R_b(C) \cap \operatorname{DC}(C)\big).$$

And further the *(brick) correction volumes* for each $D \prec C$:

$$\mathrm{w}_{bD}^{C} := \operatorname{vol}(R_b(C) \cap C \cap \operatorname{lineal}(D)).$$

Then we get the value of μ_b for C as

$$\mu_b(C) := \mathrm{v}_{bC} - \sum_{D \prec C} \mathrm{w}_{bD}^{C} \cdot \mu(D).$$

We will show below that $R_b(C)$ is a wall and thus v_{bC} is the volume of a wall. Hence, the brick relative domain volume is—in contrast to the original v_C—always a (positive) integer.

Lemma 4.1. *Let C be a full-dimensional rational cone. Then $R_b(C)$ is a wall.*

Proof. The statement is obviously true for $C = V$, since $R_b(C) = T$. Now let C be a full-dimensional cone with $V \prec C$ and we assume we have shown that $R_b(D)$ is a wall for all cones $D \prec C$. Let $y \in \Lambda$ and A be a wall. Let $X := A \cap \Lambda$ such that $A = X + T$. For $y \in \Lambda$ we have $y + A = (y + X) + T$ and since $y + X \subseteq \Lambda$, we see that translations of walls by lattice points are again walls. Since walls are completely defined by their lattice points, it is also easy to see that intersections, complements and (even infinite) unions of walls are again walls. Writing R_b as

$$R_b(C) = (\operatorname{strip}_b(C) \cap \operatorname{CDC}(C)) \setminus \bigcup_{D \prec C} \bigcup_{x \in X_D^C} (x + R_b(D))$$

thus shows the statement. □

The proof that μ_b is also a local formula is similar to the proof for μ to such a large extent that we only give a few remarks on where the proof changes.

Lemmas 1.1 to 1.3 work out exactly the same, using in the proof of Lemma 1.2 the fact that for a full-dimensional cone C we still have a tiling of V by $\{x + \text{strip}_b(C) \mid x \in \text{lat}(C)\}$. The statement of Lemma 1.4 does not hold the exact same way, since $T(C)$ is not necessarily a subset of the brick strip. But with the same arguments it holds that $\text{strip}_b(C) \cap \text{lineal}(C) \subseteq R_b(C)$ and, in particular that the origin is in $R_b(C)$, which is enough to show Lemma 1.5. Then Lemmas 1.6 to 1.9, as well as Theorems 1 and 2 hold and can be proven completely analogously.

Symmetry. The brick version is still invariant under all symmetries of the cones that can be realized as symmetries on the fundamental domains. It is, however, more sensitive regarding the boundary. That is to say there is no analogous statement along the lines of Theorem 5 as shown in Section 4.3 Example 3. The reason why this happens is that it is a zero-one decision whether a fundamental domain is in the region or not. That means that if a lattice point is on the boundary of the strip, it is in or out of the region depending on the exact boundary structure of the fundamental domain. In cases where no lattice points are on the boundary of the strip, symmetric invariance of the values of μ_b is maintained.

Even though the brick version is more sensitive to symmetry regarding the boundary of the fundamental domains, the result from Theorem 6 still holds:

Corollary 1. *Let $T(A)$ be a fundamental domain with centrally symmetric closure for each $A \subseteq V$. Let P be a lattice polytope and $F < P$ a facet. Then*

$$\mu_b(\text{fcone}(P, F)) = \frac{1}{2}.$$

Proof. In the brick version, things get even a little easier, since we can write $R_b(C)$ as

$$R_b(C) = (\text{strip}(C) \cap Y_H) + T,$$

and while most steps of the proof are essentially the same as the proof of Theorem 6 (just intersection with strip(C) and Minkowski sum with T interchanged), the proof that

$$\text{vol}((\text{strip}(C) \cap X_0) + T) = 1$$

follows immediately from the fact that the origin is the only lattice point in

$$\text{strip}(C) \cap X_0 = \text{strip}(C) \cap \text{lat}(C).$$

□

4.2 Implementation

The brick version is quite straightforward to implement. The full source code can be found in the appendix. In this section, we give a few remarks on the code. In the code we introduce the class `fcones` given by their inequalities and the class `brickRegions`. The regions can be determined by the fcone they are constructed for and their essential information are the lattice points `self._pts` and the fundamental domain `self._V0` that determine their structure. For an fcone, we can compute Dirichlet–Voronoi cells in the lineality space of the fcone via the function `Voronoi`, we can construct a brick region via the function `construct_brickRegion` and compute its μ_b-value via `brick_mu`. In order to reduce computation time, the output of all three functions is stored in maps.

The given code assumes that the lattice is $\mathbb{Z}^n$, where n is the ambient dimension of the polytope. Note that it is not necessarily the dimension of the polytope itself, as it is often useful to consider polytopes in a higher dimensional space, e.g. the permutahedra. The fundamental domains used in the code are Dirichlet–Voronoi cells with respect to the standard scalar product. A general direction to determine in which way the fundamental domains are half open is computed as a heuristic in the function `general_direction`. Optionally, a list of general directions for certain subspaces can be given as input. In that case, the program only checks whether the given directions are indeed in general position and if yes, they are taken. This can be essential when exploiting symmetries that are only respected for specific choices of the boundary.

The program has two main functions, `all_mu_values_polyhedron` and `Ehrhart_coeff`. The first function takes a polyhedron and returns the μ_b-values of the fcones of all its faces. The output is a list of tuples where the first entry codes the fcone and the second one is the μ_b-value. The fcone is given by the inner normal vectors of its facets, which is due to the way `SageMath` gives the inequalities of polyhedra. In `all_mu_values_polyhedron`, a cone can be given as input as long as it is in the base class of polyhedra.

The second main function `Ehrhart_coeff` takes a lattice polytope as input and returns the coefficients of its Ehrhart polynomial, which is computed using the local formula μ_b.

Both functions have several optional inputs. `Ehrhart_coeff` has the input `down_to` which is set to 0 by default and stops the computation of the Ehrhart coefficients at a given index. The coefficients are computed inductively starting with the coefficient of the highest exponent and going down to the constant. By this, the computation time can be reduced largely if one is only interested in higher coefficients. To compare running times, both functions have the optional input `timer`, which prints the computation time in seconds if set to `True`.

A very important input is the radius. It is an estimate for how big the regions can be. If the radius is too small, the values are simply wrong. If the radius is big, the computation time increases with exponent n, where n is the ambient dimension. When computing the Ehrhart coefficients, it is easy to check whether the chosen radius was big enough as the last coefficient, the constant should equal 1. If that is not the case, a warning is raised and the most likely problem is that the radius has to be bigger. The use of the radius in the program is a rough implementation of the bounding radius established in the proof of Lemma 1.3. The program and its running time can be improved by a more detailed realization and an automatic determination of the radius given in the proof of Lemma 1.3.

Due to the implementation of the brick version instead of the original version, the regions have the simple structure of the fundamental domain denoted by `V0` translated by a set of lattice points. Thus, many operations on regions, as, for instance, union, intersection and translation are merely a simple operation on finite sets of points. The computation of the volume as well as of the relative domain volume of a region are reduced to the counting of points. Only when intersecting with a polyhedron it is necessary to consider the actual translated fundamental domains.

Exemplary inputs and outputs for the octahedron in $\mathbb{R}^3$ with lattice $\mathbb{Z}^3$ are given in Example 3 in Section 4.3.

4.3 Examples

Using the `SageMath` implementation for the brick version, it is easy to compute examples. The figures in this section are created by translating the results of the `SageMath` program into `polymake` [GJ00], [AGH$^+$17] and plotting them via `jReality` [GHS$^+$17]. In Example 1 we revisit the example from Section 1.3 so that a direct comparison between the brick version and the original version is possible. An example for particularly easy and symmetric regions is the hypercube in any dimension, which we discuss in Example 2. As a nontrivial example in dimension 3 we look at the octahedron in Example 3.

Example 1. Constructing the brick Regions on the simplex S introduced in Section 1.3 results in the regions given in Figure 4.1. It is very easy to compute the value for the relative domain volumes of the faces as we only have to count the number of fundamental domains whose interior lattice point lies in S:

$$\begin{aligned} &\mathrm{v}_{bS} = 1, \qquad \mathrm{v}_{bf_1} = 2, \qquad \mathrm{v}_{bf_2} = 2, \qquad \mathrm{v}_{bf_3} = 1, \\ &\mathrm{v}_{bv_1} = 2, \qquad \mathrm{v}_{bv_2} = 2 \qquad \mathrm{v}_{bv_3} = 4. \end{aligned}$$

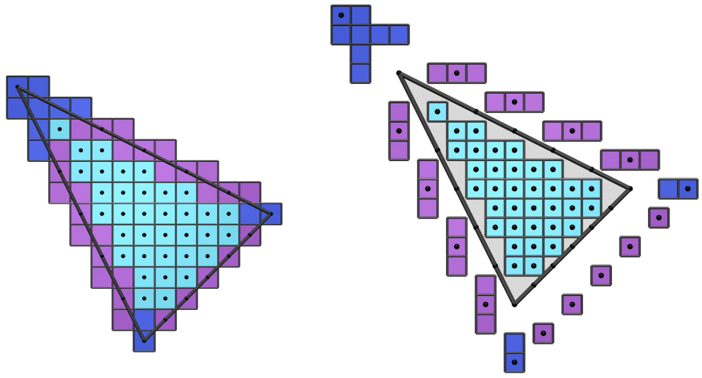


Figure 4.1: The Simplex $6 \cdot S$ with brick regions and a blow-up of the regions on the boundary from the center to help distinguish.

Together with the correction volumes we then get

$$\begin{aligned} \mu_b(S) &= 1 \\ \mu_b(f_1) = \mu_b(f_2) &= 2 - 3/2 \cdot 1 = 1/2 \\ \mu_b(f_3) &= 1 - 1/2 = 1/2 \\ \mu_b(v_1) = \mu_b(v_2) &= 2 - 1/2 \cdot 1/2 - 1/2 \cdot 1/2 - 9/8 \cdot 1 = 3/8 \\ \mu_b(v_3) &= 4 - 3/2 \cdot 1/2 - 3/2 \cdot 1/2 - 9/4 \cdot 1 = 1/4 \end{aligned}$$

Example 2. The hypercube H as the convex hull of all vectors with ± 1 entries in $\mathbb{R}^n$ with lattice $\mathbb{Z}^n$ is a particularly easy example. With the standard inner product the fundamental domain in each subspace considered is again a hypercube itself (with edge length 1 and barycenter in the origin). Moreover, there are no lattice points on the boundaries of the strip so that the values are invariant under all symmetries of the hypercube, even in the brick version. Hence, in the original version as well as in the brick version, the Regions all consist of one n-dimensional hypercube. See Figure 4.2 for examples in two and three dimensions.

Since the regions are that simple for the hypercube, it follows immediately that $v_f = \mathrm{v_b}_f = 1$ for each $f \leq H$. The correction volume is $w_g^f = \mathrm{w_b}_g^f = 1/2^{\dim(g)-\dim(f)}$ and it can be shown via induction that the resulting values of the local formulas are

$$\mu(f) = \mu_b(f) = 1/2^{n-\dim(f)}.$$

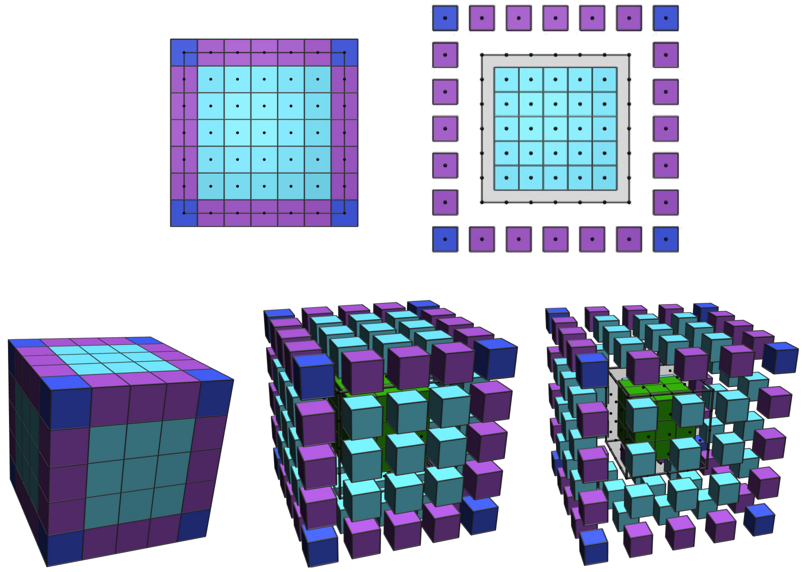


Figure 4.2: The 2-dimensional hypercube (square) and 3-dimensional hypercube (cube) with brick regions.

Example 3. An example where the symmetry of the original version is not maintained in the brick version is the octahedron O defined as the convex hull of the standard basis vectors e_i, $i \in \{1, 2, 3\}$, and their negatives. O has the same symmetries as the cube so that the μ-value of the fcones of its faces only depends on the dimension of the face and is given as

$\dim(f)$	3	2	1	0
$\mu(f)$	1	1/2	2/9	1/6

.

In the brick version, this symmetry cannot be preserved fully. To see this, we can take a look at $\text{strip}(F)$ for any facet F of O. For example, we can take $F = \text{conv}(e_1, e_2, e_3)$. The fcone $\text{fcone}(P, F)$ is the halfspace given by the outer normal vector $v := (-1, -1, -1)$. The Dirichlet–Voronoi cell $T(F)$ in the hyperplane orthogonal to v with respect to the standard inner product is a hexagon and the strip is the Minkowski sum of $T(F)$ with the line defined by v. The boundary of this strip contains lattice points, for instance the point $a = (0, 0, 1)$ and its negative. Hence, the decision of which part of the boundary is included in $T(F)$ and which one is not determines whether $a + T$ is in $\text{strip}_b(F)$ or if $-a + T$ is. Thus, the region $R_b(F)$ is not symmetrical with respect to central symmetry. The resulting brick regions are shown in Figure 4.3. The values that arise for the fcones of the faces of O for one choice of general directions are the following:

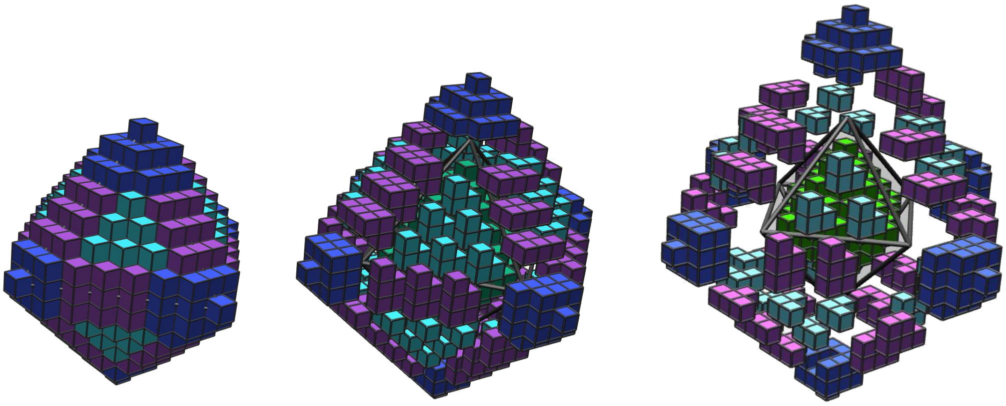


Figure 4.3: The octahedron with brick regions.

```
sage: Oct=polytopes.octahedron()
sage: all_mu_values_polyhedron(Oct, radius=6, timer=True)

timer: 166.040977001  seconds

[(set(), 1),
 ({(0, 1, 1, 1)}, 1/2),
 ({(0, 1, 1, -1)}, 1/2),
 ({(0, 1, -1, 1)}, 1/2),
 ({(0, 1, -1, -1)}, 1/2),
 ({(0, -1, 1, -1)}, 1/2),
 ({(0, -1, -1, -1)}, 1/2),
 ({(0, -1, -1, 1)}, 1/2),
 ({(0, -1, 1, 1)}, 1/2),
 ({(0, -1, 1, 1), (0, 1, 1, 1)}, 5/24),
 ({(0, 1, -1, 1), (0, 1, 1, 1)}, 1/6),
 ({(0, 1, 1, -1), (0, 1, 1, 1)}, 1/12),
 ({(0, -1, 1, -1), (0, 1, 1, -1)}, 5/24),
 ({(0, 1, -1, -1), (0, 1, 1, -1)}, 1/6),
 ({(0, -1, -1, 1), (0, 1, -1, 1)}, 5/24),
 ({(0, 1, -1, -1), (0, 1, -1, 1)}, 1/4),
 ({(0, -1, -1, -1), (0, 1, -1, -1)}, 5/24),
 ({(0, -1, 1, -1), (0, -1, 1, 1)}, 1/4),
 ({(0, -1, -1, -1), (0, -1, 1, -1)}, 7/24),
 ({(0, -1, -1, -1), (0, -1, -1, 1)}, 1/3),
 ({(0, -1, -1, 1), (0, -1, 1, 1)}, 7/24),
 ({(0,-1,-1,1), (0,-1,1,1), (0,1,-1,1), (0,1,1,1)}, 13/48),
 ({(0,-1,1,-1), (0,-1,1,1), (0,1,1,-1), (0,1,1,1)}, 5/8),
 ({(0,1,-1,-1), (0,1,-1,1), (0,1,1,-1), (0,1,1,1)}, 2/3),
 ({(0,-1,-1,-1), (0,-1,1,-1), (0,1,-1,-1), (0,1,1,-1)}, 13/48),
 ({(0,-1,-1,-1), (0,-1,-1,1), (0,1,-1,-1), (0,1,-1,1)}, -1/12),
 ({(0,-1,-1,-1), (0,-1,-1,1), (0,-1,1,-1), (0,-1,1,1)}, -3/4)]
```

The first two lines are the exact input as entered into `SageMath`, everything below is the output. The time is the time taken on a standard 2015 Laptop and the output is given as a list of tuples, where the first set decodes the fcone and the second one the μ_b-value. The fcone is given by a list of inequalities, which means that the first entry is the whole space. A tuple of the form (a, b, c, d) stands for the inequality

$$(b, c, d)^t x + a \geq 0.$$

We observe that, as expected, the values of facets (fcones given by exactly one inequality) equals 1/2, but that most of the other values are not invariant under the octahedral symmetries.

The Ehrhart polynomial computed with the function `Ehrhart_coeff` creates the following output:

```
sage: Ehrhart_coeff(Oct,radius=6, timer=True)

Ehrhart polynomial: 4/3t^3+2t^2+8/3t+1

timer: 0.714751958847  seconds

[[3, 4/3], [2, 2], [1, 8/3], [0, 1]]
```

The last line is the actual return of the function: a list of tuples (strictly speaking lists with two entries), where the first entry is the index and the second one the Ehrhart coefficient for that index. Because of the previous computations of the μ_b-values and their storing in maps, the running time here is only a fraction of what it is above.

Chapter 5

More on fundamental domains, cones and duality

This chapter is a background chapter going into more detail on two concepts that are fundamental to this work: The fundamental domains that determine our local formulas and the cones that the formulas are defined on. Due to repeated questions in talks, Section 5.1 discusses what fundamental domains can and cannot look like and what happens if the definition was to be changed in some ways. Historically, local formulas have been defined on normal cones (see, for instance, McMullen [McM83]). These concepts are basically the same and connected via duality. To facilitate a transition between both views, Section 5.2 shows their connections. Since local formulas are defined inductively, we put an emphasis on the order of fcones that was briefly introduced in Section 1.1 and examine it in relation to the order of normal cones.

5.1 More on fundamental domains

The choice of fundamental domains is the determining factor for the local formulas presented in this work. In Section 3.1, we have seen a class of examples, the Dirichlet–Voronoi cells, that enable us to exploit symmetries in polytopes. These are a natural choice of fundamental domains and are the ones mostly used in practical computations. In this section, however, we are concerned with the more unusual cases of fundamental domains to see what is still allowed and what is not. We start by recalling the definition of fundamental domains:

Definition. For a subspace $U \subseteq V$ with induced sublattice $\operatorname{lat}(U) = U \cap \Lambda$, a *fundamental domain* $T(U)$ is a bounded subset of U such that $\{x + T(U) \mid x \in \operatorname{lat}(U)\}$ is a tiling of U and that every intersection of $T(U)$ with an affine subspace of V is measurable.

The reader who is familiar with the term fundamental domain in topology or group theory might know a fundamental domain as a subset of a topological space containing exactly one point of each orbit under the action of some group G on the space. In that sense, the fact that translates of fundamental domains form a tiling of space ensures that the fundamental domains defined here are also fundamental domains in the topological sense for the Euclidean space U with the lattice $\mathrm{lat}(U)$ acting on it by translation.

Note that we do not require fundamental domains to be rational polyhedra. In practical use, rational fundamental domains seem more convenient. A setting where these non-standard fundamental domains can be useful, is, for example, if you have a particular cone and a certain value and you want to find a local formula that assigns that value to that cone.

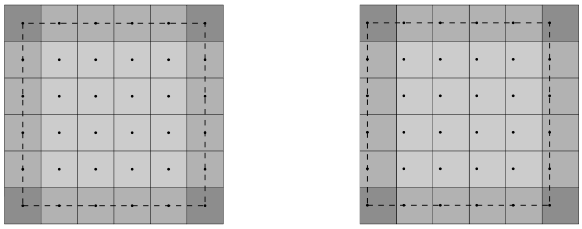


Figure 5.1: Tiling of a square by regions constructed using fundamental domains from Example 1 for $\eta = 0$ (left) and $\eta = 1/\pi$ (right).

Example 1. To create a fundamental domain that is not a rational polyhedron, we can simply take any fundamental domain and translate it by an irrational point. For example we can take

$$T_1 := \quad (\eta, 0) \quad + \quad \left[-\frac{1}{2}, \frac{1}{2}\right) \times \left[-\frac{1}{2}, \frac{1}{2}\right),$$

with $\eta \in \left(0, \frac{1}{2}\right)$. Then T_1 is a fundamental domain in $\mathbb{R}^2$ with lattice $\mathbb{Z}^2$ and local formulas can be defined based on it. A picture of the resulting regions for a square are given in Figure 5.1. It is apparent that if η is not in $\mathbb{Q}$, then T_1 is not rational. The value of μ depending on η for the fcones of the faces of the square are $1/2 - \eta$ for the edge on the left, $1/2 + \eta$ for the edge on the right, $1/2$ for the upper and lower edge, $1/4 - 1/2 \cdot \eta$ for the vertices of the left edge and $1/4 + 1/2 \cdot \eta$ for the vertices of the right edge.

Example 1 shows to which extent local formulas are not unique as it is possible to shift the weights almost arbitrarily from one side to the other.

Figure 5.2: Unbounded and, respectively, not connected fundamental domains T_2 and T_3 of $\mathbb{R}^2$.

It is quite natural to think of fundamental domains as bounded and since we need to compute the volume and do not want to take any limits, we certainly need them to be bounded. Example 2 shows that there exist subsets of $\mathbb{R}$ that fulfill all requirements of being a fundamental domain except being bounded.

Example 2. Define the fundamental domain T_2 of $\mathbb{R}^2$ with lattice $\mathbb{Z}^2$ as a union of rectangles

$$T_2 = \bigcup_{n \in \mathbb{Z}_{\geq 0}} (n, n+1] \times \left[\frac{2^n - 1}{2^n}, \frac{2^{n+1} - 1}{2^{n+1}} \right)$$

as shown in Figure 5.2. Then translates of T_2 by points $(t, 0)$ with $t \in \mathbb{Z}$ cover the strip $\mathbb{R} \times [0, 1)$ without pairwise intersections. This way, it is easy to see that the translates by all lattice points in $\mathbb{Z}^2$ form a tiling of $\mathbb{R}^2$. Thus, the only requirement T_2 does not fulfill is being bounded. With this set T_2, the regions could still be defined, but for obvious reasons they would not be bounded and there can be no results analogously to Lemma 1.3 and Theorem 1.

We do however admit fundamental domains to be not connected as all statements and proofs work just fine with not connected fundamental domains, see Example 3.

Example 3. Define the set T_3 of $\mathbb{R}^2$ with lattice $\mathbb{Z}^2$ as

$$T_3 := \left[0, \frac{1}{2}\right) \times \left[-\frac{1}{2}, \frac{1}{2}\right) \quad \cup \quad \left[\frac{3}{2}, 2\right) \times \left[-\frac{1}{2}, \frac{1}{2}\right).$$

Then T_3 is a fundamental domain but not connected.

The last property of a fundamental domain is that it is Lebesgue-measurable in every intersection with an affine subspace. We use the fundamental domain to define a relative volume by taking any measure on the considered subspace and scaling it such that a fundamental domain has volume 1. Therefore, the fundamental domain has to be measurable. Since we also take volumes of intersections with subspaces, we also need every intersection with an affine subspace to be measurable.

Example 4. To give an example for a set that is not measurable in the required way but satisfies all other conditions of a fundamental domain, one can take a any non-measurable set $\mathcal{V} \subseteq [0,1]$ (cf. remark below) and unite it with its complement $\bar{\mathcal{V}} := [0,1] \backslash \mathcal{V}$ shifted by 1:

$$T_4 = \mathcal{V} \cup (\bar{\mathcal{V}} + \{1\}).$$

Remark. In 1905, Giuseppe Vitali [Vit05] used a choice function to create a non-measurable set, in the sense that there exists no translation-invariant countably additive, positive real measure defined on the set. We briefly show the construction of Vitali as given in [Moo82]. Let $x \in \mathbb{R} \backslash \mathbb{Q}$ and define the set

$$A_x := \{x + b \mid b \in \mathbb{Q}\}.$$

For each distinct A_x, choose an element p in $A_x \cap (0, \frac{1}{2})$ and let G_0 be the set of all such p. Then for $q \in \mathbb{Q}$, the sets

$$G_q := q + G_0$$

are disjoint congruent sets. If G_0 is measurable, then so is each G_q. In particular, as G_0, $G_{1/2}$, $G_{1/3}$, $G_{1/4}, \ldots$ are all subsets of the interval $[0,1]$ such that the sum of their measures is at most one. Since there are infinitely many of them, the measure of each one and thus also the measure of each G_q, with $q \in \mathbb{Q}$, has to be zero. But then the measure of $\mathbb{R}$, which is the union of all of these, is also zero.

Any such set $\mathcal{V} \subseteq [0,1]$ with the property that for each $x \in \mathbb{R}$ there exists exactly one $y \in \mathcal{V}$ such that $y - x$ is a rational number is called a *Vitali set.*

5.2 Duality: Fcones and normal cones

Duality is a mighty and useful tool that appears in almost every corner of mathematics. Switching between one view and its dual can open up a stream of new ideas and techniques. It can also be confusing at times. The duality that appears in the context of local formulas is the one between normal cones and the cones of feasible directions of a polyhedron. Historically, a local formula is defined on the normal cones of a polytope. The local formulas presented in this work can be defined in this way, see [RS19]. However, the author chose to give the construction and all proofs from the point of view of fcones. The main reason for that is that this way the description of the construction is more straightforward so that we gain more clarity in our arguments. The aim of this section is to light up the duality relevant to this work, as well as to give a detailed description of some connections between polyhedra, cones and faces that we use at several points throughout this thesis. Most of the content in this section is well-known, for more information on this topic, see for instance Barvinok [Bar08] or Goodman et al. [GOT18].

The aim of this section is to explain the order on fcones that is used to define the local formulas in this work and to give a description of the analogy between this order and the order on normal cones. By this we want to enable the reader to see the connections and, if needed, to switch between the two. The essence of these connections are presented in Table 5.1. We further go into some more detail regarding results on fcones that are used in the proofs in Section 1.4.

Let $Q \subseteq V$ be a rational polyhedron. The *cone of feasible directions*, short *fcone*, of P at a face f was defined in Section 1.1 as

$$\mathrm{fcone}(Q, f) := \{x \in V \mid \exists \varepsilon > 0\colon s + \varepsilon x \in Q\}$$

for any point s in the relative interior $\mathrm{int}(f)$ of f. The (outer) *normal cone* N_f of Q at f is defined as

$$\mathrm{normal}(Q, f) := \{x \in V \mid \langle x, y - s\rangle \leq 0 \ \forall y \in Q\}$$

for any vector s in the *relative interior* of f. If we have a cone C with face D, the fcone of C in D has a particularly nice structure as $\mathrm{fcone}(C, D) = \{c - d \mid c \in C \text{ and } d \in D\}$, which is why the notation $C - D := \mathrm{fcone}(C, D)$ is used at times.

Given a cone $C \subseteq V$, the *dual cone* $C^\vee$ is defined as

$$C^\vee := \{x \in V \mid \langle x, y\rangle \leq 0 \ \forall y \in C\}.$$

As mentioned before, the normal cone and the fcone of a polyhedron at a face f are dual to each other. The name *cone of feasible directions* is due to the fact that it can be pictured as the cone that you get by translating P such that the origin is in the relative interior of f and then taking the cone over all directions that you can go without leaving P. Since the origin is in the relative interior of this translation of f, all directions within f are feasible and thus the lineality space of $\mathrm{fcone}(P, f)$ equals $\mathrm{lin}(f)$, the linear space parallel to the affine hull of f. That implies $\dim(f) = \dim(\mathrm{lineal}(C_f))$.

The face lattice and partial orderings on cones. For a polyhedron Q, the *face lattice* is the partially ordered set consisting of all faces of Q with order given by inclusion. We consider Q as a face of itself. This order is denoted by '$\leq$' and we write $f < g$ if we want to exclude the case $f = g$. The face lattice is a combinatorial lattice, since for every two faces f, g there exist a unique least upper bound $f \vee g$ called *join* and a unique greatest lower bound $f \wedge g$ called *meet*. $f \vee g$ is the smallest face of Q that contains both f and g, and $f \wedge g$ is given by the intersection $f \cap g$. Here, we formally consider the empty set as a face of Q.

The set of normal cones of a polyhedron Q at the faces $f \leq Q$,

$$\Sigma_Q := \{\mathrm{normal}(Q, f) \mid f \leq Q\},$$

is a *(polyhedral) fan*, called the *normal fan* of Q. That means that every face of a cone in Σ_Q is also in Σ_Q and that for any two cones N, M in Σ_Q their intersection is a face of both N and M. This fan is also *complete*, meaning that the union of all elements in Σ_Q is the whole space. The order of the faces of Q is opposite to the order on its normal fan since for faces f, g of Q we have

$$f \leq g \quad \Leftrightarrow \quad \mathrm{normal}(Q, g) \leq \mathrm{normal}(Q, f).$$

The order on cones '$\prec$' that we introduced in Section 1.1 is defined as

$$D \prec C \quad :\Leftrightarrow \quad D = \mathrm{fcone}(C, F) \text{ for a face } F \text{ with } \mathrm{lineal}(C) < F \leq C.$$

To distinguish between the to orderings, we refer to this order as the *fcone order* and to the order on the face lattice as the *face order*. The fcone order corresponds to the face order on the duals in the following way:

$$D \prec C \Leftrightarrow D^{\vee} < C^{\vee}.$$

We further have

$$\begin{aligned} D \preceq C :\Leftrightarrow\, & D = \mathrm{fcone}(C, F) \text{ for a face } F \text{ with } \mathrm{lineal}(C) \leq F \leq C \\ \Leftrightarrow\, & D^{\vee} \leq C^{\vee}. \end{aligned}$$

Since for a cone C the set of faces is finite, we also have that the set $\{D \mid D \prec C\}$ is finite. This is essential for the recursive construction of our local formulas. Whenever $\mathcal{D} \prec C$, we have $\mathrm{lineal}(C) \subsetneq \mathrm{lineal}(D)$ and thus $\dim(\mathrm{lineal}(C)) < \dim(\mathrm{lineal}(D)) \leq \dim(V)$. That means the dimension of the lineality spaces is strictly increasing when going down in the order of fcones.

Let f, g be faces of a polyhedron Q. Then we have

$$f \leq g \quad \Leftrightarrow \quad \mathrm{fcone}(Q, g) \preceq \mathrm{fcone}(Q, f).$$

The reason why we chose the order on fcones this way and not the opposite is that while a higher dimensional face f of Q has more information than its own faces, the fcone of f has less information than the fcones of the faces of f. The fcone as well as the normal cone of a polyhedron Q in a face f are *local* in the way that they only store information about the face locally around an inner point of the face and forget about the rest of the polytope including the boundary of the face. One result of this locality is that the fcones and normal cones of Q do not change under dilation, i.e. $\mathrm{fcone}(Q, f) = \mathrm{fcone}(tQ, tf)$ and $\mathrm{normal}(Q, f) = \mathrm{normal}(tQ, tf)$ for all $t \in \mathbb{R}_{>0}$.

If Q is full-dimensional, then so are all fcones of faces $f \leq Q$ and all normal cones are pointed. The fcone of a facet F of a full-dimensional polyhedron is always a halfspace and its normal cone is a ray. Due to the halfspace description of polyhedra, we have for every polyhedron Q

$$Q = \bigcap_{F \text{ facet of Q}} F + \operatorname{fcone}(Q, F). \tag{5.1}$$

In Lemma 1.5 we apply this to full-dimensional cones C to get

$$C = \bigcap_{\substack{H \prec C \\ H \text{halfspace}}} H.$$

Using that

$$\operatorname{fcone}(Q, f) = \bigcap_{f \leq F < Q} \operatorname{fcone}(Q, F),$$

for all $f < Q$, Equation (5.1) also yields that Q can be reconstructed from its vertices and their fcones by

$$Q = \bigcap_{v \text{ vertex of } Q} \operatorname{fcone}(Q, v) + v,$$

which we use in the proof of Lemma 1.6. In the same proof we use that

$$\operatorname{lineal}(\operatorname{fcone}(P, f)) \cap \operatorname{fcone}(P, v) = \operatorname{fcone}(f, v).$$

This last equation is easy to verify, since the feasible directions of f in a vertex v are the ones of P in v that are also in $\operatorname{lin}(f) = \operatorname{lineal}(\operatorname{fcone}(P, f))$.

As we have seen, the fcones and the normal cones of a polyhedron Q are in one-to-one correspondence respecting the orderings that we have given on both sets. This shows that as the set of normal cones with the face order form a combinatorial lattice, so does the set of fcones with the fcone order. The meet of two normal cones is their intersection, i.e. for $f, g \leq Q$ we have

$$\operatorname{normal}(Q, f) \wedge \operatorname{normal}(Q, g) = \operatorname{normal}(Q, f) \cap \operatorname{normal}(Q, g) = \operatorname{normal}(Q, f \vee g),$$

where $f \vee g$ is the smallest face of Q that contains both, f and g. In the fcone order we have

$$\operatorname{fcone}(Q, f) \wedge \operatorname{fcone}(Q, g) = \operatorname{fcone}(Q, f \vee g).$$

The join of two normal cones is the smallest one that contains both and we have for $f, g \leq Q$

$$\operatorname{normal}(Q, f) \vee \operatorname{normal}(Q, g) = \operatorname{normal}(Q, f \wedge g) = \operatorname{normal}(Q, f \cap g).$$

On fcones we get

$$\text{fcone}(Q, f) \vee \text{fcone}(Q, g) = \text{fcone}(Q, f \cap g). \tag{5.2}$$

Let Q be a polyhedron and let f be a face of Q. We set $C_f := \text{fcone}(Q, f)$ and $N_f := \text{normal}(Q, f)$. The following table gives an overview over some connections between normal cones and fcones:

fcone		**polyhedron**		**normal cone**
$C_f = \text{fcone}(Q, f)$	$\overset{\text{fcone}}{\longleftarrow}$	f	$\overset{\text{normal}}{\longrightarrow}$	$N_f = \text{normal}(Q, f)$
$\text{lineal}(C_f))$	$=$	$\text{lin}(f)$	$=$	${N_f}^{\perp}$
$\dim(\text{lineal}(C_f))$	$=$	$\dim(f)$	$=$	$\dim(V) - \dim(N_f)$
$\dim(C_f)$	$=$	$\dim(Q)$	$=$	$\dim(V) - \dim(\text{lineal}(N_f))$
$\dim(V) - \dim(\text{lineal}(C_f))$	$=$	$\dim(V) - \dim(f)$	$=$	$\dim(N_f)$
$\{D \mid D \prec C_f\}$	$\overset{\text{fcone}}{\longleftarrow}$	$\{g \mid f < g \leq Q\}$	$\overset{\text{normal}}{\longrightarrow}$	$\{M \mid M < N_f\}$
$\text{fcone}(Q, f) \wedge \text{fcone}(Q, g)$	$\overset{\text{fcone}}{\longleftarrow}$	$f \vee g$	$\overset{\text{normal}}{\longrightarrow}$	$\text{normal}(Q, f) \cap \text{normal}(Q, g)$
$\text{fcone}(Q, f) \vee \text{fcone}(Q, g)$	$\overset{\text{fcone}}{\longleftarrow}$	$f \wedge g = f \cap g$	$\overset{\text{normal}}{\longrightarrow}$	$\text{normal}(Q, f) \vee \text{normal}(Q, g)$

Table 5.1: Connections between faces of a polyhedron Q, their fcones and normal cones.

Appendix

Source code

```
import time
from sage.geometry.polyhedron.base import is_Polyhedron

########################################################################
################### brick regions ####################################
########################################################################
class brickRegion :
  def __init__(self,fcone, pts, V0) :
    self._fcone=fcone
    self._pts=pts
    self._pts=[vector(v) for v in self._pts]
    for v in self._pts: v.set_immutable()
    self._pts=set(pts)
    self._V0=V0

  def intersect_bickRegions(self, bReg2) :
    points=set()
    for x in self._pts :
      if x in bReg2._pts :
        points.add(x)
    return brickRegion(self._fcone, points,self._V0)

  def unite_brickRegions(self, bReg2) :
    return brickRegion(self._fcone,
                       self._pts.union(bReg2._pts),self._V0)

  def translate_brickRegion(self, pt) :
    points=set()
    for x in self._pts :
      a=x+pt
      a.set_immutable()
      points.add(a)
    return brickRegion(self._fcone, points, self._V0)

  def vol_brickRegion(self) :
```

```
    return len(self._pts)

#the volume of a Region intersected with a polyhedron
  def vol_reg_poly(self, Poly, VC) :
    vol=0
    for x in self._pts :
      part=(self._V0.translation(x)).intersection(Poly)
      if part.dim()==VC.dim():
        rel_vol_part=rel_vol(part, VC)
        vol=vol+rel_vol_part
    return vol

  def plot_brickRegion(self) :
    return sum([(x+self._V0).plot() for x in self._pts])

#test whether a region is contained in the interior of a polytope.
  def Region_interior_inside(self, Poly) :
    all_vertices=[]
    for x in self._pts :
      all_vertices=all_vertices+self._V0.translation(x).vertices_list()
    return all(Poly.interior_contains(v) for v in all_vertices)

  def Region_outside(self, H1) :
    Poly_H1=Polyhedron(ieqs=list(H1))
    for x in self._pts :
      if   Poly_H1.intersection(
                      self._V0.translation(x)).dim()<self._V0.dim() :
          return True
    return False




########################################################################
####################  fcones  ##########################################
########################################################################
class fcone :
#maps to save already computed values
  _mubmap = {}           #map of brick-mu values
  _regbmap = {}          #map of brick-regions
  _vorcellmap = {}       #map of Voronoi cells

  def __init__(self, H, LinHull, n) :
#H is a list of inequalities, LinHull the linear Hull
#and n the dimension of the ambient space
    self._n = n
#canonical choice of inequalities (important for lower dim. case):
    self._H=Polyhedron(ieqs=H, eqns=LinHull).inequalities_list()
```

```
    self._H=[[0]+h[1:] for h in self._H]
#to ensure that H and linHull can be turned into sets:
    self._H=[vector(v) for v in self._H]
    for v in self._H :        v.set_immutable()
#convention: if LinHull has no equations,
#it is set to be the whole space
    self._LinHull=LinHull
    if self._LinHull==[] :
      self._LinHull=[[0 for i in range(0,self._n+1)]]
    self._LinHull=[vector(v) for v in self._LinHull]
    for v in self._LinHull :  v.set_immutable()
    self._H = set(self._H)
    self._lowerFcones=lower_fcones(self._H,self._LinHull, self._n)
#creating an entry in each of the maps:
    if not self in fcone._mubmap.keys() :
      fcone._mubmap[self]=None
    if not self in fcone._regbmap.keys() :
      fcone._regbmap[self]=None
    if not self in fcone._vorcellmap.keys() :
      fcone._vorcellmap[self]=None

  def __hash__(self) :
    return sum(l._hash() for l in list(self._H)+self._LinHull)

  def __eq__(self, other) :
    return self._H == other._H and self._LinHull == other._LinHull


#computing the voronoi cell in the lineality space of the fcone:
  def Voronoi(self) :
    m = fcone._vorcellmap[self]
    if m != None :
      return m
    LinHull_Poly=Polyhedron(eqns=self._LinHull)
    V = VectorSpace(QQ,self._n)
    Lineal_orth=V.subspace([h[1:] for h in list(self._H)+self._LinHull])
    Lineal = Lineal_orth.complement()   #lineality space of self
    B = Lineal.intersection(FreeModule(ZZ,self._n)).basis()
#(+/-) combinations of basis vectors:
    C = reduce(lambda T,b :
      [eta*b+t for eta in range(-2,3) for t in T],B, [Lineal(0)])
#resulting inequalities for the Voronoi cell:
    VC_ieqs = [[v.inner_product(v)/2] + v.list() for v in C]
    VC = Polyhedron(eqns=list(self._H), ieqs=VC_ieqs)& LinHull_Poly
    fcone._vorcellmap[self]=VC
    return VC

  def plot_fcone(self) :
      return Polyhedron(ieqs=list(self._H), eqns=self._LinHull).plot()
```

```
################### Construction of Regions ###########################
  def construct_brickRegion(self, acc, general_directions=[]) :
    R = fcone._regbmap[self]
    if R != None  :
      if R[1]>=acc:
        return R[0]
    general_directions_list=general_directions
#Voronoi cell in LinHull:
    V0= fcone(set(), self._LinHull, self._n).Voronoi()
    zerovector = vector([0 for i in range(0,self._n)])
    zerovector.set_immutable()
    if self._H==set():
      return brickRegion(self, [zerovector], V0)
#Voronoi cell in the lineality space of self:
    VC = self.Voronoi()
    LinHull_Poly=Polyhedron(eqns=self._LinHull)
#lineality space of self:
    lineal = Polyhedron(eqns=list(self._H)+self._LinHull)
    strip = Polyhedron(vertices=VC.vertices(),
              lines=[h[1:] for h in self._H]+[h[1:]
              for h in LinHull_Poly.equations_list()])&LinHull_Poly
#search area depending on the radius 'acc':
    Area = V0.dilation(acc)
    int_strip = (strip & Area).integral_points()
    for x in int_strip: x.set_immutable()
    int_strip=set(int_strip)
    allX=set()
#Computing the set of generic points for all lower cones in the
#order of fcones
    for f in self._lowerFcones:
      Regf=fcone(f._H, self._LinHull, self._n).construct_brickRegion(acc)
      ############## (I) inside #####################
      #output list of points x in lat(H) that fulfil property (I)
      #and that are outside of the covering domain complex
      XHf=self.inside(f, Regf, V0, acc)
      ############## (II) overlapping ##############
      #output: list of points x such that x+Reg(f) is inside
      #AND does not intersect any y+Reg(g) for g incomparable to f
      for g in self._lowerFcones:
        if not akin(f,g) :
          XHf=fcone.non_intersect(f, g, XHf, V0, acc)
      for x in XHf:
        allX=allX.union(Regf.translate_brickRegion(x)._pts)
    points_of_region=int_strip.difference(allX)
#to account for the fact that strip is not bounded but half open
#we take out all visible points from a general direction
#the general direction is either given in general_directions_list
```

```
#or computed here.
    if strip.inequalities_list()!=[] :
      if self._H in [L[0] for L in general_directions_list] :
        for [H1,v] in general_directions_list :
          if H1==self._H:
            gen_direc=v
      else:
        gen_direc=general_direction(strip.inequalities_list())
      visible=[h for h in strip.inequalities_list()
               if gen_direc.inner_product(vector(h[1:]))<0]
      visible_halfsps=[Polyhedron(ieqs=[h]) for h in visible]
      visible_pts=[]
      for x in points_of_region :
        if not all(halfsp.interior_contains(x)
                   for halfsp in visible_halfsps) :
          visible_pts=visible_pts+[x]
      points_of_region=points_of_region.difference(set(visible_pts))
    Regb=brickRegion(self, points_of_region,V0)
    fcone._regbmap[self] = (Regb, acc)
    return Regb

################### (I) inside ########################################
  def inside(self,f, Regf, V0, acc) :
    XHf1=[]
    lineal_f=Polyhedron(eqns=list(f._H)+f._LinHull)
#all facets of self that are no facets of f:
    H1=Polyhedron(ieqs=list(self._H.difference(f._H)))
#a rough radius in which to test
    pts_to_test_inside=(lineal_f & H1 &
                    (V0.dilation(2*acc))).integral_points()
    if f._H==set():
#to end up with a region in CDC,
#we take out all V0 that are completely outside the cone:
      pts_to_test_outside=(lineal_f &
                       (V0.dilation(2*acc))).integral_points()
      for x in pts_to_test_outside:
        if Regf.translate_brickRegion(x).Region_outside(self._H) :
          XHf1=XHf1+[x]
    for x in pts_to_test_inside:
      if Regf.translate_brickRegion(x).Region_interior_inside(H1) :
        XHf1=XHf1+[x]
    return XHf1

################### (II) non_intersect ################################
  def non_intersect(f,g,XHf, V0, acc) :
    lineal_g=Polyhedron(eqns=list(g._H)+g._LinHull)
    pts_test_inters=(lineal_g & (V0.dilation(2*acc))).integral_points()
    Regf=f.construct_brickRegion(acc)
    Regg=g.construct_brickRegion(acc)
```

```
    for y in pts_test_inters :
      for x in XHf :
        Regf_pts=Regf.translate_brickRegion(x)._pts
        if Regf_pts.intersection(
                           Regg.translate_brickRegion(y)._pts)!=set() :
          XHf.remove(x)
    return XHf


#################### Computing mub #####################################
  def brick_mu(self, radius=4, general_directions=[]) :
    m = fcone._mubmap[self]
    if m != None :
      if m[1]>=radius :
        return m[0]
    gen=general_directions
    lineal=Polyhedron(eqns=list(self._H)+self._LinHull)
    fcone_Poly=Polyhedron(ieqs=self._H, eqns=self._LinHull)
    Region1=self.construct_brickRegion(radius, general_directions=gen)
#for the relative domain volume it suffices to count the lattice points
#in Region1 that are also in fcone_Poly:
    DC_vol=0
    for x in Region1._pts :
      if fcone_Poly.contains(x) :
        DC_vol=DC_vol+1
#Computing the correction volumes:
    CorVols=[]
    for f in self._lowerFcones :
      lineal_f=Polyhedron(eqns=list(f._H)+self._LinHull)
      VC_f=f.Voronoi()
      CorVol_f= Region1.vol_reg_poly(lineal_f & fcone_Poly, VC_f)
      brick_mu_f=f.brick_mu()
      CorVols = CorVols + [(f, f.brick_mu() , CorVol_f)]
    mub=DC_vol-sum(a*b for (c, a,b) in CorVols)
    fcone._mubmap[self] = (mub, radius)
    return mub


########################################################################
##################### MAIN  ############################################
########################################################################

#### Computing all mub values ###############
#Input: Polyhedron, radius (optional), timer (optional, takes the time),
#       general_directions (optional, can be given here manually)
#Caution, if radius is too small, values are wrong!
def all_mu_values_polyhedron(Poly, radius=4,
                           timer=False, general_directions=[]) :
  if timer==1 :
```

```
    start=begintimer()
  gen=general_directions
  r=radius
  n=Poly.ambient_dim()
  LinHull=Poly.equations_list()
  if LinHull==[] :
    LinHull= [[0 for i in range(0,n+1)]]
  LinHull=[[0]+h[1:] for h in LinHull]
  mu=[]
  All_fcones_of_Poly=lower_fcones(Poly.inequalities_list(),LinHull,n)
  for f in All_fcones_of_Poly:
    mu=mu+[(f._H,f.brick_mu(radius=r, general_directions=gen))]
#ending timer, prints the time measured in seconds
  if timer==1 :
    endtimer(start)
  return mu

#################### Computing Ehrhart coefficients ####################
#Input: Polytope,
#      down_to (optional, down to which index the coefficients
#                          should be computed)
#      radius (an estimate for a bound of the region -> needs to be big
#              enough to get a correct result, but smaller is faster)
#      timer  (optional, time of computations in seconds)
def Ehrhart_coeff(Polytope, down_to=0, radius=4,
                  timer=False, general_directions=[]):
  if timer==1 :
    start=begintimer()
  gen=general_directions
  k=down_to
  r=radius
  n=Polytope.ambient_dim()
  d=Polytope.dim()
  AffHull=Polytope.equations_list()
  if AffHull==[] :
    AffHull= [[0 for i in range(0,n+1)]]
  LinHull=[[0]+h[1:] for h in AffHull]
  All_Ineqs=Polytope.inequalities_list()
  fcone_of_Polytope=fcone(set(), LinHull, n)
  V0=fcone_of_Polytope.Voronoi()
  Ehr=[[d,fcone(set(),LinHull, n).brick_mu(radius=r)*rel_vol(Polytope,V0)]]
#going through all faces of dimension d-1,...,k to compute the coefficient:
  L=range(k,d)
  L.reverse()
  Ehr=Ehr+[[i,0] for i in L]
  for (F, affine_F) in lower_fcones(All_Ineqs, LinHull, n, aff=True ) :
    face_F=Polyhedron(ieqs=All_Ineqs, eqns=affine_F)
    i=face_F.dim()
    if i in L:
```

```
      mu_F=F.brick_mu(radius=r, general_directions=gen)
      print 'dim(F)=', i
      print 'mu_F', mu_F
      vol_F=rel_vol(face_F, F.Voronoi() )
      Ehr[d-i][1]=Ehr[d-i][1]+mu_F*vol_F
  if k==0:
    string='\n'
    string+='Ehrhart␣polynomial:␣'
    for j in range(0,len(Ehr)-1):
      string+= str(Ehr[j][1])
      if Ehr[j][0]==1 : string+='t'
      else :
        string+= 't^'
        string+=str(Ehr[j][0])
      string+='+'
    string+= str(Ehr[len(Ehr)-1][1])
    print string
    if Ehr[d][1]!=1:
      print 'Warning,␣for␣lattice␣polytopes␣c_0␣should␣be␣1!'
      print 'Possible␣solution:␣a␣bigger␣radius:␣Ehrhart(Poly,radius=_)'
  if timer==1 :
    endtimer(start)
  return Ehr

########################################################################
################### auxiliary functions ################################
########################################################################

def begintimer() :
  return time.time()

def endtimer(start) :
  end = time.time()
  print '\n', 'timer:',  end-start, '␣seconds', '\n'
  return end-start

#computing the relative volume of a polytope w.r.t. a fundamental domain
def rel_vol(Poly, V0):
  d1 = V0.dim()
  d2 = Poly.dim()
  if d1>d2: return 0
  if d2>d1:
    raise Exception("dim(Poly)>dim(fund.dom.).␣No␣relative␣volume")
  if d2==0 :
    return 1
  return Poly.affine_hull().volume()/V0.affine_hull().volume()

#generating a list of those fcones that are lower in the order of fcones
def lower_fcones(H,LinHull, n, aff=False):
```

```
  if H==set(): return []
  Poly_LinHull=Polyhedron(eqns=LinHull)
  P=Polyhedron(ieqs=H) & Poly_LinHull
  P1=Polyhedron(eqns=H) & Poly_LinHull
  L=range(P1.dim()+1, Poly_LinHull.dim())
  L.reverse()
  lower_fcones=[]
  if aff==True:
    for i in L:
      for f in P.faces(i) :
        f=Polyhedron(ieqs=f.ambient_Hrepresentation())
        F=f.inequalities_list()
        lower_fcones=lower_fcones+[(fcone(F, LinHull, n), F)]
    return lower_fcones
  lower_fcones=[fcone(set(), LinHull, n)]
  for i in L:
    for f in P.faces(i) :
      f=Polyhedron(ieqs=f.ambient_Hrepresentation())
      F=f.inequalities_list()
      lower_fcones=lower_fcones+[fcone(F, LinHull, n)]
  return lower_fcones

#test for comparability of two cones
def akin(f,g)  :
  if f._H.issubset(g._H) :
    return True
  if g._H.issubset(f._H) :
    return True
  return False

#heuristically computing a general direction:
#general means not parallel to the facets
#if given!=[], testing if it really is a general direction, otherwise
#computing a new one
def general_direction(vectors, given=[]) :
  bad_orth_directions=[vector(v[1:]) for v in vectors]
  if given!=[]:
    gen_direc=given
    if all(gen_direc.inner_product(v)!=0 for v in bad_orth_directions) :
      return gen_direc
    else :
      print 'given␣direction␣not␣a␣genral␣direction.␣Compute␣different␣one'
  gen_direc_heur=vector(bad_orth_directions[0])
  for i in range(0,len(bad_orth_directions)) :
    gen_direc_heur=gen_direc_heur+ i* bad_orth_directions[i]
    if all(gen_direc_heur.inner_product(v)!=0 for v in bad_orth_directions) :
      return gen_direc_heur
#second heuristical try of a 'random' combination:
  gen_direc_heur=vector(bad_orth_directions[0])
```

```
for i in range(0,len(bad_orth_directions)) :
  gen_direc_heur=gen_direc_heur+ (i+1)* bad_orth_directions[i]
  if all(gen_direc_heur.inner_product(v)!=0 for v in bad_orth_directions) :
    return gen_direc_heur
raise Exception("Computing␣general␣direction␣failed")
```

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