Character Correspondences in Finite Groups

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Abstract

We consider character correspondences between the characters of a group lying over a fixed character of a normal subgroup, and a similar defined set of characters of a subgroup. This situation occurs in many applications, for example in the proof of important character correspondences found by Glauberman, Dade and Isaacs. With our methods we can give more transparent proofs of the results of Dade and Isaacs. We also consider rationality questions and generalizations to modular representation theory. We show that the Isaacs part of the Glauberman-Isaacs correspondence preserves Schur indices.

Contents

Introduction

Character correspondences play an important role in the character theory of finite groups. Often they allow to obtain information about the character theory of a group from information about a smaller group. An elementary and well known example is the following: Let N be a normal subgroup of the finite group G and let $\vartheta \in \operatorname{Irr} N$. Let $T = G_\vartheta$ be the inertia group of ϑ in G. Then induction provides a bijection between $\operatorname{Irr}(T \mid \vartheta)$ and $\operatorname{Irr}(G \mid \vartheta)$, the so called Clifford correspondence.

A less simple example is a correspondence found by Glauberman in 1968: Let G be a finite group and A a finite solvable group acting on G and of order coprime to the order of G. Then there is a canonical bijection between the A-invariant characters of G and the characters of $\mathbf{C}_G(A)$, the centralizer of A in G. If A is a p-group, then the correspondence is particularly easy to describe: The Glauberman correspondent of $\chi \in \operatorname{Irr}_A G$ is the unique constituent of $\chi_{\mathbf{C}_G(A)}$ occurring with multiplicity not divisible by p [34, Theorem 13.14], [26, §18]. When A is a p-group, the Glauberman correspondence can be obtained by using the Brauer homomorphism from modular representation theory, as was pointed out by Alperin. (See also the expositions of Navarro [50] or Huppert [26, §18].)

When A is not solvable, then by the Odd Order Theorem of Feit and Thompson A has even order and so G has odd order. In this case Isaacs [27] constructed a canonical correspondence between $\operatorname{Irr}_A G$ and $\operatorname{Irr} \mathbf{C}_G(A)$ by completely different methods. The Isaacs correspondence is defined whenever $|G|$ is odd. Later, Thomas R. Wolf [69] showed that the Isaacs and the Glauberman correspondence agree whenever both are defined (that is, when $|G|$ is odd and A is solvable). Various other properties of the Glauberman-Isaacs Correspondence have been found in the meantime (for examples, see [16, 37, 69, 70]). There is also an expository paper of Navarro [51] listing some open problems connected with the Glauberman-Isaacs Correspondence. Some of these problems have been solved since the paper was published in 1994.

This work arose from an attempt to better understand character correspondences connected with fully ramified sections of a group G. These play a major role in the proof of the Isaacs correspondence, but also in other questions concerning the character theory of solvable groups. Let $L \trianglelefteq K$ and suppose $\vartheta \in \operatorname{Irr} K$ with $\vartheta_L = n\varphi$ for some $\varphi \in \operatorname{Irr} L$ such that $n^2 = |K/L|$. Then φ is said to be fully ramified in K and ϑ is said to be fully ramified over L.

The importance of fully ramified sections for the character theory of solvable groups comes from the so-called "Going Down" Theorem on characters of chief sections of finite groups: Suppose K/L is an abelian chief factor of the group G and $\vartheta \in \operatorname{Irr} K$ is invariant in G. Then either $\vartheta_L \in \operatorname{Irr} L$ or $\vartheta = \varphi^K$ for some $\varphi \in \operatorname{Irr} L$ or ϑ is fully ramified over L [27, Theorem 6.18]. In the first two situations, when studying a specific problem, it is usually rather easy to proceed by inductive arguments. The last situation, however, requires deep investigations. Most of Isaacs' important paper is devoted to such investigations.

Now suppose that K/L is some section of a group G and $\varphi \in \operatorname{Irr} L$ is fully ramified in K, and $\vartheta \in \operatorname{Irr}(K \mid \varphi)$. Under certain mild conditions there is a (proper) subgroup H and a correspondence between $\operatorname{Irr}(G \mid \vartheta)$ and $\operatorname{Irr}(H \mid \varphi)$ such that for corresponding χ and ξ one has $\chi(1)/\xi(1) = n$. Results of this type for abelian K/L have been proved by Dade and Isaacs [27, Theorem 9.1], [6, Theorem 5.10], [30, Theorem B]. Lewis [43, 44] has generalized their results to not necessarily abelian fully ramified sections under additional restrictions (essentially coprimeness).

The results of Isaacs and Dade have been applied in a great variety of problems, the most prominent being the Isaacs half of the Glauberman-Isaacs Correspondence and the solution of the McKay Conjecture for solvable groups [27, 30]. Other applications include Dade's construction of a injective map from the irreducible characters of a Sylow system normalizer of a solvable group G into the irreducible characters of that group G [7]. This example is certainly less known, but it is the origin of the π-special characters invented by Gajendragadkar [17, 18].

The results are stronger when K/L is of odd order, and have been used to prove special properties of characters of odd groups [11, 13, 29, 33, 35, 52].

We now motivate the main idea of this thesis. In the situation above, let $e_\varphi = e_\vartheta$ be the central idempotent of $\mathbb{C}K$ belonging to φ (and to ϑ, by assumption). We have

$$\mathbb{C}Ge_\varphi \cong \bigoplus_{\chi \in \operatorname{Irr}(G|\vartheta)} \mathbf{M}_{\chi(1)}(\mathbb{C}) \text{ and } \mathbb{C}He_\varphi \cong \bigoplus_{\xi \in \operatorname{Irr}(H|\varphi)} \mathbf{M}_{\xi(1)}(\mathbb{C}).$$

If a correspondence, degree proportional as above, is given then we can arrange the summands in pairs and we see (since $\mathbf{M}_{mn}(\mathbb{C}) \cong \mathbf{M}_n(\mathbf{M}_m(\mathbb{C}))$) that $\mathbb{C}Ge_\varphi \cong \mathbf{M}_n(\mathbb{C}He_\varphi)$. Of course an isomorphism is far from being unique given just the correspondence above. Is it possible to prove directly that $\mathbb{C}Ge_\varphi \cong \mathbf{M}_n(\mathbb{C}He_\varphi)$? This would provide a somewhat different proof of the cited correspondences, and a more transparent explanation "why" there is such a correspondence.

The main result of this work is that there is indeed a nearly canonical isomorphism from $\mathbb{C}Ge_\vartheta$ to $\mathbf{M}_n(\mathbb{C}He_\varphi)$, and we can describe it without assuming the existence of a degree proportional correspondence. This means that we obtain alternative proofs of these correspondences.

We now describe the general idea that leads to the isomorphism above. The idea works in a situation that is more general than that of an abelian fully ramified section of a group. Suppose that $K \trianglelefteq G$ and $H \leqslant G$ with $G = HK$ and let $L = H \cap K$. Suppose that $\vartheta \in \operatorname{Irr} K$ is G-invariant, that $\varphi \in \operatorname{Irr} L$ is H-invariant and that $(\vartheta_L, \varphi) = n > 0$. (See Figure 1.) Let $i = e_\vartheta e_\varphi$, an idempotent invariant under the action of H. We will

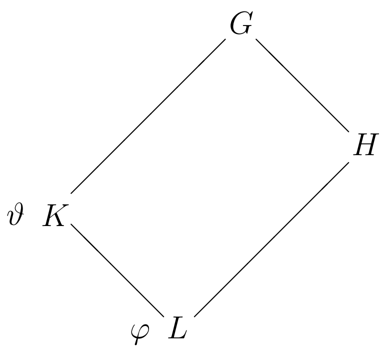

Figure 1: The Basic Configuration

show, and it is not difficult to do so, that $S = \mathbf{C}_{i\mathbb{C}Ki}(L) \cong \mathbf{M}_n(\mathbb{C})$. By general elementary ring theoretic principles, it follows that $i\mathbb{C}Gi \cong \mathbf{M}_n(C)$, where C is the centralizer of S in $i\mathbb{C}Gi$. The action of H/L on S defines a projective representation $\sigma : H/L \to S$. If this representation turns out to be linear, we get an isomorphism from $\mathbb{C}He_\varphi$ to C that is uniquely determined by σ, and thus in general is determined up to multiplication with a linear character of H/L. Therefore we have an isomorphism $i\mathbb{C}Gi \cong \mathbf{M}_n(\mathbb{C}He_\varphi)$. The characters of $\mathbb{C}Ge_\vartheta$ are determined by their values on $i\mathbb{C}Gi$, and thus we can derive that there is a bijection between $\operatorname{Irr}(G \mid \vartheta)$ and $\operatorname{Irr}(H \mid \varphi)$ which is defined uniquely up to multiplication with a linear character of $H/L \cong G/K$. For the exact statement, see Theorem 2.8 one page 10.

From this point of view, proving the existence of the character correspondences boils down to showing that a certain projective representation is an ordinary linear one, or, in cohomological terms, that a certain 2-cocycle is a coboundary. This remains a nontrivial task, but at least in the case where K/L is odd abelian and fully ramified, we obtain a significant simplification of Isaacs' original proof [27]. We also generalize the result for "strong" abelian fully ramified sections [6, 30] from our point of view to show the power of the methods used here.

We mention that the general situation of Figure 1 is also studied in a paper of Isaacs, Malle and Navarro on the McKay conjecture [36], but the methods and results are different.

We generalize our main result in different directions: First, not necessarily algebraically closed fields are considered, and the existence of correspondences preserving fields of values of the characters and Schur indices is examined. The condition that the characters are invariant in H is relaxed to "semi-invariance" (see Section 2.2). As an application we show in Section 4.6 that the Isaacs correspondence preserves Schur indices. This result seems to be new, although not particulary difficult. Turull [61, 63, 65] has done related work concerning correspondences connected with the McKay conjecture; he showed that there is a bijection between the characters of G with degree not divisible by p, and the characters of the normalizer of a Sylow p-subgroup with the same property, such that this bijection preserves, among other things, Schur indices and fields of values over a relatively small field, when G is solvable. Recently, he extended this result to p-solvable groups over the field of p-adic numbers [64]. Some of his arguments certainly overlap the ones given here. But while he makes heavy use of his theory of Clifford classes [60, 62] respective of the Brauer-Clifford group, the proof given here is independent of this theory. Instead we use some ideas of Riese and Schmid [55].

Second, we also consider discrete valuation rings and give a modular version of some results. Work of Watanabe [67], [68], Hoshimoto [24] and Harris and Koshitani [20] shows that the Isaacs correspondence preserves certain p-blocks, and indeed that there are Morita equivalences behind the correspondences of the characters of these blocks.

We begin this thesis with a short chapter collecting some more or less well known results on matrix rings and rings containing a central simple subalgebra which we need.

In Chapter 2, we develop the general main idea described above (Figure 1), first for fields containing the values of the characters (Section 2.1), and then for general fields (Section 2.2). We introduce the magic representations and crossed magic representations which are crucial for our work.

Moreover we prove some general properties of magic representations and give some simple examples.

In Chapter 3 we describe the correspondence or, to put it more exactly, the category equivalence behind the correspondence, for discrete valuation rings and fields of prime characteristic, and consider reduction and lifting modulo a prime of the correspondence. We do this only for φ and ϑ characters of p-defect zero.

Chapter 4 is mainly concerned with fully ramified sections, that is we assume that K/L is a fully ramified section of G. We call the figuration $(G, K, L, \vartheta, \varphi)$ a character five. After some preliminaries on a bilinear form of Isaacs and Dade, we prove more properties of magic representations in this special situation. The very short section 4.3 considers character fives with G/K and K/L of coprime order. It includes a very short proof of the main results of two papers of M. Lewis [43, 44]. Then we reprove Isaacs' fundamental results on character fives with K/L odd abelian (Section 4.4). We extend a result of Dade [6], which we can reduce quickly with our methods to the situation of a character five. Finally, we show that the Isaacs correspondence preserves Schur indices (Section 4.6).

Chapter 5 considers the situation where φ and ϑ are Glauberman correspondents. The result is known, and wa s strengthened recently by Turull [64].

In an appendix, we review shortly the theory of the Brauer-Clifford group, and sketch another proof of our main result using that theory.

Chapter 1

Central Simple Algebras

We collect some easy facts about a ring which is a matrix ring over some other ring.

1.1 Definition. Let $n \in \mathbb{N}$. A subset $\{E_{ij} \mid i, j = 1, \ldots, n\}$ of a ring A is a full set of matrix units in A if

$$E_{ij}E_{kl} = \delta_{jk}E_{il} \quad \text{and} \quad \sum_{i=1}^{n} E_{ii} = 1_A.$$

If C is a ring, then the $n \times n$-matrix ring $\mathbf{M}_n(C)$ has a "canonical" set of matrix units, namely the set of matrices with exactly one entry equal to 1 and all the other entries zero. Let I be the $n \times n$-identity matrix. The set $\{aI \mid a \in C\}$ is a subring of $\mathbf{M}_n(C)$ which is naturally isomorphic with C. An easy computation shows that this subring is the centralizer of the full set of matrix units. The following well known lemma can be viewed as a converse to the last remarks. We formulate it for algebras in view of the applications we have in mind.

1.2 Lemma. *Let A be an algebra over some commutative ring $\mathbb{F}$. Suppose $\{E_{ij} \mid i, j = 1, \ldots, n\}$ is a full set of matrix units in A. Let*

$$C = \mathbf{C}_A(\{E_{ij} \mid i, j = 1, \ldots, n\}).$$

Then $A \cong \mathbf{M}_n(C)$. Let $S = \sum_{i,j} \mathbb{F}E_{ij}$ be the $\mathbb{F}$-subalgebra generated by the set of matrix units. Then $S \cap C = \mathbb{F} \cdot 1_A$ and $S \otimes_{\mathbb{F}} C \cong A$ via $s \otimes a \mapsto sa$.

Proof. This is fairly well known (see, for example, [40, Theorem 17.5]). The map

$$\mathbf{M}_n(C) \ni (c_{ij})_{i,j} \mapsto \sum_{i,j} c_{ij}E_{ij} \in A$$

is an isomorphism: Its inverse is the map

$$A \ni a \mapsto (a_{ij})_{i,j} \quad \text{where} \quad a_{ij} = \sum_{k=1}^{n} E_{ki} a E_{jk}.$$

Indeed it sends a to an element in $\mathbf{M}_n(C)$ (that is, $a_{ij} \in C$), and is the inverse of the map defined before. The isomorphism from $\mathbf{M}_n(C)$ to A sends $\mathbf{M}_n(\mathbb{F} \cdot 1_A)$ onto S, and thus $S \cong \mathbf{M}_n(\mathbb{F} \cdot 1_A)$. It follows that $C \cap S = \mathbf{Z}(S) = \mathbb{F} \cdot 1_A$. Since $S \otimes C = \bigoplus_{i,j} E_{ij} \otimes C$, it is not difficult to see that $s \otimes c \mapsto sc$ is an isomorphism. □

Suppose $\mathbb{F}$ is a field. If an $\mathbb{F}$-algebra A contains a full set of matrix units of size $n \times n$, then A contains the split central simple algebra $S \cong \mathbf{M}_n(\mathbb{F})$ as above. Next we will consider algebras containing a (not necessarily split) central simple algebra over a field. By a central simple $\mathbb{F}$-algebra, we mean an algebra simple as a ring with center $\mathbb{F}$ and finite dimensional over $\mathbb{F}$. We need an easy, well known lemma.

1.3 Lemma. *Let A and B be algebras over a field $\mathbb{F}$ and $X \leqslant A$ and $Y \leqslant B$ subspaces containing 1_A and 1_B, respectively. Then*

$$\mathbf{C}_{A \otimes_{\mathbb{F}} B}(X \otimes_{\mathbb{F}} Y) = \mathbf{C}_A(X) \otimes_{\mathbb{F}} \mathbf{C}_B(Y).$$

Proof. Let $\{a_i \mid i \in I\}$ a basis of A such that $\{a_i \mid i \in I_0\}$, where $I_0 \subseteq I$, is a basis of $\mathbf{C}_A(X)$. In the same way, let $\{b_j \mid j \in J\}$ be a basis of B containing the basis $\{b_j \mid j \in J_0\}$ of $\mathbf{C}_B(Y)$. If

$$c = \sum_{\substack{i \in I \\ j \in J}} \lambda_{ij} a_i \otimes b_j \in \mathbf{C}_{A \otimes B}(X \otimes Y),$$

then from $c(x \otimes 1) = (x \otimes 1)c$ we see

$$\sum_{i,j} \lambda_{ij} a_i x \otimes b_j = \sum_{ij} \lambda_{ij} x a_i \otimes b_j.$$

It follows that $\lambda_{ij} = 0$ when $i \notin I_0$. Analogously, $\lambda_{ij} = 0$ when $j \notin J_0$. Thus $c \in \mathbf{C}_A(X) \otimes \mathbf{C}_B(Y)$ as claimed. □

The following lemma generalizes Lemma 1.2.

1.4 Lemma. *Let A be an $\mathbb{F}$-algebra containing a unitary subalgebra S which is central simple. Let $C = \mathbf{C}_A(S)$. Then $A \cong S \otimes_{\mathbb{F}} C$ canonically (via $s \otimes c \mapsto sc$).*

Proof. Let $(s \otimes c)^\kappa = sc \in A$. This defines an algebra homomorphism from $S \otimes_{\mathbb{F}} C$ into A. Now let $\mathbb{E}$ be a splitting field for S, so that $S \otimes_{\mathbb{F}} \mathbb{E} \cong \mathbf{M}_n(\mathbb{E})$ where $n^2 = \dim_{\mathbb{F}} S$. By Lemma 1.3, $\mathbf{C}_{A\otimes_{\mathbb{F}}\mathbb{E}}(S \otimes_{\mathbb{F}} \mathbb{E}) = C \otimes_{\mathbb{F}} \mathbb{E}$. From the splitting case (Lemma 1.2) we see that $A \otimes_{\mathbb{F}} \mathbb{E} \cong (S \otimes_{\mathbb{F}} \mathbb{E}) \otimes_{\mathbb{E}} (C \otimes_{\mathbb{F}} \mathbb{E})$. Since $(S \otimes_{\mathbb{F}} C) \otimes_{\mathbb{F}} \mathbb{E} \cong (S \otimes_{\mathbb{F}} \mathbb{E}) \otimes_{\mathbb{E}} (C \otimes_{\mathbb{F}} \mathbb{E})$ canonically, it follows that $\kappa \otimes 1\colon (S \otimes_{\mathbb{F}} C) \otimes_{\mathbb{F}} \mathbb{E} \to A \otimes_{\mathbb{F}} \mathbb{E}$ is an isomorphism, and thus is κ. □

Notation. Let A be an $\mathbb{F}$-algebra. We denote by $\mathrm{ZF}(A, \mathbb{F})$ the set of central $\mathbb{F}$-forms on A, that is the set of $\mathbb{F}$-linear maps $\tau\colon A \to \mathbb{F}$ with $\tau(ab) = \tau(ba)$ for all $a, b \in A$.

For an algebra A, let

$$[A, A] = \langle ab - ba \mid a, b \in A\rangle$$

be the $\mathbb{F}$-subspace generated by the additive commutators. Then $\mathrm{ZF}(A, \mathbb{F})$ is just the set of linear maps from A to $\mathbb{F}$ with $[A, A]$ in the kernel.

Let A and B be two $\mathbb{F}$-algebras. Since

$$[A \otimes_{\mathbb{F}} B, A \otimes_{\mathbb{F}} B] = A \otimes_{\mathbb{F}} [B, B] + [A, A] \otimes_{\mathbb{F}} B,$$

we have $A \otimes_{\mathbb{F}} B/[A \otimes_{\mathbb{F}} B, A \otimes_{\mathbb{F}} B] \cong A/[A, A] \otimes_{\mathbb{F}} B/[B, B]$. It follows that

$$\mathrm{ZF}(A, \mathbb{F}) \otimes_{\mathbb{F}} \mathrm{ZF}(B, \mathbb{F}) \cong \mathrm{ZF}(A \otimes_{\mathbb{F}} B, \mathbb{F}).$$

To describe a canonical isomorphism concretely, let $\sigma \in \mathrm{ZF}(A, \mathbb{F})$ and $\tau \in \mathrm{ZF}(B, \mathbb{F})$ be central forms. Then $\sigma \otimes \tau$ becomes a central form of $A \otimes_{\mathbb{F}} B$ by setting $(\sigma \otimes \tau)(a \otimes b) = \sigma(a)\tau(b)$.

We denote the reduced trace of a central simple $\mathbb{F}$-algebra S by $\mathrm{tr}_{S/\mathbb{F}}$ or simply tr, if no confusion can arise. Remember that the reduced trace is computed as follows: first choose a splitting field $\mathbb{E}$ of S and an isomorphism $\varepsilon\colon S \otimes_{\mathbb{F}} \mathbb{E} \to \mathbf{M}_n(\mathbb{E})$, then let $\mathrm{tr}_{S/\mathbb{F}}(s)$ be the ordinary matrix trace of $(s \otimes 1)^\varepsilon$. Then indeed $\mathrm{tr}_{S/\mathbb{F}}(s)$ is a welldefined element of $\mathbb{F}$ [54, Section 9a]. The kernel of $\mathrm{tr}_{S/\mathbb{F}}$ is exactly the subspace $[S, S]$. Thus $\mathrm{ZF}(S, \mathbb{F}) = \mathbb{F} \cdot \mathrm{tr}_{S/\mathbb{F}} \cong \mathbb{F}$.

Combining the results of the last two paragraphs, we get that

$$\mathrm{ZF}(S \otimes_{\mathbb{F}} C, \mathbb{F}) \cong \mathrm{ZF}(C, \mathbb{F})$$

for a central simple $\mathbb{F}$-algebra S and any algebra C. Any central form $\chi \in \mathrm{ZF}(S \otimes_{\mathbb{F}} C, \mathbb{F})$ can be written as $\mathrm{tr}_{S/\mathbb{F}} \otimes \tau$ for some $\tau \in \mathrm{ZF}(C, \mathbb{F})$.

1.5 Lemma. *Let A be an $\mathbb{F}$-algebra containing the central simple subalgebra S and let $C = \mathbf{C}_A(S)$. Define $\varepsilon\colon \mathrm{ZF}(A, \mathbb{F}) \to \mathrm{ZF}(C, \mathbb{F})$ by*

$$\chi^\varepsilon(c) = \chi(s_0 c) \quad \textit{for any} \quad s_0 \in S \quad \textit{with} \quad \mathrm{tr}_{S/\mathbb{F}}(s_0) = 1.$$

Then ε is an isomorphism and

$$\chi(sc) = \mathrm{tr}_{S/\mathbb{F}}(s)\chi^{\varepsilon}(c) \quad \text{for all } s \in S \text{ and } c \in C.$$

Proof. By Lemma 1.4, $A \cong S \otimes_{\mathbb{F}} C$. The remarks above show that for every $\chi \in \mathrm{ZF}(A,\mathbb{F})$ there is $\tau \in \mathrm{ZF}(C,\mathbb{F})$ such that $\chi(sc) = \mathrm{tr}_{S/\mathbb{F}}(s)\tau(c)$ for all $s \in S$ and $c \in C$. It is clear that then $\tau = \chi^{\varepsilon}$. In particular, ε is an isomorphism. The proof is complete. □

1.6 Lemma. *Let the assumptions be as in Lemma 1.5. Let V be the simple right S-module, and let $m^2 = \dim_{\mathbb{F}}(\mathrm{End}\, V_S)$. Let $\chi \in \mathrm{ZF}(A,\mathbb{F})$. If χ^{ε} is the character of an C-module M, then $m\chi$ is the character of the A-module $V \otimes_{\mathbb{F}} M$. If S is split, then χ^{ε} affords a C-module if and only if χ affords an A-module.*

Proof. Identify A with $S \otimes C$. The trace of S on V is $m\,\mathrm{tr}$ and the trace of $s \otimes c$ on $V \otimes_{\mathbb{F}} M$ is thus $m\,\mathrm{tr}(s)\chi^{\varepsilon}(c) = m\chi(sc)$ as claimed.

That S is split means that $m = 1$. Thus if χ^{ε} is the character of the module M, then χ is the character of the module $V \otimes_{\mathbb{F}} M$. Conversely, suppose χ is the character of the $S \otimes_{\mathbb{F}} C$-module N. Since S is split, there is an idempotent $e \in S$ of trace 1. Then the character of Ne as C-module is obviously χ^{ε}. □

Next we will consider field automorphisms and central forms. Suppose that A is an $\mathbb{F}$-algebra and $\mathbb{E}$ is a field extension of $\mathbb{F}$. Let α be a field automorphism of $\mathbb{E}$ fixing the elements of $\mathbb{F}$. We have the algebra $A \otimes_{\mathbb{F}} \mathbb{E} = A^{\mathbb{E}}$ obtained by scalar extension from A, and we may extend α to a ring automorphism of $A^{\mathbb{E}}$ that sends $a \otimes \lambda$ to $a \otimes \lambda^{\alpha}$ for $a \in A$ and $\lambda \in \mathbb{E}$. For simplicity of notation, we denote this algebra automorphism by α, too. Now suppose that $\tau\colon A^{\mathbb{E}} \to \mathbb{E}$ is a central form. Then τ^{α} defined by $\tau^{\alpha}(a^{\alpha}) = \tau(a)^{\alpha}$ is a central form on $A^{\mathbb{E}}$, too. In this way, $\mathrm{Aut}_{\mathbb{F}}\,\mathbb{E}$ acts on the central forms of $A^{\mathbb{E}}$.

1.7 Lemma. *In the situation of Lemma 1.5, let $\mathbb{E} \geqslant \mathbb{F}$ be a field extension and $\alpha \in \mathrm{Aut}_{\mathbb{F}}\,\mathbb{E}$. Then $S \otimes_{\mathbb{F}} \mathbb{E}$ is a central simple $\mathbb{E}$-subalgebra of $A \otimes_{\mathbb{F}} \mathbb{E}$ with centralizer $C \otimes_{\mathbb{F}} \mathbb{E}$, and $(\chi^{\varepsilon})^{\alpha} = (\chi^{\alpha})^{\varepsilon}$ for $\chi \in \mathrm{ZF}(A \otimes_{\mathbb{F}} \mathbb{E}, \mathbb{E})$.*

Proof. Since $\mathrm{tr}_{S\otimes_{\mathbb{F}}\mathbb{E}/\mathbb{E}} = \mathrm{tr}_{S/\mathbb{F}} \otimes 1$ by the definition of the reduced trace, we see that there is $s_0 \in S \otimes_{\mathbb{F}} \mathbb{E}$ of trace 1 with $s_0^{\alpha} = s_0$. Thus

$$(\chi^{\varepsilon})^{\alpha}(c^{\alpha}) = \chi^{\varepsilon}(c)^{\alpha} = \chi(s_0 c)^{\alpha} = \chi^{\alpha}(s_0^{\alpha} c^{\alpha}) = (\chi^{\alpha})^{\varepsilon}(c^{\alpha})$$

as claimed. □

We also need a result concerning central forms and full idempotents of an algebra. An idempotent $i \in A$ is called a *full idempotent* of A if $AiA = A$. It is well known that A and iAi are Morita equivalent when i is full [40, 18.30].

1.8 Lemma. *Let A be an $\mathbb{F}$-algebra and $i \in A$ an idempotent with $A = AiA$. Set $C = iAi$. Then restriction defines an isomorphism $\delta\colon \mathrm{ZF}(A,\mathbb{F}) \to \mathrm{ZF}(C,\mathbb{F})$. Here τ^δ is the character of a C-module if and only if τ is the character of an A-module.*

Proof. Suppose $1_A = \sum_{k=1}^r x_k i y_k$ with $x_k, y_k \in A$. For $\xi \in \mathrm{ZF}(C,\mathbb{F})$ define $\widehat{\xi}(b) = \sum_{k=1}^r \xi(iy_k b x_k i)$. Then

$$\begin{aligned}\widehat{\xi}(bc) &= \sum_k \xi(iy_k bcx_k i) = \sum_k \xi(iy_k b\big(\sum_l x_l i y_l\big) c x_k i)\\ &= \sum_{k,l} \xi(iy_k b x_l i \cdot iy_l c x_k i) = \sum_{k,l} \xi(iy_l c x_k i \cdot iy_k b x_l i)\\ &= \sum_l \xi(iy_l cb x_l i) = \widehat{\xi}(cb),\end{aligned}$$

so that $\widehat{\xi} \in \mathrm{ZF}(A,\mathbb{F})$. It is now easy to see that $\xi \mapsto \widehat{\xi}$ is the inverse of δ.

It is clear that if χ is the character of the A-module M, then $\chi_{|C}$ is the character of Mi as C-module. Conversely if $\chi_{|C}$ is the character of the C-module N, then χ is the character of the A-module $N \otimes_C iA$. □

Again, the isomorphism is compatible with field automorphisms:

1.9 Lemma. *Let A be an $\mathbb{F}$-algebra, $i \in A$ a full idempotent and $\mathbb{F} \leqslant \mathbb{E}$ a field extension, and $\alpha \in \mathrm{Aut}_{\mathbb{F}} \mathbb{E}$. Then $i \otimes 1$ is a full idempotent in $A \otimes_{\mathbb{F}} \mathbb{E}$, and the isomorphism of Lemma 1.8 commutes with α.*

Proof. This is clear since $(i \otimes 1)^\alpha = i \otimes 1$. □

The following simple observation is sometimes useful.

1.10 Lemma. *Let $\mathbb{F}$ be a field and A a simple $\mathbb{F}$-algebra which is separable over $\mathbb{F}$, and suppose $\mathbb{E} = \mathbf{Z}(A)$ is a Galois extension of $\mathbb{F}$. Let e be a central primitive idempotent of $A^{\mathbb{E}} = A \otimes_{\mathbb{F}} \mathbb{E}$. Then $A \cong A^{\mathbb{E}}e$ via $a \mapsto ae$, and the inverse is $1 \otimes \mathrm{T}_{\mathbb{F}}^{\mathbb{E}}$ restricted to $A^{\mathbb{E}}e$.*

Proof. Since A is separable, $A^{\mathbb{E}}$ is semisimple. Clearly $\mathbf{Z}(A^{\mathbb{E}}) = \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E}$ has $|\mathbb{E} : \mathbb{F}|$ different primitive idempotents which are Galois conjugate. Thus if $1 \neq \alpha \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})$ and $b \in (A^{\mathbb{E}})e$, then $b^\alpha e = 0$. Therefore, $\mathrm{T}_{\mathbb{F}}^{\mathbb{E}}(b)e = be = b$ for $b \in (A^{\mathbb{E}})e$. Conversely, $\mathrm{T}_{\mathbb{F}}^{\mathbb{E}}(ae) = a\,\mathrm{T}_{\mathbb{F}}^{\mathbb{E}}(e) = a1_A = a$ for $a \in A$. □

Chapter 2

A Character Correspondence

2.1 Magic Representations

Throughout this section we assume the following situation:

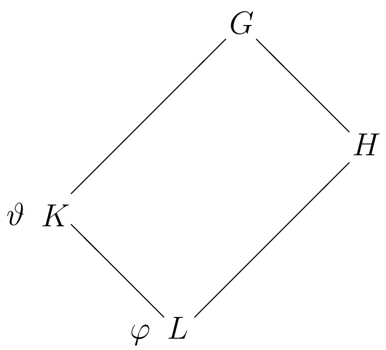

Figure 2.1: The Basic Configuration

2.1 Hypothesis. Let $\mathbf{C}$ be an algebraically closed field (not necessarily of characteristic zero). Suppose $G = HK$ is a finite group where $K \trianglelefteq G$ and $H \leqslant G$, and set $L = H \cap K$. Let $\vartheta \in \operatorname{Irr}_{\mathbf{C}} K$ be the character of a simple, projective $\mathbf{C}K$-module, and $\varphi \in \operatorname{Irr}_{\mathbf{C}} L$ the character of a simple, projective $\mathbf{C}L$-module. Assume that ϑ and φ are invariant in H, and let $n > 0$ be the multiplicity of φ as a constituent of ϑ_L. Let $\mathbb{F} \subseteq \mathbf{C}$ be a field containing the values of ϑ and φ. Let e_ϑ and e_φ be the central idempotents of $\mathbb{F}K$ and $\mathbb{F}L$ belonging to ϑ and φ. We set $i = e_\vartheta e_\varphi$ and $S = (i\mathbb{F}Ki)^L = (\mathbb{F}K)^L i$. We call the six-tuple $(G, H, K, L, \vartheta, \varphi)$ the *Basic Configuration.*

If the characteristic of $\mathbf{C}$ is a prime $p > 0$, the assumption of projectiveness means that ϑ and φ belong to simple modules of p-defect zero, and thus can be lifted to ordinary complex characters of p-defect zero. Note that in any case, $\mathbb{F}Ke_\vartheta$ and $\mathbb{F}Le_\varphi$ are central simple $\mathbb{F}$-algebras, in the prime characteristic case even matrix algebras over $\mathbb{F}$.

Since $\mathbb{F}Ke_\vartheta$ is a simple ring, $\mathbb{F}Ke_\vartheta = \mathbb{F}Ki\mathbb{F}K$. We can thus apply Lemma 1.8 with $A = \mathbb{F}Ge_\vartheta$ and $C = i\mathbb{F}Gi$. Our aim is to show that under certain conditions $i\mathbb{F}Gi \cong S \otimes_{\mathbb{F}} \mathbb{F}He_\varphi$.

2.2 Lemma. *S is a central simple $\mathbb{F}$-algebra of dimension n^2 over $\mathbb{F}$ and $\mathbf{C}_{i\mathbb{F}Ki}(S) = \mathbb{F}Li \cong \mathbb{F}Le_\varphi$.*

Proof. Since i is not zero by assumption, the ring homomorphism $\alpha \mapsto \alpha i = \alpha e_\vartheta$ from $\mathbb{F}Le_\varphi$ to $\mathbb{F}Li$ is not zero. As $\mathbb{F}Le_\varphi$ is simple, the map $\alpha \mapsto \alpha i$ is injective. Thus $\mathbb{F}Le_\varphi \cong \mathbb{F}Li$. The algebra $\mathbb{F}Ke_\vartheta$ is central simple, as is $i\mathbb{F}Ki$. By definition, S is just the centralizer of $\mathbb{F}Li$ in $i\mathbb{F}Ki$. By the Centralizer Theorem [15, Theorem 3.15][49, Theorem 2.4.6], S is central simple, too, and the centralizer of S is again $\mathbb{F}Li$. Also $i\mathbb{F}Ki \cong S \otimes_{\mathbb{F}} \mathbb{F}Li$. To show the statement on the dimension, we have to show that $\dim_{\mathbb{F}}(i\mathbb{F}Ki) = n^2\varphi(1)^2$. To do this we can extend the field and assume that $\mathbb{F}$ is a splitting field of the groups involved. If V is a $\mathbb{F}K$-module affording ϑ, then $Vi = Ve_\varphi$ is the φ-part of $V_{\mathbb{F}L}$, and thus $Vi \cong nU$ for some simple $\mathbb{F}L$-module affording φ. As $i\mathbb{F}Ki \cong \operatorname{End}_{\mathbb{F}}(Vi) \cong \mathbf{M}_{n\varphi(1)}(\mathbb{F})$, the result follows. □

2.3 Lemma. *Let $C = \mathbf{C}_{i\mathbb{F}Gi}(S)$. Then $i\mathbb{F}Gi \cong S \otimes_{\mathbb{F}} C$. If S is split, then even $i\mathbb{F}Gi \cong \mathbf{M}_n(C)$.*

Proof. Straightforward application of Lemma 1.4. □

The last lemma reduces the study of the characters of $i\mathbb{F}Gi$ to the study of the characters of C. Of course this is only of use when we know C better than $i\mathbb{F}Gi$. The following simple observation is essential for all that follows:

2.4 Lemma. *Let T be the inertia group of φ in $\mathbf{N}_G(L)$. Then T/L acts on $S = (i\mathbb{F}Ki)^L$ (by conjugation). There is a projective representation $\sigma\colon T/L \to S$ such that $s^g = s^{\sigma(Lg)}$ for all $s \in S$ and $g \in T$.*

Observe that by Hypothesis 2.1, we have $H \leqslant T$.

Proof. As ϑ is G-invariant, T acts on S and clearly L acts trivial on S. Since S is a central simple $\mathbb{F}$-algebra, all automorphisms are inner by the Skolem-Noether Theorem [15, Theorem 3.14][49, Theorem 2.4.6]. We can thus choose $\sigma(Lg) \in S$ for every $g \in T$ such that $s^g = s^{\sigma(Lg)}$ for all $s \in S$. This determines $\sigma(Lg)$ up to multiplication with an element of $\mathbf{Z}(S) = \mathbb{F}i$. In this way we get a projective representation σ from T/L to S with the desired property. □

2.5 Definition. Assume Hypothesis 2.1. We say that $\sigma\colon H/L \to S = (i\mathbb{F}Ki)^L$ is a magic representation (for $G, H, K, L, \vartheta, \varphi$), if

1. $\sigma(Lh_1h_2) = \sigma(Lh_1)\sigma(Lh_2)$ for all $h_1, h_2 \in H$ and
2. $s^h = s^{\sigma(Lh)}$ for all $s \in S$ and $h \in H$.

The character of a magic representation, that is the function $\psi\colon H/L \to \mathbb{F}$ with $\psi(Lh) = \mathrm{tr}_{S/\mathbb{F}}(\sigma(Lh))$, is called a magic character.

Here $\mathrm{tr}_{S/\mathbb{F}}$ is the reduced trace of S. As $S \otimes_{\mathbb{F}} \mathbb{E} \cong \mathbf{M}_n(\mathbb{E})$ for some field $\mathbb{E} \geqslant \mathbb{F}$, a magic representation is a representation in the classical sense. If a magic representation exists, we have the following:

2.6 Theorem. *Assume Hypothesis 2.1 and let $\sigma\colon H/L \to S$ be a magic representation. Then the linear map*

$$\kappa\colon \mathbb{F}H \to C = \mathbf{C}_{i\mathbb{F}Gi}(S), \quad \text{defined by} \quad h \mapsto h\sigma(Lh)^{-1} \text{ for } h \in H,$$

is an algebra-homomorphism and induces an isomorphism $\mathbb{F}He_\varphi \cong C$.

Proof. For $h \in H$ let $c_h = h\sigma(Lh)^{-1} = \sigma(Lh)^{-1}h$. (The inverse $\sigma(Lh)^{-1}$ is the inverse in S, so $\sigma(Lh)\sigma(Lh)^{-1} = i$.) Clearly $c_h \in C$. Note that

$$\begin{aligned} c_gc_h &= g\sigma(Lg)^{-1}h\sigma(Lh)^{-1} = gh(\sigma(Lg)^{-1})^{\sigma(Lh)}\sigma(Lh)^{-1} \\ &= gh\sigma(Lh)^{-1}\sigma(Lg)^{-1} = gh\sigma(Lgh)^{-1} = c_{gh}. \end{aligned}$$

Thus extending the map $h \mapsto c_h$ linearly to $\mathbb{F}H$ defines an algebra homomorphism $\kappa\colon \mathbb{F}H \to C$. For $l \in L$ we have $l \mapsto l\sigma(L)^{-1} = li = l \cdot 1_C$. Thus κ restricted to $\mathbb{F}L$ is just multiplication with i, so that $e_\varphi\kappa = e_\varphi i = i = 1_C$, and any other central idempotent of $\mathbb{F}L$ maps to zero. Therefore

$$(\mathbb{F}L)\kappa = (\mathbb{F}Le_\varphi)\kappa = \mathbb{F}Li.$$

For any $h \in H$ we have $(\mathbb{F}Le_\varphi h)\kappa = \mathbb{F}Lic_h$. Let T be a transversal for the cosets of L in H. As $\mathbb{F}He_\varphi = \bigoplus_{t\in T} \mathbb{F}Le_\varphi t$, the proof will be finished if we show that $C = \bigoplus_{t \in T} \mathbb{F}Lic_t$. The decomposition $\mathbb{F}G = \bigoplus_{Kg\in G/K} \mathbb{F}Kg$ yields the decomposition $i\mathbb{F}Gi = \bigoplus_{Kg \in G/K} i\mathbb{F}Kgi$ which defines a G/K-grading[1] on $i\mathbb{F}Gi$. For $Kg \in G/K$ set $C_{Kg} = C \cap \mathbb{F}Kg = \mathbf{C}_{i\mathbb{F}Kgi}(S)$. Since $S \subseteq i\mathbb{F}Ki$ we conclude that

$$C = \bigoplus_{Kg\in G/K} C_{Kg}.$$

[1] See Definition A.1

As $c_h \in C \cap i\mathbb{F}Kih = C \cap i\mathbb{F}Khi = C_{Kh}$ and c_h is a unit of C, we conclude that

$$C_{Kh} = C_{Kh}c_h^{-1}c_h \subseteq C_K c_h \subseteq C_{Kh},$$

so equality holds throughout. As $C_K = \mathbf{C}_{i\mathbb{F}Ki}(S) = \mathbb{F}Li$ by assumption, the proof follows. □

Remark. If the projective representation of Lemma 2.4 is not equivalent with an ordinary representation, we still get some result. Let α be a factor set associated with σ, that is, $\sigma(x)\sigma(y) = \sigma(xy)\alpha(x,y)$ for $x, y \in H/L$. Nearly the same proof as above shows that $C \cong \mathbb{F}^{\alpha^{-1}}[H]e_\varphi$. Here we view α as factor set of H which is constant on cosets of L. The ordinary group algebra $\mathbb{F}L$ can be embedded in the twisted group algebra $\mathbb{F}^{\alpha^{-1}}[H]$ in an obvious way, and thus it is meaningful to view e_φ as an idempotent in $\mathbb{F}^{\alpha^{-1}}[H]$.

The following is an immediate consequence:

2.7 Corollary. *Assume Hypothesis 2.1 and that there is a magic representation for this configuration. Then* $i\mathbb{F}Gi \cong S \otimes \mathbb{F}He_\varphi$ *and if* $S \cong \mathbf{M}_n(\mathbb{F})$*, then* $\mathbb{F}Ge_\vartheta$ *and* $\mathbb{F}He_\varphi$ *are Morita equivalent.*

Proof. The first assertion follows by combining Theorem 2.6 with Lemma 2.3. If $S \cong \mathbf{M}_n(\mathbb{F})$, then $\mathbb{F}He_\varphi$ and $i\mathbb{F}Gi \cong \mathbf{M}_n(\mathbb{F}He_\varphi)$ are Morita-equivalent, and $\mathbb{F}Ge_\vartheta$ and $i\mathbb{F}Gi$ are Morita-equivalent, as we observed earlier. □

This corollary maybe justifies the terminology of "magic" representations and characters. The existence of a magic representation has as a consequence that the character theories of G over ϑ and of H over φ are "the same". This will be stated more precisely below. We now give a list of situations where a magic representation is known to exist:

1. H/L and $n = (\vartheta_L, \varphi)$ are coprime (Proposition 2.15).

2. $L \trianglelefteq G$, K/L is abelian of odd order and φ is fully ramified in K. Then at least for *some* H there is a magic representation σ (see Theorem 4.23). The magic character and the character correspondence in this case were first described by Isaacs [27]. There even exists a canonical magic character, and this canonical character is called "magic" by some authors (Navarro [52], Lewis [44]).

3. $L \triangleleft G$ and there is $L \leqslant M \trianglelefteq H$ such that $(|K/L|, |M/L|) = 1$ and $\mathbf{C}_{K/L}(M) = 1$ (see Theorem 4.38). This result is in essence due to Dade [6].

4. Assume that there is $M \leqslant G$ such that $MK \trianglelefteq G$ and $(|K|,|M|) = 1$, and let $H = \mathbf{N}_G(M)$ and $L = H \cap K = \mathbf{C}_K(M)$ as usual. Assume that M is solvable. If $\varphi \in \operatorname{Irr} L$ is the Glauberman correspondent of ϑ with respect to M, then there is a magic representation for this configuration (Theorem 5.1). Again this follows from work of Dade [8], including deep results on endo-permutation-modules.

Observe that via the canonical isomorphism $G/K \cong H/L$ we can view ψ as character of G/K. We also can view ψ as character of G or H and we will do so if convenient.

Next we state the main theorem. We use the following notation: For $\gamma\colon G \to \mathbb{C}$ a class function and ϑ an irreducible character of some (normal) subgroup, set $\gamma_\vartheta = \sum_{\chi \in \operatorname{Irr}(G|\vartheta)} (\gamma, \chi)\chi$.

2.8 Theorem. *Assume Hypothesis 2.1 with $\mathbb{F} \leqslant \mathbb{C}$. Every magic representation $\sigma\colon H/L \to S^*$ determines a linear isometry $\iota = \iota(\sigma)$ from $\mathbb{C}[\operatorname{Irr}(G \mid \vartheta)]$ to $\mathbb{C}[\operatorname{Irr}(H \mid \varphi)]$. Let ψ be the character of σ, and let $\chi \in \mathbb{C}[\operatorname{Irr}(G \mid \vartheta)]$. The correspondence ι has the following properties:*

1. *$\chi \in \operatorname{Irr}(G \mid \vartheta)$ if and only if $\chi^\iota \in \operatorname{Irr}(H \mid \varphi)$.*
2. *$\chi(1)/\vartheta(1) = \chi^\iota(1)/\varphi(1)$.*
3. *If α is a field automorphism fixing $\mathbb{F}$, then $(\chi^\alpha)^\iota = (\chi^\iota)^\alpha$.*
4. *$\mathbb{F}(\chi) = \mathbb{F}(\chi^\iota)$.*
5. *If S is split and $\chi \in \operatorname{Irr}(G \mid \vartheta)$, the Schur index of χ over $\mathbb{F}$ and that of χ^ι are the same.*
6. *$(\beta\chi)^\iota = \beta\chi^\iota$ for all $\beta \in \mathbb{C}[\operatorname{Irr}(G/K)]$.*
7. *(In this and the next property, U is a subgroup with $K \leqslant U \leqslant G$, and $V = H \cap U$.) $(\chi_U)^\iota = (\chi^\iota)_V$.*
8. *$(\tau^G)^\iota = (\tau^\iota)^H$ for $\tau \in \mathbb{C}[\operatorname{Irr}(U \mid \vartheta)]$.*
9. *$(\chi_H)_\varphi = \psi\chi^\iota$.*
10. *$(\xi^G)_\vartheta = \overline{\psi}\xi^{\iota^{-1}}$ for all $\xi \in \mathbb{C}[\operatorname{Irr}(H \mid \varphi)]$.*

Proof. Let $\mathbb{E} \geqslant \mathbb{F}$ be any extension field. By Lemma 1.8, we have an isomorphism from $\operatorname{ZF}(\mathbb{E}Ge_\vartheta, \mathbb{E})$ to $\operatorname{ZF}(i\mathbb{E}Gi, \mathbb{E})$, which commutes with field automorphisms over $\mathbb{F}$ by Lemma 1.9. By Lemma 1.5, we have an isomorphism from $\mathrm{ZF}(i\mathbb{E}Gi, \mathbb{E})$ to $\mathrm{ZF}(\mathbf{C}_{i\mathbb{E}Gi}(S\mathbb{E}), \mathbb{E})$, again commuting with field automorphisms by Lemma 1.7. The isomorphism $\kappa\colon \mathbb{F}He_\varphi \to \mathbf{C}_{\mathbb{F}Gi}(S)$ of Theorem 2.6 yields, by scalar extension, an isomorphism $\kappa \otimes 1$ from $\mathbb{E}He_\varphi$ onto $\mathbf{C}_{i\mathbb{E}Gi}(S)$. This yields an isomorphism $(\kappa\otimes 1)^*\colon \mathrm{ZF}(\mathbf{C}_{i\mathbb{E}Gi}(S), \mathbb{E}) \to \mathrm{ZF}(\mathbb{E}He_\varphi, \mathbb{E})$ which commutes with field automorphisms over $\mathbb{F}$. Identifying $\mathrm{ZF}(\mathbb{C}Ge_\vartheta, \mathbb{C})$ with $\mathbb{C}[\mathrm{Irr}(G \mid \vartheta)]$ and $\mathrm{ZF}(\mathbb{C}He_\varphi, \mathbb{C})$ with $\mathbb{C}[\mathrm{Irr}(H \mid \varphi)]$, we get the desired isomorphism ι. It commutes with field automorphisms, which is Property 3. To compute χ^ι explicitly, choose an element $s_0 \in S$ with reduced trace 1: then for $h \in H$ we have $\chi^\iota(h) = \chi^{\delta\varepsilon\kappa^*}(h) = \chi(s_0\sigma(Lh)^{-1}h)$ (see Lemma 1.5 and Lemma 1.8).

By Lemma 1.8 applied to $\mathbb{C}Ge_\vartheta$ and Lemma 1.6 applied to $i\mathbb{C}Gi$, irreducible characters are sent to irreducible characters. This proves Property 1 and shows that ι is an isometry with respect to the usual inner product on the space of class functions (it sends an orthonormal basis to an orthonormal basis). To show that 2. holds, we choose $s_0 = (1/n)i = (1/n)e_\vartheta e_\varphi$. Remember that $n = (\vartheta_L, \varphi)_L$, so that $\vartheta(e_\varphi) = n\varphi(1)$. Thus

$$\frac{\chi^\iota(1)}{\varphi(1)} = \frac{\chi((1/n)e_\vartheta e_\varphi)}{\varphi(1)} = \frac{\chi(e_\varphi)}{n\varphi(1)} = \frac{(\chi_K, \vartheta)\vartheta(e_\varphi)}{n\varphi(1)} = (\chi_K, \vartheta) = \frac{\chi(1)}{\vartheta(1)}.$$

Property 3 was proved above, and Property 4 immediately follows from it.

Suppose that $S \cong \mathbf{M}_n(\mathbb{F})$. Then by Lemma 1.6 the isomorphism ε of Lemma 1.5 maps characters of $i\mathbb{F}Gi$-modules to characters of C-modules. The isomorphisms δ and κ^* (see above) fulfill this condition anyway, and thus does ι. To show Property 5, we may assume that $\mathbb{F}(\chi) = \mathbb{F}(\chi^\iota) = \mathbb{F}$, by Property 4. Then the Schur index of χ is the smallest positive integer m such that $m\chi$ is afforded by an $\mathbb{F}G$-module, and similar for χ^ι. The result now follows.

Now suppose that $\beta \in \mathbb{C}[\mathrm{Irr}(G/K)]$. Let $s_0 \in S$ be any element of S with $\mathrm{tr}(s_0) = 1$. Then

$$(\beta\chi)^\iota(h) = (\beta\chi)(s_0\sigma(h)^{-1}h).$$

Letting $s_0\sigma(h)^{-1} = \sum_{k\in K} \lambda_k k$, we get

$$\begin{aligned}(\beta\chi)^\iota(h) &= \sum_{k\in K} \lambda_k \beta(kh)\chi(kh) = \beta(h)\sum_{k\in K}\lambda_k\chi(kh) \\ &= \beta(h)\chi(s_0\sigma(h)^{-1}h) = \beta(h)\chi^\iota(h).\end{aligned}$$

This proves 6.

In the next property, it is understood that $(\chi_U)^\iota$ is defined by the restriction of σ to V/L. The property is now immediate from the definition of the correspondence: $\chi^\iota(v) = \chi(s_0\sigma(Lv)^{-1}v)$ and here $s_0\sigma(Lv)^{-1}v$ is in $\mathbb{C}U$.

Property 8 follows since induction and restriction are adjoint maps and ι is an isometry: Letting $\chi \in \operatorname{Irr}(G \mid \vartheta)$, on has

$$((\tau^G)^\iota, \chi^\iota) = (\tau^G, \chi) = (\tau, \chi_U) = (\tau^\iota, (\chi_U)^\iota) = (\tau^\iota, (\chi^\iota)_V) = ((\tau^\iota)^H, \chi^\iota).$$

Since every element of $\operatorname{Irr}(H \mid \varphi)$ is of the form χ^ι, it follows that $(\tau^G)^\iota = (\tau^\iota)^H$ as claimed.

Property 9: For $\chi \in \mathbb{C}[\operatorname{Irr}(G \mid \vartheta)]$ we have

$$(\chi_H)_\varphi(h) = \chi(he_\varphi) = \chi(he_\varphi e_\vartheta) = \chi(hi).$$

Now note that $hi = ih = \sigma(Lh)\sigma(Lh)^{-1}h$ for $h \in H$, and thus by Lemma 1.5 and the definition of ι,

$$\chi(hi) = \chi^\delta(hi) = \operatorname{tr}(\sigma(Lh))\chi^{\delta\varepsilon}(\sigma(Lh)^{-1}h) = \psi(h)\chi^\iota(h)$$

as claimed in 9.

To prove 10, it suffices to show that $(\overline{\psi}\xi^{\iota^{-1}}, \chi)_G = (\xi^G, \chi)_G$ for all $\chi \in \operatorname{Irr}(G \mid \vartheta)$. Using what we have already proved, we get

$$\begin{aligned}
(\overline{\psi}\xi^{\iota^{-1}}, \chi)_G &= (\xi^{\iota^{-1}}, \psi\chi)_G \\
&= (\xi, (\psi\chi)^\iota)_H \quad \text{(as } \iota \text{ is an isometry)} \\
&= (\xi, \psi\chi^\iota)_H \quad \text{(by 6.)} \\
&= (\xi, (\chi_H)_\varphi)_H \quad \text{(by 9.)} \\
&= (\xi, \chi_H)_H \quad \text{(as } \xi \in \mathbb{C}[\operatorname{Irr}(H \mid \varphi)]) \\
&= (\xi^G, \chi)_G
\end{aligned}$$

as was to be shown. The proof is complete. □

2.9 Remark. Property 5 can be generalized: Let $\chi \in \operatorname{Irr} G$. Then χ defines uniquely a division algebra over $\mathbb{F}$ with center isomorphic to $\mathbb{F}(\chi)$. We write $[[\chi]]_\mathbb{F}$ to denote the equivalence class of this division algebra in the Brauer group of $\mathbb{F}(\chi)$. It is also the class of $\mathbb{F}(\chi)Ge_\chi \cong \mathbb{F}Ge$, where $e := e_{\mathbb{F},\chi}$ is the central primitive idempotent of $\mathbb{F}G$ with $\chi(e) \neq 0$. If $\xi = \chi^\iota$, then we have $i\mathbb{F}Ge_{\mathbb{F},\chi}i \cong S \otimes_\mathbb{F} \mathbb{F}He_{\mathbb{F},\xi}$. This means that $[[\chi]]_\mathbb{F} = [[S \otimes_\mathbb{F} \mathbb{F}(\chi)]] \cdot [[\xi]]_\mathbb{F}$ in the Brauer group of $\mathbb{F}(\chi) = \mathbb{F}(\chi^\iota)$.

Suppose $\alpha \in \operatorname{Aut} \mathbb{F}$. Then α extends naturally to an automorphism of the group algebra $\mathbb{F}G$. More generally, if $\alpha\colon \mathbb{F} \to \mathbb{E}$ is a field isomorphism, we get a ring isomorphism $\mathbb{F}G \to \mathbb{E}G$ which we denote by α, too.

Similarly, if $\alpha\colon G \to G^\alpha$ is a group isomorphism, then we get naturally an algebra isomorphism $\alpha\colon \mathbb{F}G \to \mathbb{F}G^\alpha$.

If we have any ring isomorphism $\alpha\colon \mathbb{F}G \to A$, where A is another ring, then we get by restriction a field isomorphism from $\mathbb{F}$ to $\mathbb{F}^\alpha$ and a group isomorphism from G to G^α, and $A \cong \mathbb{F}^\alpha G^\alpha$. So ring isomorphisms from the group algebra to another ring generalize both field isomorphisms and group isomorphisms. When χ is a character of some subgroup $X \leqslant G$ with values in $\mathbb{F}$, then χ^α defined by $\chi^\alpha(x^\alpha) = \chi(x)^\alpha$ is a character of X^α with values in $\mathbb{F}^\alpha$. Keep this in mind for the following proposition.

2.10 Proposition. *Let $\mathcal{B} = (G, H, K, L, \vartheta, \varphi)$ be a basic configuration over the field $\mathbb{F}$, and let $\alpha\colon \mathbb{F}G \to \mathbb{F}^\alpha G^\alpha$ be a ring isomorphism. If $\sigma\colon H/L \to S$ is a magic representation for $\mathcal{B}$, then*

$$\sigma^\alpha\colon H^\alpha/L^\alpha \to S^\alpha \quad \textit{defined by} \quad \sigma^\alpha(h^\alpha) = \sigma(h)^\alpha$$

is a magic representation for $\mathcal{B}^\alpha$. Let $\iota(\sigma)$ and $\iota(\sigma^\alpha)$ be the associated character correspondences. Then

$$\chi^{\alpha\iota(\sigma^\alpha)} = \chi^{\iota(\sigma)\alpha} \quad \textit{for} \quad \chi \in \operatorname{Irr}(G \mid \vartheta).$$

Proof. Note that $e_\vartheta^\alpha = e_{\vartheta^\alpha}$ for the central primitive idempotent belonging to ϑ. Thus $\sigma^\alpha\colon H^\alpha/L^\alpha \to S^\alpha = (i^\alpha \mathbb{F}^\alpha K^\alpha i^\alpha)^{L^\alpha}$ with $i^\alpha = e_{\vartheta^\alpha} e_{\varphi^\alpha}$ is a magic representation for the basic configuration $\mathcal{B}^\alpha$.

Let $\chi \in \operatorname{Irr}(G \mid \vartheta)$ and pick $s_0 \in S$ with reduced trace 1. The isomorphism α maps the reduced trace of S to the reduced trace of S^α, by uniqueness of the reduced trace. Thus $\operatorname{tr}_{S^\alpha/\mathbb{F}^\alpha}(s_o^\alpha) = 1$. Therefore

$$\begin{aligned}(\chi^\alpha)^{\iota(\sigma\alpha)}(h^\alpha) &= \chi^\alpha(s_0^\alpha \sigma^\alpha(L^\alpha h^\alpha)^{-1} h^\alpha) = \chi^\alpha\left((s_0\sigma(Lh)^{-1}h)^\alpha\right) \\ &= \chi(s_0\sigma(Lh)^{-1}h)^\alpha = \left(\chi^{\iota(\sigma)}(h)\right)^\alpha = \chi^{\iota(\sigma)\alpha}(h^\alpha),\end{aligned}$$

as was to be shown. □

Note that if α is a field automorphism, this generalizes Property 3 of Theorem 2.8.

2.11 Remark. Let π be the set of prime divisors of n. If there is any magic representation, then there is a magic representation σ such that $\det\sigma$ has order a π-number.

Proof. Suppose $\sigma\colon H/L \to S$ is given. Let $\lambda = \det\sigma$, a linear character of H/L. Let b be the π'-part of $\operatorname{ord}(\lambda)$. As $n = \dim\sigma$ is π, there is $r \in \mathbb{Z}$ with $rn + 1 \equiv 0 \mod b$. Then $\det(\lambda^r\sigma) = \lambda^{rn+1}$ has π-order. □

The following consequence of Theorem 2.8 is clear in view of Properties 7 and 8.

2.12 Corollary. *Assume Hypothesis 2.1 with a magic representation σ and associated character correspondence ι. Assume $K \leqslant U \leqslant G$. Pick $\tau \in \operatorname{Irr} U$. Then the restriction of ι to $\mathbb{C}[\operatorname{Irr}(G \mid \tau)]$ is an isometry onto $\mathbb{C}[\operatorname{Irr}(H \mid \tau^{\iota})]$.*

Note that when $U \trianglelefteq G$ and τ invariant in G, this corollary implies that the character triples (G, U, τ) and $(H, H \cap U, \tau^{\iota})$ are isomorphic in the sense of Isaacs [34, Definition 11.23].

The bijection of the theorem depends on the magic representation σ. If such a representation exists, it is unique up to multiplication with a linear character of H/L (with values in $\mathbb{F}$). Different choices of σ give bijections which differ by multiplication with a linear character of H/L. Note that if $\psi(h) \neq 0$ for all $h \in H$, then χ^{ι} is determined by the equation $(\chi_H)_{\varphi} = \psi\chi^{\iota}$. Otherwise on needs the representation σ to compute χ^{ι}.

The theorem depends on the hypothesis that a certain projective representation turns out to be linear. It is possible that over some field $\mathbb{F}$ containing the values of φ and ϑ, the projective representation is not equivalent to a linear one, while over some bigger field Theorem 2.8 applies. The following example illustrates this and other points. It is taken from a paper of Dade [6].

2.13 Example. Let $K = \langle x, y \rangle$ be the quaternion group of order 8. The symmetric group on 3 elements acts on K as follows: let t be the automorphism permuting x, y and xy cyclically, and u the one with $x^u = y^{-1}$ and $y^u = x^{-1}$. Then $C = \langle t, u \rangle \cong S_3$. Let G be the semidirect product of C and K. (In fact $G \cong GL(2,3)$.) Let $L = \mathbf{Z}(K)$ and $H = CL$. Further let φ be the linear character of L of order 2 and ϑ the irreducible character of K lying above φ. Here Theorem 2.8 applies, at least if we work over an algebraically closed field. This follows from a general result which we will prove later and which is essentially equivalent to a result of Dade [6, 30]. We can see this directly here as follows: Let $\mathbb{F} = \mathbb{Q}(\sqrt{-2})$ and $S = \mathbb{F}Ke_{\varphi} = (\mathbb{F}Ke_{\varphi})^L$. Set $\sigma(t) = (1/2)(-1 + x + y + xy)e_{\varphi}$ and $\sigma(u) = (1/\sqrt{-2})(x - y)e_{\varphi}$. This can be extended to an homomorphism $\sigma\colon C \to S$, and then $s^h = s^{\sigma(h)}$ for $s \in S$ and $h \in C$. (It suffices to check this for $s = xe_{\varphi}, ye_{\varphi}$ and $h = t, u$.) As $H/L \cong C$, we see that Theorem 2.8 applies for $\mathbb{F} = \mathbb{Q}(\sqrt{-2})$. Now since φ and ϑ are rational valued, it is natural to ask wether Theorem 2.8 applies over $\mathbb{Q}$, too. However, we cannot find $\sigma_0(u) \in \mathbb{Q}Ke_{\varphi}$ with $s^{\sigma_0(u)} = s^u$ and $\sigma_0(u)^2 = e_{\varphi}$, since this would imply $\sigma_0(u) = \pm\sigma(u)$. Thus Theorem 2.8 does not apply over $\mathbb{Q}$. Observe that the character of σ is the irreducible character of $H/L \cong C \cong S_3$ of degree 2, which is afforded by a rational representation.

There are two different magic representations here, as $|\mathrm{Lin}(H/L)| = 2$, but both have the same magic character.

That Theorem 2.8 can not apply over $\mathbb{Q}$ can also seen from the character table of G: The characters in $\mathrm{Irr}(H \mid \varphi)$ are rational, while the two irreducible characters of G of degree 2, that lie above ϑ, are not: They are complex conjugate. Therefore we see that $\mathbb{Q}He_\varphi$ has three nonisomorphic (absolutely) irreducible modules while $\mathbb{Q}Ge_\vartheta$ has two nonisomorphic irreducible modules. It is also clear now that none of the two possible bijections is better than the other, so that no canonical bijection can be defined. In Dade's paper, the example served to illustrate this lack of uniqueness.

In general it is difficult to decide wether a magic representation exists, given only the data of Hypothesis 2.1. A big part of this work will be devoted to give sufficient conditions, when K/L is a fully ramified section of G. In this way we will reprove and generalize the classical results of Dade and Isaacs [6, 27, 30].

However there is an easy special case that we can prove already in our quite general setting. We need the following fact:

2.14 Lemma. *Let $\mathbb{F}$ be a field, X a group and $\alpha \in Z^2(X, \mathbb{F}^*)$. Suppose that S is a central simple $\mathbb{F}$-algebra and $\sigma\colon X \to S^*$ a map with $\sigma(x)\sigma(y) = \alpha(x,y)\sigma(xy)$. If $\dim_{\mathbb{F}} S = n^2$, then $\alpha^n \in B^2(X, \mathbb{F}^*)$. If $(|X|, n) = 1$, then $\alpha \in B^2(X, \mathbb{F}^*)$.*

Proof. This is well known, at least in the case where $S \cong \mathbf{M}_n(\mathbb{F})$ [26, 20.5 and 20.6]. In the general case, let $\mathbb{E} \geqslant \mathbb{F}$ be a field such that $S \otimes_{\mathbb{F}} \mathbb{E} \cong \mathbf{M}_n(\mathbb{E})$. The restriction of the determinant to S gives a multiplicative map $\det\colon S \to \mathbb{F}$, also known as reduced norm of S over $\mathbb{F}$ [54, Section 9a]. Then as usual we see that $\alpha(x,y)^n = \det(\sigma(x))\det(\sigma(y))\det(\sigma(xy))^{-1}$. □

2.15 Proposition. *Assume Hypothesis 2.1 with $(|H/L|, n) = 1$. Then there is a unique magic representation σ with $\det(\sigma) = 1$, and the corresponding character bijection between $\mathrm{Irr}(G \mid \vartheta)$ and $\mathrm{Irr}(H \mid \varphi)$ is invariant under all automorphisms of G fixing H, ϑ and φ. The magic character is nonzero on every element of prime power order mod L.*

That the Clifford extensions associated with the characters ϑ invariant in G and φ invariant in H are isomorphic was already proved by Dade [5, 0.4] in a more general situation, but over an algebraically closed field. Schmid [58] has generalized it to arbitrary fields, under additional assumptions. The description using the magic character ψ seems to be new, however. We remark that it would be nice to have a purely character theoretic description of the correspondence. The last part of the proposition yields that for H/L a

p-group, one needs only the magic character (not the magic representation) to describe the correspondence. In the general case, no simply character theoretic description seems to be known, although the existence und uniqueness of the correspondence was proved by Dade [5] in 1970.

Proof of Proposition 2.15. The action of H/L on S defines a projective representation σ. As $(|H/L|, n) = 1$, we see that the associated factor set is trivial. Thus we can assume that σ is a magic representation. Multiplication of σ with $\lambda \in \mathrm{Lin}_{\mathbb{F}}(H/L)$ causes a multiplication of $\det \sigma$ with λ^n and $\lambda \mapsto \lambda^n$ is a permutation of $\mathrm{Lin}_{\mathbb{F}}(H/L)$, since $|\mathrm{Lin}_{\mathbb{F}}(H/L)|$ divides $|H/L|$. Thus there is a unique magic representation σ that has determinant 1. It is clear now that the correspondence is invariant under automorphisms fixing H, ϑ and φ. If $\mathrm{ord}(Lh) = p^r$ with p a prime, then $\psi(Lh) \equiv \psi(1) = n \mod p$. (This is well known, it follows since $\omega - 1 \in \mathfrak{P}$ where ω is a primitive p^r-th root of unity and $\mathfrak{P}$ any prime ideal of $\mathbb{Z}[\omega]$ lying over $p\mathbb{Z}$.) It follows that $\psi(Lh) \neq 0$. The result follows. □

The assumption of the last proposition obviously holds if $n = 1$, that is, $(\vartheta_L, \varphi) = 1$. In this case the natural choice is $\psi = \sigma = 1$. If χ and ξ correspond, then by Theorem 2.8(9), $(\chi_H)_\varphi = \xi$. The last equation in fact defines then the correspondence. It follows that ξ is the unique element in $\mathrm{Irr}(H \mid \varphi)$ with $(\chi_H, \xi) \neq 0$ and the correspondence can also be defined by this condition. This fact is known and can be proved just using elementary character theory [32, Lemma 4.1]. Theorem 2.8 also implies that the map $\chi \mapsto (\chi_H)_\varphi$ preserves Schur indices if $n = 1$: Because then clearly $S \cong \mathbb{F}$ is split. In fact, $\mathbb{F}He_\varphi \cong i\mathbb{F}Gi$ when $n = 1$, where the isomorphism is given by multiplication with i (or e_ϑ).

If $n = \psi(1) > 1$ in Proposition 2.15, then ψ is reducible (at least over a field big enough) since its degree is prime to $|H/L|$. It follows that $(\chi_H)_\varphi$ is reducible for any $\chi \in \mathrm{Irr}(G \mid \vartheta)$.

We return to the general situation and give some more properties of the correspondence of Theorem 2.8.

2.16 Proposition. *Assume Hypothesis 2.1 and let $C \leqslant \mathbf{C}_H(S)$ with $L \leqslant C \trianglelefteq H$. For $\chi \in \mathrm{Irr}(KC \mid \vartheta)$ and $\xi \in \mathrm{Irr}(C \mid \varphi)$ define $\chi \leftrightarrow \xi$ if and only if $(\chi_C, \xi)_C > 0$.*

1. *"$\leftrightarrow$" defines a bijection between $\mathrm{Irr}(KC \mid \vartheta)$ and $\mathrm{Irr}(C \mid \varphi)$, which has all the properties of Theorem 2.8 with $\psi = n1_{C/L}$. So if $\chi \leftrightarrow \xi$, then $(\chi_C, \xi)_C = n$.*

2. *The bijection is H-invariant, that is for $h \in H$ we have $\chi \leftrightarrow \xi$ if and only if $\chi^h \leftrightarrow \xi^h$.*

3. *Assume that* $\xi \in \operatorname{Irr}(C \mid \varphi)$ *is* H*-stable and* $\chi \leftrightarrow \xi$*. Then* $T = (e_\xi \mathbb{F}KCe_\chi e_\xi)^C$ *and* $S = (e_\varphi \mathbb{F}Ke_\vartheta e_\varphi)^L$ *are isomorphic as* H*-algebras.*

Note that if $n = 1$, then $\mathbf{C}_H(S) = H$.

Proof. Theorem 2.8 applies to the configuration $(KC, C, K, L, \vartheta, \varphi)$ with $\sigma\colon C/L \to S$, $\sigma(c) = 1_S = i$ for all $c \in C$. Observe that then $\psi = n1_C$. From Property 9 in Theorem 2.8 it now follows that the restriction χ_C of every $\chi \in \operatorname{Irr}(KC \mid \vartheta)$ has a unique constituent in $\operatorname{Irr}(C \mid \varphi)$, which occurs with multiplicity n, as claimed. Conversely, for $\xi \in \operatorname{Irr}(C \mid \varphi)$, the induced character ξ^G has a unique constituent lying in $\operatorname{Irr}(KC \mid \vartheta)$, by Property 10. The correspondence $\leftrightarrow$ is thus just the correspondence ι of Theorem 2.8. The first claim follows. That $\leftrightarrow$ is compatible with the action of H is clear from the definition. Let $j = e_\chi e_\xi$, where we assume that $\xi \leftrightarrow \chi$ are H-invariant. Thus $T = (j\mathbb{F}KCj)^C$. The idempotent j centralizes S, as e_χ is in the center of $\mathbb{F}KC$, and $e_\xi \in C$ centralizes S by assumption. It follows that for every $s \in S$, we have $sj = js = jsj \in T$. As $e_\chi e_\vartheta = e_\chi$ and $e_\xi e_\varphi = e_\xi$, it follows that $ij = j$. The map $s \mapsto sj$ is thus an algebra homomorphism from S into T. Since S is simple and $\dim_\mathbb{F} S = n = \dim_\mathbb{F} T$, the map is an isomorphism. It is compatible with the action of H as j is H-stable. $\square$

Next we show a result that may be useful in inductive proofs.

2.17 Proposition. *Assume Hypothesis 2.1. Let* $H \leqslant U \leqslant G$ *and* $N = K \cap U$*. Let* $\eta \in \operatorname{Irr} N$ *be invariant in* U *with* $n_1 = (\vartheta_N, \eta) > 0$ *and* $n_2 = (\eta_L, \varphi) > 0$*, and* $\mathbb{F}(\eta) = \mathbb{F}$*. Assume that there are magic representations*

$$\sigma_1\colon U/N \to S_1 = (e_\eta \mathbb{F}Ke_\vartheta e_\eta)^N \quad \textit{and} \quad \sigma_2\colon H/L \to S_2 = (e_\varphi \mathbb{F}Ne_\eta e_\varphi)^L.$$

By Theorem 2.8 we have bijections

$$\begin{aligned} \iota(\sigma_1)&\colon \mathbb{C}[\operatorname{Irr}(G \mid \vartheta)] \to \mathbb{C}[\operatorname{Irr}(U \mid \eta)] \quad \textit{and} \\ \iota(\sigma_2)&\colon \mathbb{C}[\operatorname{Irr}(U \mid \eta)] \to \mathbb{C}[\operatorname{Irr}(H \mid \varphi)]. \end{aligned}$$

Then there is a magic representation

$$\sigma\colon H/L \to S = (e_\varphi \mathbb{F}Ke_\vartheta e_\varphi)^L$$

such that

$$\iota(\sigma) = \iota(\sigma_1)\iota(\sigma_2).$$

We think of the configuration $(G, H, K, L, \vartheta, \varphi)$ as composed of the configurations $(G, U, K, N, \vartheta, \eta)$ and $(U, H, N, L, \eta, \varphi)$. The proposition says that if Theorem 2.8 applies to the smaller configurations, then it applies to the composed configuration, and the resulting character correspondence is just composition of the correspondences of the smaller configurations.

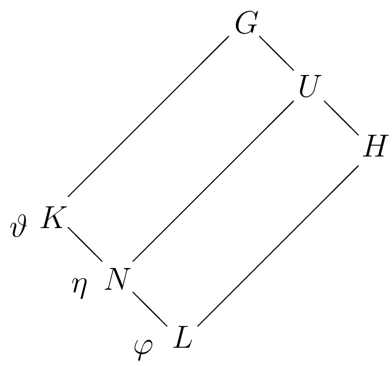

Figure 2.2: A composed basic configuration

Proof. Consider the map

$$\mu\colon S_1 \otimes_{\mathbb{F}} S_2 \to S, \quad s_1 \otimes s_2 \mapsto s_1 s_2.$$

Then $\mu(1_{S_1} \otimes 1_{S_2}) = \mu(e_\vartheta e_\eta \otimes e_\eta e_\varphi) = e_\vartheta e_\eta e_\varphi = e_\eta \cdot 1_S \neq 0$. Clearly $\mu(S_1 \otimes 1) = S_1 e_\varphi$ and $\mu(1 \otimes S_2) = e_\vartheta S_2$ commute. Thus μ is an algebra homomorphism, in general not unitary. Since $S_1 \otimes S_2$ is simple, it follows that μ is injective. Clearly $\mu(S_1 \otimes S_2) \subseteq e_\eta S e_\eta$. We claim that equality holds. To show this, we compute the dimension of $S_0 := e_\eta S e_\eta$. For this computation, we can assume that $\mathbb{F}$ is a splitting field for the characters involved. Let V be a $\mathbb{F}K$-module affording ϑ. Then $Ve_\eta \cong W^{n_1}$ as $\mathbb{F}N$-module, where W affords η, and $Ve_\eta e_\varphi \cong X^{n_1 n_2}$ as $\mathbb{F}L$-module, where X affords φ. Thus $S_0 = (e_\eta e_\varphi \mathbb{F}K e_\vartheta e_\eta e_\varphi)^L \cong \operatorname{End}_{\mathbb{F}L}(Ve_\eta e_\varphi) \cong \mathbf{M}_{n_1 n_2}(\mathbb{F})$. Therefore $\dim_{\mathbb{F}}(S_0) = (n_1 n_2)^2$. As also $\dim_{\mathbb{F}}(S_1 \otimes S_2) = n_1^2 n_2^2$, we conclude that $S_1 S_2 = \mu(S_1 \otimes S_2) = S_0$ as claimed.

Let $\sigma_0(h) = \sigma_1(h)\sigma_2(h) \in S_0$. For $s_1 \in S_1$ and $s_2 \in S_2$ we have then $(s_1 s_2)^{\sigma_0(h)} = s_1^{\sigma_1(h)} s_2^{\sigma_2(h)}$, as the images of S_1 and S_2 in S commute. (The inverses of $\sigma_i(h)$ are to be computed in S_i.) We also have $\sigma_0(h_1 h_2) = \sigma_0(h_1)\sigma_0(h_2)$, again since $\mu(S_1)$ and $\mu(S_2)$ commute.

On the other hand, there is a projective representation $\sigma\colon H/L \to S$ such that $s^h = s^{\sigma(h)}$ for all $s \in S$. In particular, this holds for elements of S_0. The idempotent $e_\eta e_\vartheta e_\varphi = e_\eta 1_S$ is H-invariant by assumption, so that $\sigma(h)$ centralizes e_η for every $h \in H$. Thus $e_\eta \sigma(h) = \sigma(h) e_\eta \in S_0$, and this element is invertible in S_0. For $s \in S_0$ we have thus $s^{e_\eta \sigma(h)} = s^{\sigma(h)} = s^h = s^{\sigma_0(h)}$. As $S_0 = e_\eta S e_\eta$ is simple, it follows that $e_\eta \sigma(h) = \lambda_h \sigma_0(h)$ for some $\lambda_h \in \mathbb{F}$. Therefore σ is projectively equivalent with a representation.

From now on, assume that $\sigma\colon H/L \to S$ is such that $e_\eta \sigma(h) = \sigma_0(h) =$

$\sigma_1(h)\sigma_2(h)$. We want to show that $\iota(\sigma) = \iota(\sigma_1)\iota(\sigma_2)$. Choose $s_i \in S_i$ with $\mathrm{tr}_{S_i}(s_i) = 1$. Then $\mathrm{tr}_S(s_1s_2) = \mathrm{tr}_{S_0}(s_1s_2) = \mathrm{tr}_{S_1}(s_1)\,\mathrm{tr}_{S_2}(s_2) = 1$. Let $\chi \in \mathbb{C}[\mathrm{Irr}(G \mid \vartheta)]$. Let $u \in U$ and $\alpha \in \mathbb{F}N$. Writing $\alpha = \sum_{n\in N} \alpha_n n$ and using $N \leqslant \ker\sigma_1$, we see that $\chi^{\iota(\sigma_1)}(\alpha u) = \chi(s_1\sigma_1(u)^{-1}\alpha u)$. Using this, we get for $h \in H$

$$\begin{aligned}
\chi^{\iota(\sigma_1)\iota(\sigma_2)}(h) &= \chi^{\iota(\sigma_1)}(s_2\sigma_2(h)^{-1}h) \\
&= \chi(s_1\sigma_1(h)^{-1}s_2\sigma_2(h)^{-1}h) && \text{(see above)} \\
&= \chi(s_1s_2\sigma_0(h)^{-1}h) \\
&= \chi(s_1s_2\sigma(h)^{-1}h) && (\text{as } s_1s_2 \in e_\eta S e_\eta) \\
&= \chi^{\iota(\sigma)}(h) && (\text{as } \mathrm{tr}_S(s_1s_2) = 1)\,. \quad \square
\end{aligned}$$

2.2 The Character Correspondence for Semi-Invariant Characters

Let $K \trianglelefteq G$ and $\vartheta \in \mathrm{Irr}\,K$. Classical Clifford theory reduces the study of $\mathrm{Irr}(G \mid \vartheta)$ to the study of $\mathrm{Irr}(G_\vartheta \mid \vartheta)$ where G_ϑ is the inertia group of ϑ in G. Namely, Frobenius induction provides a bijection between $\mathrm{Irr}(G_\vartheta \mid \vartheta)$ and $\mathrm{Irr}(G \mid \vartheta)$.

Suppose that φ and ϑ in Hypothesis 2.1 are not H-invariant. In the applications often the actions of H on the orbit of φ and on the orbit of ϑ are isomorphic, so that $H_\varphi = H_\vartheta$. (For example, this is the case if ϑ and φ are fully ramified with respect to each other.) Observe that always

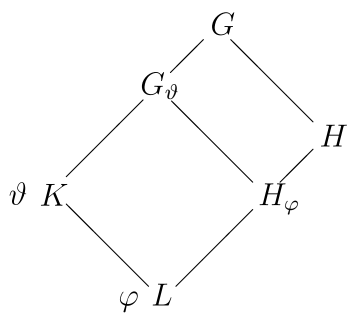

Figure 2.3: Generalized Basic Configuration

$G_\vartheta = KH_\vartheta$ if $G = KH$. Thus if Theorem 2.8 applies to G_ϑ, H_φ and so on, we may compose the resulting character correspondence with the Clifford correspondences between $\mathrm{Irr}(G \mid \vartheta)$ and $\mathrm{Irr}(G_\vartheta \mid \vartheta)$, and between $\mathrm{Irr}(H \mid \varphi)$ and $\mathrm{Irr}(H_\varphi \mid \varphi)$ and we obtain a bijection between $\mathrm{Irr}(G \mid \vartheta)$ and $\mathrm{Irr}(H \mid \varphi)$.

However, the Clifford correspondence is not well behaved with respect to fields of values and Schur indices: Let $\mathbb{F}$ be a field (a subfield of $\mathbb{C}$, say). If $\tau \in \operatorname{Irr}(T \mid \vartheta)$, we may have that $\mathbb{F}(\tau^G) < \mathbb{F}(\tau)$. Also, the Schur index of τ^G over the field $\mathbb{F}$ may be bigger than that of τ. Our aim in this section is to relax the condition of H-invariance of the characters ϑ and φ, and to obtain correspondences preserving Schur indices and fields of values over smaller fields than those generated by the values of ϑ and φ. This will be applied in Section 4.6 to prove that the Isaacs correspondence preserves Schur indices.

Clifford Theory with Fields

Most of the following proposition is well known [55, Theorem 1]:

2.18 Proposition. *Let $K \trianglelefteq G$. Let $\mathbb{E}$ be algebraically closed and $\vartheta \in \operatorname{Irr}_{\mathbb{E}} K$ the character of a simple $\mathbb{E}K$-module. Suppose $\mathbb{F} \leqslant \mathbb{E}$ and U is an $\mathbb{F}K$-module whose character contains ϑ as constituent. Let T be the inertia group of U in G. Then induction defines a bijection between $\operatorname{Irr}_{\mathbb{E}}(T \mid \vartheta)$ and $\operatorname{Irr}_{\mathbb{E}}(G \mid \vartheta)$. For $\tau \in \operatorname{Irr}_{\mathbb{E}}(T \mid \vartheta)$ on has*

$$\mathbb{F}(\tau) = \mathbb{F}(\tau^G) \quad \textit{and} \quad [[\tau]]_{\mathbb{F}} = [[\tau^G]]_{\mathbb{F}}.$$

Proof. Let I be the maximal G-invariant ideal of $\mathbb{F}K$ contained in the annihilator of U. (Thus $I = \operatorname{Ann}_{\mathbb{F}K}(U^G)$.) Regard $A = \mathbb{F}G/I\mathbb{F}G$ as G/K-graded algebra. Then A_1 is semisimple and its blocks are permuted transitively by G. The inertia group T is the stabilizer of one of these blocks, with central primitive idempotent e, say. By Proposition A.14 in the appendix, we get that Frobenius induction is a category equivalence between A_Te-modules and A-modules. The same proposition may be applied to $A \otimes_{\mathbb{F}} \mathbb{E}$.

Let $\tau \in \operatorname{Irr}_{\mathbb{E}}(T \mid \vartheta)$ and let V be the unique simple $\mathbb{F}T$-module whose character contains τ as a constituent. Then $Ve = V$, so that V is an A_Te-module. As $[[\tau]]_{\mathbb{F}}$ is the Brauer equivalence class of $\operatorname{End}_{\mathbb{F}T}(V, V)$, it follows $[[\tau]]_{\mathbb{F}} = [[\tau^G]]_{\mathbb{F}}$ from Proposition A.14.

As $\mathbb{F}(\tau) \cong \mathbf{Z}(\operatorname{Hom}_{\mathbb{F}T}(V, V))$ and $\mathbb{F}(\tau^G) \cong \mathbf{Z}(\operatorname{Hom}_{\mathbb{F}G}(V^G, V^G))$, it follows that $\mathbb{F}(\tau) \cong \mathbb{F}(\tau^G)$. Since $\mathbb{F}(\tau^G) \leqslant \mathbb{F}(\tau)$ and both field extensions are finite, equality follows. □

For an elementary proof, see [55], but note that there only equality of Schur indices is proved. Our next purpose is to elucidate the relation between the inertia group of a simple $\mathbb{F}K$-module and the inertia groups of its absolutely irreducible constituents.

First we review the central character associated with an absolutely irreducible $\mathbb{E}$-character ϑ of K. Let $D\colon \mathbb{E}K \to \mathbf{M}_{\vartheta(1)}(\mathbb{E})$ be a representation affording ϑ. Then $D(\mathbb{E}K) = \mathbf{M}_{\vartheta(1)}(\mathbb{E})$ [49, Theorem 2.3.3]. If $z \in \mathbb{E}K$ is such that $z + \ker D \in \mathbf{Z}(\mathbb{E}K/\ker D)$, then $D(z) = \omega_\vartheta(z)I$ with $\omega_\vartheta(z) \in \mathbb{E}$. We call $\omega_\vartheta(z)$ the central character associated with ϑ. From the definition it follows that $\omega_\vartheta(z) = \vartheta(az)$ where $a \in \mathbb{E}K$ is any element with $\vartheta(a) = 1$.

If $\mathbb{F} \leqslant \mathbb{E}$ and V is a simple $\mathbb{F}K$-module whose character contains ϑ as constituent, then ω_ϑ induces an isomorphism from $\mathbf{Z}(\mathbb{F}K/\operatorname{ann}_{\mathbb{F}K} V)$ onto $\mathbb{F}(\vartheta)$, as is well known [49, Theorem 2.6.2].

The next result is known [28, Lemma 2.1], except perhaps the last part.

2.19 Lemma. *Let V be a simple $\mathbb{F}K$-module where $K \trianglelefteq G$. Let $\mathbb{E} \geqslant \mathbb{F}$ be a splitting field of K and $\vartheta \in \operatorname{Irr}_{\mathbb{E}} K$ an absolutely irreducible constituent of the character of V. Let T be the inertia group of V in G. Set $Z = \mathbf{Z}(\mathbb{F}K/\operatorname{ann}_{\mathbb{F}K} V)$. Then*

(a) $T = \{g \in G \mid$ *there is* $\gamma \in \operatorname{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ *such that* $\vartheta^{g\gamma} = \vartheta\}$.

(b) For every $g \in T$, there is a unique $\gamma_g \in \operatorname{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ with $\vartheta^{g\gamma_g} = \vartheta$. The map $g \mapsto \gamma_g$ is a homomorphism with kernel G_ϑ.

(c) $G_\vartheta = \{t \in T \mid z^t = z$ *for all* $z \in Z\}$.

(d) For $z \in Z$ and $g \in T$, we have $\omega_\vartheta(z^g) = \omega_\vartheta(z)^{\gamma_g}$.

Proof. Let α be the character of V. Then $\alpha = m\sum_{\gamma\in\operatorname{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})} \vartheta^\gamma$ [26, 38.1]. If $g \in T$, then $\alpha^g = \alpha$ and thus $\vartheta^g = \vartheta^\gamma$ for some γ. Conversely, if $\vartheta^g = \vartheta^\gamma$ with $\gamma \in \operatorname{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$, then

$$\alpha^g = m\sum_{\gamma'} \vartheta^{\gamma' g} = m\sum_{\gamma'} \vartheta^{g\gamma'} = m\sum_{\gamma'} \vartheta^{\gamma\gamma'} = \alpha,$$

so $g \in T$. This proves (a).

Since only the identity of $\operatorname{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ can fix ϑ, the γ_g with $\vartheta^{g\gamma_g} = \vartheta$ is unique. For $g, h \in T$ one has

$$\vartheta^{gh\gamma_g\gamma_h} = \vartheta^{g\gamma_g h\gamma_h} = \vartheta$$

and so $\gamma_{gh} = \gamma_g\gamma_h$. The kernel of $g \mapsto \gamma_g$ is obviously G_ϑ.

Next we prove (d). Suppose $g \in T$. We claim that there is $e \in \mathbb{F}K$ with $\vartheta(e) = \vartheta(geg^{-1}) = 1$. First, there is $e_0 \in \mathbb{F}(\vartheta)K$ such that $\vartheta(e_0) = 1$ and $\widetilde{\vartheta}(e_0) = 0$ for any absolutely irreducible character $\widetilde{\vartheta}$ different from ϑ. Now set $e = \mathrm{T}_{\mathbb{F}}^{\mathbb{F}(\vartheta)}(e_0) \in \mathbb{F}K$. Then $\vartheta(e) = \vartheta(e_0) = 1$ and $\vartheta(geg^{-1}) = \vartheta^g(e) = \vartheta^{\gamma_g^{-1}}(e) = 1$. Now we get for $z \in Z$

$$\begin{aligned}\omega_\vartheta(z^g) = \vartheta(ez^g) &= \vartheta^{g^{-1}}(geg^{-1}z) = \vartheta^{\gamma_g}(geg^{-1}z) \\ &= (\vartheta(geg^{-1}z))^{\gamma_g} = (\omega_\vartheta(z))^{\gamma_g},\end{aligned}$$

since $\vartheta(geg^{-1}) = 1$. This shows (d). Now (c) follows from (b) and (d) and the fact that ω_ϑ is an isomorphism: If $t \in G_\vartheta$ then $\gamma_t = \mathrm{Id}$ and thus $\omega_\vartheta(z^t) = \omega_\vartheta(z)$ for all $z \in Z$. This yields $z^t = z$ for all z. Conversely, if $z^t = z$ for $z \in Z$, then $\omega_\vartheta(z) = \omega_\vartheta(z^t) = \omega_\vartheta(z)^{\gamma_t}$ and thus $\gamma_t = \mathrm{Id}$. By Part (b) we get $t \in G_\vartheta$. □

2.20 Corollary. *Let $\Gamma \leqslant \mathrm{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ be the image of the map $t \mapsto \gamma_t$ from the last lemma. Set*

$$\Theta_0 = \sum_{t \in [T:G_\vartheta]} \vartheta^t \quad \textit{and} \quad \Theta_1 = \sum_{g \in [G:G_\vartheta]} \vartheta^g.$$

Then $\mathbb{F}(\Theta_0) = \mathbb{F}(\Theta_1) = (\mathbb{F}(\vartheta))^\Gamma$, so that $\Gamma = \mathrm{Gal}(\mathbb{F}(\vartheta)/\mathbb{F}(\Theta_0))$.

Proof. As $\mathbb{F}(\vartheta^g) = \mathbb{F}(\vartheta)$, we clearly have $\mathbb{F}(\Theta_1) \leqslant \mathbb{F}(\Theta_0) \leqslant \mathbb{F}(\vartheta)$. Let $\gamma \in \Gamma$. Then $\vartheta^\gamma = \vartheta^t$ for some $t \in T$ and thus γ permutes the constituents of Θ_0. It follows $\Theta_0^\gamma = \Theta_0$ and so $\mathbb{F}(\Theta_0) \leqslant (\mathbb{F}(\vartheta))^\Gamma$. It remains to show $(\mathbb{F}(\vartheta))^\Gamma \leqslant \mathbb{F}(\Theta_1)$, and this is equivalent to $\mathbf{C}_{\mathrm{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})}\big(\mathbb{F}(\Theta_1)\big) \leqslant \Gamma$. But this follows from Lemma 2.19, Part (a). □

When interested in the fields of values and Schur indices of characters of G lying over ϑ, it is thus no loss to work over the field $\mathbb{F}(\Theta_0)$. This is sometimes convenient, but seldom really necessary.

Following Isaacs [28], we say that $\vartheta \in \mathrm{Irr}_{\mathbb{E}} K$ is $\mathbb{F}$-*semi-invariant* in G, where $K \trianglelefteq G$, if for every $g \in G$ there is $\gamma \in \mathrm{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ with $\vartheta^{g\gamma} = \vartheta$.

The results so far yield that it is no loss to assume that ϑ is semi-invariant in G, when interested in questions of Schur indices and fields of values. For completeness we mention the following result of Riese and Schmid [55, Theorem 1]. Here $m_{\mathbb{F}}(\chi)$ denotes the Schur index of the character χ over the field $\mathbb{F}$.

2.21 Proposition. *Let $K \trianglelefteq G$ and suppose $\vartheta \in \mathrm{Irr}\, K$ is $\mathbb{F}$-semi-invariant in G. Let $\eta \in \mathrm{Irr}(G_\vartheta \mid \vartheta)$ and $\chi = \eta^G$. Then*

1. *$\mathbb{F}(\eta) = \mathbb{F}(\chi, \vartheta)$.*
2. *$m_{\mathbb{F}}(\chi) = t m_{\mathbb{F}}(\eta)$ where $t \in \mathbb{N}$ divides $|G : G_\vartheta|$.*

Magic Crossed Representations

We collect the assumptions and fix the notation for this section:

2.22 Hypothesis. Let G be a group, $K \trianglelefteq G$ and $H \leqslant G$ with $G = HK$ and set $L = H \cap K$. Let $\mathbb{E}$ be an algebraically closed field and let $\varphi \in \mathrm{Irr}_{\mathbb{E}} L$ and $\vartheta \in \mathrm{Irr}_{\mathbb{E}} K$ be characters of simple, projective modules over $\mathbb{E}L$ respective $\mathbb{E}K$ and $\mathbb{F} \subseteq \mathbb{E}$ a field such that the following conditions hold:

1. $n = (\vartheta_L, \varphi) > 0$.
2. $\mathbb{F}(\varphi) = \mathbb{F}(\vartheta)$.
3. For every $h \in H$ there is $\gamma = \gamma_h \in \operatorname{Gal}(\mathbb{F}(\varphi)/\mathbb{F})$ such that $\vartheta^{h\gamma} = \vartheta$ and $\varphi^{h\gamma} = \varphi$.

For later use, we fix the following notation: Let e be the central primitive idempotent of $\mathbb{F}K$ with $\vartheta(e) \neq 0$, and f the central primitive idempotent of $\mathbb{F}L$ with $\varphi(f) \neq 0$. Let e_ϑ and e_φ be the central primitive idempotents of $\mathbb{E}K$ and $\mathbb{E}L$ corresponding to ϑ and φ, and set $i = \sum_{\gamma \in \operatorname{Gal}(\mathbb{F}(\varphi)/\mathbb{F})} e_\varphi^\gamma e_\vartheta^\gamma$. Set $S = (i\mathbb{F}Ki)^L$, $Z = \mathbf{Z}(i\mathbb{F}Ki)$ and $Z_0 = Z^H = \mathbf{C}_Z(H)$.

Observe that $h \mapsto \gamma_h$ is a group homomorphism by Lemma 2.19. By the third condition, e and f are H-invariant.

2.23 Lemma. *Assume Hypothesis 2.22.*

1. *The H-sets*
$$\{\varphi^\gamma \mid \gamma \in \operatorname{Gal}(\mathbb{F}(\varphi)/\mathbb{F})\} \quad \text{and} \quad \{\vartheta^\gamma \mid \gamma \in \operatorname{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})\}$$
are isomorphic via $\varphi^\gamma \mapsto \vartheta^\gamma$.
2. *$\mathbf{Z}(\mathbb{F}Ke) \cong \mathbf{Z}(\mathbb{F}Lf)$ as fields with H/L-action.*

Proof. By assumption, $\mathbb{F}(\varphi) = \mathbb{F}(\vartheta)$. As only the identity of $\operatorname{Gal}(\mathbb{F}(\varphi)/\mathbb{F}) = \operatorname{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ can fix φ, the map $\varphi^\gamma \mapsto \vartheta^\gamma$ is well defined. It is a bijection between the sets in the statement of the lemma. If $h \in H$, then
$$(\varphi^\gamma)^h = \varphi^{h\gamma} = \varphi^{\gamma_h\gamma} \mapsto \vartheta^{\gamma_h\gamma} = \vartheta^{h\gamma} = (\vartheta^\gamma)^h.$$
This means that $\varphi^\gamma \mapsto \vartheta^\gamma$ commutes with the action of H, as desired.

For the last part, consider the diagramm
$$\mathbf{Z}(\mathbb{F}Ke) \xrightarrow{\omega_\vartheta} \mathbb{F}(\vartheta) = \mathbb{F}(\varphi) \xleftarrow{\omega_\varphi} \mathbf{Z}(\mathbb{F}Lf).$$
Every map here is an isomorphism and commutes with the action of H by Part (d) of Lemma 2.19. This completes the proof. □

2.24 Lemma. *Let e_φ and e_ϑ be the central idempotents corresponding to φ and ϑ, respectively. Then*
$$i = \sum_{\gamma \in \operatorname{Gal}(\mathbb{F}(\varphi)/\mathbb{F})} e_\varphi^\gamma e_\vartheta^\gamma$$
is a H-stable nonzero idempotent in $\mathbb{F}Ke$, and we have $ei = i = ie$ and $fi = i = if$.

Proof. Clearly, $e = \sum_{\gamma \in \mathrm{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})} e_\vartheta^\gamma$. Thus $ie = i = ie$ follows from $e_\vartheta^\gamma e_\vartheta^{\gamma'} = 0$ for $\gamma \neq \gamma' \in \mathrm{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$. A similar argument shows that $fi = i = if$. By assumption, $e_\varphi e_\vartheta \neq 0$ and thus $i \neq 0$, as $e_\vartheta i = e_\varphi e_\vartheta$. For $h \in H$, we have

$$i^h = \sum_\gamma e_\varphi^{h\gamma} e_\vartheta^{h\gamma} = \sum_\gamma e_\varphi^{\gamma_h\gamma} e_\vartheta^{\gamma_h\gamma} = i$$

as desired. □

As $\mathbb{F}Ke$ is simple, it follows that $\mathbb{F}Ki\mathbb{F}K = \mathbb{F}Ke$ and thus $i\mathbb{F}Ki$ and $\mathbb{F}Ke$ are Morita equivalent. We have $\mathbf{Z}(i\mathbb{F}Ki) \cong \mathbf{Z}(\mathbb{F}Ke) \cong \mathbb{F}(\vartheta) \cong \mathbf{Z}(\mathbb{F}Lf)$ as H-algebras.

2.25 Lemma. *Set $Z = \mathbf{Z}(i\mathbb{F}Ki)$ and let $S = (i\mathbb{F}Ki)^L$. Then S is a simple subalgebra of $i\mathbb{F}Ki$ with center Z, and dimension n^2 over Z, and*

$$\mathbf{C}_{i\mathbb{F}Ki}(S) = \mathbb{F}Li \cong \mathbb{F}Lf.$$

Proof. Set $i_0 = e_\varphi e_\vartheta$. The following diagram is commutative:

$$\begin{array}{ccccc}
\mathbb{F}Lf & \xrightarrow{\cdot i} & i\mathbb{F}Ki & \xrightarrow{\subseteq} & \mathbb{F}Ke \\
{\scriptstyle \cdot e_\varphi}\downarrow & & {\scriptstyle \cdot i_0}\downarrow & & {\scriptstyle \cdot e_\vartheta}\downarrow \\
\mathbb{F}(\vartheta)Le_\varphi & \xrightarrow{\cdot i_0} & i_0\mathbb{F}(\vartheta)Ki_0 & \xrightarrow{\subseteq} & \mathbb{F}(\vartheta)Ke_\vartheta.
\end{array}$$

By Lemma 1.10, its vertical maps are isomorphisms. The result now follows from the corresponding result when $\mathbb{F}$ contains the values of ϑ and φ, Lemma 2.2. □

As before, we have that $i\mathbb{F}Ki \cong S \otimes_Z \mathbb{F}Lf$. But now H may act nontrivially on Z, so Z is in general not in the center of $i\mathbb{F}Gi$.

2.26 Lemma. $\mathbf{Z}(i\mathbb{F}Gi) \cap i\mathbb{F}Ki = Z^H$.

Proof. Let $H = \dot{\bigcup}_{t \in T} Lt$. Then $i\mathbb{F}Gi = \bigoplus_{t \in T} i\mathbb{F}Kit$. Thus $\mathbf{C}_{i\mathbb{F}Ki}(i\mathbb{F}Gi) = Z^H$ as claimed. □

What we need is a subalgebra S_0 of S such that $\mathbf{Z}(S_0) = Z_0$ and $S = S_0 Z$. We now add to Hypothesis 2.22 the assumption that there is such a subalgebra S_0 in S. We emphasize that S_0 need not be invariant under H.

We have now collected all the ideas necessary to generalize Theorem 2.8 to the situation of Hypothesis 2.22. In particular, the reader will see that i and S are the right objects to work with (and not the idempotent ef, for example).

2.27 Lemma. *Suppose Hypothesis 2.22 and let $S_0 \subseteq S$ with $\mathbf{Z}(S_0) = Z_0$ and $S = S_0 Z$. Define $\varepsilon\colon H \to \operatorname{Aut} S$ by $(s_0 z)^{\varepsilon(x)} = s_0 z^x$. For any $x \in H/L$ there is $\sigma(x) \in S$ such that for every $s \in S_0$ we have $s^x = s^{\sigma(x)}$. For $x, y \in H/L$ we have*

$$\sigma(x)^{\varepsilon(y)}\sigma(y) = \alpha(x,y)\sigma(xy) \text{ for some } \alpha(x,y) \in Z^*,$$

and $\alpha \in Z^2(H/L, Z^*)$.

Proof. Let $x \in H$. The action of x on S, restricted to S_0, gives an injection $\mu\colon S_0 \to S$. We may extend μ uniquely to a Z-algebra automorphism $\widehat{\mu}$ of S. We apply the Skolem-Noether Theorem to μ to get $\sigma(x) \in S^*$ with $s\widehat{\mu} = s^{\sigma(x)}$ for all $s \in S$. Note that then for $s_0 \in S_0$ and $z \in Z$, we have $(s_0 z)^{\sigma(x)} = (s_0 z)\widehat{\mu} = s_0^x z$ and thus

$$(s_0 z)^x = s_0^x z^x = s_0^{\sigma(x)} z^x = (s_0 z^x)^{\sigma(x)} = (s_0 z)^{\varepsilon(x)\sigma(x)}.$$

Thus

$$s_0^{\sigma(xy)} = (s_0^x)^y = (s_0^{\sigma(x)})^y = (s_0^{\sigma(x)})^{\varepsilon(y)\sigma(y)} = s_0^{\sigma(x)^{\varepsilon(y)}\sigma(y)}.$$

Since $\mathbf{C}_S(S_0) = Z$, it follows that $\sigma(x)^{\varepsilon(y)}\sigma(y) = \alpha(x,y)\sigma(xy)$ for some $\alpha(x,y) \in Z^*$. Comparing

$$\left(\sigma(x)^{\varepsilon(y)}\sigma(y)\right)^{\varepsilon(z)}\sigma(z) = \alpha(x,y)^z\sigma(xy)^{\varepsilon(z)}\sigma(z) = \alpha(x,y)^z\alpha(xy,z)\sigma(xyz)$$

with

$$\sigma(x)^{\varepsilon(yz)}\left(\sigma(y)^{\varepsilon(z)}\sigma(z)\right) = \sigma(x)^{\varepsilon(yz)}\alpha(y,z)\sigma(yz) = \alpha(x,yz)\alpha(y,z)\sigma(xyz)$$

yields that $\alpha \in Z^2(H/L, Z^*)$. □

We may call $\sigma\colon H/L \to S$ a "crossed projective representation". Note that $S_0 = \mathbf{C}_S(\varepsilon(H))$, so that S_0 is determined by ε.

The image of α in $H^2(H/L, Z^*)$ *does* depend on the choice of S_0. Details can be found in the next subsection.

2.28 Definition. We call $\sigma\colon H/L \to S$ a *magic* (ε-) crossed representation for the configuration of Hypothesis 2.22, if

1. $\sigma(x)^{\varepsilon(y)}\sigma(y) = \sigma(xy)$ for all $x, y \in H/L$ and
2. $s^x = s^{\varepsilon(x)\sigma(x)}$ for all $x \in H/L$ and $s \in S$.

2.29 Remark. In particular, $\sigma(x)\sigma(y) = \sigma(xy)$ and $s^x = s^{\sigma(x)}$ for $x, y \in H_\varphi$. This looks like a magic representation of H_φ, but we defined magic representations only when the ground field contains the values of φ and ϑ. On the other side, we have the canonical isomorphism $\mathbb{F}Ke \to \mathbb{F}(\varphi)Ke_\vartheta$, $a \mapsto ae_\vartheta$. If $g \in G_\vartheta$, then obviously $(ae_\vartheta)^g = a^g e_\vartheta$. If $s \in S$, then $se_\vartheta = se_\varphi e_\vartheta \in e_\vartheta \mathbb{F}(\vartheta)Ke_\varphi e_\vartheta$. Thus $H_\varphi/L \ni x \mapsto \sigma(x)e_\vartheta$ is a magic representation in the sense of Definition 2.5.

Here is the main result of this section:

2.30 Theorem. *Assume Hypothesis 2.22 and let $\sigma\colon H/L \to S$ be a magic crossed representation, with respect to $S_0 \subseteq S$. Then $i\mathbb{F}Gi \cong S_0 \otimes_{Z_0} \mathbb{F}Hf$. If $S_0 \cong \mathbf{M}_n(Z_0)$, then $i\mathbb{F}Gi \cong \mathbf{M}_n(\mathbb{F}Hf)$ and $\mathbb{F}Ge$ and $\mathbb{F}Hf$ are Morita equivalent.*

Proof. All assertions follow from the first. Let $C = \mathbf{C}_{i\mathbb{F}Gi}(S_0)$. Then by Lemma 1.4 we have $i\mathbb{F}Gi \cong S_0 \otimes_{Z_0} C$ (Remember that $Z_0 \subseteq \mathbf{Z}(i\mathbb{F}Gi)$). We define a map κ from $\mathbb{F}H$ to C by extending the map $h \mapsto c_h = h\sigma(Lh)^{-1}$ linearly. It is easy to see that indeed $c_h \in C$. We compute

$$\begin{aligned} c_h c_g &= h\sigma(Lh)^{-1} g\sigma(Lg)^{-1} = hg\left(\sigma(Lh)^{-1}\right)^{\varepsilon(g)\sigma(Lg)} \sigma(Lg)^{-1} \\ &= hg\sigma(Lg)^{-1}\left(\sigma(Lh)^{\varepsilon(g)}\right)^{-1} = hg\left(\sigma(Lh)^{\varepsilon(g)}\sigma(Lg)\right)^{-1} \\ &= hg\sigma(Lhg)^{-1} = c_{hg}. \end{aligned}$$

Thus κ is an algebra homomorphism. From $\sigma(1)^{\varepsilon(1)}\sigma(1) = \sigma(1)$ we see that $\sigma(1) = 1_S = i$, and thus $\kappa(\mathbb{F}Lf) = \mathbb{F}Li$ and $\kappa(f') = 0$ for any other central idempotent f' of $\mathbb{F}L$. Let $H = \dot{\bigcup}_{t\in T} Lt$ be the partition in left cosets. Then

$$C = \bigoplus_{t\in T} C \cap \mathbb{F}Lt = \bigoplus_{t\in T} \mathbb{F}Lic_t.$$

Therefore κ maps $\mathbb{F}Lft$ onto $\mathbb{F}Lic_t$. It follows that κ induces an isomorphism from $\mathbb{F}Hf$ onto C as claimed. The proof is finished. □

As for magic representations, we have

2.31 Corollary. *Assume Hypothesis 2.22 with $\mathbb{F} \leqslant \mathbb{C}$. Every magic crossed representation σ defines a linear isometry $\iota = \iota(\sigma)$ from $\mathbb{C}[\operatorname{Irr}(G \mid \vartheta)]$ to $\mathbb{C}[\operatorname{Irr}(H \mid \varphi)]$ with Properties 1–8 from Theorem 2.8 (where S has to be replaced by S_0).*

Proof. By Hypothesis, $\mathbb{E} = \mathbb{F}(\sum_{g\in G}\vartheta^g) = \mathbb{F}(\sum_{h\in H}\varphi^h)$, and $\mathbb{F}(\chi)$ contains $\mathbb{E}$ for $\chi \in \operatorname{Irr}(G \mid \vartheta) \cup \operatorname{Irr}(H \mid \varphi)$. Also φ and ϑ remain semi-invariant over

$\mathbb{E}$. Let $e_0 = \sum_{\alpha\in\mathrm{Gal}(\mathbb{F}(\vartheta)/\mathbb{E})} e_\vartheta^\alpha$, then $\mathbb{F}Ge \cong \mathbb{E}Ge_0$ via multiplication with e_0. In a similar way, $\mathbb{F}Hf \cong \mathbb{E}Hf_0$, where $f_0 = \sum_{\alpha\in\mathrm{Gal}(\mathbb{F}(\vartheta)/\mathbb{E})} e_\varphi^\alpha$. Via these isomorphisms we may replace $\mathbb{F}$ by $\mathbb{E}$. It is thus no loss of generality to assume that $\mathbb{F} = \mathbb{E}$. Then $Z_0 \cong \mathbb{F}$, and we have $i\mathbb{F}Gi \cong S_0 \otimes_{\mathbb{F}} \mathbb{F}Hf$. The proof of Theorem 2.8 now carries over verbatim. □

For the next result, compare with the introduction to this section and Figure 2.3 on Page 19.

2.32 Proposition. *Assume Hypothesis 2.22 and let $\sigma\colon H/L \to S$ be a magic crossed representation. Let $j = e_\vartheta e_\varphi$ and $T = (j\mathbb{F}(\varphi)Kj)^L$. Define $\tau\colon H_\varphi/L \to T$ by $\tau(h) = \sigma(h)j$. Then τ is a magic representation for the configuration $(G_\vartheta, H_\varphi, K, L, \vartheta, \varphi)$ and for $\chi \in \mathrm{Irr}(G_\vartheta \mid \vartheta)$ we have*

$$(\chi^G)^{\iota(\sigma)} = (\chi^{\iota(\tau)})^H.$$

Proof. That τ is a magic representation was remarked earlier. Take $h \in H$. If $h \notin H_\varphi$, then all $\xi \in \mathrm{Irr}(H \mid \varphi)$ vanish on h, since every such ξ is induced from H_φ and $H_\varphi \trianglelefteq H$. So the equation holds in this case.

Now assume $h \in H_\varphi$. Choose $s_0 \in S_0$ with $\mathrm{tr}_{S_0/Z_0}(s_0) = 1 = \mathrm{tr}_{S/Z}(s_0)$. Then $\mathrm{tr}_{T/\mathbb{F}(\vartheta)}(s_0 j) = 1$. Note that for $h \in H_\varphi$ and $x \in H$, one has

$$\sigma(h^x) = \sigma(x^{-1})^{\varepsilon(hx)}\sigma(h)^{\varepsilon(x)}\sigma(x) = \sigma(x)^{-1}\sigma(h)^{\varepsilon(x)}\sigma(x) = \sigma(h)^x.$$

Also note that for $s \in S$ one has $js = jis = e_\varphi s = e_\vartheta s$. We get

$$\begin{aligned}
(\chi^{\iota(\tau)})^H(h) &= \sum_x \chi^{\iota(\tau)}(xhx^{-1}) \\
&= \sum_x \chi(s_0 j \sigma(xhx^{-1})^{-1} xhx^{-1}) \\
&= \sum_x \chi(s_0 e_\vartheta (\sigma(h)^{-1}h)^{x^{-1}}) \\
&= \sum_x \chi(s_0 (\sigma(h)^{-1}h)^{x^{-1}}) && \text{as } \chi \text{ lies over } \vartheta \\
&= \sum_x \chi(s_0^{\sigma(x^{-1})} (\sigma(h)^{-1}h)^{x^{-1}\sigma(x^{-1})}) && \text{as } \chi \text{ is a central form} \\
&= \sum_x \chi(s_0^{x^{-1}} (\sigma(h)^{-1}h)^{x^{-1}}) && \text{as } \sigma(h^{-1})h \text{ centralizes } S \\
&= \chi^G(s_0\sigma(h^{-1}h)) = (\chi^G)^{\iota(\sigma)}(h),
\end{aligned}$$

as desired. □

This result just means that we get the correspondence $\iota(\sigma)$ by composing the Clifford correspondences associated with ϑ and φ and a correspondence induced by a magic representation.

As in the invariant case, we have the following application:

2.33 Proposition. *Assume Hypothesis 2.22 with $(n, |H/L|) = 1$. Then there is a central simple Z_0-algebra $S_0 \subseteq S$ unique up to conjugacy in S, such that $i\mathbb{F}Gi \cong S_0 \otimes_{Z_0} \mathbb{F}Hf$.*

Proof. As Γ is a factor group of H/L, it follows that $|\Gamma|$ and $\dim_{\mathbb{F}(\varphi)} S$ are coprime. Thus by a result of Dade [10, Theorem 4.4] (see Proposition 2.36 below), there is $S_0 \subseteq S$ with $\mathbf{Z}(S_0) = Z_0$ and $S_0 Z = S$, and S_0 is unique up to inner automorphisms of S. By Lemma 2.27 there is an ε-crossed projective representation with factor set $\alpha \in Z^2(H/L, Z^*)$, say. But as n is coprime to $|H/L|$, it follows that the cohomology class of α is trivial. Thus there exists a magic crossed representation and the proposition follows from Theorem 2.30. □

Digression: Group Action on Simple Algebras

In the last section we introduced rather ad hoc the assumption that S is obtained by scalar extension from a central simple Z_0-algebra S_0. Here we consider the following questions:

1. When does such a subalgebra S_0 exist?
2. How many really different such subalgebras do exist?
3. How does the cocycle defined in Lemma 2.27 depend on the choice of S_0?

The results of this subsection will not be needed elsewhere in this thesis.

The following result is due to Hochschild [23, Lemma 1.2]:

2.34 Proposition. *Let S be a simple algebra with center Z and X a finite subgroup of the automorphism group of S with $X \cap \operatorname{Inn} S = \{\mathrm{Id}\}$. Then $S_0 = S^X$ is a simple algebra with center $Z_0 = Z^X$ and $S \cong S_0 \otimes_{Z_0} Z$.*

This gives the following corollary:

2.35 Corollary. *Let S be a simple algebra with center Z and let $Z_0 \leqslant Z$ be a subfield such that Z is a Galois extension of Z_0 with Galois group Γ. Then S contains a unitary central simple Z_0-algebra S_0 with $S \cong S_0 \otimes_{Z_0} Z$ if and only if the restriction-homomorphism $\operatorname{Aut}_{Z_0} S \to \Gamma$ is surjective and splits.*

Proof. If S_0 is such a subalgebra, then for $\tau \in \Gamma$ we may define $\varepsilon(\tau) \in \operatorname{Aut} S$ by $(s_0 \otimes z)^{\varepsilon(\tau)} = s_0 \otimes z^\tau$, and this gives a splitting $\varepsilon\colon \Gamma \to \operatorname{Aut}_{Z_0} S$. Conversely, if $\varepsilon\colon \Gamma \to \operatorname{Aut}_{Z_0} S$ with $\varepsilon(\tau)_{|Z} = \tau$ for all $\tau \in \Gamma$ is given, we may apply the result of Hochschild with $X = \operatorname{Im} \varepsilon$. □

Observe that both statements are trivially true if S is a matrix ring over Z. More generally, we have the following result, which can be found in papers of Dade [10, Theorem 4.4] and Schmid [57, Theorem 2]:

2.36 Proposition. *Let Z/Z_0 be a Galois extension with Galois group Γ and S a central simple Z-algebra such that $\operatorname{Aut}_{Z_0} S \ni \alpha \mapsto \alpha_{|Z} \in \Gamma$ is surjective. If the Schur index of S is prime to $|\Gamma|$, then the last homomorphism splits. If Γ is prime to $\dim_Z S$ then the splitting is unique up to conjugacy.*

This can be derived from Teichmüller's work on noncommutative Galois theory, as Schmid [57] has pointed out. (Teichmüller considered simple algebras such that $\operatorname{Aut}_{Z_0} S \to \operatorname{Gal}(Z/Z_0)$ is surjective in general. See [14] for an exposition and related results.)

This is all we have to say about the first question. Before considering the two other questions, we develop some general theory.

Let X be a group and S a simple artinian ring with center Z. Suppose that $\varepsilon\colon X \to \operatorname{Aut} S$ is a group homomorphism, that is, ε describes an action of X on S as ring. If X centralizes Z, so that X acts on S as Z-algebra, then every automorphism is inner by the Skolem-Noether Theorem, and we may choose $\sigma(x) \in S^*$ with $s^{\varepsilon(x)} = s^{\sigma(x)}$. Then the cohomology class of $f(x,y) = \sigma(xy)^{-1}\sigma(x)\sigma(y)$ is determined uniquely by ε. It follows that we may associate a unique element of $H^2(X, Z^*)$ to ε.

The situation is different if we drop the assumption that X centralizes Z. Then in general there is no unique way to associate a cohomology class to the action of X on S. The group action of X on S yields an action of X on Z, and we can form the cohomology group $H^2(X, Z^*)$ with respect to that action. We will consider different actions of X on S, but all will agree on Z^*, so that formulas like $H^2(X, Z^*)$, $Z^2(X, Z^*)$ will be unambiguous.

For the rest of this section, S is a central simple Z-algebra. Now suppose that $\varepsilon, \eta\colon X \to \operatorname{Aut} S$ are two group homomorphisms, which induce the same action of X on Z. This means that the compositions

$$X \underset{\eta}{\overset{\varepsilon}{\longrightarrow}} \operatorname{Aut} S \xrightarrow{\operatorname{Res}_Z} \operatorname{Aut} Z$$

are the same. We now show that an element of $H^2(X, Z^*)$ may be associated uniquely to the pair (η, ε). Since $\varepsilon(x)$ and $\eta(x)$ agree on $Z = \mathbf{Z}(S)$, $\varepsilon(x)^{-1}\eta(x)$ is a Z-algebra homomorphism, and thus by the Skolem-Noether Theorem inner. Therefore we may choose $u_x \in S^*$ with $s^{\eta(x)} = s^{\varepsilon(x)u_x}$ for all $s \in S$. From

$$s^{\varepsilon(xy)u_{xy}} = s^{\eta(x)\eta(y)} = s^{\varepsilon(x)u_x\varepsilon(y)u_y} = s^{\varepsilon(xy)u_x^{\varepsilon(y)}u_y}$$

it follows that $f(x,y) = u_x^{\varepsilon(y)}u_y u_{xy}^{-1} \in Z^*$. Comparing

$$(u_x^{\varepsilon(y)}u_y)^{\varepsilon(z)}u_z = f(x,y)^z u_{xy}^{\varepsilon(z)}u_z = f(x,y)^z f(xy,z)u_{xyz}$$

with

$$u_x^{\varepsilon(y)\varepsilon(z)}(u_y^{\varepsilon(z)}u_z) = u_x^{\varepsilon(yz)}f(y,z)u_{yz} = f(y,z)f(x,yz)u_{xyz}$$

shows that $f\colon X \times X \to Z^*$ is a 2-cocycle. (The reader may think of the map $x \mapsto u_x$ as a projective crossed representation with associated factor set f.)

2.37 Definition. We write $[\eta/\varepsilon]_S$ to denote the cohomology class of the cocycle f just constructed.

Before we show that $[\eta/\varepsilon]_S$ is well defined and give some simple properties, we review the concept of inflation in cohomology. Suppose $\gamma\colon G \to X$ is a group homomorphism. Then also G acts on Z^*. We have a natural map $\gamma^*\colon Z^2(X,Z^*) \to Z^2(G,Z^*)$, sending f to $f\gamma^*$ defined by $f\gamma^*(g,h) = f(\gamma(g),\gamma(h))$. γ^* sends cohomologous elements to cohomologous elements and thus induces a map $H^2(X,Z^*) \to H^2(G,Z^*)$ which we also call γ^*. (Note that in the situation above, $\ker\gamma$ acts trivial on Z^*. Inflation can be defined without this restriction [59, p. 124], but we will only need the above special case.)

2.38 Lemma. *Let $\varepsilon,\eta,\zeta\colon X \to \operatorname{Aut} S$ be group homomorphisms with $z^{\varepsilon(x)} = z^{\eta(x)} = z^{\zeta(x)}$ for all $x \in X$ and $z \in Z = \mathbf{Z}(S)$. The element $[\eta/\varepsilon]_S \in H^2(X,Z^*)$ only depends on the pair (η,ε). It has the following properties:*

1. *$[\zeta/\eta]_S[\eta/\varepsilon]_S = [\zeta/\varepsilon]_S$.*
2. *If η and ε are conjugate under $\operatorname{Inn} S$, then $[\eta/\varepsilon]_S = 1$.*
3. *Suppose $\gamma\colon G \to X$ is a group homomorphism. Then $[\eta\circ\gamma/\varepsilon\circ\gamma]_S = [\eta/\varepsilon]_S\gamma^*$.*
4. *$[\eta/\varepsilon]_S^n = 1$, where $n^2 = \dim_Z S$.*

Proof. The choice of the u_x in the construction above is unique up to scalars from Z, and thus $[\eta/\varepsilon]_S$ is independent of this choice.

To prove the first property, choose units v_x with $s^{\zeta(x)} = s^{\eta(x)v_x}$. Set $g(x,y) = v_x^{\eta(y)} v_y v_{xy}^{-1}$, so that g is a cocycle in $[\zeta/\eta]_S$. Now

$$\begin{aligned} h(x,y) :&= (u_x v_x)^{\varepsilon(y)}\, u_y v_y\, (u_{xy} v_{xy})^{-1} \\ &= u_x^{\varepsilon(y)} u_y v_x^{\varepsilon(y) u_y}\, v_y\, v_{xy}^{-1} u_{xy}^{-1} \\ &= u_x^{\varepsilon(y)} u_y\, v_x^{\eta(y)} v_y v_{xy}^{-1}\, u_{xy}^{-1} \\ &= f(x,y) g(x,y). \end{aligned}$$

As $s^{\zeta(x)} = s^{\varepsilon(x) u_x v_x}$, we see that the cohomology class of h is in $[\zeta/\varepsilon]_S$. Thus $[\zeta/\varepsilon]_S = [\zeta/\eta]_S [\eta/\varepsilon]_S$.

Now suppose η and ε are conjugate under $\operatorname{Inn} S$, so that for all $x \in X$ one has $\eta(x) = \kappa^{-1}\varepsilon(x)\kappa$, where κ is conjugation with some $u \in S^*$. Thus

$$s^{\eta(x)} = s^{u^{-1}\varepsilon(x) u} = s^{\varepsilon(x)[\varepsilon(x),u]}.$$

As $[\varepsilon(xy), u] = [\varepsilon(x), u]^{\varepsilon(y)} [\varepsilon(y), u]$, we see that $[\eta/\varepsilon]_S = 1$ as wanted.

For the third assertion, observe that for $g \in G$ we have

$$s^{\eta(\gamma(g))} = s^{\varepsilon(\gamma(g)) u_{\gamma(g)}}.$$

Thus

$$(f\gamma^*)(g,h) = f(\gamma(g), \gamma(h)) = u_{\gamma(g)}^{\varepsilon(\gamma(h))} u_{\gamma(h)} u_{\gamma(gh)}^{-1}$$

is a cocycle in $[\eta \circ \gamma / \varepsilon \circ \gamma]_S$ and the assertion follows.

To show the last assertion, we use the reduced norm of S over Z which we denote by $\operatorname{nr} = \operatorname{nr}_{S/Z}$. We have then

$$f(x,y)^n = \operatorname{nr}(f(x,y)\cdot 1_S) = \operatorname{nr}(u_x^{\varepsilon(y)} u_y u_{xy}^{-1}) = \operatorname{nr}(u_x)^{\varepsilon(y)} \operatorname{nr}(u_y) \operatorname{nr}(u_{xy})^{-1}.$$

Thus f^n is a coboundary, so that $[\eta/\varepsilon]_S^n = 1$. The proof is complete. □

Note that the cocycle in Lemma 2.27 associated with the subalgebra S_0 is just $[\kappa/\varepsilon]$, where $\kappa\colon H/L \to \operatorname{Aut} S$ is given by the conjugation action of H/L on S. (ε was defined in Lemma 2.27.)

Next we examine the uniqueness of the central simple Z_0-algebra S_0.

2.39 Lemma. *Let S be a simple algebra with center Z. Let $\Gamma \leqslant \operatorname{Aut} Z$ with $|\Gamma| < \infty$ and $Z_0 = Z^\Gamma$. Let ε and $\eta\colon \Gamma \to \operatorname{Aut}_{Z_0} S$ extend the action of Γ on Z. Equivalent are:*

(i) There is $\varphi \in \operatorname{Aut}_Z S$ such that $\eta(x) = \varepsilon(x)^\varphi$ for all $x \in \Gamma$.

(ii) $S^{\varepsilon(\Gamma)}$ and $S^{\eta(\Gamma)}$ are conjugate in S.

(iii) $S^{\varepsilon(\Gamma)}$ *and* $S^{\eta(\Gamma)}$ *are isomorphic as* Z_0*-algebras.*

Proof. Assume (i). By the Skolem-Noether Theorem, φ is conjugation with $u \in S^*$, say. Thus $s^{\eta(x)} = s^{u^{-1}\varepsilon(x)u}$ for all $s \in S$. It follows $S^{\eta(\Gamma)} = S^{\varepsilon(\Gamma)u}$, which is (ii). That (ii) implies (iii) is clear. Suppose $\alpha \colon S^{\varepsilon(\Gamma)} \to S^{\eta(\Gamma)}$ is an isomorphism. By Hochschild's result (Proposition 2.34) we have $S = S^{\varepsilon(\Gamma)}Z \cong S^{\varepsilon(\Gamma)} \otimes_{Z_0} Z$ and analogously for η, so that α can be extended uniquely to an Z-algebra automorphism of S which is inner by the Skolem-Noether Theorem. Thus there is $u \in S^*$ with $S^{\varepsilon(\Gamma)u} = S^{\eta(\Gamma)}$. Then for $s \in S^{\varepsilon(\Gamma)}$, $z \in Z$ and $x \in \Gamma$ we get

$$(s^u z)^{\eta(x)} = s^{u\eta(x)} z^x = s^u z^x = s^{\varepsilon(x)u} z^x = (s^u z)^{u^{-1}\varepsilon(x)u}.$$

As $S = S^{\varepsilon(\Gamma)u} Z$, it follows that $\eta = u^{-1}\varepsilon u$ as claimed. □

2.40 Corollary. *There is a bijection between the* Inn S*-conjugacy classes of homomorphisms* $\Gamma \to \mathrm{Aut}_{Z_0} S$ *splitting* $\mathrm{Aut}_{Z_0} S \to \Gamma$*, and the isomorphism classes of central simple* Z_0*-algebras* S_0 *that are contained in* S *such that* $S = S_0 Z$*.*

We may apply the construction preceding Definition 2.37 to the various splittings of $\mathrm{Aut}_{Z_0} S \to \Gamma$. We get a sharper result then.

2.41 Proposition. *Hold the assumptions above, and set*

$$\mathcal{H} = \{\eta \colon \Gamma \to \mathrm{Aut}_{Z_0} S \mid z^{\eta(x)} = z^x \text{ for all } z \in Z \text{ and } x \in \Gamma\}.$$

Let $\varepsilon \in \mathcal{H}$ *be fixed. Then* $\eta \mapsto [\eta/\varepsilon, S]$ *induces an* injective *map from the set of* Inn S*-conjugacy classes of* $\mathcal{H}$ *into* $H^2(\Gamma, Z^*)$*.*

Proof. We will need some general properties of cohomology theory. As Γ acts on S^*, Z^* and S^*/Z^* via ε, we can form $Z^1(\Gamma, S^*/Z^*)$ and $H^1(\Gamma, S^*/Z^*)$ with respect to this action [59, Chapitre VII, Annexe]. For the convenience of the reader we review the definitions here. Set, for the moment, $P = S^*/Z^*$. Then

$$Z^1(\Gamma, P) = \{u \colon \Gamma \to P \mid u_{xy} = u_x^{\varepsilon(y)} u_y \text{ for all } x, y \in \Gamma\}.$$

The elements of $Z^1(\Gamma, P)$ are called cocycles. Two cocycles u and v are said to be cohomologous, if there is $a \in P$ with $v_x = (a^{\varepsilon(x)})^{-1} u_x a$. Then $H^1(\Gamma, P)$ is the factor set of $Z^1(\Gamma, P)$ modulo this equivalence relation. The exact sequence

$$1 \longrightarrow Z^* \longrightarrow S^* \longrightarrow P \longrightarrow 1$$

yields an exact sequence

$$H^1(\Gamma, Z^*) \longrightarrow H^1(\Gamma, S^*) \longrightarrow H^1(\Gamma, P) \xrightarrow{\ \delta\ } H^2(\Gamma, Z^*).$$

The connecting homomorphism δ is defined as follows: Suppose $\bar{u} \in Z^1(\Gamma, P)$. We may lift $\bar{u}$ to some map $u\colon \Gamma \to S^*$. Then $u_x^{\varepsilon(y)} u_y = u_{xy} f(x, y)$ for some $f(x, y) \in Z^*$. We define $\delta(\bar{u}) = f B^2(\Gamma, Z^*)$. It is easily shown that $\delta(\bar{u})$ is independent of all choices made and that cohomologous cocycles are mapped to the same element. As here Γ is the Galois group of Z/Z_0, by a result of Galois cohomology, one has $H^1(\Gamma, S^*) = 1$ [59, p. 160, Exercice 2]. Thus $\delta\colon H^1(\Gamma, S^*/Z^*) \to H^2(\Gamma, Z^*)$ is an injection.

To show the proposition, remember the construction of $[\eta/\varepsilon]_S$: We chose units $u_x \in S^*$ with the property that $s^{\eta(x)} = s^{\varepsilon(x) u_x}$. The image $u_x Z^*$ in $P = S^*/Z^*$ is uniquely determined by this property. With the above notations at hand, we now see that $x \mapsto \bar{u}_x = u_x Z^*$ is a cocycle, that is, $\bar{u} \in H^1(\Gamma, P)$: Namely, we have $f(x, y) = u_{xy}^{-1} u_x^{\varepsilon(y)} u_y \in Z^*$. Moreover, we now see that $[\eta/\varepsilon]_S = \delta(\bar{u})$. As δ is injective, it suffices to show that if η and ζ yield cohomologous 1-cocycles $\bar{u}$ and $\bar{v}$, then ζ and η are conjugate in $\operatorname{Inn} S$. So, suppose there is $a \in P$ with $\bar{v}_x = (a^{\varepsilon(x)})^{-1} \bar{u}_x a$ for all $x \in \Gamma$. Then

$$s^{\zeta(x)} = s^{\varepsilon(x)(a^{\varepsilon(x)})^{-1} u_x a} = s^{a^{-1} \varepsilon(x) u_x a} = s^{a^{-1} \eta(x) a}.$$

This shows that ζ and η are conjugate under $\operatorname{Inn} S$. The proof is complete. □

Combing the last result with Lemma 2.38, we see that the cocycle defined in Lemma 2.27 is unique up to multiplication with elements of a certain subset of $H^2(\Gamma, Z^*)$, where here $\Gamma \cong H/H_\varphi$.

2.3 Clifford Extensions

In this section we review the theory of Clifford extensions. First suppose $N \trianglelefteq G$ and $\mu \in \operatorname{Irr} N$ is G-invariant. Suppose that $\mathbb{F} \subseteq \mathbb{C}$ is a field containing the values of μ. It is well known that μ determines an element in the cohomology group $H^2(G/N, \mathbb{F}^*)$. This can be defined as follows: Let $e = e_\mu$ be the central primitive idempotent of $\mathbb{F}N$ associated with μ. Let $C = (\mathbb{F}Ge)^N = \mathbf{C}_{\mathbb{F}Ge}(\mathbb{F}N)$ be the centralizer of $\mathbb{F}N$ in $\mathbb{F}Ge$. For $x \in G/N$ set $C_x = C \cap \mathbb{F}x$, where $\mathbb{F}x$ is the $\mathbb{F}$-subspace of $\mathbb{F}G$ generated by the elements of x. It can be shown that every C_x has dimension 1 over $\mathbb{F}$ and contains units of C. Thus

$$C = \bigoplus_{x \in G/N} C_x = \bigoplus_{x \in G/N} u_x \mathbb{F}$$

for some units $u_x \in C_x$. Then $u_x u_y = u_{xy} f(x, y)$ for some $f(x, y) \in \mathbb{F}$, and the cohomology class of f does not depend on the choice of the u_x.

We write $[\mu, \mathbb{F}]_{G/N}$ to denote the corresponding element of $H^2(G/N, \mathbb{F}^*)$. Of course, if $\mathbb{E} \geqslant \mathbb{F}$ is a field extension of $\mathbb{F}$, then $[\mu, \mathbb{E}]_{G/N}$ is the image of $[\mu, \mathbb{F}]_{G/N}$ under the natural map from $H^2(G/N, \mathbb{F}^*)$ to $H^2(G/N, \mathbb{E}^*)$. If the field is clear from context, we simply write $[\mu]_{G/N}$. A word of caution is maybe appropriate: The more classical way of associating factor sets to μ is the following: An $\mathbb{F}$-representation affording μ is extended to a projective representation of G. One can do this so that the factor set associated with the representation is constant on cosets of N and thus defines a factor set of G/N. The resulting cohomology class obtained thus is the *inverse* of $[\mu]_{G/N}$. Under our assumptions, there may be no $\mathbb{F}$-representation affording μ, but an approach working with $\mathbb{F}Ne$ itself still goes through: As $\mathbb{F}Ne$ is simple with center $\mathbb{F}$, there is $a_g \in \mathbb{F}Ne$ for every $g \in G$ such that $a^{a_g} = a^g$ for all $a \in \mathbb{F}Ne$. (This follows from the Skolem-Noether Theorem.) These units are unique up to multiplication with scalars, and thus $a_g a_h = a_{gh}\alpha(g,h)$ with $\alpha(g,h) \in \mathbb{F}^*$. Observe that then $a_g^{-1}g \in C_{Ng}$. A suitable choice of the a_g's ensures that $u_{Ng} = a_g^{-1}g$ is well defined. It is then easy to see that $\alpha = f^{-1}$.

The importance of $[\mu]_{G/N}$ lies in the fact that it determines to a large extent the character theory over μ. Namely, we have (see Lemma 1.4)

$$\mathbb{F}Ge = \mathbb{F}Ne \cdot C \cong \mathbb{F}Ne \otimes_{\mathbb{F}} C,$$

and C is a twisted group algebra which is determined by $[\mu]_{G/N}$ up to isomorphism. If there is an $\mathbb{F}$-representation affording μ, then $\mathbb{F}Ne \cong \mathbf{M}_{\mu(1)}(\mathbb{F})$ and thus even $\mathbb{F}Ge \cong \mathbf{M}_{\mu(1)}(C)$.

Let $\Omega = \bigcup_{x \in G/N} \mathbb{F}x \cap C^*$ be the set of graded units in C. The central extension

$$1 \longrightarrow \mathbb{F}^* \longrightarrow \Omega \longrightarrow G/N \longrightarrow 1$$

is called the Clifford extension [4] associated with (G, N, μ). We write $G_{\mathbb{F}}\langle\mu\rangle$ to denote it, and simply $G\langle\mu\rangle$ if the field is clear from context or algebraically closed. Let (G^o, N^o, μ^o) be another character triple. An isomorphism between the two Clifford extensions corresponding to these two character triples is a pair of isomorphisms $\kappa \colon G/N \to G^o/N^o$ and $\omega \colon \Omega \to \Omega^o$ such that

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{F}^* & \longrightarrow & \Omega & \longrightarrow & G/N & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\kappa} & & \\
1 & \longrightarrow & \mathbb{F}^* & \longrightarrow & \Omega^o & \longrightarrow & G^o/N^o & \longrightarrow & 1
\end{array}$$

is commutative. The following fact is well known and not difficult to prove: When an isomorphism $\kappa \colon G/N \to G^o/N^o$ is given, then there is an isomorphism of Clifford extensions from $G_{\mathbb{F}}\langle\mu\rangle$ to $G^o_{\mathbb{F}}\langle\mu^o\rangle$ if and only if $[\mu]_{G/N} =$

$\left([\mu^o]_{G^o/N^o}\right)\kappa^*$. Here $\kappa^*\colon H^2(G^o/N^o,\mathbb{F}^*) \to H^2(G/N,\mathbb{F}^*)$ is the isomorphism induced by κ as usual.

It is quite clear that if $o\colon G_{\mathbb{F}}\langle\mu\rangle \to G^o_{\mathbb{F}}\langle\mu^o\rangle$ is an isomorphism of Clifford extensions and $\mathbb{F} \leqslant \mathbb{E}$, then o extends uniquely to an isomorphism of Clifford extensions $G_{\mathbb{E}}\langle\mu\rangle \to G^o_{\mathbb{E}}\langle\mu^o\rangle$. (The converse is false.) An isomorphism of Clifford extensions determines various bijections, which we now describe.

Subgroups: If $N \leqslant H \leqslant G$, then there is a unique subgroup H^o with $N^o \leqslant H^o \leqslant G^o$ and $(H/N)\kappa = H^o/N^o$.

The twisted group algebra: Write $C = (\mathbb{F}Ge_\mu)^N$ as before and $C^o = (\mathbb{F}G^oe_{\mu^o})^{N^o}$. Then ω can be extended to an $\mathbb{F}$-algebra isomorphism $C \to C^o$, and this extension is unique. Abusing notation, we write o for this isomorphism. Now C is naturally G/N-graded, and we have $(C_x)^o = (C^o)_{x\kappa}$.

Characters: Let $N \leqslant H \leqslant G$, and let $\tau\colon \mathbb{F}He_\mu \to \mathbb{F}$ be a central form. Let $C_H = C \cap \mathbb{F}H$. Since $\mathbb{F}He_\mu \cong \mathbb{F}Ne_\mu \otimes_{\mathbb{F}} C_H$, and $\mathbb{F}Ne_\mu$ is central simple, we have a bijection between the central $\mathbb{F}$-forms on $\mathbb{F}He_\mu$ and those on C_H (Lemma 1.5). The algebra isomorphism of the last item restricts to an isomorphism $C_H \to (C^o)_{H^o} = (C_H)^o$, and induces an isomorphism from $\mathrm{ZF}(C_H,\mathbb{F})$ to $\mathrm{ZF}((C_H)^o,\mathbb{F})$. Again by Lemma 1.5, a canonical isomorphism $\mathrm{ZF}((C_H)^o,\mathbb{F}) \to \mathrm{ZF}(\mathbb{F}H^oe_{\mu^o},\mathbb{F})$ is given. The composition of these isomorphisms is again denoted o, so that now τ^o is a central form on $\mathbb{F}H^oe_{\mu^o}$. In different terms: There is a bijection from class functions of H lying over μ to the class functions of H^o lying over μ^o.

If $\mathbb{F} \leqslant \mathbb{E}$, then the extension of o to $G_{\mathbb{E}}\langle\mu\rangle$ defines maps $C \otimes_{\mathbb{F}} \mathbb{E} \to C^o \otimes_{\mathbb{F}} \mathbb{E}$ and $\mathrm{ZF}(\mathbb{E}He_\mu,\mathbb{E}) \to \mathrm{ZF}(\mathbb{E}H^oe_{\mu^o},\mathbb{E})$, and these are obtained from the above maps by scalar extension, as is to be expected. The following properties are well known.

2.42 Proposition. *Let $o\colon G_{\mathbb{F}}\langle\mu\rangle \to G_{\mathbb{F}}\langle\mu^o\rangle$ be an isomorphism of Clifford extensions. Let $N \leqslant K \leqslant H \leqslant G$, $\vartheta \in \mathbb{C}[\mathrm{Irr}(K \mid \mu)]$ and $\tau, \sigma \in \mathbb{C}[\mathrm{Irr}(H \mid \mu)]$. Then*

1. $(\tau+\sigma)^o = \tau^o + \sigma^o$.
2. $(\tau^o,\sigma^o)_{H^o} = (\tau,\sigma)_H$.
3. $(\tau_K)^o = (\tau^o)_K$.
4. $(\vartheta^H)^o = (\vartheta^o)^H$.

5. $(\beta\tau)^o = \beta^\kappa \tau^o$ *for* $\beta \in \mathbb{C}[\mathrm{Irr}(H/N)]$.

6. $\mathbb{F}(\tau^o) = \mathbb{F}(\tau)$.

The next result is also well known. Isaacs [27] proves a similar result for character triples instead of Clifford extensions. However, it should be remarked that Isaacs' notion of "Character triple isomorphism" is weaker than that of Clifford extension isomorphism. We include here a proof for the sake of completeness although it is in essence a translation of Isaacs' proof from character triple isomorphisms to Clifford extension isomorphisms. We work over an algebraically closed field, $\mathbb{C}$, say.

2.43 Proposition. *Let* (G, N, μ) *be a character triple. There exists a character triple* (G^o, N^o, μ^o) *such that*

(a) $G\langle\mu\rangle \cong G^o\langle\mu^o\rangle$,

(b) μ^o *is linear and faithful and*

(c) every coset of N^o *in* G^o *contains an element* x*, such that* $N \cap \langle x\rangle = 1$.

Proof. Let $C = (\mathbb{C}Ge_\mu)^N$ and $C_x = C \cap \mathbb{C}x$ as before. For each coset $x \in G/N$ choose a nonzero element $u_x \in C_x$, and make the choice such that for $k = \mathrm{ord}(x)$ one has $u_x^k = 1_C (= e_\mu)$. This can be done since the field is algebraically closed. As $C_x C_y = C_{xy}$, we have $u_x u_y = \alpha(x, y)u_{xy}$ for some $\alpha(x, y) \in \mathbb{C}^*$, and α is a factor set of G/N.

Let N^o be the subgroup of $\mathbb{C}^*$ generated by the values of α. We claim that N^o has finite order: Let $\delta(x)$ be the determinant of u_x acting by right multiplication on C. Then $\delta(x)^{|G/N|} = 1$. We have $\delta(x)\delta(y) = \alpha(x, y)^{|G/N|}\delta(xy)$ and it follows that $\alpha(x, y)^{|G/N|^2} = 1$. This proves the claim.

Set $G^o = \langle u_x \mid x \in G/N\rangle$. From the claim it follows that G^o has finite order and $G^o \cap \mathbb{F}^* = N^o$. Let μ^o be the inclusion $N^o \hookrightarrow \mathbb{F}^*$. It is clear that μ^o is a linear faithful character of N^o, and that $N^o \subseteq \mathbf{Z}(G^o)$. It is also clear that the Clifford extensions $G\langle\mu\rangle$ and $G^o\langle\mu^o\rangle$ are isomorphic, and that $G^o = \dot{\bigcup}_{x\in G/N} N^o u_x$, where $\mathrm{ord}(N^o u_x) = \mathrm{ord}(x) = \mathrm{ord}(u_x)$. □

The following result will be used later with $N = L$.

2.44 Proposition. *Let* $\mathcal{B}$ *be a basic configuration as in Hypothesis 2.1 and let* $N \leqslant L$ *with* $N \trianglelefteq G$. *Suppose that* $\mu \in \mathrm{Irr}\, N$ *is invariant in* G *and* $(\varphi_N, \mu) \neq 0$. *Let* $o\colon G\langle\mu\rangle \to G^o\langle\mu^o\rangle$ *be an isomorphism of Clifford extensions. Then* o *induces in a natural way a bijection between the magic representations for the basic configuration* $\mathcal{B}$ *and the magic representations for the configuration* $\mathcal{B}^o$. *For the corresponding magic characters* ψ *and* ψ^o *we have* $\psi^o = \psi^\kappa$.

Proof. Note that $S = (e_\varphi \mathbb{F} K e_\vartheta e_\varphi)^L \subseteq (\mathbb{F} G e_\mu)^N$. Let $x \in H$ and $c_x \in (\mathbb{F} G e_\mu)^N \cap \mathbb{F} N x$. As N centralizes S, we get $s^x = s^{c_x}$, if c_x is a unit. Suppose $\sigma \colon H/L \to S$ is magic. Then define $\hat{\sigma} \colon H^o/L^o \to S^o$ by $\hat{\sigma}(x^\kappa) = \sigma(x)^o$. It is clear that $\hat{\sigma}$ is multiplicative. Let $s \in S$ so that $s^o \in S^o$. Then

$$(s^o)^{\hat{\sigma}(x^\kappa)} = (s^o)^{\sigma(x)^o} = (s^{\sigma(x)})^o = (s^x)^o = (s^{c_x})^o = (s^o)^{(c_x)^o} = (s^o)^{x^\kappa},$$

as $(c_x)^o \in (\mathbb{F} G^o e_\mu)^N \cap \mathbb{F} x^\kappa$. This shows that $\hat{\sigma}$ is magic. The inverse of the Clifford extension isomorphism o defines in a similar way the inverse of the map $\sigma \mapsto \hat{\sigma}$. The part on the characters is clear. □

Chapter 3

Magic Representations and Discrete Valuation Rings

3.1 Reducing Magic Representations Modulo a Prime

We now turn our attention to group algebras over discrete valuation rings. In this section, we assume that in Hypothesis 2.1, $\mathbb{F}$ is a field with a discrete valuation $\nu\colon \mathbb{F} \to \mathbb{Z}$ and valuation ring R, and that the residue class field of R has characteristic $p > 0$. Moreover we assume that $e_\vartheta \in RK$ and $e_\varphi \in RL$.

If $S \cong \mathbf{M}_n(\mathbb{F})$ and if there is a magic representation σ, then $\mathbb{F}Ge_\vartheta$ and $\mathbb{F}He_\varphi$ are Morita equivalent. We want to show that then also RGe_ϑ and RHe_φ are Morita equivalent. There is a quite general result of Broué of this kind [2], but verifying the premises of Broué's result is nearly the same amount of work as proving the desired result directly.

We denote reduction modulo the maximal ideal of R by "$\overline{}$", thus $\overline{}\colon R \to \overline{R} = R/\operatorname{J}(R)$. We use the same symbol for its extension $\overline{}\colon RG \to \overline{R}G$. As in Hypothesis 2.1, set $i = e_\varphi e_\vartheta$.

3.1 Lemma. $RKe_\vartheta = RKiRK$.

Proof. As $\overline{R}K\overline{e_\vartheta}$ is central simple, we have $\overline{R}K\overline{e_\vartheta} = \overline{R}K\overline{i}\overline{R}K$ and thus $RKe_\vartheta = RKiRK + \operatorname{J}(R)RKe_\vartheta$. By Nakayama's lemma, the proof follows. □

It follows that RGe_ϑ and $iRGi$ are Morita equivalent. We now assume that a magic representation $\sigma\colon H/L \to S \cong \mathbf{M}_n(\mathbb{F})$ exists. We know that then $i\mathbb{F}Gi \cong \mathbf{M}_n(\mathbb{F}He_\varphi)$, and we want to show that the same is true if we replace $\mathbb{F}$ by R. We split the proof into two lemmas.

3.2 Lemma. *If $S \cong \mathbf{M}_n(\mathbb{F})$, then $\Sigma = (iRKi)^L \cong \mathbf{M}_n(R)$.*

Proof. Since $\overline{R}$ has finite characteristic, we have $\overline{R}K\overline{e_\vartheta} \cong \mathbf{M}_{\vartheta(1)}(\overline{R})$ and $\overline{R}L\overline{e_\varphi} \cong \mathbf{M}_{\varphi(1)}(\overline{R})$. Thus $\overline{\Sigma} = (\bar{i}\overline{R}K\bar{i})^L \cong \mathbf{M}_n(\overline{R})$ by Lemma 2.2 applied with $\overline{R}$ instead of $\mathbb{F}$. If R is complete, then we can lift the matrix units of $\overline{\Sigma}$ to matrix units of Σ and the result follows.

Otherwise, let $\widehat{\mathbb{F}}$ be the completion of $\mathbb{F}$ and let $\widehat{S} = (i\widehat{\mathbb{F}}Ki)^L \cong \widehat{\mathbb{F}} \otimes_{\mathbb{F}} S$ and $\widehat{\Sigma} = (i\widehat{R}Ki)^L \cong \widehat{R} \otimes_R \Sigma$. We have seen that $\widehat{\Sigma} \cong \mathbf{M}_n(\widehat{R})$. Thus it is a maximal $\widehat{R}$-order in $\widehat{S}$ [54, Theorem 8.7]. But then also Σ is a maximal R-order in S [54, Theorem 11.5]. Thus it follows that $\Sigma \cong \mathbf{M}_n(R)$ [54, Theorem 18.7]. □

Remark. If ϑ and φ are afforded by $\mathbb{F}K$- respective $\mathbb{F}L$-modules, one can give a proof independent of lifting idempotents (respective matrix units) and the theory of maximal orders: Just assume that $R \subseteq \mathbb{F}$ is a ring with quotient field $\mathbb{F}$, and such that every finitely generated torsionfree R-module is free. As before, $e_\vartheta \in RK$ and $e_\varphi \in RL$. Let V be a $\mathbb{F}K$-module affording ϑ and U an $\mathbb{F}L$-module affording φ. There are R-free nonzero submodules $M \leqslant V_{RK}$ and $N \leqslant U_{RL}$, by assumption on R. Let $\varepsilon\colon \mathbb{F}K \to \operatorname{End}_{\mathbb{F}} V = E$ be the corresponding representation and $\tau\colon E \to \mathbb{F}Ke_\vartheta$ be defined by

$$\alpha\tau = \frac{\vartheta(1)}{|K|} \sum_{k \in K} \operatorname{tr}_V(\alpha k \varepsilon) k^{-1}.$$

The representation ε makes E into a K-K-bimodule and a routine calculation shows that τ is a bimodule homomorphism. Since ε is surjective, τ is an algebra homomorphism from E to $\mathbb{F}K$. Now it is clear that $\varepsilon\tau$ is just multiplication with $e_\vartheta = 1_E\tau$, and that $\tau\varepsilon = \mathrm{Id}_E$. Let Λ be the subset of E mapping M into itself. Then $\Lambda \cong \operatorname{End}_R M \cong \mathbf{M}_{\vartheta(1)}(R)$, the first isomorphism being just restriction to M. (As $\Lambda\tau \subseteq RKe_\vartheta$, we see, by the way, that $RKe_\vartheta\varepsilon = \Lambda$.) Now $Me_\varphi = Mi$ is a direct summand of M and an RL-lattice affording $n\varphi$. Since $RLe_\varphi \cong \mathbf{M}_{\varphi(1)}(R)$, there is, up to isomorphism, only one RL-lattice affording φ, namely N. Thus $Me_\varphi \cong N^n$ as RL-modules. We have $\operatorname{End}_R(Me_\varphi) \cong e_\varphi\Lambda e_\varphi$ and $\operatorname{End}(Me_\varphi)_{RL} \cong (e_\varphi\Lambda e_\varphi)^L$ canonically. The last is isomorphic with $\mathbf{M}_n(R)$. Since $(iRKi)^L = \big((e_\varphi\Lambda e_\varphi)^L\big)\tau$, the result follows.

We return to our original assumption. As $\Sigma \cong \mathbf{M}_n(R)$, we have that $iRGi \cong \mathbf{M}_n(\Gamma)$, where $\Gamma = \mathbf{C}_{iRGi}(\Sigma)$. It is clear that $\Gamma = RG \cap C$, where $C = \mathbf{C}_{i\mathbb{F}Gi}(S)$. We have to show that the isomorphism of Theorem 2.6 sends RHe_φ onto Γ.

3.3 Lemma. *Keep the notation above and assume that $\sigma\colon H/L \to S$ is magic. Then the homomorphism κ of that theorem maps RHe_φ onto Γ.*

Proof. We first show that $\sigma(H) \subseteq \Sigma = (iRKi)^L$. Let $h \in H$. Then $\Sigma^h = \Sigma$, so conjugation with h or $\sigma(h)$ induces an automorphism of Σ. As Σ is a matrix ring over a local ring, this automorphism is inner [49, Theorem 2.4.8], and thus $\lambda\sigma(h) \in \Sigma^*$ for some $\lambda \in \mathbb{F}$. This means that also $(\lambda\sigma(h))^{-1} = \lambda^{-1}\sigma(h^{-1}) \in \Sigma^*$. As λ or λ^{-1} is in R, it follows that $\sigma(h^{-1})$ or $\sigma(h)$ is in Σ. But as σ is a representation and h has finite order, both are in Σ. Thus $\sigma(H) \subseteq \Sigma$ as claimed.

It follows that $a_h = h\sigma(Lh)^{-1} \in C \cap RG = \Gamma$. Therefore $(RHe_\varphi)\kappa \subseteq \Gamma$. As in the proof of Theorem 2.6, we see that $\Gamma = \bigoplus_{t\in T} \Gamma_1 a_t$, where $\Gamma_1 = \Gamma \cap RK$ and T is a set of representative of the cosets of L in H. Thus it suffices to show that $(RLe_\varphi)\kappa = \Gamma_1$. We know that $(RLe_\varphi)\kappa = RLi$ and that $\Gamma_1 = \mathbb{F}Li \cap RK$. But RLe_φ is a maximal R-order in $\mathbb{F}Le_\varphi$ (by Jacobinski's formula [54, Theorem 41.3], say, or for a splitting field $\mathbb{F}$ see the remark above). The same is thus true for RLi in $\mathbb{F}Li$. As $\mathbb{F}Li \cap RK$ is also an R-order in $\mathbb{F}Li$, it follows that $RLi = \mathbb{F}Li \cap RK$, as was to be shown. □

3.4 Remark. The conclusion that $\operatorname{Im}\sigma \subset (iRKi)^L$ holds for all rings R integrally closed in $\mathbb{F}$ (and with $i \in RK$), since such a ring is the intersection of the valuation rings of $\mathbb{F}$ containing it [41, p. 302].

From what we have done so far it follows that, the hypothesis above given, RHe_φ and RGe_ϑ are Morita equivalent, and the equivalence induces the character bijection of Theorem 2.8. The equivalence can be described more concretely: First, choose an primitive idempotent $j \in \Sigma = (iRKi)^L$. Then we have $jRGj = jiRGij \cong \Gamma \cong RHe_\varphi$, where an isomorphism from RHe_φ onto $jRGj$ is induced by the map sending $h \in H$ to $jh\sigma(Lh)^{-1} = h\sigma(Lh)^{-1}j = jh\sigma(Lh)^{-1}j$. Also we have $i = 1_S \in \Sigma j \Sigma$ and $e_\vartheta \in RGiRG = RGe_\vartheta$, so that $RGjRG = RGe_\vartheta$. The idempotent j is thus full in RGe_ϑ and we have a Morita equivalence between RGe_ϑ and $jRGj = jRGe_\vartheta j$ sending an RGe_ϑ-module V to Vj and an $jRGj$-module U to $U \otimes_{jRGj} jRG$ [40, Example 18.30]. Since $jRGj \cong RHe_\varphi$, this gives also an Morita equivalence between RGe_ϑ and RHe_φ. We now have proved:

3.5 Theorem. *Assume Hypothesis 2.1 and let $\sigma\colon H/L \to S$ be a magic representation. Furthermore, assume that $\nu\colon \mathbb{F} \to \mathbb{Z}$ is a discrete valuation of $\mathbb{F}$ with valuation ring R and residue class field of characteristic $p > 0$. Suppose that ϑ and φ have p-defect 0 and that $S \cong \mathbf{M}_n(\mathbb{F})$. Then there is an idempotent $j \in (iRKi)^L = \Sigma$ such that $RHe_\varphi \cong jRGj$ via the map defined by $h \mapsto jh\sigma(Lh)^{-1}$, and $RGjRG = RGe_\vartheta$. The rings RHe_φ and RGe_ϑ are Morita equivalent.*

We record some properties of the Morita equivalence:

3.6 Proposition. *Assumptions as before, and let U be a subgroup with $K \leqslant U \leqslant G$ and $V = H \cap U$. Let M be an RGe_ϑ-module, N an RUe_ϑ-module and X an $R[G/K]$-module. The category equivalence associated with j has the following additional properties:*

6. $Mj \otimes_R X \cong (M \otimes_R X)j$ *as RH-modules,*

7. $M_U j \cong (Mj)_V$ *as RV-modules,*

8. $(N^G)j \cong (Nj)^H$ *as RH-modules,*

9. $Me_\varphi \cong Mj \otimes_R j\Sigma$ *as RH-modules, where $j\Sigma$ is viewed as RH-module via σ.*

Proof. Maybe some explanations are appropriate: First, since $G/K \cong H/L$, any G/K-module can be view as H/L-module and vice versa, and also as G-module with K acting trivial respective as H-module with L acting trivial.

Second, if M and X are G-modules, the action of G on $M \otimes_R X$ is defined by $(m \otimes x)g = mg \otimes xg$, this is extended linearly to RG. If K acts trivial on X, then $(M \otimes_R X)e_\vartheta = (Me_\vartheta) \otimes_R X$: in fact, for *any* $a = \sum_{k \in K} r_k k \in RK$ we have $(m \otimes x)a = \sum_{k \in K} r_k(mk \otimes x) = ma \otimes x$. Thus tensoring with X over R maps RGe_ϑ-modules to RGe_ϑ-modules. An analogous statement holds for RHe_φ-modules and X viewed as H/L-module.

Remember that Mj is an RHe_φ-module via the homomorphism $\kappa\colon RH \to \Gamma$. To distinguish this action from the action of h on M coming from the RG-module-structure, we denote it by '$\circ$': thus $mj \circ h = mj\sigma(h)^{-1}h = m\sigma(h)^{-1}hj$.

We begin with Property 6. The injection $Mj \hookrightarrow M$ is split (as R-homomorphism) and thus induces an injection $\mu\colon Mj \otimes_R X \hookrightarrow M \otimes X$. (Maybe this is a good place to emphasize that we do not assume M or X to be R-free.) As $j \in RK$, the image of this injection is just $(M \otimes X)j$. We have to show that μ is compatible with the action of H. Let $m \in M$, $x \in X$ and $h \in H$. Then

$$\begin{aligned}((mj \otimes x)h)\mu = (mj \circ h \otimes xh)\mu = mj\sigma(h^{-1})h \otimes xh = (mj\sigma(h^{-1}) \otimes x)h \\ = (m \otimes x)j\sigma(h)^{-1}h = (mj \otimes x)\mu \circ h.\end{aligned}$$

Thus μ is H-linear. Property 6 is proved.

Property 7 is trivial. For Property 8 we can not argue as in the proof of Theorem 2.8, so we need to find another argument. Define a map

$$\varphi\colon Nj \otimes_{RV} RH \to N \otimes_{RU} RG \quad \text{by} \quad (n \otimes h)\varphi = n\sigma(h^{-1}) \otimes h$$

for $n \in Nj$ and $h \in H$. This is well defined: If $v \in V$ then

$$\begin{aligned}(n \otimes vh)\varphi &= n\sigma(h^{-1}v^{-1}) \otimes vh \\ &= n\sigma(h^{-1})\sigma(v^{-1})v \otimes h \\ &= n\sigma(v^{-1})v\sigma(h^{-1}) \otimes h \qquad (\text{as } \sigma(v^{-1})v \text{ centralizes } \sigma(h^{-1}) \in S) \\ &= \big((n \circ v)\sigma(h^{-1})\big) \otimes h \\ &= (n \circ v \otimes h)\varphi.\end{aligned}$$

Furthermore, $(Nj \otimes_{RV} RH)\varphi \subseteq (N \otimes_{RU} RG)j$:

$$\begin{aligned}(n \otimes h)\varphi j &= n\sigma(h^{-1}) \otimes hj\,h^{-1}h = n\sigma(h^{-1}) \otimes j^{\sigma(h^{-1})}h \\ &= n\sigma(h^{-1})j^{\sigma(h^{-1})} \otimes h = nj\sigma(h^{-1}) \otimes h \\ &= n\sigma(h^{-1}) \otimes h = (n \otimes h)\varphi\end{aligned}$$

as $n \in Nj$. Thus $\varphi\colon Nj \otimes_{RV} RH \to (N \otimes_{RU} RG)j$. The last is an RH-module via the '$\circ$'-action. We now show that φ is H-linear:

$$\begin{aligned}(n \otimes h)\varphi \circ \tilde{h} &= (n\sigma(h^{-1}) \otimes h)\sigma(\tilde{h}^{-1})\tilde{h} = n\sigma(h^{-1}) \otimes h\sigma(\tilde{h}^{-1})h^{-1}h\tilde{h} \\ &= n\sigma(h^{-1})\sigma(\tilde{h}^{-1})^{\sigma(h^{-1})} \otimes h\tilde{h} = n\sigma(\tilde{h}^{-1}h^{-1}) \otimes h\tilde{h} \\ &= (n \otimes h\tilde{h})\varphi.\end{aligned}$$

To show that φ is bijective we describe the inverse. Let $H = \dot{\bigcup}_{t\in T} Vt$. Then $G = \dot{\bigcup}_{t \in T} Ut$ and thus $N \otimes_{RU} RG = \bigoplus_{t\in T} N \otimes t$. Define

$$\psi\colon N \otimes_{RU} RG \to Nj \otimes_{RV} RH \text{ by } (\sum_{t\in T} n_t \otimes t)\psi = \sum_{t\in T} n_t\sigma(t)j \otimes t.$$

That $\varphi\psi = 1_{Nj\otimes_{RV}RH}$ is easy. In the other direction we have

$$(\sum_{t\in T} n_t \otimes t)\psi\varphi = \sum_{t\in T} n_t\sigma(t)j\sigma(t^{-1}) \otimes t = \sum_{t\in T} n_t \otimes j^{t^{-1}}t = (\sum_{t\in T} n_t \otimes t)j$$

and thus the restriction of ψ to $(N \otimes_{RU} RG)j$ is the inverse of φ. This shows that $Nj \otimes_{RV} RH \cong (N \otimes_{RU} RG)j$ and finishes the proof of Property 8.

For Property 9, remember that $\Sigma \cong \mathbf{M}_n(R)$. We may choose a full set of matrix units $\{E_{kl} \mid k, l = 1, \dots, n\}$ with $j = E_{11}$. Then $i = \sum_k E_{kk}$ and $j\Sigma = \bigoplus_{k=1}^n E_{1k}R$. Define a map $\mu\colon Mj \otimes_R j\Sigma \to Mi = Me_\varphi$ by $m \otimes s \mapsto ms$. If $\sum_k m_k \otimes E_{1k} \mapsto 0$, then $0 = \sum_k m_k E_{1k}E_{l1} = m_l E_{11} = m_l$, so μ is injective. If $m \in Me_\varphi$, then $m = mi = \sum_k mE_{k1}E_{1k} = (\sum_k mE_{k1} \otimes E_{1k})\mu$, so μ is surjective. Finally, let us see that μ is compatible with the H-action:

$$\begin{aligned}((mj \otimes js)h)\mu &= (mj \circ h \otimes js\sigma(h))\mu = mh\sigma(h)^{-1}js\sigma(h) = mjsh \\ &= (mj \otimes js)\mu h.\end{aligned}$$

The proof is finished. □

Remark. The author has been unable to decide wether the appropriate generalization of Property 10 in Theorem 2.8 holds.

3.2 Lifting Magic Representations

Assume that $\mathbb{F}$ is a field of characteristic $p > 0$, and R a discrete valuation ring with $R/\operatorname{J}(R) \cong \mathbb{F}$. We want to show that then a magic representation over $\mathbb{F}$ can be lifted to one over R, if certain additional conditions hold. First of all, we have to assume that R is complete. We do this from now on. Remember that then $R^* \cong \mathbb{F}^* \times (1 + \operatorname{J}(R))$ [59, Chap. II, Prop. 8]. We need the following lemma.

3.7 Lemma. *Let R be a complete valuation ring with residue class field of characteristic p and let X be a group. Let $\alpha \in Z^2(X, 1 + \operatorname{J}(R))$. Then the order of the cohomology class of α is a p-number (or ∞, if $|X|$ is not finite).*

Proof. It suffices to show that if the cohomology class of α has p'-order, then α is a coboundary. So suppose that α is such an element with order k a p'-number. Thus $\alpha(x,y)^k = \nu(x)\nu(y)\nu(xy)^{-1}$ for some $\nu\colon X \to R^*$. Let π a prime element of R. We can assume that $\nu(x) \in 1 + R\pi$, since $R^* = (R/R\pi)^* \times (1 + R\pi)$ and $\alpha(x,y) \in 1 + R\pi$ by assumption. We now construct a sequence of maps $\mu_n\colon X \to R^*$ $(n = 1, 2, \dots)$ such that

$$\begin{aligned} \alpha(x,y)\mu_n(xy) &\equiv \mu_n(x)\mu_n(y) \mod \pi^n \quad \text{and} \\ \mu_{n+1}(x) &\equiv \mu_n(x) \mod \pi^n \end{aligned}$$

for all n. Set $\mu_1(x) = 1_R$ for all $x \in X$. Choose $a, b \in \mathbb{Z}$ with $ak + bp = 1$. We define μ_n recursively by setting $\mu_{n+1}(x) = \nu(x)^a \mu_n(x)^{bp}$. Use induction to prove the above properties: By assumption $\alpha(x,y) \equiv 1 \mod \pi$ for all $x, y \in X$. Also $\mu_2(x) = \nu(x)^a \equiv 1 = \mu_1(x) \mod \pi$ since $\nu(x) \in 1 + R\pi$ by assumption. From $\mu_{n+1}(x) \equiv \mu_n(x) \mod \pi^n$ it follows $\mu_{n+1}(x)^p \equiv \mu_n(x)^p \mod \pi^{n+1}$ and thus

$$\begin{aligned} \mu_{n+2}(x) &= \nu(x)^a \mu_{n+1}(x)^{bp} \\ &\equiv \nu(x)^a \mu_n(x)^{bp} \mod \pi^{n+1} \\ &= \mu_{n+1}(x). \end{aligned}$$

Assuming that $\alpha(x,y) \equiv \mu_n(x)\mu_n(y)\mu_n(xy)^{-1} \mod \pi^n$ by induction, we get

$$\begin{aligned} \alpha(x,y) = \alpha(x,y)^{ak+bp} &= \nu(x)^a\nu(y)^a\nu(xy)^{-a}\alpha(x,y)^{bp} \\ &\equiv \nu(x)^a\nu(y)^a\nu(xy)^{-a}\mu_n(x)^{bp}\mu_n(y)^{bp}\mu_n(xy)^{-bp} \mod \pi^{n+1} \\ &= \mu_{n+1}(x)\mu_{n+1}(y)\mu_{n+1}(xy)^{-1}. \end{aligned}$$

Thus for $\mu(x) = \lim_{n\to\infty} \mu_n(x)$, we have $\alpha(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}$, as was to be shown. □

3.8 Theorem. *Assume Hypothesis 2.1 with $\mathbb{F}$ a field of characteristic $p > 0$ and R a complete discrete valuation ring with residue class field $\mathbb{F}$. Let $\bar{\ }\colon RG \to \mathbb{F}G$ be the natural ring epimorphism. Let $e_\vartheta \in RK$ and $e_\varphi \in RL$ be the central idempotents as usual and $i = e_\vartheta e_\varphi$. Suppose that there is a magic representation $\sigma\colon H/L \to S = (\bar{i}\mathbb{F}K\bar{i})^L$. Then for every p'-subgroup $V/L \leqslant H/L$ there is a magic representation $\widehat{\sigma}\colon V/L \to (iRKi)^L$ lifting $\sigma_{V/L}$. If $n = (\vartheta_L, \varphi) \not\equiv 0 \mod p$, then there is a magic representation $\widehat{\sigma}\colon H/L \to (iRKi)^L$ lifting σ.*

Proof. Since R is complete, we have $(iRKi)^L \cong \mathbf{M}_n(R)$ (see the proof of Lemma 3.2). Since $(iRKi)^L$ is a matrix ring over the local ring R, automorphisms of $(iRKi)^L$ are inner. Thus there is a projective representation $\widehat{\sigma}\colon H/L \to (iRKi)^L$ with $s^h = s^{\widehat{\sigma}(h)}$ for all $s \in (iRKi)^L$. We can choose $\widehat{\sigma}$ such that $\overline{\widehat{\sigma}(h)} = \sigma(h)$ for $h \in H$. Let $\alpha \in Z^2(H/L, R^*)$ be the cocycle associated with $\widehat{\sigma}$. Then α has values in $1 + \mathrm{J}(R)$, since σ is multiplicative. By Lemma 3.7, the cohomology class of α has p-order. In particular, $\alpha_{V/L} \sim 1$ for any p'-group V/L. Thus $\widehat{\sigma}_{V/L}$ is projectively equivalent with an ordinary representation. If $n \not\equiv 0 \mod p$, then it follows $\alpha \sim 1$, since the class of α has order dividing n. The proof is finished. □

3.3 Vertices and Defect Groups

In this section, R is either a complete discrete valuation ring or a field of characteristic $p > 0$, and we assume Hypothesis 2.1 for the quotient field of R with $e_\vartheta \in RK$ and $e_\varphi \in RL$. Moreover, we assume that there is a magic representation $\sigma\colon H/L \to (iRKi)^L \cong \mathbf{M}_n(R)$, and we let j be a primitive idempotent of $(iRKi)^L$.

3.9 Proposition. *Let M be an indecomposable RGe_ϑ-module. Let D be a vertex of M and E a vertex of Mj (as RH-module). Then $DK =_G EK$, $D \cap K = 1 = E \cap L$ and $D \cong E$. If K is a p'-group, then even $D =_G E$.*

Proof. Let b be the block of M. Then a defect group of b has trivial intersection with K, since b covers a block of defect zero of K, namely e_ϑ. Since a vertex of M is contained in a defect group of the block b, it follows $D \cap K = 1$, and for the same reason we have $E \cap L = 1$. Now let U be a minimal subgroup of G containing K such that M is U-projective (a "vertex modulo K"). From proposition 3.6, Property 8, we see that $V = U \cap H$ is a minimal subgroup containing L such that Mj is V-projective. We may assume that $D \leqslant U$ and $E \leqslant V$. But since clearly M is DK-projective, we have $U = DK$. In the same way we see that $V = EL$. Therefore $DK = EK$. As $D \cap K = 1 = E \cap K$, it follows $D \cong DK/K = EK/K \cong E$. If K is a p'-group, then D and E are conjugate in DK by the conjugacy part of the Schur-Zassenhaus Theorem. The proof is complete. □

As RGe_ϑ and RHe_φ are Morita-equivalent, there is a correspondence between blocks of G covering e_ϑ and blocks of H covering e_φ. Since every block contains modules with vertex a defect group of the block, we have the following corollary:

3.10 Corollary. *Assumptions as before, let b and c corresponding block idempotents of G and H, respectively. If D is a defect group of b and E a defect group of c, then $DK =_G EK$, $D \cap K = 1 = E \cap L$ and $D \cong E$. If K is a p'-group, then even $D =_G E$.*

3.11 Proposition. *In the situation of the last corollary, assume that there is $s \in \Sigma^H$ with $\operatorname{tr}_S(s)$ invertible in R. Then $D =_G E$.*

This assumption holds, if, for example, $(\vartheta_L, \varphi) \not\equiv 0 \mod p$. We can then take simply $s = i$.

Proof. First we show that $\mathrm{T}_L^K(s)$ is invertible in RKe_ϑ. From $\vartheta(\mathrm{T}_L^K(s)) = |K : L|\vartheta(s) = |K : L| \operatorname{tr}_S(s)\varphi(1)$ we see that $\nu_p(\vartheta(\mathrm{T}_L^K(s))) = \nu_p|K : L| + \nu_p(\operatorname{tr}_S(s)) + \nu_p|L| = \nu_p|K| = \nu_p(\vartheta(1))$, and since $(RKe_\vartheta)^K = Re_\vartheta$ it follows that $\mathrm{T}_L^K(s) = \lambda e_\vartheta$ with $\lambda \in R$ invertible. So we can and will assume that $e_\vartheta = \mathrm{T}_L^K(s) = \mathrm{T}_H^G(s)$ for some $s \in \Sigma^H$.

Next, observe that the isomorphism $\kappa\colon RHe_\varphi \to C = \mathbf{C}_{iRGi}(S)$ preserves conjugation with elements of H:

$$(x^h)\kappa = (x\kappa)^{h\kappa} = (x\kappa)^{\sigma(h^{-1})h} = (x\kappa)^h$$

since $x\kappa \in C$ centralizes $\sigma(h^{-1}) \in S$. So if $c = \mathrm{T}_E^H(x)$ for some $x \in (RHe_\varphi)^E$, then $c\kappa = \mathrm{T}_E^H(x\kappa)$ and $x\kappa \in C^E$. Since c and b correspond, we have $c\kappa = ibi = bi$. It follows

$$b = be_\vartheta = b\,\mathrm{T}_H^G(s) = \mathrm{T}_H^G(bis) = \mathrm{T}_H^G(\mathrm{T}_E^H(x\kappa)s) = \mathrm{T}_E^G(x\kappa s).$$

Thus E contains a defect group of b, and since these defect groups have the same order, equality follows. The proof is finished. □

Chapter 4

Character Fives

4.1 Good Elements

We now assume that $L \trianglelefteq G$ in Hypothesis 2.1. Then we can use a bilinear form, introduced by Dade and Isaacs [27], to obtain more information about the character ψ. Suppose that $\varphi \in \operatorname{Irr} L$ is invariant in G and let $\mathbb{F}$ be a field containing the values of φ. For every $g \in G$ we can choose a $c_g \in \mathbb{F}Le_\varphi$ such that $\alpha^g = \alpha^{c_g}$ for all $\alpha \in \mathbb{F}Le_\varphi$, since $\mathbb{F}Le_\varphi$ is central simple. If $x, y \in G$ with $[x, y] \in L$ then $[x, y]e_\varphi$ and $[c_x, c_y]$ induce the same action (by conjugation) on $\mathbb{F}Le_\varphi$ and so these elements differ by some scalar. We denote this scalar by $\langle x, y\rangle_\varphi \in \mathbb{F}$. So by definition,

$$\langle x, y\rangle_\varphi e_\varphi = [x, y][c_y, c_x].$$

It is easy to see that $\langle x, y\rangle_\varphi$ does not depend on the choice of c_x or c_y. Alternatively, given a representation $\rho\colon L \to \mathbf{M}_{\varphi(1)}(\mathbb{F})$ affording φ, choose $\gamma_g \in \mathbf{M}_{\varphi(1)}(\mathbb{F})$ with $\rho(l^g) = \rho(l)^{\gamma_g}$ for all $l \in L$ and define $\langle x, y\rangle_\varphi = \rho([x, y])[\gamma_y, \gamma_x]$. Since the restriction of ρ to $\mathbb{F}Le_\varphi$ is an isomorphism between $\mathbb{F}Le_\varphi$ and $\mathbf{M}_{\varphi(1)}(\mathbb{F})$, both definitions agree. From the first definition we see, however, that $\langle x, y\rangle_\varphi \in \mathbb{Q}(\varphi)$, while for the second we have to assume that $\mathbb{F}$ splits φ. On the other hand the second definition works for absolutely irreducible representations over fields of any characteristic.

In most of this work, φ will be fixed, and so we drop the index if no confusion can arise. The original definition is different, but from the definition given here it is easier to prove that $\langle\ ,\ \rangle$ is indeed a bilinear alternating form. (I learned this definition from Knörr.)

4.1 Lemma. *Let $g, x, x_1, x_2, y \in G$ with $[x, y], [x_i, y] \in L$ and $l_1, l_2 \in L$, and define $\langle x, y\rangle_\varphi = \langle x, y\rangle$ as above. Then*

1. *$\langle x, y\rangle$ does not depend on the choice of c_x or c_y.*

2. *$\langle x_1x_2, y\rangle = \langle x_1, y\rangle\langle x_2, y\rangle$.*

3. *$\langle y, x\rangle = \langle x, y\rangle^{-1}$.*

4. *$\langle xl_1, yl_2\rangle = \langle x, y\rangle$.*

5. *$\langle x^g, y^g\rangle = \langle x, y\rangle$.*

Proof. The first part follows since c_x is determined up to multiplication with a nonzero scalar. The rest can be verified by routine calculations using commutator identities. □

The definition given by Isaacs [27, p. 596] was from the next lemma for $H = \langle L, h\rangle$. It shows that the form can be computed using only characters.

4.2 Lemma. *Let $L \leqslant H \leqslant G$ and χ be a class function of H with all its irreducible constituents lying over φ. Let $h \in H$ and $g \in G$ with $[h, g] \in L$. Then $\chi(h^g) = \chi(h)\langle h, g\rangle$.*

Proof. We work in the subgroup $\langle L, h\rangle$ of H. Writing $\chi_{\langle L,h\rangle}$ as a linear combination of irreducible characters lying above φ, we see that it is no loss to assume that $H = \langle L, h\rangle$, and that χ is irreducible and extends φ. Let $\widehat{\rho}\colon H \to \mathbf{M}_{\varphi(1)}(\mathbb{C})$ be a representation affording χ that extends the representation ρ affording φ. We may choose $c_h = \widehat{\rho}(h)$. Then

$$\begin{aligned}\widehat{\rho}(h^g) &= \widehat{\rho}(h[h, g]) = \widehat{\rho}(h)\rho[h, g][c_g, c_h][c_h, c_g] = \widehat{\rho}(h)\langle h, g\rangle[\widehat{\rho}(h), c_g]\\ &= \widehat{\rho}(h)^{c_g}\langle h, g\rangle.\end{aligned}$$

Taking the trace yields the desired result. □

4.3 Definition. Let $H \leqslant G$ and $g \in G$. Then g is called H-φ-good if $\langle c, g\rangle_\varphi = 1$ for all $c \in \mathbf{C}_H(Lg)$. We drop φ if it is clear from context. We also drop H if $H = G$.

By Lemma 4.1, Part 4., g is (H-φ-) good if and only if any other element of Lg is. Also if g is H-good, then any H-conjugate of g is H-good. We can thus speak of good conjugacy-classes of G/L. Gallagher [19] has shown that $|\mathrm{Irr}(G \mid \varphi)|$ equals the number of φ-good conjugacy classes of G/L. From the last lemma we see that class functions belonging to $\mathbb{C}[\mathrm{Irr}(G \mid \varphi)]$ vanish on elements that are not good. Even more is true: there is a $\mathbb{C}$-basis of $\mathbb{C}[\mathrm{Irr}(G \mid \varphi)]$ where every basis-element vanishes on every but one conjugacy class of G/L.

A slightly different view on the form $\langle\ ,\ \rangle$ gives the following lemma.

4.4 Lemma. *Let $L \trianglelefteq G$ and $\varphi \in \operatorname{Irr} L$ invariant in G.*

1. $(\mathbb{F}Ge_\varphi)^L = \bigoplus_{x\in G/L} \mathbb{F}s_x$ *with units $s_x \in \mathbb{F}Le_\varphi x$.*

2. *If $[x,y] \in L$, then $\langle x,y\rangle_\varphi e_\varphi = [s_x, s_y] = s_x^{-1}s_x^y$.*

Proof. The first assertion means that $(\mathbb{F}Ge_\varphi)^L$ is a twisted group algebra of G/L over $\mathbb{F}$ and is well known. We get the s_x as follows: For $x \in G$ choose $c_x \in \mathbb{F}Le_\varphi$ with $a^x = a^{c_x}$ for all $a \in \mathbb{F}Le_\varphi$. Then $s_x = xc_x^{-1} = c_x^{-1}x \in (\mathbb{F}Ge_\varphi)^L \cap \mathbb{F}Le_\varphi x$. Suppose that $[x,y] \in L$. As c_y centralizes s_x, we have $s_x^{s_y} = s_x^y$ and

$$\begin{aligned}[s_x, s_y] = s_x^{-1}s_x^y = c_x x^{-1}x^y(c_x^{-1})^y = [x,y]c_x^{[c_x,c_y]}(c_x^{-1})^{c_y}\\ = [x,y][c_y,c_x] = \langle x,y\rangle_\varphi e_\varphi.\end{aligned}$$ □

It will sometimes be convenient to speak of H-φ-good elements even if φ is not H-invariant. We say that $g \in G_\varphi$ is H-φ-good, if $\langle c,g\rangle_\varphi = 1$ for all $c \in \mathbf{C}_H(Lg) \cap H_\varphi$.

4.5 Lemma. *Let $L \trianglelefteq G$ and $\varphi \in \operatorname{Irr} L$. Assume $g \in G_\varphi$ and $L \leqslant H \leqslant G$. Then g is H-φ-good if and only if g centralizes $(e_\varphi \mathbb{F}\,\mathbf{C}_H(Lg)e_\varphi)^L$.*

Proof. If $h \notin H_\varphi$, then $e_\varphi^h \neq e_\varphi$ and thus $e_\varphi^h e_\varphi = 0$. It follows $e_\varphi h e_\varphi = 0$. Therefore $(e_\varphi \mathbb{F}\,\mathbf{C}_H(Lg)e_\varphi)^L = (\mathbb{F}\,\mathbf{C}_{H_\varphi}(Lg)e_\varphi)^L$ and the result follows from Lemma 4.4. □

For later use, we state the following simple lemma [27, p. 600]:

4.6 Lemma. *Let $g \in G_\varphi$ and $K \geqslant L$. If g^m is K-good where $(m, |K/L|) = 1$ then g is K-good.*

Proof. For $c \in \mathbf{C}_K(Lg)$, we have $\langle c,g\rangle^{|K/L|} = \langle c^{|K/L|}, g\rangle = 1 = \langle c, g^m\rangle = \langle c,g\rangle^m$. Thus $\langle c,g\rangle = 1$. □

We return to the character correspondence of Theorem 2.8, but now we assume that (L,φ) is invariant in G. As we remarked earlier, if $\psi(h) = 0$ for some $h \in H$ then it is more difficult to compute the correspondent of a character. The next result shows that elements that are not K-good behave "bad" in this sense.

4.7 Lemma. *Assume Hypothesis 2.1 with $L \triangleleft G$ and φ invariant in G. Let σ be the projective representation of Lemma 2.4. If $h \in H$ is not K-φ-good, then $\operatorname{tr}(\sigma(h)) = 0$.*

Proof. We can assume that $G = \langle K, h\rangle$. Thus $H/L = \langle Lh\rangle$. Observe that then the projective representation $\sigma\colon H/L \to S$ is automatically equivalent to an ordinary representation since H/L is cyclic. We can thus assume that Theorem 2.8 applies (with $\mathbb{F} = \mathbb{C}$, say). Let ψ be the magic character. Observe that $\vartheta_L = n\varphi$ and thus $(\chi_H)_\varphi = \chi_H$ for class functions χ lying above ϑ. Let χ be an extension of ϑ to G. Then $\chi_H = \psi\xi$ for some extension ξ of φ. As h is not φ-good, $\chi(hx) = 0$ for all $x \in L$. Thus

$$0 = \sum_{x\in L} |\chi(hx)|^2 = \sum_{x\in L} |\psi(hx)\xi(hx)|^2 = |\psi(h)|^2 \sum_{x\in L} |\xi(hx)|^2 = |\psi(h)|^2|L|$$

where the last equation follows from a result of Gallagher [19, Lemma on p. 178], as $\xi_L = \varphi$ is irreducible. The result follows. □

In our still quite general setting it may happen that $\psi(h) = 0$ even if h is φ-good. For an example, let $G = GL(2,5)$, $K = SL(2,5)$, $L = \mathbf{Z}(K) = \{1, -1\}$ and $H = \langle L, h\rangle$ where h is the diagonal matrix with entries 2 and 1. Let φ be the faithful character of L and ϑ the character of degree 4 lying above φ. (The character table of $SL(2,5)$ can be found in [26, p. 140].) As φ is linear, we have $\langle x, y\rangle_\varphi = \varphi([x, y])$. Since h and $-h$ are not conjugate in G, we have $\mathbf{C}_{K/L}(h) = \mathbf{C}_K(h)/L$. It follows that h is φ-good. However, the extensions from ϑ to G vanish on h, while the extensions of φ to H do not vanish (they are linear). This forces $\psi(h) = 0$.

4.2 Generalities on Character Fives

Examples like the one at the end of the last section do not occur if $\operatorname{Irr}(K \mid \varphi) = \{\vartheta\}$, that is if φ is fully ramified in K. First we remind the reader of some easy and well known equivalent conditions for a character to be fully ramified. (The last condition is a consequence of the result of Gallagher [19] about $|\operatorname{Irr}(K \mid \varphi)|$ mentioned before.)

4.8 Lemma. *Let $L \trianglelefteq K$, $\varphi \in \operatorname{Irr} L$ and $\vartheta \in \operatorname{Irr}(K \mid \varphi)$. Then the following are equivalent:*

1. *$\vartheta_L = n\varphi$ with $n^2 = |K : L|$,*
2. *$\varphi^K = n\vartheta$ with $n^2 = |K : L|$,*
3. *φ is invariant in K and $\operatorname{Irr}(K \mid \varphi) = \{\vartheta\}$,*
4. *φ is invariant in K and ϑ vanishes outside L,*

5. $e_\varphi = e_\vartheta$,

6. *φ is invariant in K and $\{L\}$ is the only φ-good conjugacy class of K/L.*

If φ has these properties, we say that φ is fully ramified in K. We remark that Howlett and Isaacs [25] have proved, using the classification of finite simple groups, that K/L is solvable if some $\varphi \in \operatorname{Irr} L$ is fully ramified in K.

An interesting consequence of the last condition of the lemma is the following:

4.9 Corollary. *Suppose $\varphi \in \operatorname{Irr} L$ is fully ramified in K, where K/L is abelian. Let e be the exponent of K/L. Then $\mathbb{Q}(\varphi)$ contains a primitive e-th root of unity.*

Proof. The last condition of the lemma implies that $\langle\ ,\ \rangle_\varphi$ is a nondegenerate alternating form on $K/L \times K/L$. Since it has values in $\mathbb{Q}(\varphi)^*$, this enforces $\mathbb{Q}(\varphi)$ to contain a primitive e-th root of unity. □

The following definition describes the situation we will be concerned with in the rest of this chapter:

4.10 Definition. A *character five* is a quintuple $(G, K, L, \vartheta, \varphi)$ where G is a finite group, $L \leqslant K$ are normal subgroups of G, φ and ϑ are G-invariant irreducible characters of L and K respectively and $\vartheta_L = n\varphi$ where $n^2 = |K : L|$. An abelian (nilpotent, solvable) character five is a character five $(G, K, L, \vartheta, \varphi)$ with K/L abelian (nilpotent, solvable[1]).

The term *character five* is due to Isaacs [27], but observe that he defines a character five to be abelian, and he only considers character fives where K/L is abelian. Since some of our results are valid when K/L is not abelian, we drop the hypothesis of commutativity of K/L from the definition of a character five. We hope that this change of definition will not cause too much confusion.

The next result generalizes a result of Lewis [43, Theorem 4.3], which itself generalizes a former result of Isaacs [27, Theorem 3.2]. The proof given here is, however, completely different from their proofs, is shorter than Lewis' proof, and does not depend on these former results.

4.11 Theorem. *Let $(G, K, L, \vartheta, \varphi)$ be a character five and $g \in G$. Then the K-φ-good cosets of L contained in Kg are all conjugate under K.*

[1]Of course, by the before-mentioned result of Howlett and Isaacs [25], every character five is solvable.

Proof. We may assume $G = \langle K, g\rangle$. Let $x \in K$. Then xg is K-good if and only if it is good in G, since $\mathbf{C}_G(Lxg) = \mathbf{C}_K(Lxg)\langle xg\rangle$ and $\langle xg, xg\rangle = 1$. Let $\widehat{\vartheta} \in \mathrm{Irr}(G \mid \vartheta)$. Then $\widehat{\vartheta}$ is an extension of ϑ and $\sum_{x\in K}|\widehat{\vartheta}(xg^j)|^2 = |K|$ for every $j \in \mathbb{N}$ [19, Lemma on p.178]. Therefore $\widehat{\vartheta}(xg^j) \neq 0$ for some $x \in K$. But then the element xg^j is φ-good. Thus every coset of K contains at least one φ-good conjugacy class of G/L (more precisely, the pre-image of such a conjugacy class in G). The number of φ-good conjugacy classes of G/L is $|\mathrm{Irr}(G \mid \varphi)| = |\mathrm{Irr}(G \mid \vartheta)| = |G/K|$. It follows that every coset of K contains exactly one conjugacy class of φ-good cosets of L. In particular, the good cosets of L in the coset Kg are conjugate under G. Since $G = \langle K, xg\rangle$ for any $x \in K$, they are conjugate under K. □

Let $(G, K, L, \vartheta, \varphi)$ be a character five and $S = (e_\varphi \mathbb{F}K e_\vartheta e_\varphi)^L$, where $\mathbb{F}$ is a field containing the values of φ. As we saw earlier in the more general situation of Hypothesis 2.1, the group G/L acts on S and S is central simple, so that we can choose $\sigma(g) = \sigma(Lg) \in S$ with $s^g = s^{\sigma(g)}$ for all $s \in S$. This defines a projective representation from G/L into S. We now prove some simple facts about this projective representation. We keep the notation just introduced.

4.12 Lemma. *$S = (\mathbb{F}Ke_\varphi)^L = \bigoplus_{x\in K/L} \mathbb{F}s_x$ with units $s_x \in S \cap \mathbb{F}Le_\varphi x$. If $g \in G$ and $x \in \mathbf{C}_{K/L}(g)$ then $s_x^g = \langle x, g\rangle s_x$.*

Proof. Since φ is fully ramified in K, we have in fact $e_\vartheta = e_\varphi$. The rest of the lemma now follows from Lemma 4.4. □

The next result describes the character of σ.

4.13 Proposition. *Let $g \in G$ and choose $\sigma(g) \in S$ with $s^g = s^{\sigma(g)}$ for all $s \in S$. Then*

$$\mathrm{tr}(\sigma(g)^{-1})\,\mathrm{tr}(\sigma(g)) = \sum_{x\in\mathbf{C}_{K/L}(g)} \langle x, g\rangle = \begin{cases} |\mathbf{C}_{K/L}(g)| & \text{if } g \text{ is } K\text{-good} \\ 0 & \text{otherwise.}\end{cases}$$

Proof. The second equation is clear. Without loss of generality, we can assume that S splits, that is $S \cong \mathbf{M}_n(\mathbb{F})$. The $\mathbb{F}$-linear map κ from S to S sending s to $s^{\sigma(g)}$ has trace $\mathrm{tr}(\sigma(g)^{-1})\,\mathrm{tr}(\sigma(g))$, as an easy computation with matrix units shows. Now we use as basis of S the s_x, $x \in K/L$, of the last lemma. If $x \notin \mathbf{C}_{K/L}(g)$ then $(s_x)^g$ is a multiple of another basis element and so it contributes nothing to the trace of κ. If $x \in \mathbf{C}_{K/L}(g)$, the contribution is $\langle x, g\rangle$ by the last lemma. The result now follows. □

It is interesting that for a K-good element g we can give a $\sigma(g)$ in terms of the s_x explicitly. We use here an idea of Ward [66, Proposition 2.1].

4.14 Proposition. *Suppose $g \in G$ is K-good. Set*

$$\sigma(g) := \sum_{x \in K/L} s_x^{-1} s_x^g.$$

Then $\sigma(g)$ is an invertible element of S independent of the choice of the s_x and we have $s^g = s^{\sigma(g)}$ for all $s \in S$.

Proof. The s_x are unique up to scalars and so $\sigma(g)$ is independent of their choice.

From the definition of $\sigma(g)$ we get at once $s_y^{-1}\sigma(g)s_y^g = \sigma(g)$ for all $y \in K/L$. As the s_y form a basis of S, we have $\sigma(g)s^g = s\sigma(g)$ for all $s \in S$. It follows that $\sigma(g)\sigma(g^{-1}) \in \mathbf{Z}(S)$. Therefore $\sigma(g)\sigma(g^{-1}) = \lambda 1_S = \lambda s_1$ for some $\lambda \in \mathbb{F}$. We compute

$$\begin{aligned}
\lambda &= \sum_{x \in K/L} \sum_{\substack{y \in K/L \\ [x,g][y,g^{-1}]=1}} s_x^{-1} s_x^g s_y^{-1} s_y^{g^{-1}} = \sum_{\substack{x \in K/L \\ c \in \mathbf{C}_{K/L}(g)}} s_x^{-1} s_x^g s_{cx^g}^{-1} s_{cx^g}^{g^{-1}} \\
&= \sum_{\substack{x \in K/L \\ c \in \mathbf{C}_{K/L}(g)}} s_x^{-1} s_x^g (s_x^g)^{-1} s_c^{-1} s_c^{g^{-1}} s_x = \sum_{\substack{x \in K/L \\ c \in \mathbf{C}_{K/L}(g)}} s_x^{-1} \langle c, g^{-1} \rangle s_x \\
&= |K/L||\mathbf{C}_{K/L}(g)| \neq 0.
\end{aligned}$$

Thus $\sigma(g)$ is invertible. The proof is complete. □

If g is not K-good then by Theorem 4.11 there is $x \in K$ sucht that xg is K-good. Then we can define $\sigma(xg)$ as above, and set $\sigma(g) = s_x^{-1}\sigma(xg)$.

In view of Theorem 2.8 we are interested in the case where the projective representation is in fact a *linear* representation when restricted to a complement H/L of K/L, so that we have a magic representation.

We now prove some properties of magic characters and the associated character correspondences for character fives. The first restates part of Theorem 2.8.

4.15 Corollary. *Let $(G, K, L, \vartheta, \varphi)$ be a character five with a magic character ψ defined on H. Suppose $\chi \in \operatorname{Irr}(G \mid \vartheta)$ and $\xi \in \operatorname{Irr}(H \mid \varphi)$ correspond to each other under the bijection of Theorem 2.8. Then*

$$\chi_H = \psi\xi \text{ and } \xi^G = \overline{\psi}\chi.$$

Proof. Follows from Theorem 2.8, since now $(\chi_H)_\varphi = \chi_H$ and $(\xi^G)_\vartheta = \xi^G$. □

The following is an additional property of the character correspondence:

4.16 Proposition. *Let $(G, K, L, \vartheta, \varphi)$ be a character five where G is π-separable and K/L a π-group. Let ψ be a magic character with $\mathrm{ord}(\psi)$ a π-number, and ι the character correspondence of Theorem 2.8. Then $\chi \in \mathrm{Irr}(G \mid \vartheta)$ is π-special if and only if χ^{ι} is.*

We will need the following properties of π-special characters in the proof: Suppose that G is π-separable and $N \trianglelefteq G$. Then

1. If G/N is a π-group, $\zeta \in \mathrm{Irr}\, N$ and $\chi \in \mathrm{Irr}(G \mid \zeta)$, then χ is π-special if and only if ζ is π-special.

2. If G/N is a π'-group and $\zeta \in \mathrm{Irr}\, N$ is invariant in G and π-special, then the unique extension χ of ζ to G with $(|G/N|, \mathrm{ord}(\chi)) = 1$ is the only π-special character in $\mathrm{Irr}(G \mid \vartheta)$. Conversely, if G/N is a π'-group and $\chi \in \mathrm{Irr}\, G$ is π-special, then $\zeta = \chi_N$ is irreducible and π-special.

See [26, §40].

Proof of Proposition 4.16. By induction on $|G/K|$. If $G = K$ then the result is clear. Suppose $K < G$ and let $N \lhd G$ be a maximal normal subgroup of G containing K. Set $\xi = \chi^{\iota}$. Let ζ be a irreducible constituent of χ_N. Then $\tau = \zeta^{\iota}$ is a constituent of $\xi_{N\cap H} = (\chi^{\iota})_{N\cap H}$ by Property 7 in Theorem 2.8. By induction, ζ is π-special if and only if τ is. If G/N is a π-group the result follows. Suppose that G/N is a π'-group. From $\chi_H = \psi\xi$ we get by computing determinants

$$\det(\chi_H) = (\det\psi)^{\xi(1)}(\det\xi)^{\psi(1)}.$$

From this formula it follows that $\mathrm{ord}(\xi)$ is a π-number if and only if $\mathrm{ord}(\chi_H)$ is a π-number: remember that $\mathrm{ord}(\psi)$ and $\psi(1)$ are π-numbers. As $|G : H|$ is also a π-number, $\mathrm{ord}(\chi_H)$ is a π-number if and only if $\mathrm{ord}(\chi)$ is a π-number. Thus ξ has π-order if and only if χ has π-order.

If now χ is π-special, it has π-order. Also $\zeta = \chi_N$ is π-special. By induction, τ is π-special. We can conclude that ξ is the unique extension of τ that has π-order and as such it is π-special. The same argument works in the other direction. □

4.17 Proposition. *Assume that the magic character ψ has π-order, where π is the set of prime divisors of $|K/L|$. If $\mathrm{ord}(hL)$ is π', then $\psi(h)$ is rational.*

Proof. By Propositions 2.43 and 2.44 we can assume that $L \subseteq \mathbf{Z}(G)$ and that $\varphi \in \operatorname{Lin} L$. Now let P be a Hall π-subgroup of K and set $Q = P \cap L$. As $K/\mathbf{Z}(K)$ is a π-group, such a Hall subgroup exists and is characteristic in K, thus normal in G. As $P\,\mathbf{Z}(K) = K$, ϑ_P is irreducible. It follows that

$$\mathbb{C}Pe_{\vartheta_P} \ni a \mapsto ae_\vartheta \in \mathbb{C}Ke_\vartheta$$

is an isomorphism. Let $\mathbb{F}$ be the field that is obtained by adjoining a $|P|$-th root of unity to $\mathbb{Q}$, and set $T = \mathbb{F}Pe_{\vartheta_P}$. Since $\mathbb{F}$ is a splitting field for P, we have $T \cong \mathbf{M}_n(\mathbb{F})$. Let σ be a magic representation with character ψ. Now conjugation with $\sigma(h)$, $h \in H$, leaves $T \subset S$ invariant. Thus there is $\lambda \in \mathbb{C}$ with $\lambda\sigma(h) \in T$. Let $d = \operatorname{ord}(hL)$. Then $(\lambda\sigma(h))^d = \lambda^d 1_T \in T$, that is $\lambda^d \in \mathbb{F}$. Also $\det(\lambda\sigma(h)) = \lambda^n \det(\sigma(h)) \in \mathbb{F}$. If d is π' then $\det(\sigma(h)) = 1$ and so $\lambda^n \in \mathbb{F}$. Since $(d,n) = 1$ we get $\lambda \in \mathbb{F}$. It follows that $\sigma(h) \in T \cong \mathbf{M}_n(\mathbb{F})$ and $\psi(h) \in \mathbb{F}$. But the eigenvalues of $\sigma(h)$ lie in a field $\mathbb{E}$ obtained by adjoining a primitive d-th root of unity to $\mathbb{Q}$. Again from $(d, |P|) = 1$ we get $\mathbb{F} \cap \mathbb{E} = \mathbb{Q}$ [41, Corollary on p. 204]. Thus $\psi(h) \in \mathbb{Q}$ as claimed. □

4.18 Remark. Actually, we have shown more than stated in the proposition: if $U/L \leqslant H/L$ is coprime with K/L, then σ_U is afforded by an $\mathbb{F}$-representation.

4.3 Coprime Character Fives

When does a character five admit a magic representation? In the next few sections we give some sufficient conditions. The first one is essentially Proposition 2.15 applied to character fives.

4.19 Proposition. *Let $(G, K, L, \vartheta, \varphi)$ be a coprime character five (that is, $(|G : K|, |K : L|) = 1$). Then there is $H \leqslant G$ with $G = HK$ and $H \cap K = L$ and a rational magic character ψ of H/L vanishing nowhere such that the equation $\chi_H = \psi\xi$ defines an isometry between $CF(G \mid \vartheta)$ and $CF(H \mid \varphi)$.*

Proof. By the Schur-Zassenhaus Theorem, there is a complement H/L of K/L in G/L. Apply Proposition 2.15 to get a magic character ψ. By Lemma 4.6, every $h \in H$ is K-good and thus $\psi(h) \neq 0$ for all $h \in H$ by Proposition 4.13. We can choose a ψ with $\det \psi = 1$. By Proposition 4.17, this ψ is rational. The character correspondence of Theorem 2.8 is determined by the equation $\chi_H = \psi\xi$ since ψ has no zeros. □

4.20 Remark. Suppose $x \in H/L$ has order p^r where p is a prime. Let $\omega \in \mathbb{C}$ be a primitive p^r-th root of 1. Then it is well known that $\omega - 1 \in \mathfrak{P}$ where

$\mathfrak{P}$ is the prime ideal of $\mathbb{Z}[\omega]$ with $\mathfrak{P} \cap \mathbb{Z} = p\mathbb{Z}$. It follows that $\psi(x) \equiv \psi(1) \mod \mathfrak{P}$. This holds for any character and is well known. Since here $\psi(x)$ is rational, we even have that $\psi(x) \equiv \psi(1) \mod p$. If p is an odd prime, then $\psi(x)$ is completely determined by the two conditions

$$\psi(x)^2 = |\mathbf{C}_{K/L}(x)| \text{ and } \psi(x) \equiv n \mod p.$$

We emphasize that we need only the character ψ to compute the correspondence: The correspondent of $\chi \in \operatorname{Irr}(G \mid \vartheta)$ is $(1/\psi)\chi_H$ and the correspondent of $\xi \in \operatorname{Irr}(H \mid \varphi)$ is $(1/\overline{\psi})\xi^G$. If $|K/L|$ is odd, even more can be said.

4.21 Corollary. *In the situation of Proposition 4.19 assume that $|K/L|$ is odd. Let H/L be a complement of K/L in G/L and ψ the unique magic character with* $\det\psi = 1$*. Then for every $U/L \leqslant H/L$ with $|U/L|$ odd, 1_U is the unique constituent of ψ_U with odd multiplicity.*

Proof. By Proposition 4.17 we know that the magic character ψ with $\det\psi = 1$ is rational. By Proposition 4.13, $|\psi(h)|^2 = |\mathbf{C}_{K/L}(h)|$ for all $h \in H$. Since $|K/L|$ is odd, $\psi(h) \in \mathbb{Z}$ is odd for all $h \in H$. For $U \leqslant H$ with $|U/L|$ odd, let $\beta = \psi_U - 1_U$, a (generalized) character of U with $L \leqslant \ker\beta$. For $\tau \in \operatorname{Irr}(U/L)$ we have

$$|U/L|(\beta,\tau)_{U/L} = \sum_{u \in U/L} \beta(u)\overline{\tau(u)} \in 2\mathbb{Z}$$

since $\beta(u)$ is even for all $u \in U$. As $|U/L|$ is odd, we conclude that $(\beta,\tau)_U$ is even. Thus every $\tau \in \operatorname{Irr}(U/L)$ occurs with even multiplicity in β. Thus 1_U occurs with odd multiplicity in $\psi = 1_U + \beta$, while all other constituents occur with even multiplicity, as claimed. □

We remark that in the proof we have shown that β can be divided by 2. For this we could have appealed to a more general result of Knörr [39, Proposition 1.1(iii)], but for the convenience of the reader we have repeated the simple argument here.

The following result includes two related results of Lewis [43, 44, Theorem A in both]:

4.22 Corollary. *Let $(G,K,L,\vartheta,\varphi)$ be a coprime character five with $|G : L|$ odd and H/L a complement of K/L in G/L. In the bijection of Proposition 4.19, $\chi \in \operatorname{Irr}(G \mid \vartheta)$ and $\xi \in \operatorname{Irr}(H \mid \varphi)$ correspond if and only if (χ_H, ξ) is odd.*

Proof. From the last result follows that $\psi = 1 + 2\gamma$ for some character γ of H/L. From $\chi_H = \psi\xi$ we get $\chi_H = \xi + 2\gamma\xi$. Thus ξ is the only constituent of χ_H occurring with odd multiplicity. □

In the next section we will see that we can remove the hypothesis of coprimeness when we add the hypothesis that K/L is abelian (and odd).

4.4 Odd Abelian Character Fives

The main goal of this section is to give alternative proofs of some results due to Isaacs [27].

4.23 Theorem. *Let $(G, K, L, \vartheta, \varphi)$ be an odd abelian character five*[2] *and $\mathbb{F}$ a field containing the values of φ. Then there is $H \leqslant G$ with $G = HK$ and $L = H \cap K$, such that every element of H is K-good, and a magic representation $\sigma\colon H/L \to (\mathbb{F}Ke_\varphi)^L$.*

We will first prove the theorem for an algebraically closed field. Then we will show that there is a canonical magic character, and finally we will compute the field of values of this canonical character. As this field of values happens to be contained in $\mathbb{Q}(\varphi)$, we will be able to conclude that the magic representation belonging to the canonical magic character has image in $(\mathbb{F}Ke_\varphi)^L$.

We will need the following proposition in the proof.

4.24 Proposition. *Let $(G, K, L, \vartheta, \varphi)$ be a character five. Assume that there is $\tau \in \operatorname{Aut} G$ that leaves invariant K, L and ϑ, and that $\mathbf{C}_{K/L}(\tau) = 1$. Set $U/L = \mathbf{C}_{G/L}(\tau)$. Then every element of U is K-good and there is a magic representation $\sigma\colon U/L \to S = (\mathbb{C}Ke_\varphi)^L$.*

Proof. First we show that U contains only K-good elements. Let $u \in U$ and $C = \mathbf{C}_{K/L}(u)$. As $(Lu)^\tau = Lu$, it follows that $C^\tau = C$. From $\mathbf{C}_C(\tau) = 1$ we get $[C, \tau] = C$. Thus every element in C has the form $c^{-1}c^\tau$ for some $c \in C$. We have

$$\langle c^{-1}c^\tau, u\rangle_\varphi = \langle c, u\rangle_\varphi^{-1}\langle c^\tau, u\rangle_\varphi = \langle c, u\rangle_\varphi^{-1}\langle c^\tau, u^\tau\rangle_\varphi = \langle c, u\rangle_\varphi^{-1}\langle c, u\rangle_\varphi = 1$$

since $Lu^\tau = Lu$ and $\langle c^\tau, u^\tau\rangle_\varphi = \langle c, u\rangle_\varphi$. It follows that u is K-good. (This is in fact exactly the same proof as that of Isaacs [27, Lemma 3.7] for abelian K/L.)

We now work towards the existence of a magic representation. For convenience, we let notation be as if $\tau \in G$. This is no loss of generality, since we can work in the semidirect product $\langle\tau\rangle \ltimes G$. As S is isomorphic to a matrix ring, we can choose, for every $Lu \in U/L$, a $\widetilde{\sigma}(Lu) \in S$ with $s^u = s^{\widetilde{\sigma}(Lu)}$ for

[2] This means that K/L is abelian of odd order

all $s \in S$. We have then $\widetilde{\sigma}(Lu)\widetilde{\sigma}(Lv) = \alpha(Lu, Lv)\widetilde{\sigma}(Luv)$ for some factor set α of U/L. Let $k = \operatorname{ord}(\tau)$. Then $\widetilde{\sigma}(\tau)^k$ centralizes S, and thus is some scalar $a \in \mathbb{C}$. Let α be a k-th root of a and set $t = \alpha^{-1}\widetilde{\sigma}(\tau)$.

Let V be the simple S-module (unique up to isomorphism). We claim that t has the following properties:

1. $t^k = 1$,
2. $|\operatorname{tr}_V t| = 1$,
3. the eigenspaces of t are invariant under $\widetilde{\sigma}(u)$ for all $u \in U$.

That $t^k = 1$ is clear from the definition. As $t^k = 1$, we have

$$|\operatorname{tr}(t)|^2 = \operatorname{tr}(t^{-1})\operatorname{tr}(t) = |\mathbf{C}_{K/L}(\tau)| = 1,$$

where the second equation follows from Proposition 4.13. To show that the eigenspaces of t are $\widetilde{\sigma}(U)$-invariant, it suffices to show that $\widetilde{\sigma}(u)t = t\widetilde{\sigma}(u)$ for all $u \in U$. As $(Lu)^\tau = Lu$, we have $\widetilde{\sigma}(u)^t = \lambda\widetilde{\sigma}(u)$ for a scalar λ. But $\operatorname{tr}(\widetilde{\sigma}(u)) \neq 0$ since u is good. Therefore $\lambda = 1$ and $\widetilde{\sigma}(u)t = t\widetilde{\sigma}(u)$. The claim is shown.

Now let $\varepsilon_1, \ldots, \varepsilon_r$ be the different eigenvectors of t. Let $V_j = \{v \in V \mid vt = \varepsilon_j v\}$ be the space of eigenvectors of t with eigenvalue ε_j. Then V is the direct sum of the V_j (as $t^k = 1$). Set $d_j = \dim V_j$ and $\delta_j(u) = \det(\widetilde{\sigma}(u)$ on $V_j)$. From $\widetilde{\sigma}(u)\widetilde{\sigma}(v) = \alpha(u,v)\widetilde{\sigma}(uv)$ we get $\delta_j(u)\delta_j(v) = \alpha(u,v)^{d_j}\delta_j(uv)$. As $\operatorname{tr}(t) = \sum_j d_j\varepsilon_j$ is an algebraic integer of norm 1, it follows that the greatest common divisor of the d_j in $\mathbb{Z}$ is 1. Choose $k_j \in \mathbb{Z}$ with $\sum_j k_j d_j = -1$ and put

$$\sigma(u) = \left(\prod_{j=1}^r \delta_j(u)^{k_j}\right)\widetilde{\sigma}(u).$$

Then

$$\begin{aligned}
\sigma(u)\sigma(v) &= \left(\prod_{j=1}^r \delta_j(u)^{k_j}\right)\widetilde{\sigma}(u)\left(\prod_{j=1}^r \delta_j(v)^{k_j}\right)\widetilde{\sigma}(v)\\
&= \left(\prod_{j=1}^r \left(\delta_j(uv)\alpha(u,v)^{d_j}\right)^{k_j}\right)\alpha(u,v)\widetilde{\sigma}(uv)\\
&= \left(\prod_{j=1}^r \delta_j(uv)^{k_j}\right)\widetilde{\sigma}(uv) = \sigma(uv).
\end{aligned}$$

Thus σ is the required magic representation. □

4.25 Remark. The proposition is also true for every field $\mathbb{F}$ containing a primitive k-th root of unity, and such that $(\mathbb{F}Ke_\varphi)^L \cong \mathbf{M}_n(\mathbb{F})$. (Here as before $k = \operatorname{ord}(\tau)$.)

Proof. The only step in the proof of the proposition that needs further justification is the definition of t. Let $t_0 = \widetilde{\sigma}(\tau)$ be any element with $s^t = s^\tau$. We have $t_0^k = a \in \mathbb{F}$. Let α be a root of the polynomial $X^k - a$. We have to show that $\alpha \in \mathbb{F}$. By assumption, $\mathbb{F}$ contains a primitive kth root of unity, ε. It follows that $\mathbb{F}(\alpha)$ has cyclic Galois group over $\mathbb{F}$ of order d which divides k and that $\alpha^d \in \mathbb{F}$ [41, Chapter VIII, Theorem 10]. Let $\alpha^d = b$, say. The minimal polynomial of t_0 over $\mathbb{F}$ divides

$$X^k - a = \prod_{i=0}^{(k/d)-1} X^d - \varepsilon^{di} b.$$

A matrix of t_0 in canonic rational form contains thus only blocks of the form

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ \varepsilon^{di} b & 0 & \dots & \dots & 0 \end{pmatrix}.$$

However, τ is K-good and thus $\operatorname{tr}(t_0) \neq 0$ by Proposition 4.13. This is only possible when $d = 1$, which means $\alpha \in \mathbb{F}$. □

In our proof of Theorem 4.23 we will also need the following result:

4.26 Lemma. *[27, Corollary 4.4], [31, Lemma 3.3 and proof of Theorem 3.1] Let $K \trianglelefteq G$ with $K' \leqslant L = \mathbf{Z}(K) \leqslant \mathbf{Z}(G)$, $|K/L|$ odd, and L cyclic, and suppose that every coset of L in K contains an element, k, such that $L \cap \langle k \rangle = 1$. Then there is $\tau \in \operatorname{Aut} G$ of order 2 which inverts K/L such that for $H = \mathbf{C}_G(\tau)$ we have $G = HK$ and $H \cap K = L$.*

Proof of Theorem 4.23 for $\mathbb{F} = \mathbb{C}$. Using Lemma 2.43, we can assume that $L \leqslant \mathbf{Z}(G)$, that L is cyclic and that every coset of L in K contains an element, x, such that $L \cap \langle x \rangle = 1$. As K/L is odd, we can apply Lemma 4.26. Let H be as in that lemma. Now Proposition 4.24 applies so that there is a magic representation $\sigma\colon H/L \to (\mathbb{C}Ke_\varphi)^L$. □

The proof for a general field $\mathbb{F}$ will be given at the end of this section.

In the odd abelian case, it is possible to choose a canonical ψ, as Isaacs has shown. The existence and the most important properties of this canonical magic character can be derived from what we have done so far, with (I hope) simpler proofs. Some of the arguments we need are taken from the original proof, but for the convenience of the reader and the sake of completeness we repeat them here. The following is an adaption of Isaacs' definition of "canonical" [27, Definition 5.2] to our purposes.

4.27 Definition. Let $(G, K, L, \vartheta, \varphi)$ be a character five with $|K/L|$ odd. Let π be the set of prime divisors of $|K/L|$. A magic character $\psi \in \mathbb{Z}[\mathrm{Irr}(H/L)]$ is called *canonical* if

1. $\mathrm{ord}(\psi)$ is a π-number and
2. If $p \in \pi$ and $P \in \mathrm{Syl}_p(H)$, then 1_P is the unique irreducible constituent of ψ_P which appears with odd multiplicity.

4.28 Remark. Assume Hypothesis 2.1 with $(G, K, L, \vartheta, \varphi)$ a character five. If a canonical magic character $\psi\colon H \to \mathbb{C}$ exists, then all $h \in H$ are good.

Proof. Let $h \in H$. We have to show that $\langle h, c\rangle_\varphi = 1$ for all $c \in \mathbf{C}_{K/L}(h)$. Write $h = \prod_p h_p$ as product of its p-parts. Since $\mathbf{C}_{K/L}(h) = \bigcap_p \mathbf{C}_{K/L}(h_p)$, we may assume that h itself has prime power order. If p does not divide $|K/L|$, then h is good by Lemma 4.6. If p divides $|K/L|$, then let $P \in \mathrm{Syl}_p(H)$ be a Sylow p-subgroup containing h. Then, by canonicalness, $\psi_P = 1_P + 2\beta$ for some character β. It follows that $\psi(h) \neq 0$ and thus h is good by Proposition 4.13. □

If K/L is not abelian, it may happen that there is no canonical ψ even if there is a magic character. For example it may be that there are p-elements in a complement that are not good. An example where this occurs has been given by Lewis [42]. In his example, K/L is a p-group, and the complement H is unique up to conjugacy.

4.29 Lemma. *The complement H given, there is at most one canonical magic character ψ.*

Proof. [27, p. 610] Suppose ψ and ψ_1 are canonical. Then $\psi_1 = \lambda\psi$ for some $\lambda \in \mathrm{Lin}(H/L)$. Let π be the set of primes dividing $|K/L|$. If some prime $q \notin \pi$ divides $\mathrm{ord}(\lambda)$, then from $\det \psi_1 = \lambda^n \det \psi$ and $(q, n) = 1$ we conclude that q divides $\mathrm{ord}(\psi_1)$, but this contradicts ψ_1 being canonical. Therefore $\mathrm{ord}(\lambda)$ is a π-number. Let $p \in \pi$ and $P \in \mathrm{Syl}_p H$. Then $[\lambda_P, (\psi_1)_P] = [1_P, \psi_P]$ and the last is odd by the definition of canonical. From the assumption that ψ_1 is canonical too we conclude that $\lambda_P = 1_P$. This holds in fact for all π-subgroups of H. As $\mathrm{ord}(\lambda)$ is a π-number, we have $\lambda = 1_H$ and $\psi_1 = \psi$. □

4.30 Theorem. *If $(G, K, L, \vartheta, \varphi)$ is an odd abelian character five, then there is a canonical magic character ψ. Let H/L be the complement of K/L in G/L on which ψ is defined. For every subgroup $U/L \leqslant H/L$ with $|U/L|$ odd, 1_U is the unique irreducible constituent of ψ which appears with odd multiplicity.*

Proof. (cf. Isaacs [27, Theorem 5.3].) As in the proof of Theorem 4.23 we can assume that G has an automorphism τ of order 2 inverting K/L and leaving K, L and φ invariant. After replacing G by the semidirect product of $\langle\tau\rangle$ with G, we can assume that $\tau \in G$. By Theorem 4.23 there is a magic character ψ, and by Lemma 2.11 we can assume that ψ has π-order. As $L\tau \in \mathbf{Z}(H/L)$, we can write $\psi = \psi_+ + \psi_-$ where $\psi_+(\tau) = \psi_+(1)$ and $\psi_-(\tau) = -\psi_-(1)$.

Let $L \leqslant U \leqslant H$ with $|U/L|$ odd and take $u \in U$. As τ centralizes Lu and Lu has odd order, we have $L\tau = (L\tau u)^{\mathrm{ord}(u)}$. Thus $\mathbf{C}_{K/L}(\tau u) \subseteq \mathbf{C}_{K/L}(\tau) = 1$. Therefore

$$1 = |\psi(\tau u)| = |\psi_+(u) - \psi_-(u)|$$

for every $u \in U$. This yields $(\psi_+ - \psi_-, \psi_+ - \psi_-)_U = 1$ and hence $(\psi_+ - \psi_-)_U = \pm\lambda$ where $\lambda \in \mathrm{Lin}\, U/L$. The sign depends not on U, but only on wether $\psi_+(1) > \psi_-(1)$ or $\psi_+(1) < \psi_-(1)$. We conclude

$$\psi_U = 2\gamma_U + \lambda, \text{ where } \gamma = \begin{cases} \psi_- & \text{if } \psi_+(1) > \psi_-(1) \\ \psi_+ & \text{if } \psi_+(1) < \psi_-(1). \end{cases}$$

This equation shows that λ is the only constituent of ψ_U occuring with odd multiplicity. Taking determinants in the equation yields $\det \psi_U = (\det \gamma_U)^2 \lambda$. Thus λ can be extended to a linear character of H/L, namely to $\mu = \det \psi (\det \gamma)^{-2}$. Write $\mu = \mu_\pi \mu_{\pi'}$ where μ_π is the π-part of μ. Then $\overline{\mu_\pi}\psi$ still has determinantal order a π-number. For $P \in \mathrm{Syl}_p(H)$ where $p \in \pi$ we have $(\mu_\pi)_P = \mu_P$ and thus the unique irreducible constituent of $\overline{\mu_\pi}\psi$ with odd multiplicity is 1_P. This shows that $\overline{\mu_\pi}\psi$ is canonical and completes the proof of the existence of a canonical magic character.

Now assume that ψ is canonical. We have already seen that for $|U/L|$ odd, ψ has a unique constituent λ of odd multiplicity and that this constituent is linear. To show that $\lambda = 1_U$ it suffices to show that $\lambda_P = 1_P$ if P/L is a p-subgroup of U/L. If $p \in \pi$ this is clear from the definition of canonical. If $p \notin \pi$, then $(|P/L|, |K/L|) = 1$ and ψ_P is rational by Corollary 4.17. Thus every field automorphism of the complex numbers fixes ψ_P and, a fortiori, its unique irreducible constituent occurring with odd multiplicity, λ. Since p is odd if $P/L \neq 1$, we conclude $\lambda_P = 1_P$. The proof is complete. □

We remark that the last few sentences of the proof can be replaced by an appeal to Corollary 4.21, applied to P.

As in the coprime case, we get as a corollary:

4.31 Corollary. *Let $(G, K, L, \vartheta, \varphi)$ be an abelian character five with $|G : L|$ odd. Then there is a complement H/L of K/L in G/L and a bijection between $\operatorname{Irr}(G \mid \vartheta)$ and $\operatorname{Irr}(H \mid \varphi)$ where χ and ξ correspond if and only if (χ_H, ξ) is odd.*

We have now proved the most important properties of the character correspondence. In his paper, Isaacs describes an algorithm for computing the values of the canonical character ψ. Actually, this is part of his proof of the existence of the character correspondence. We now give alternative proofs of some of his results. These are strong enough to obtain the value field of the canonical magic character.

We assume throughout that $(G, K, L, \vartheta, \varphi)$ is a character five with K/L odd abelian and that there is $\tau \in G$ with $x^\tau = x^{-1}$ for all $x \in K/L$, and $H = \mathbf{C}_{K/L}(\tau)$. We need a lemma.

4.32 Lemma. *Hold the assumptions just introduced. Then for every $x \in K/L$ there is a unique element*

$$s_x \in (\mathbb{C}Ke_\varphi)^L \cap \mathbb{C}x \text{ with } (s_x)^\tau = s_x^{-1} \text{ and } (s_x)^{\operatorname{ord}(x)} = e_\varphi.$$

For these elements, we have

$$s_x s_y = \alpha(x,y) s_{xy} \text{ with } \alpha(x,y)^2 = \langle x, y\rangle_\varphi \text{ and } \alpha(x,y) = \langle x, y\rangle_\varphi^{\frac{n+1}{2}}.$$

In particular, $\alpha\colon K/L \times K/L \to \mathbb{C}$ is bilinear and alternating.

Proof. Let $x \in K/L$ and $t_x \in (\mathbb{C}Ke_\varphi)^L \cap \mathbb{C}x$. As $\dim_{\mathbb{C}}\left((\mathbb{C}Ke_\varphi)^L \cap \mathbb{C}x^{-1}\right) = 1$, we have $t_x^\tau = \lambda t_x^{-1}$ for some $\lambda \in \mathbb{C}$. Take α with $\alpha^2 = \lambda$. Then $(\alpha^{-1}t_x)^\tau = \alpha^{-1}\lambda t_x^{-1} = \alpha t_x^{-1} = (\alpha^{-1}t_x)^{-1}$. Set $k = \operatorname{ord}(x)$. Then $(\alpha^{-1}t_x)^k = \varepsilon e_\varphi \in \mathbb{C}e_\varphi$. We have

$$\varepsilon e_\varphi = (\varepsilon e_\varphi)^\tau = (\alpha^{-1}t_x)^{k\tau} = (\alpha^{-1}t_x)^{-k} = \varepsilon^{-1} e_\varphi.$$

Thus $\varepsilon = \pm 1$. As k is odd, $s_x = \varepsilon\alpha^{-1}t_x$ is an element as required. Uniqueness is also clear now.

Let α be the factor set of K/L defined by the s_x's. Then

$$\begin{aligned}\alpha(x,y)s_{xy}^{-1} = (\alpha(x,y)s_{xy})^\tau = (s_x s_y)^\tau = s_x^{-1}s_y^{-1} &= \langle x,y\rangle_\varphi s_y^{-1}s_x^{-1} \\ &= \langle x,y\rangle_\varphi (s_x s_y)^{-1} = \langle x,y\rangle \alpha(x,y)^{-1}s_{xy}^{-1},\end{aligned}$$

where we have used Lemma 4.4. It follows that $\alpha(x,y)^2 = \langle x,y\rangle$.

We claim that $\alpha(x,y)^{n^2} = 1$ for all $x, y \in K/L$: Taking determinants, we first get $\det(s_x)\det(s_y) = \alpha(x,y)^n \det(s_{xy})$. The abelian group K/L of order

n^2 carries a nondegenerate bilinear form, and thus has exponent dividing n. Thus $s_x^n = 1$, and the claim follows. As n is odd, it now follows that $\alpha(x,y) = \langle x,y\rangle_\varphi^{\frac{n^2+1}{2}}$. So *a posteriori* we get $\alpha(x,y)^n = 1$. The proof now follows. □

4.33 Theorem. *Let ψ be a magic character of an odd abelian character five $(G,K,L,\vartheta,\varphi)$, and let v and $w \in H/L$. Then*

$$\psi(vw) = \frac{\psi(v)\psi(w)}{\psi(1)|\mathbf{C}_{K/L}(v)||\mathbf{C}_{K/L}(w)|} \sum_{\substack{x,y\in K/L\\ [x,v][y,w]=1}} \langle x^v, x\rangle_\varphi^2 \langle y^w, y\rangle_\varphi^2.$$

Proof. Let σ be the magic representation affording the magic character ψ. We may assume that Lemma 4.32 applies, so choose the s_x as done there. Observe that $\mathrm{tr}_S(s_x) = 0$ if $x \neq 1$ and $\mathrm{tr}_S(s_1) = \mathrm{tr}_S(e_\varphi) = \psi(1) = n$. From this and Proposition 4.14 it follows that for v good, we have

$$\sigma(v) = \frac{\psi(v)}{\psi(1)|C_{K/L}(v)|} \sum_{x\in K/L} s_x^{-1} s_x^v, \qquad (*)$$

and similar for w. To compute $\psi(vw) = \mathrm{tr}_S(\sigma(vw))$, we determine the coefficient of $1_S = s_1$ in $\sigma(v)\sigma(w)$. Observe that our special choice of the s_x ensures that $s_x^{-1} = s_{x^{-1}}$ and $s_x^v = s_{x^v}$. This gives

$$\begin{aligned}\psi(vw) = \psi(1)\cdot &\frac{\psi(v)\psi(w)}{\psi(1)^2|\mathbf{C}_{K/L}(v)||\mathbf{C}_{K/L}(w)|} \sum_{\substack{x,y\in K/L\\ [x,v][y,w]=1}} s_x^{-1}s_x^v \cdot s_y^{-1}s_y^w \\ = \quad &\frac{\psi(v)\psi(w)}{\psi(1)|\mathbf{C}_{K/L}(v)||\mathbf{C}_{K/L}(w)|} \sum_{\substack{x,y\in K/L\\ [x,v][y,w]=1}} \alpha(x^{-1},x^v)\alpha(y^{-1},y^w).\end{aligned}$$

As $|K/L|$ is odd, the map $x \mapsto x^2$ is a bijection of K/L on itself. Also, if $[x,v][y,w] = 1$, then $[x^2,v][y^2,w] = [x,v]^2[y,w]^2 = 1$, as K/L is abelian. Thus

$$\begin{aligned}\sum_{\substack{x,y\in K/L\\ [x,v][y,w]=1}} \alpha(x^{-1},x^v)\alpha(y^{-1},y^w) &= \sum_{\substack{x,y\in K/L\\ [x,v][y,w]=1}} \alpha(x^{-2},(x^2)^v)\alpha(y^{-2},(y^2)^w) \\ &= \sum_{\substack{x,y\in K/L\\ [x,v][y,w]=1}} \alpha(x,x^v)^{-4}\alpha(y,y^w)^{-4} \\ &= \sum_{\substack{x,y\in K/L\\ [x,v][y,w]=1}} \langle x^v,x\rangle_\varphi^2\langle y^w,y\rangle_\varphi^2,\end{aligned}$$

as required. □

The following corollary generalizes a result of Isaacs [27, Theorem 6.1].

4.34 Corollary. *Let ψ be the canonical magic character of a odd abelian character five, and let $u \in H$ have odd order modulo $\mathbf{C}_H(K/L)$. Then*

$$\psi(u) = \frac{1}{\psi(1)} \sum_{x \in K/L} \langle x, x^u \rangle_\varphi^2.$$

Proof. We may assume that there is $\tau \in G$ inverting K/L and centralizing H/L, where ψ is defined on H/L. In the proof of Theorem 4.30 it was shown that

$$\psi(\tau u) = \psi_+(u) - \psi_-(u) = \pm 1 = \psi(\tau).$$

We now apply the last theorem with $v = \tau$ and $w = \tau u$ and get

$$\begin{aligned} \psi(u) &= \frac{\psi(\tau)\psi(\tau u)}{n} \sum_{\substack{x,y \in K/L \\ [x,\tau][y,\tau u]=1}} \langle x^\tau, x \rangle_\varphi^2 \langle y^{\tau u}, y \rangle_\varphi^2 \\ &= \frac{1}{n} \sum_{\substack{x,y \in K/L \\ x^{-2}[y,\tau u]=1}} \langle x^{-1}, x \rangle^2 \langle (y^{-1})^u, y \rangle^2 \\ &= \frac{1}{n} \sum_{y \in K/L} \langle y, y^u \rangle_\varphi^2, \end{aligned}$$

since $x \mapsto [x, \tau] = x^{-2}$ is a bijection of K/L. □

4.35 Corollary. *Assumptions as in the last corollary, and let ε be a primitive n-th root of unity. Let $\Gamma = \mathrm{Gal}(\mathbb{Q}(\varepsilon)/\mathbb{Q})$. Let $\mathbb{F}$ be the fixed field of Γ^2. Then the canonical character ψ has values in $\mathbb{F}$.*

Proof. An automorphism α of $\mathbb{Q}(\varepsilon)$ sends ε to ε^k, where $(k, n) = 1$. It follows $\langle x, y \rangle_\varphi^{\alpha^2} = \langle x^k, y^k \rangle_\varphi$. If $u \in H$ has odd order modulo L, then it follows from the formula in the last corollary that $\psi(u)^{\alpha^2} = \psi(u)$, since x^k runs through K/L if x does. If the order of t modulo L is a power of 2, then $\psi(t) \in \mathbb{Q}$ by Proposition 4.17. For arbitrary h, write $h = tu$ with t of order 2^r and u of odd order, and apply Theorem 4.33. Consider the set

$$E = \{(x, y) \in K/L \times K/L \mid [x, t][y, u] = 1\}.$$

Sending (x, y) to (x^k, y^k) permutes this set. It follows that

$$\sum_{(x,y) \in E} \langle x^t, x \rangle_\varphi^2 \langle y^u, y \rangle_\varphi^2 \in \mathbb{F}.$$

Thus $\psi(h) \in \mathbb{F}$ as claimed. □

Remark. $\mathbb{F}$ depends on the prime divisors of n, but not on their multiplicity. Instead of n we could have taken the exponent of K/L. Note that

$$\mathrm{Gal}(\mathbb{F}/\mathbb{Q}) \cong \Gamma/\Gamma^2 \cong C_2^{\{p|p \text{ prime and divides } n\}}.$$

Finally, let us show that the canonical magic representation has image in $(\mathbb{Q}(\varphi)Ke_\varphi)^L$.

Proof of Theorem 4.23 for arbitrary field. Let $\mathbb{F}$ be any field containing the values of φ. Let e be the exponent of K/L. By Corollary 4.9, $\mathbb{F}$ contains primitive e-th roots of unity. By Corollary 4.35, $\mathbb{F}$ contains the values of the canonical magic character ψ. From equation $(*)$ on page 62, we see that $\sigma(h) \in (\mathbb{F}Ke_\varphi)^L$ for all $h \in H$. This finishes the proof of Theorem 4.23. □

4.5 Strong Sections

We give now another sufficient condition for the existence of magic representations. This result is in essence due to Dade [6], but we can considerably generalize it without to much effort.

First we adapt Isaacs' definition of a "strong section" [30] to our purposes:

4.36 Definition. Let K/L be a section of G, assume that $\vartheta \in \mathrm{Irr}\, K$ is G-invariant and let $\varphi \in \mathrm{Irr}\, L$ be a constituent of ϑ_L. Then $(G, K, L, \vartheta, \varphi)$ is *strong* if there is $M \leqslant G_\varphi$, such that the following conditions hold:

1. $MK \trianglelefteq G$,
2. $\mathbf{C}_{K/L}(M) = 1$,
3. $M \cap K = L$,
4. $(|M/\mathbf{C}_M(K/L)|, |K/L|) = 1$,
5. Every element of $\mathbf{C}_M(K/L)$ is K-φ-good.

If we replace the last three conditions by the condition $(|M/L|, |K/L|) = 1$, we get a special case. By Lemma 4.6, the last condition is equivalent to every element of M being K-φ-good. By Lemma 4.5, it is equivalent to the condition that $\mathbf{C}_M(K/L)$ centralizes $(\mathbb{C}Ke_\varphi)^L$. (We do not assume that φ is invariant in K.) The hypotheses mean that M induces a coprime fixed point free operator group of K/L. It follows from the classification of finite simple groups that K/L is solvable [56]. The solvability of K/L will be assumed in what follows. (The reader may simply consider this to be an additional hypothesis in Definition 4.36.)

The original definition does not depend on the character φ given, but is completely group theoretic. However, I found that unavoidable. We digress to show that our definition extends Isaacs' definition. Recall that an abelian character five is a character five $(G, K, L, \vartheta, \varphi)$ with K/L abelian.

4.37 Proposition. *An abelian character five $(G, K, L, \vartheta, \varphi)$ is strong in the sense of Definition 4.36 if and only if there is $N \lhd G$ such that the group $A \cong N/\mathbf{C}_N(K/L)$ of automorphisms of K/L induced by N satisfies*

1. $\mathbf{C}_{K/L}(A) = 1$ *and*
2. $(|A|, |K/L|) = 1$.

Proof. First suppose $(G, K, L, \vartheta, \varphi)$ is strong as defined in Definition 4.36. Let $N = MK$. Since K/L is abelian, we have $K \leqslant \mathbf{C}_G(K/L)$. Thus $\mathbf{C}_N(K/L) = \mathbf{C}_M(K/L)K$. This yields

$$\begin{aligned} N/\mathbf{C}_N(K/L) &= MK/\mathbf{C}_M(K/L)K \cong M/(M \cap \mathbf{C}_M(K/L)K) \\ &= M/\mathbf{C}_M(K/L), \end{aligned}$$

as $M \cap K = L$. The conditions on A follow.

Conversely, let N as above be given. Let $C = \mathbf{C}_G(K/L)$ and $B = \{c \in C \mid \langle c, x\rangle_\varphi = 1 \text{ for all } x \in K\}$. Thus $B = K^\perp$ with respect to the form $\langle\ ,\ \rangle_\varphi \colon K \times C \to \mathbb{C}$. We conclude $|C/B| \leqslant |K/L|$. Since elements of B are good, we have $B \cap K = L$. Therefore $|K/L| = |BK/B| \leqslant |C/B|$. Thus $C = BK$ and $|C/B| = |K/L|$. We may assume $C \leqslant N$ (otherwise replace N by NC). By assumption, $|N/C|$ is prime to $|K/L| = |C/B|$. Therefore C/B is a normal Hall subgroup of N/B. Let M/B be a complement. Then $N = MC = MBK = MK \trianglelefteq G$ (Condition 1 in Definition 4.36) and $\mathbf{C}_M(K/L) = M \cap C = B$ (Condition 5). Thus $M/\mathbf{C}_M(K/L) \cong N/C \cong A$, which acts fixed point freely and coprimely on K/L. Thus Conditions 2 and 4 hold. Finally, Condition 3 follows from $M \cap K = \mathbf{C}_M(K/L) \cap K = B \cap K = L$. The proof is finished. □

The following is essentially Theorem B of Isaacs [30], but without the hypothesis that K/L is abelian or φ is fully ramified in K.

4.38 Theorem. *Suppose $(G, K, L, \vartheta, \varphi)$ is strong. Then there is $H \leqslant G$ with $G = HK$ and $L = H \cap K$ and a magic representation $\sigma \colon H/L \to S = (\mathbb{C}Ke_\varphi)^L$.*

To prove his Theorem B, Isaacs uses an intermediate result [30, Theorem 6.1] saying that if φ can be extended to H in the situation of Theorem 4.38, then ϑ can be extended to G. Our proof of Theorem 4.38 follows the lines of Isaacs' proof of this result. While Isaacs assumes K/L to be abelian and φ fully ramified in K, we will suppose only that K/L is solvable. As remarked above, this follows, since by assumption K/L has a coprime fixed point free operator group. However, the first very easy step of the proof will be to reduce the situation to one where $(G, K, L, \vartheta, \varphi)$ is an abelian character five.

First we prove the following technical lemma, which will be used in the proof of Theorem 4.38. It is another induction-type result.

4.39 Lemma. *Let $(G, K, L, \vartheta, \varphi)$ be a character five with K/L abelian and let $H \leqslant G$ such that $G = HK$ and $L = H \cap K$. Let A be a subgroup with $L \leqslant A \leqslant K$ and set $U = \mathbf{N}_H(A)$. Suppose we have the following:*

1. *$K/L = \prod_{h\in[H:U]} A^h/L$.*
2. *Different conjugates of A are orthogonal with respect to $\langle \ , \ \rangle_\varphi$.*

Then $|\mathrm{Irr}(A \mid \varphi)| = 1$. Let $\mathrm{Irr}(A \mid \varphi) = \{\eta\}$, say, and $T = (\mathbb{C}Ae_\varphi)^L$. If there is a magic representation $\tau\colon U/L \to T$ for $(AU, A, L, \eta, \varphi)$, then there is a magic representation $\sigma\colon H/L \to S$ for $(G, K, L, \vartheta, \varphi)$.

Proof. We first show that K/L is the direct product of the different conjugates of A/L. Since K/L is abelian and φ fully ramified in K, the form $\langle \ , \ \rangle_\varphi$ is defined on all of K/L and nondegenerate. If $a \in A \cap \prod_{h\notin \mathbf{N}_H(A)} A^h$, then $\langle a, x\rangle = 1$ for all $x \in K$ and thus $a \in L$. The same argument holds if A is replaced by one of its conjugates. It follows that K/L is the direct product of the different conjugates of A/L and that the restriction of the form to A is nondegenerate. From this it follows that φ is fully ramified in A.

It remains to construct a magic representation $\sigma\colon H/L \to S$, if τ is given. For the readers knowing about tensor induction, we remark that the σ we are going to construct is isomorphic with the tensor induced representation of τ.

As in Lemma 4.12, choose $s_x \in S^* \cap \mathbb{C}Lx$ for $x \in K/L$. For $a \in A$ and $b \in A^h \neq A$ we have $s_a^{s_b} = s_a^b = \langle a, b\rangle s_a = s_a$ by Lemma 4.12 and the assumption. Therefore T and T^h centralize each other if $h \notin U$. Let R be a transversal of U in H. For $x = \prod_{r\in R} a_r^r \in K/L = \prod_{r\in R} A^r/L$ we have $s_x = \lambda \prod_{r\in R} s_{a_r}^r$ with some $\lambda \in \mathbb{C}$. Thus S is generated by the T^r, $r \in R$. (In fact, $S \cong \bigotimes_{r\in R} T^r$ canonically, but we will not need this.)

Since φ is fully ramified in A, we have that $T \cong \mathbf{M}_d(\mathbb{C})$ for some d. Let $\{E_{ij} \mid i,j = 1,\dots,d\}$ be a full set of matrix units in T. For maps $k,l\colon R \to \{1,\dots,d\} = [d]$ we define

$$E_{k,l} := \prod_{r\in R} E^r_{k(r),l(r)} \in S = \prod_{r\in R} T^r.$$

The $E_{k,l}$, where k,l run through all the maps from R to $[d]$, form a full set of matrix units of S: Namely, let k, l, $\tilde{k}$ and $\tilde{l} \in [d]^R$. Using that the different conjugates of T centralize each other, we get

$$\begin{aligned} E_{k,l}E_{\tilde{k},\tilde{l}} &= \prod_{r\in R} E^r_{k(r),l(r)} \prod_{r\in R} E^r_{\tilde{k}(r),\tilde{l}(r)} = \prod_{r\in R}\left(E^r_{k(r),l(r)}E^r_{\tilde{k}(r),\tilde{l}(r)}\right) \\ &= \delta_{l,\tilde{k}}E_{k,\tilde{l}}. \end{aligned}$$

Furthermore,

$$1_S = \prod_{r\in R}(1_T)^r = \prod_{r\in R}\sum_{k(r)=1}^{d} E^r_{k(r),k(r)} = \sum_{k\in[d]^R}\prod_{r\in R} E^r_{k(r),k(r)} = \sum_{k\in[d]^R} E_{k,k}.$$

Thus the $E_{k,l}$ form a full set of matrix units.

The permutations on R act on $[d]^R$ by $k^\pi(r) = k(r\pi^{-1})$. For a permutation π of R we now define

$$\alpha(\pi) = \sum_{k\in[d]^R} E_{k,k^\pi}.$$

The straightforward computation

$$\alpha(\pi_1)\alpha(\pi_2) = \sum_{k,l\in[d]^R} E_{k,k^{\pi_1}}E_{l,l^{\pi_2}} = \sum_{k\in[d]^R} E_{k,k^{\pi_1\pi_2}} = \alpha(\pi_1\pi_2)$$

shows that α is a homomorphism from the group of permutations of R into S^*.

We claim that for $r_0 \in R$ and $t \in T$ we have $t^{r_0\alpha(\pi)} = t^{r_0\pi}$. It suffices to prove this for a matrix unit $t = E_{ij}$. For this,

$$\begin{aligned}
E_{ij}^{r_0\alpha(\pi)} &= \sum_{k\in[d]^R} E_{k^\pi,k}\, E_{ij}^{r_0} \sum_{l\in[d]^R} E_{l,l^\pi} \\
&= \sum_k \prod_{r\in R} E^r_{k(r\pi^{-1}),k(r)}\, E_{ij}^{r_0} \sum_l \prod_{r\in R} E^r_{l(r),l(r\pi^{-1})} \\
&= \sum_{\substack{k\in[d]^R \\ k(r_0)=i}} \sum_{\substack{l\in[d]^R \\ l(r_0)=j}} \prod_r E^r_{k(r\pi^{-1}),k(r)} E^r_{l(r),l(r\pi^{-1})} \\
&= E_{ij}^{r_0\pi} \sum_{k\in[d]^{R\setminus\{r_0\}}} \prod_{r\neq r_0} E^{r\pi}_{k(r),k(r)} \\
&= E_{ij}^{r_0\pi} \prod_{r\neq r_0} \sum_{\nu\in[d]} E^{r\pi}_{\nu\nu} \\
&= E_{ij}^{r_0\pi},
\end{aligned}$$

as desired.

For $h \in H$ and $r \in R$ we write $r \bullet h$ for the unique element in $R \cap Urh$. This defines an action of H on R. We view α as being defined on H (via that action), and we just have proved that $t^{r\alpha(h)} = t^{r\bullet h}$.

Now suppose $\tau\colon U/L \to T$ is a magic representation. We define

$$\sigma(h) = \left(\prod_{r\in R} \tau(rh(r\bullet h)^{-1})^r\right)\alpha(h).$$

First we show that this is an homomorphism. Let $g, h \in H$ and set $u_r = rg(r\bullet g)^{-1}$ and $v_r = rh(r\bullet h)^{-1}$ for $r \in R$. Then

$$rgh = u_r(r\bullet g)h = u_r v_{r\bullet g}(r\bullet g\bullet h) = u_r v_{r\bullet g}(r\bullet(gh)).$$

We get

$$\begin{aligned}
\sigma(gh) &= \left(\prod_{r\in R} \tau\Big(rgh\big(r \bullet (gh)\big)^{-1}\Big)^{r}\right) \alpha(gh) \\
&= \left(\prod_{r\in R} \tau(u_r v_{r\bullet g})^{r}\right) \alpha(g)\alpha(h) \\
&= \left(\prod_{r\in R} \tau(u_r)^{r}\right) \left(\prod_{r\in R} \tau(v_{r\bullet g})^{r}\right) \alpha(g)\alpha(h) \\
&= \left(\prod_{r\in R} \tau(u_r)^{r}\right) \alpha(g) \left(\prod_{r\in R} \tau(v_{r\bullet g})^{r\alpha(g)}\right) \alpha(h) \\
&= \sigma(g) \left(\prod_{r\in R} \tau(v_{r\bullet g})^{r\bullet g}\right) \alpha(h) \\
&= \sigma(g)\sigma(h).
\end{aligned}$$

Second we have to show that $s^h = s^{\sigma(h)}$ for all $s \in S$. It suffices to do this for elements t^r where $t \in T$ and $r \in R$. For these we have

$$\begin{aligned}
t^{r\sigma(h)} &= t^{r \prod_{r'\in R} \tau\left(r'h(r' \bullet h)^{-1}\right)^{r'} \alpha(h)} \\
&= t^{r\tau(rh(r\bullet h)^{-1})^{r}\alpha(h)} \\
&= t^{\tau(rh(r\bullet h)^{-1})r\alpha(h)} \\
&= t^{rh(r\bullet h)^{-1}(r\bullet h)} = t^{rh}
\end{aligned}$$

as desired. This finishes the proof that σ is a magic representation. □

As we remarked earlier, we have in fact constructed the tensor induced representation of τ within S.

While the construction of σ in the proof apparently depends on the choice of the E_{ij} and of R, it turns out that $s^{\sigma(h)} = s^h$ for all $s \in S$ and the last property determines $\sigma(h)$ up to a scalar.

In the proof of Theorem 4.38, we will also need some standard facts on coprime action, namely:

4.40 Lemma. *Let A act on X and leave $Y \trianglelefteq X$ invariant, and assume $(|A|, |Y|) = 1$. Then $\mathbf{C}_{X/Y}(A) = \mathbf{C}_X(A)Y/Y$.*

4.41 Lemma. *[30, Lemmas 2.4 and 2.5] Suppose A acts on K leaves $L \trianglelefteq K$ invariant, and $(|A|, |K/L|) = 1$.*

1. *If $\vartheta \in \mathrm{Irr}_A(K)$, then $C_{K/L}(A)$ acts transitively on $\mathrm{Irr}_A(\vartheta_L)$, which is not empty. (That is, ϑ_L has an A-invariant constituent.)*

2. *Let K/L be abelian. If $\varphi \in \mathrm{Irr}_A(L)$, then φ^K has an A-invariant constituent ϑ; if $\mathbf{C}_{K/L}(A) = 1$, then ϑ is unique.*

Both follow from a lemma of Glauberman [34, Lemma 13.8 and Corollary 13.9]. The second part of Lemma 4.41 holds without the assumption that K/L is abelian, but becomes more difficult to prove.

Proof of Theorem 4.38. We begin by producing the subgroup H. We have $\mathbf{C}_M(K/L) \trianglelefteq KM$ and also $K\,\mathbf{C}_M(K/L) \trianglelefteq KM$. The definition of strong ensures that

$$|K\,\mathbf{C}_M(K/L)/\,\mathbf{C}_M(K/L)| = |K/L|$$
$$\text{and}\qquad |KM/K\,\mathbf{C}_M(K/L)| = |M/\,\mathbf{C}_M(K/L)|$$

are coprime. Thus $KC_M(K/L)/C_M(K/L)$ is a normal Hall subgroup of $KM/C_M(K/L)$ with complement $M/C_M(K/L)$. Let $H = \mathbf{N}_G(M)$. The Frattini argument yields $G = HMK = HK$. We have $[H\cap K, M] \subseteq M\cap K = L$ and thus $(H \cap K)/L \subseteq \mathbf{C}_{K/L}(M) = 1$. Thus $H \cap K = L$.

Since $\mathbf{C}_{K/L}(M) = 1$, it follows from Lemma 4.41 that φ is the only M-invariant constituent of ϑ_L. Thus φ is invariant under H.

We now show that there is a magic representation $\sigma\colon H/L \to S$. (The proof will also show that $\mathbf{C}_H(K/L) \leqslant \ker\sigma$.) Suppose that there is not always a magic representation and let $(G, K, L, \vartheta, \varphi)$ be a counterexample with $|G : L|$ minimal. We derive properties of $(G, K, L, \vartheta, \varphi)$ until we reach a contradiction.

Step 1. $(G, K, L, \vartheta, \varphi)$ is a character five and K/L a chief factor of G.

Proof. Suppose N/L is a chief factor of G with $N < K$. As the action of $A = M/\,\mathbf{C}_M(K/L)$ on K/L is coprime and fixed point free, Lemma 4.40 yields that the A acts coprime and fixed point freely on K/N and N/L. The first part of Lemma 4.41 implies the existence of a unique irreducible M-invariant constituent η, say, of ϑ_N. The unique M-invariant constituent of η_L must be φ, since this is the unique M-invariant constituent of ϑ_L. By uniqueness, η is H-invariant.

Let $U = HN$. We wish to apply the inductive hypothesis to the configurations $(G, U, K, N, \vartheta, \eta)$ and $(U, H, N, L, \eta, \varphi)$.

First we show that $(G, K, N, \vartheta, \eta)$ is strong. Let $\widetilde{M} = MN$. We verify the conditions of Definition 4.36. The first is clear since $\widetilde{M}K = MK \trianglelefteq G$. That $\mathbf{C}_{K/N}(\widetilde{M}) = 1$ follows from Lemma 4.40. Condition 3 follows from

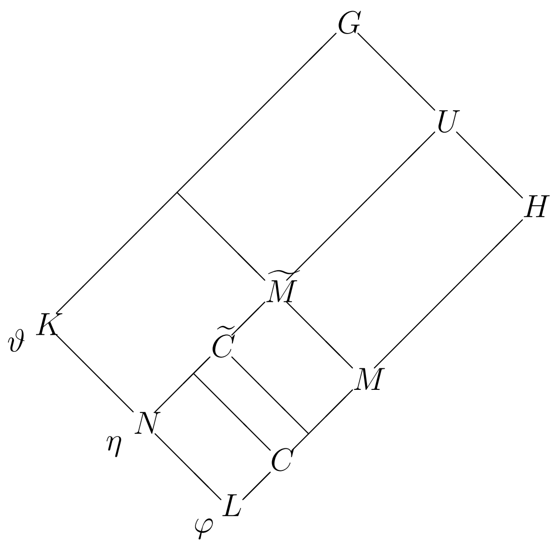

Figure 4.1: Proof of Step 1

$\widetilde{M} \cap K = MN \cap K = (M \cap K)N = N$, and Condition 4 from $\widetilde{C} := \mathbf{C}_{\widetilde{M}}(K/N) = \mathbf{C}_M(K/N)N$. Finally, we verify that elements of $\widetilde{C}$ are good. As $(e_\eta \mathbb{C}Ke_\eta)^N$ embeds into $(e_\varphi \mathbb{C}Ke_\varphi)^L$, the group $\mathbf{C}_M(K/L)$ centralizes $(e_\eta \mathbb{C}Ke_\eta)^N$ and thus elements of $\mathbf{C}_M(K/L)$ are N-η-good. By Lemma 4.6 and since $|M/\mathbf{C}_M(K/L)|$ is coprime to $|N/L|$, also elements of $\mathbf{C}_M(K/N)$ are N-η-good. As $\widetilde{C} = \mathbf{C}_M(K/N)N$, elements of $\widetilde{C}$ are good, so that Condition 5 in the definition of "strong" holds for $(G, K, N, \vartheta, \eta)$. Thus we can apply induction to $(G, U, K, N, \vartheta, \eta)$.

That (U, N, L, η, φ) is strong is seen as follows: First, $MN = MK \cap U \trianglelefteq U$, so Condition 1 holds. Conditions 2, 3 and 4 are clear. It was remarked earlier that in fact the elements of M are K-φ-good, and thus the elements of M are N-φ-good, so Condition 5 holds.

By Proposition 2.17, (G, H, K, L, L, φ) can not be a minimal counterexample. Thus K/L is chief. As K/L is solvable, it is elementary abelian. Then one of three possibilities occurs [34, Theorem 6.18]: Either $\vartheta_L = \varphi$, or $\vartheta = \varphi^K$ or φ is fully ramified in K. In the first two cases, however, we have $(\vartheta_L, \varphi) = 1$ and thus Proposition 2.15 would apply in these cases. Thus φ is fully ramified in K, as claimed. □

Step 2. All elements of $C = \mathbf{C}_H(K/L)$ are K-good, that is, C centralizes $S = (\mathbb{C}Ke_\varphi)^L$.

Proof. We have to show that $\langle x, c \rangle = 1$ for all $x \in K/L$ and $c \in C$. Since $M/(M \cap C)$ acts coprimely and fixed point freely on K/L, we have $K/L =$

$[K/L, M]$. We may thus assume that $x = y^{-1}y^m$ for some $y \in K/L$ and $m \in M$. Then

$$\langle y^{-1}y^m, c\rangle = \langle y^{-1}, c\rangle\langle y^m, c\rangle = \langle y, c\rangle^{-1}\langle y^m, c^m\rangle\langle y^m, [m, c]\rangle = \langle y^m, [m, c]\rangle.$$

Since $[M, C] \subseteq M \cap C$ and elements of M are good, $\langle y^m, [m, c]\rangle = 1$, as was to be shown. □

As a result, we can replace M by MC and assume $C \leqslant M$.

Step 3. We may assume that M/C is a chief factor of H.

Proof. Let M_0/C be a chief factor of H with $M_0 \subseteq M$. As $M_0 \trianglelefteq H$ we see that $\mathbf{C}_{K/L}(M_0)$ is invariant under H. As K/L is chief, we have $\mathbf{C}_{K/L}(M_0) = 1$. Also $M_0K \trianglelefteq G$ since H normalizes M_0 and $G = HK$. We may thus replace M by M_0. □

Step 4. H/M is a noncyclic p-group where p is the prime dividing $|K/L|$.

Proof. The action of H/C on S induces a projective representation $H/C \to S$ with associated cohomology class $\alpha \in H^2(H/C, \mathbb{C}^*)$, say. Then $\alpha = 1$ if and only if the restriction of α to every Sylow subgroup of H/C is 1. (The "if" part follows by using corestriction [26, 20.10-20.12 and proof of 21.4].) Since α comes from a projective representation with degree a power of p, for every group $Q/C \leqslant H/C$ of p'-order, $\alpha_{Q/C} = 1$. Since by assumption $\alpha \neq 1$, there is $P/C \in \mathrm{Syl}_p(H/C)$ with $\alpha_{P/C} \neq 1$. By minimality of $|G : L|$, we have $H = PM$. As $H/M \cong P/P \cap M = P/C$ has a nontrivial cocycle, it is noncyclic. (Here we use that we are working over an algebraically closed field.) □

From now on we assume that $A = M/C$ is solvable. We can do so by using the Feit-Thompson theorem: If A is not solvable, then $|A|$ is even and so $|K/L|$ is odd. We have already reduced to the case where K/L is abelian (even chief) and thus can apply Theorem 4.23 to produce a complement $\widetilde{H}$ and a magic representation. All elements of $\widetilde{H}$ are good and thus $C \leqslant \widetilde{H}$. It follows that $(MK \cap \widetilde{H})/C$ is a complement of CK/C in MK/C and therefore conjugate to M/C. We conclude that $\widetilde{H}$ and H are also conjugate. Thus Theorem 4.23 finishes the proof in the case $|K/L|$ odd.

We remark that Dade [6] proves a result about the extendibility of a nonlinear character of a normal extraspecial subgroup E in a similar situation (a complement H of E is given with a normal subgroup acting coprimely and fixed-point-freely on E) without using a solvability hypothesis or the Feit-Thompson theorem.

Step 5. M/C is noncyclic.

Proof. Let p and $P \in \mathrm{Syl}_p(H/C)$ as in the proof of the last step. If M/C is cyclic, then it has prime order by step 3 and solvability. Since P acts on M/C this would imply that $P/\mathbf{C}_P(M/C)$ is cyclic. Let $T = \mathbf{C}_P(M/C)$. As $H = PM$, we have $T \lhd H$ and thus $C_{K/L}(T)$ is H-invariant. Since T/C and K/L are p-groups, $C_{K/L}(T) > 1$ and thus $C_{K/L}(T) = K/L$. But then $T = C$ and this contradicts P/C noncyclic. □

Step 6. There is A with $L < A < K$ such that

1. $M \leqslant U := \mathbf{N}_H(A)$,
2. $K/L = \prod_{h \in H:U} A^h/L$,
3. Different conjugates of A are orthogonal with respect to $\langle\ ,\ \rangle_\varphi$.

Proof. Let A_0/L be a simple M-submodule of K/L and set $N = \mathbf{C}_M(A_0/L)$. Then $N < M$ and M/N is cyclic since M/C is abelian. We set $A/L = \mathbf{C}_{K/L}(N)$. Then we have $L < A_0 \leqslant A$ and also $A < K$, since $N > C$ by step 5. As K/L is chief and $\prod_{h \in H} A^h$ is H-invariant, we conclude that $K/L = \prod_{h \in H} A^h$. If $h \notin \mathbf{N}_H(A)$ then $N^h \neq N$, so $M = N^h N$. Therefore $\mathbf{C}_{A^h/L}(N) = \mathbf{C}_{A^h/L}(M) = 1$. As M acts coprime on K/L, we conclude $A^h/L = [A^h/L, N]$. Let $a \in A^h/L$, $n \in N$ and $b \in A/L$. Then $\langle a^{-1}a^n, b\rangle = \langle a, b\rangle^{-1}\langle a, b\rangle = 1$ as $b^n = b$ and the form is invariant under n. Since $[A^h/L, N] = A^h/L$ we conclude that A^h and A are orthogonal (with respect to $\langle\ ,\ \rangle$). □

Step 7. Contradiction.

Proof. Let A and U be as in the last step and let $\eta \in \mathrm{Irr}(A \mid \varphi)$. By Lemma 4.39, $\mathrm{Irr}(A \mid \varphi) = \{\eta\}$. The character five $(AU, A, L, \eta, \varphi)$ is strong, and thus by minimality there is a magic representation $\tau\colon U/L \to (\mathbb{C}Ae_\varphi)$. By Lemma 4.39, there is a magic representation $\sigma\colon H \to S$, and this contradicts G being a counterexample. This finishes the proof of Theorem 4.38. □

4.6 Strong Odd Sections and Schur indices

Suppose $\varphi \in \mathrm{Irr}\, L$ is fully ramified in K, and $\vartheta \in \mathrm{Irr}\, K$ is the unique irreducible character lying above φ. Then $\mathbb{F}(\varphi) = \mathbb{F}(\vartheta)$ for any field $\mathbb{F}$ (of characteristic zero), since ϑ vanishes outside L.

Now assume that $L, K \trianglelefteq G$. Let $\mathbb{E}$ be some splitting field of G and $\Gamma = \mathrm{Gal}(\mathbb{E}/\mathbb{Q})$ be its Galois group. Then the direct product $G \times \Gamma$ acts on

$\operatorname{Irr} K$ and $\operatorname{Irr} L$. Let $\mathcal{O}$ and $\mathcal{P}$ be the $G\times\Gamma$-orbits of ϑ and φ, respectively, and $\mathcal{O}_0$ and $\mathcal{P}_0$ the G-orbits. Then the actions of $G\times\Gamma$ on $\mathcal{O}$ and $\mathcal{P}$ are naturally isomorphic, the isomorphism sending $\vartheta' \in \mathcal{O}$ to the unique constituent of ϑ'_L. If K/L has a complement H/L in G/L, then clearly $H_\varphi = H_\vartheta$. Observe that also $e_\varphi = e_\vartheta$. We have seen results to the end that $\mathbb{E}G_\vartheta e_\varphi \cong \mathbf{M}_n(\mathbb{E}H_\varphi e_\varphi)$ when additional conditions are given (for example if K/L is abelian of odd order). Clifford theory yields that also $\mathbb{E}Gf_0 \cong \mathbf{M}_n(\mathbb{E}Hf_0)$, where $f_0 = \sum_{\varphi'\in\mathcal{P}_0} e_{\varphi'}$. This also holds for $f = \sum_{\varphi'\in\mathcal{P}} e_{\varphi'}$. But as $f \in \mathbb{Q}K$, it is natural to ask if even $\mathbb{Q}Ge \cong \mathbf{M}_n(\mathbb{Q}Hf)$ holds.

We show this here for odd abelian character fives under the additional condition that the section K/L is strong. I do not know wether this condition is really necessary. As an application, we will prove that the Isaacs correspondence preserves Schur indices. Apparently this has gone unnoticed in the literature until now, although similar arguments are present in papers of Turull [61, 65] on character correspondences in solvable groups and refinements of the McKay conjecture. However, we do not use Turull's theory of Clifford classes. Character correspondences have already been used for Schur index computations [21, 22].

4.42 Theorem. *Let L, $K \trianglelefteq G$ with $L \leqslant K$ and K/L abelian of odd order. Assume $\varphi \in \operatorname{Irr} L$ is fully ramified in K and semi-invariant in G. Suppose there is $N \trianglelefteq G$ with $K \leqslant N \leqslant G_\varphi$ and $N/\mathbf{C}_N(K/L)$ acting coprimely and fixed point freely on K/L. Let f be the primitive idempotent in $\mathbf{Z}(\mathbb{Q}L)$ with $\varphi(f) \neq 0$, and $n = \sqrt{|K/L|}$. Then there is $H \leqslant G$ such that $HK = G$, $H \cap K = L$, every element of H_φ is K-φ-good, and $\mathbb{Q}Gf \cong \mathbf{M}_n(\mathbb{Q}Hf)$ as G/K-graded algebras.*

It is important to note that H is determined by the assumptions up to conjugacy:

4.43 Proposition. *Assume the situation of Theorem 4.42. Then there is a unique conjugacy class of subgroups $H \leqslant G$ satisfying:*

1. *$HK = G$ and $H \cap K = L$,*
2. *every element of $H \cap \mathbf{C}_N(K/L)$ is K-φ-good.*

We need a general lemma to show this.

4.44 Lemma. *Let $L \trianglelefteq G$, let $\varphi \in \operatorname{Irr} G$ and $x, y \in G_\varphi$ with $[x, y] \in L$.*

(i) If $g \in G$, then $\langle x^g, y^g\rangle_{\varphi^g} = \langle x, y\rangle_\varphi$.

(ii) If α is a field automorphism, then $\langle x, y\rangle_{\varphi^\alpha} = \langle x, y\rangle_\varphi^\alpha$.

Proof. Let $\mathbb{E} = \mathbb{Q}(\varphi)$. Remember that $\langle x, y\rangle_\varphi e_\varphi = [x, y][c_y, c_x]$, where $c_x \in \mathbb{E}Le_\varphi$ is such that $a^x = a^{c_x}$ for all $a \in \mathbb{E}Le_\varphi$, and similar for c_y. If $g \in G$, then $c_x^g \in (\mathbb{E}Le_\varphi)^g = \mathbb{E}Le_{\varphi^g}$. Any $b \in \mathbb{E}Le_{\varphi^g}$ can be written as $b = a^g$. Thus

$$b^{c_x^g} = (a^g)^{c_x^g} = a^{c_x g} = a^{xg} = a^{g x^g} = b^{x^g}.$$

It follows that

$$\langle x^g, y^g\rangle_{\varphi^g} e_{\varphi^g} = [x^g, y^g][c_y^g, c_x^g] = ([x, y][c_y, c_x])^g = (\langle x, y\rangle_\varphi e_\varphi)^g = \langle x, y\rangle_\varphi e_{\varphi^g}.$$

The first assertion follows. The proof of the second is similar: We may extend α naturally to an automorphism of $\mathbb{E}G$, acting trivially on G. Replacing g with α in the above argument, we see that

$$\langle x, y\rangle_{\varphi^\alpha} e_{\varphi^\alpha} = [x, y][c_y^\alpha, c_x^\alpha] = ([x, y][c_y, c_x])^\alpha = (\langle x, y\rangle_\varphi e_\varphi)^\alpha = \langle x, y\rangle_\varphi^\alpha e_{\varphi^\alpha}.$$

The proof follows. □

Proof of Proposition 4.43. Let $C = \mathbf{C}_N(K/L)$. Observe that then $\langle\ ,\ \rangle_\varphi$ is defined on $C/L \times K/L$. Let

$$B = \{c \in C \mid \langle c, x\rangle_\varphi = 1 \text{ for all } x \in K\}.$$

We claim that $B \trianglelefteq G$. Let $b \in B$ and $g \in G$. There is $\alpha \in \mathrm{Aut}(\mathbb{Q}(\varphi))$ such that $\varphi^{\alpha g} = \varphi$. Let $k \in K$. Using Lemma 4.44 twice, we get

$$\langle b^g, k^g\rangle_\varphi = \langle b^g, k^g\rangle_{\varphi^{\alpha g}} = \langle b, k\rangle_{\varphi^\alpha} = \langle b, k\rangle_\varphi^\alpha = 1.$$

Since $k^g \in K^g = K$ was arbitrary, it follows that $b^g \in B$. This estabilshes the claim.

Via the form $\langle\ ,\ \rangle$, the factor group C/B is isomorphic to a subgroup of $\mathrm{Lin}(K/L)$, and so $|C/B| \leqslant |K/L|$. Since φ is fully ramified in K/L, $\langle\ ,\ \rangle_\varphi$ on K/L is nondegenerate, and thus we have $B \cap K = L$. Therefore $|K/L| = |BK/B| \leqslant |C/B|$. It follows $BK = C$ and $C/B \cong K/L$. Since $|C/B| = |K/L|$ and $|N/C|$ are coprime, there is a complement, M/B. Let $H = \mathbf{N}_G(M)$. By the Frattini-argument, $G = HN = HMC = HBK = HK$. Moreover, we have $[H \cap K, M] \leqslant K \cap M = L$, and so $(H \cap K)/L \leqslant \mathbf{C}_{K/L}(M) = \mathbf{C}_{K/L}(N/C) = 1$, so that $H \cap K = L$. Since $K/L \cong C/B$ as group with M-action, the same argument shows that $H \cap C = B$, and every element of B is good.

Now let U be another subgroup with the properties given above. Then $U \cap C \leqslant B$, since $U \cap C$ is good. Since $C = C \cap UK = (C \cap U)K$, it follows that $C \cap U = B$. Since $N = N \cap UC = (N \cap U)C$, it follows that $(N \cap U)/B$ is a complement of C/B in N/B. By the conjugacy part of the Schur-Zassenhaus Theorem, $N \cap U = M^c$ with $c \in C$. It follows that $H^c = \mathbf{N}_G(M^c) \geqslant U$, and thus $\mathbf{N}_G(M^c) = U$ as claimed. □

4.45 Corollary. *If $(|N/K|, |K/L|) = 1$ in Theorem 4.42, then H is determined up to conjugacy by the properties $HK = G$ and $H \cap K = L$.*

Note. One can prove this special case directly using standard, purely group theoretic methods.

Proof. For such a subgroup, we have $(H \cap N)K = HK \cap N = N$, and $|(H \cap N)/L| = |N/K|$ is prime to $|K/L|$. From Lemma 4.6 it follows that every element of $H \cap N$ is good. Thus the proposition yields the result. □

Theorem 4.42 will be derived from the following more precise proposition.

4.46 Proposition. *In the situation of Theorem 4.42, set $S = (\mathbb{Q}Kf)^L$ and $\mathbb{E} = \mathbf{Z}(\mathbb{Q}Kf)$ $(= \mathbf{Z}(\mathbb{Q}Lf) = \mathbf{Z}(S) \cong \mathbb{Q}(\varphi))$. Then $S \cong \mathbf{M}_n(\mathbb{E})$, and there is a complement H/L of K/L in G/L and a magic crossed representation $\sigma\colon H/L \to S$, such that $H_\varphi \ni h \mapsto \sigma(Lh)e_\varphi$ is a canonical magic representation.*

The magic crossed representation is crossed with respect to the map $H \to \operatorname{Aut}\mathbb{E} \to \operatorname{Aut} S$, where the last map comes from an isomorphism $S \cong \mathbf{M}_n(\mathbb{E})$, that is, $\operatorname{Aut}\mathbb{E}$ acts on the entries of $\mathbf{M}_n(\mathbb{E})$.

Theorem 4.42 follows from the proposition by Theorem 2.30.

Proof of Proposition 4.46. In the proof it will be convenient to view the bilinear form $\langle\ ,\ \rangle_\varphi$ defined earlier as a form with values in $\mathbb{E}$. Namely, every $x \in G_\varphi$ acts trivially on $\mathbf{Z}(\mathbb{Q}Lf) = \mathbb{E}$ and thus there is $a_x \in \mathbb{Q}Lf$ with $b^{a_x} = b^x$ for all $b \in \mathbb{Q}Lf$. If $[x,y] \in L$, then $\langle x, y\rangle := [x,y][a_y, a_x] \in \mathbb{E}$. As in Lemma 4.4, on shows that $\langle x,y\rangle = [xa_x^{-1}, ya_y^{-1}] = (xa_x^{-1})^{-1}(xa_x^{-1})^y$.

We divide the proof in a number of steps.

Step 1. Let

$$\Omega = \bigcup_{k\in K} (S^* \cap \mathbb{Q}Lk)$$

be the set of graded units of S and set

$$P = [\Omega, N] \quad \text{and} \quad C = P \cap \mathbb{E}.$$

Then $P/C \cong K/L$ and $P\mathbb{E}^* = \Omega$.

Proof. By Lemma 4.12, $S \cong (\mathbb{Q}(\varphi)Ke_\varphi)^L$ is a twisted group algebra of K/L over $\mathbb{E}$, so that $\Omega/\mathbb{E}^* \cong K/L$. Thus $P/C \cong K/L$ will follow, if we can show $P\mathbb{E}^* = \Omega$.

The group N acts on Ω and centralizes $\mathbb{E}$, since $N \leqslant G_\varphi$. As $N/\mathbf{C}_N(K/L)$ acts coprimely and fixed point freely on $K/L \cong \Omega/\mathbb{E}^*$, we have $[\Omega/\mathbb{E}^*, N] = \Omega/\mathbb{E}^*$. It follows that $[\Omega, N]\mathbb{E}^* = \Omega$, as claimed. □

Step 2. P is finite and every coset of C in P contains an element u with $|C \cap \langle u \rangle| = 1$.

Proof. For $r \in \mathbb{N}$, let $B_r = \{x \in K/L \mid x^r = 1\}$. Then B_r is a characteristic subgroup of K/L and thus is normalized by N. It follows that $[B_r, N] = B_r$. Let Ω_r be the inverse image of B_r in Ω under the natural map $\Omega \to K/L$. It follows that $[\Omega_r, N]\mathbb{E}^* = \Omega_r$, and thus $[\Omega_r, N]C = \Omega_r \cap P$.

Suppose r divides $|K/L|$. We claim that $[\Omega_r, N]$ has exponent r. Let $x, y \in B_r$ and choose $s_x \in S^* \cap \mathbb{Q}x$ and $s_y \in S^* \cap \mathbb{Q}y$. Then

$$(s_x s_y)^r = s_x^r s_y^r [s_x, s_y]^{\binom{r}{2}} = s_x^r s_y^r \langle x, y \rangle^{\binom{r}{2}} = s_x^r s_y^r$$

as r is odd and $x^r = 1$.

Let $a \in N$. Then

$$[s_x, a]^r = (s_x^{-1} s_x^a)^r = s_x^{-r} (s_x^r)^a = 1,$$

as N centralizes $\mathbb{E}^*$. It follows that $[\Omega_r, N]$ is generated by elements of order r. The claim follows, since we saw $(s_x s_y)^r = s_x^r s_y^r$ before.

Taking for r the exponent of K/L yields that P has the same exponent as K/L, and thus C is finite (and cyclic). Then also $|P| = |K/L||C|$ is finite.

To show the last assertion, let $v \in P$ and set $r = \operatorname{ord}(Cv)$. Then $u \in Cv \cap [\Omega_r, N]$ has order r as desired. This finishes the proof. □

Step 3. View the inclusion $\mu\colon C \to \mathbb{E}$ as a linear character and let e_μ be the corresponding central primitive idempotent of the group algebra $\mathbb{E}C$. Then $S \cong \mathbb{E}Pe_\mu$ naturally. The character μ is fully ramified in P, and $P' \leqslant C = \mathbf{Z}(P)$, and the commutator map yields a nondegenerate alternating form on $P/C \times P/C$ with values in C.

Proof. The map $\mathbb{E}P \to S$ induced by the inclusion $P \subset S$ sends e_μ to $1_S = f$. It follows $S \cong \mathbb{E}Pe_\mu$. As S is central simple, μ is fully ramified in P. The other assertions follow now. □

Remark. Indeed, the character triple (P, C, μ) is isomorphic to (K, L, φ) over $\mathbb{E} \cong \mathbb{Q}(\varphi)$.

Step 4. The action of G on S defines an homomorphism

$$\kappa\colon G \to \widehat{G} := \{\alpha \in \operatorname{Aut} S \mid P\alpha = P\}$$

with L in the kernel.

Proof. Clear. (Here $\operatorname{Aut} S$ are the ring automorphisms of S.) □

Step 5. Let $I = \{\alpha \in \mathbf{C}_{\widehat{G}}(\mathbb{E}) \mid \alpha_P \in \operatorname{Inn} P\}$. Then $I \cong \operatorname{Inn} P$, and I is the kernel of the natural map $\widehat{G} \to \operatorname{Aut}(P/C) \times \operatorname{Aut} \mathbb{E}$.

Proof. The map $\mathbf{C}_{\widehat{G}}(\mathbb{E}) \to \operatorname{Aut} P$ is injective, since $\mathbb{E}$ and P generate S and $\widehat{G} \leqslant \operatorname{Aut} S$. By definition, I maps to $\operatorname{Inn} P$. As $S \cong \mathbb{E}Pe_\mu$, where $\mu \in \operatorname{Lin} C$, we see that every inner automorphism of P induces an automorphism of S, which is in $\mathbf{C}_{\widehat{G}}(\mathbb{E})$.

Clearly I is in the kernel of $\widehat{G} \to \operatorname{Aut}(P/C) \times \operatorname{Aut} \mathbb{E}$. Conversely, suppose $\alpha \in \widehat{G}$ acts trivially on P/C and on $\mathbb{E}$. Then α centralizes also C. It is well known [27, 4.2] and not difficult to show that an automorphism of P centralizing P/C and C is inner. Thus $\alpha \in I$ as claimed. □

Step 6. There is $\tau \in \widehat{G}$ such that τ inverts P/C, centralizes $\mathbb{E}$, $\tau^2 = 1$, and $U = \mathbf{C}_{\widehat{G}}(\tau)$ is a complement of I in $\widehat{G}$.

Proof. There is $\tau_0 \in \operatorname{Aut} P$ of order 2, inverting P/C and centralizing C [27, Lemmas 4.2, 4.3]. This τ_0 can be extended to an element τ of $\widehat{G}$ of order 2 and centralizing $\mathbb{E}$. Observe that τ maps to a central element of $\operatorname{Aut}(P/C) \times \operatorname{Aut} \mathbb{E}$. The rest of Isaacs' proof of a slightly less general result [27, 4.3] now runs through: $\langle I, \tau \rangle \trianglelefteq \widehat{G}$ and $\langle \tau \rangle \in \operatorname{Syl}_2 \langle I, \tau \rangle$, and thus by the Frattini argument $\widehat{G} = I\, \mathbf{C}_{\widehat{G}}(\tau)$. As τ inverts P/C, it follows $\mathbf{C}_I(\tau) = 1$, as desired. □

Step 7. Let $R = \{r \in P \mid r^\tau = r^{-1}\}$ with τ the automorphism of the last step. Then $P = \dot{\bigcup}_{r \in R} Cr$.

Proof. For arbitrary $x \in P$ one has $x^\tau = cx^{-1}$ for some $c \in C$. There is a unique $d \in C$ with $d^2 = c$ as C has odd order. For this d we get $d^{-1}x \in R$. Thus every coset of C contains exactly one element of R, as claimed. □

Step 8. Let $X, Y \leqslant P$ be two maximal abelian subgroups of P with $X \cap Y = C$ and $XY = P$. There is a unique element $\lambda \in \operatorname{Irr}(X \mid \mu)$ that is fixed by τ. Then

$$\{E_{r,s} = r^{-1} e_\lambda s \mid r, s \in R \cap Y\}$$

is a full set of matrix units in $\mathbb{E}Pe_\mu \cong S$.

Proof. First note that X is a maximal abelian subgroup of P if X/C is a maximal isotropic subspace of P/C with respect to the commutator form $P/C \times P/C \to C$. Thus X and Y as above exist.

As X is abelian, it follows that $R \cap X$ is a subgroup: We have $(rs)^\tau = r^{-1}s^{-1} = (rs)^{-1}$ for $r, s \in R \cap X$. It follows that $X = (X \cap R) \times C$.

As τ inverts X/C, there is a unique $\lambda \in \mathrm{Lin}(X \mid \mu)$ that is fixed by τ, namely $\lambda = 1_{X\cap R} \times \mu$. In particular, λ has values in $\mathbb{E}$. Then $\lambda^P \in \mathrm{Irr}\, P$, as μ is fully ramified in P and $|P/X| = |X/C|$. (It follows $\mathbb{E}Pe_\mu \cong \mathbf{M}_{|P:X|}(\mathbb{E}Xe_\lambda) \cong \mathbf{M}_{|P:X|}(\mathbb{E})$ from general Clifford theory, see Proposition A.14.) Since $\lambda^y \neq \lambda$ for $y \in P \setminus X$, it follows $e^r_\lambda e^s_\lambda = 0$ if $rs^{-1} \notin X$. Since $P = \dot\bigcup_{r\in R\cap Y} Xr$, it follows $E_{r,s}E_{u,v} = \delta_{s,u}E_{r,v}$ for $r, s, u, v \in R\cap Y$ and $e_\mu = \sum_{r\in R\cap Y} e^r_\lambda$. □

Step 9. There is a group homomorphism $\varepsilon\colon \mathrm{Aut}\,\mathbb{E} \to U = \mathbf{C}_{\widehat{G}}(\tau)$ such that $\varepsilon(\alpha)_{|\mathbb{E}} = \alpha$ for all $\alpha \in \mathrm{Aut}\,\mathbb{E}$ and $S^{\varepsilon(\mathrm{Aut}\,\mathbb{E})} \cong \mathbf{M}_n(\mathbb{Q})$.

Proof. Let $\{E_{r,s}\}$ be the set of matrix units of the last step. Now define $\varepsilon\colon \mathrm{Aut}\,\mathbb{E} \to \mathrm{Aut}\, S$ by

$$\left(\sum_{x,y\in Y} c_{x,y}E_{x,y}\right)^{\varepsilon(\alpha)} = \sum_{x,y\in Y} c^\alpha_{x,y}E_{x,y} \quad \text{for} \quad c_{x,y} \in \mathbb{E}.$$

It is clear that $\varepsilon(\alpha)_{\mathbb{E}} = \alpha$ and that the centralizer of the image of ε contains the full set of matrix units $\{E_{r,s}\}$.

As $E^\tau_{r,s} = E_{r^{-1},s^{-1}}$, it follows that $\varepsilon(\alpha)$ centralizes τ for all $\alpha \in \mathrm{Aut}\,\mathbb{E}$.

It remains to show that the image of ε is in $\widehat{G}$, that is, $\varepsilon(\alpha)$ maps P onto itself for $\alpha \in \mathrm{Aut}\,\mathbb{E}$. As C is a finite subgroup of $\mathbb{E}$, there is $k \in \mathbb{N}$ with $c^\alpha = c^k$ for all $c \in C$. Let $x \in X$. Then

$$x^{\varepsilon(\alpha)} = \left(x \sum_{r\in R\cap Y} e^r_\lambda\right)^{\varepsilon(\alpha)} = \sum_{r\in R\cap Y} (\lambda^r(x))^\alpha e^r_\lambda = \sum_{r\in R\cap Y} \lambda^r(x^k)e^r_\lambda = x^k.$$

Thus $\varepsilon(\alpha)$ maps X onto X.

Now we show that $\varepsilon(\alpha)$ maps $R\cap Y$ onto itself:

$$r^{\varepsilon(\alpha)} = \left(r \sum_{s\in R\cap Y} e^s_\lambda\right)^{\varepsilon(\alpha)} = \left(\sum_{s\in R\cap Y} E_{sr^{-1},s}\right)^{\varepsilon(\alpha)} = \sum_{s\in R\cap Y} E_{sr^{-1},s} = r,$$

so in fact $\varepsilon(\alpha)$ centralizes $R\cap Y$. As $P = (R\cap Y)X$, it follows that $\varepsilon(\alpha)$ maps P onto itself, and thus $\varepsilon(\alpha) \in \widehat{G}$. □

Step 10. Let $\gamma\colon \widehat{G} \to \mathrm{Aut}\,\mathbb{E}$ be the homomorphism defined by restriction from S to $\mathbb{E} = \mathbf{Z}(S)$. There is an $\varepsilon\circ\gamma$-crossed representation $\widetilde{\sigma}\colon U \to S^*$ with $s^u = s^{\varepsilon(\gamma(u))\widetilde{\sigma}(u)}$ for all $u \in U$ and $\widetilde{\sigma}_{|C_U(\mathbb{E})}$ canonical in the sense of Definition 4.27.

Proof. First we show the existence of $\widetilde{\sigma}_{C_U(\mathbb{E})}$. Form the semidirect product $\widetilde{G} = \mathbf{C}_U(\mathbb{E})P$. Then $\mu \in \operatorname{Irr} C$ is invariant in $\widetilde{G}$ and fully ramified in P, and we may apply the results of Section 4.4. We let $\widetilde{\sigma}\colon \mathbf{C}_U(\mathbb{E}) \to S \cong \mathbb{E}Pe_\mu$ be the canonical magic representation that exists by these results.

Now we want to extend this to the whole of U. Observe that U acts on S and on $\mathbf{C}_U(\mathbb{E})$. We claim that for all $u \in \mathbf{C}_U(\mathbb{E})$ and $v \in U$,

$$\widetilde{\sigma}(u^v) = \widetilde{\sigma}(u)^v.$$

View v as fixed and consider the map $u \mapsto \widetilde{\sigma}^v(u) = \widetilde{\sigma}(u^{v^{-1}})^v$. We will show that this is also a canonical magic representation. The claim will then follow from uniqueness. It is clear that $\widetilde{\sigma}^v$ is magic. Also the order of the determinant is unchanged. Let ψ be the character of $\widetilde{\sigma}_{\mathbf{C}_U(\mathbb{E})}$ with values in $\mathbb{E}$ on which U acts. Then ψ^v defined by $\psi^v(u^v) = \psi(u)^v$ is the character of $\widetilde{\sigma}^v$. Now observe that for $Q \leqslant \mathbf{C}_U(\mathbb{E})$ we have

$$(\psi^v_{Q^v}, 1_Q) = \frac{1}{|Q|}\sum_{q\in Q}\psi^v(q^v) = \frac{1}{|Q|}\sum_{q\in Q}\psi(q)^v = (\psi_Q, 1_Q)^v = (\psi_Q, 1_Q).$$

Applying this to $Q \in \operatorname{Syl}_q(\mathbf{C}_U(\mathbb{E}))$, we see that ψ^v is canonical. The claim follows.

Now we extend $\widetilde{\sigma}$ to U by defining $\widetilde{\sigma}(vu) = \widetilde{\sigma}(u)$ for $v \in \varepsilon(\Gamma)$, and claim that $\widetilde{\sigma}$ has the required properties.

Let u_1, $u_2 \in U_\varphi$ and v_1, v_2 in $\varepsilon(\Gamma)$. Then

$$\widetilde{\sigma}(v_1u_1)^{v_2}\widetilde{\sigma}(v_2u_2) = \widetilde{\sigma}(u_1)^{v_2}\widetilde{\sigma}(u_2) = \widetilde{\sigma}(u_1^{v_2}u_2) = \widetilde{\sigma}(v_1v_2u_1^{v_2}u_2) = \widetilde{\sigma}(v_1u_1v_2u_2).$$

Suppose $v = \varepsilon(\gamma)$. Then

$$s^{vu} = s^{\varepsilon(\gamma)u} = s^{\varepsilon(\gamma)\widetilde{\sigma}(u)} = s^{\varepsilon(\gamma)\widetilde{\sigma}(vu)}.$$

This finishes the proof. □

Step 11. Let $H = \kappa^{-1}(U)$. Then $G = HK$ and $H \cap K = L$.

Proof. As the bilinear form on K/L is nondegenerate, we see that K maps onto I under κ. Thus $I = K\kappa \subseteq G\kappa$. Let $g \in G$. Since $\widehat{G} = UI$, we get $g\kappa = u \cdot k\kappa$ with $u \in U$ and $k \in K$. Thus $(gk^{-1})\kappa \in U$, and it follows that $G = HK$. If $k \in H \cap K$, then $k\kappa \in U \cap I = 1$, so that k centralizes S. As φ is fully ramified in K, it follows $k \in L$. □

Step 12. There is a magic crossed representation $\sigma\colon H \to S$ extending the canonical magic representation $H_\varphi/L \to S$.

Proof. Set $\sigma = \widetilde{\sigma} \circ \kappa$. Then σ has the required properties. This step finishes the proof of Proposition 4.46 and Theorem 4.42. ☐

Together with Corollary 2.31 we get:

4.47 Corollary. *Assume the situation of the theorem. Then there is a choice-free bijection between* $\operatorname{Irr}(G \mid \vartheta)$ *and* $\operatorname{Irr}(H \mid \varphi)$ *preserving fields of values and Schur indices over any field (of characteristic zero).*

Next we prove the following theorem.

4.48 Theorem. *Let* $L, K \trianglelefteq G$ *with* $L \leqslant K$ *and* K/L *of odd order. Suppose there is* $M \leqslant G$ *with* $MK \trianglelefteq G$, $(|M/L|, |K/L|) = 1$ *and* $\mathbf{C}_{K/L}(M) = 1$. *Let* $\varphi \in \operatorname{Irr}_M L$ *and* $\vartheta \in \operatorname{Irr}_M K$ *with* $(\vartheta_L, \varphi) > 0$ *and set* $H = \mathbf{N}_G(M)$. *Then there is a natural choice free bijection between* $\operatorname{Irr}(G \mid \vartheta)$ *and* $\operatorname{Irr}(H \mid \varphi)$ *which commutes with field automorphisms and preserves Schur indices.*

Note. The solvability of K/L will be assumed.

We need the following version of the Going Down Theorem [34, Theorem 6.18] for semi-invariant characters:

4.49 Lemma. *Let* K/L *be an abelian chief factor of* G *and suppose* $\vartheta \in \operatorname{Irr} K$ *is* $\mathbb{F}$*-semi-invariant in* G. *Then one of the following possibilities occurs:*

1. $\vartheta = \varphi^K$ *with* $\varphi \in \operatorname{Irr} L$, *and either* φ *is* $\mathbb{F}$*-semi-invariant in* G *and* $K/L \cong \operatorname{Gal}(\mathbb{F}(\varphi)/\mathbb{F}(\vartheta))$, *or* $\mathbb{F}(\varphi) = \mathbb{F}(\vartheta)$ *and induction defines a bijection from* $\{\varphi^\alpha \mid \alpha \in \operatorname{Gal}(\mathbb{F}(\varphi)/\mathbb{F})\}$ *onto* $\{\vartheta^\alpha \mid \alpha \in \operatorname{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})\}$.
2. $\vartheta_L \in \operatorname{Irr} L$.
3. $\vartheta_L = e\varphi$ *with* $\varphi \in \operatorname{Irr} L$ *and* $e^2 = |K/L|$, *and* $\mathbb{F}(\vartheta) = \mathbb{F}(\varphi)$.

Proof. Let φ be an irreducible constituent of ϑ_L. Let

$$T = \{g \in G \mid \varphi^g \text{ is Galois conjugate to } \varphi \text{ over } \mathbb{F}\}.$$

Let $g \in G$ and pick $\alpha \in \operatorname{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ with $\vartheta^g = \vartheta^\alpha$. Then

$$((\vartheta^\alpha)_L, \varphi^\alpha) = (\vartheta_L, \varphi) = (\vartheta^g_L, \varphi^g) = (\vartheta^\alpha_L, \varphi^g),$$

and thus $\varphi^\alpha = \varphi^{gk}$ for some $k \in K$. It follows that $G = KT$. Since K/L is abelian, $K \cap T \trianglelefteq KT = G$ and thus either $K \cap T = L$ or $K \cap T = K$.

If $K \cap T = L$, then $\varphi^K = \vartheta$. Thus clearly $\mathbb{F}(\vartheta) \leqslant \mathbb{F}(\varphi)$. Let $\alpha \in \operatorname{Gal}(\mathbb{F}(\varphi)/\mathbb{F}(\vartheta))$. Then $(\vartheta_L, \varphi^\alpha) = (\vartheta_L, \varphi) > 0$ and thus $\varphi^\alpha = \varphi^k$ for some $k \in K \cap T = L$, so that $\varphi^\alpha = \varphi$. It follows that $\operatorname{Gal}(\mathbb{F}(\varphi)/\mathbb{F}(\vartheta)) = 1$, and thus $\mathbb{F}(\vartheta) = \mathbb{F}(\varphi)$. Then induction is a bijection from the Galois orbit of φ onto that of ϑ.

Now suppose $K \cap T = K$, that is $T = G$ and φ is semi-invariant in G. Consider the inertia group K_φ. For $g \in G$, there is $\alpha_g \in \operatorname{Aut}\mathbb{F}(\varphi)$ with $\varphi^g = \varphi^{\alpha_g}$, so that

$$K_\varphi^g = K_{\varphi^g} = K_{\varphi^{\alpha_g}} = K_\varphi.$$

It follows that $K_\varphi \trianglelefteq G$. Again, either $K_\varphi = K$ or $K_\varphi = L$.

If $K_\varphi = L$, then again $\varphi^K = \vartheta \in \operatorname{Irr} K$. By Lemma 2.19, K/L is isomorphic with a subgroup of the Galois group of $\mathbb{F}(\varphi)$. The fixed field of this subgroup is, by Corollary 2.20, the field generated by the sum of the K-conjugates of φ. However, $\mathbb{F}(\sum_{k\in K} \varphi^k) = \mathbb{F}(\vartheta)$ as ϑ vanishes on $K \setminus L$. Thus $K/L \cong \operatorname{Gal}(\mathbb{F}(\varphi)/\mathbb{F}(\vartheta))$.

Now assume $K_\varphi = K$, so that φ is invariant in K. Set

$$\Lambda = \{\lambda \in \operatorname{Lin}(K/L) \mid \lambda\vartheta = \vartheta\} \quad \text{and} \quad U = \bigcap_{\lambda\in\Lambda} \ker\lambda.$$

We claim that $U \trianglelefteq G$. If $\vartheta\lambda = \vartheta$, then $\vartheta^\alpha\lambda = \vartheta^\alpha$ for field automorphisms α, as ϑ^α and ϑ have the same zeros. Let $g \in G$ and $\lambda \in \Lambda$. From the semi-invariance of ϑ it follows that there is $\alpha \in \operatorname{Aut}\mathbb{F}(\vartheta)$ with $\vartheta^{\alpha g} = \vartheta$. Thus

$$\vartheta\lambda^g = (\vartheta^\alpha\lambda)^g = \vartheta^{\alpha g} = \vartheta.$$

Thus $\lambda^g \in \Lambda$, so that Λ is invariant in G. It follows that $U \trianglelefteq G$. Thus either $U = K$ or $U = L$. Now the proof is finished as the proof of the original Going Down Theorem: If $U = K$, then $\Lambda = \{1\}$ and thus the $\vartheta\lambda$ with $\lambda \in \operatorname{Lin}(K/L)$ are $|K/L|$ different constituents of φ^K occurring with the same multiplicity, e, so that

$$|K/L|\varphi(1) = \varphi^K(1) = e|K/L|\vartheta(1) = e^2|K/L|\varphi(1),$$

and it follows $e = 1$.

If $U = L$, then it follows that ϑ vanishes on $K \setminus L$, and thus $\vartheta_L = e\varphi$ with $e^2 = |K/L|$. It is clear that then $\mathbb{F}(\vartheta) = \mathbb{F}(\varphi)$. □

Proof of Theorem 4.48. As M/L acts coprimely and fixed point freely on K/L, it follows that over every $\varphi \in \operatorname{Irr}_M L$ lies a unique $\vartheta \in \operatorname{Irr}_M K$, and conversely (see Lemma 4.41). This bijection commutes with the action of H and with Galois action. In particular, $\mathbb{Q}(\vartheta) = \mathbb{Q}(\varphi)$ and $H_\vartheta = H_\varphi$. It is thus no loss to assume that φ is semi-invariant in H and ϑ is semi-invariant in G.

We use induction on $|G/L|$. Let $L < N \leqslant K$ with N/L a chief factor. By Lemma 4.41, there is a unique $\eta \in \operatorname{Irr}_M N$ that lies above φ, and is a constituent of ϑ_N. This η is also semi-invariant in H and has the same field of values as φ and ϑ. Let $U = \mathbf{N}_G(MN)$. If $N < K$, then induction applies to yield natural bijections between $\operatorname{Irr}(G \mid \vartheta)$ and $\operatorname{Irr}(U \mid \eta)$ with the required properties, and between $\operatorname{Irr}(U \mid \eta)$ and $\operatorname{Irr}(H \mid \varphi)$. We may thus assume that K/L is a chief factor of G. Then, by our Going Down Theorem for semi-invariant characters, one of three possibilities occurs.

First, suppose that $\varphi^K = \vartheta$. Here φ can not be semi-invariant in K, since this would imply $\mathbb{Q}(\vartheta) < \mathbb{Q}(\varphi)$ which is impossible. It follows that H is the inertia group of the Galois orbit of φ. Now Clifford correspondence (see Proposition 2.18) gives the desired bijection.

Now suppose that $\vartheta_L = \varphi \in \operatorname{Irr} L$. As $\mathbb{Q}(\vartheta) = \mathbb{Q}(\varphi)$, Proposition 2.33 applies with $n = 1$ and yields the bijection. Here the bijection is just restriction from G to H.

Thus we assume that φ is fully ramified in K. Now Corollary 4.47 (with $N = MK$) applies and yields the result. □

Note that oddness of $|K/L|$ was only applied in the last sentence of the proof (if solvability is assumed). Nevertheless the result is false for $|K/L|$ even.

The following corollary about the Isaacs correspondence is not true for the Glauberman correspondence.

4.50 Corollary. *Let A act coprimely on G, where $|G|$ is odd. Let $C = \mathbf{C}_G(A)$ and $\chi \in \operatorname{Irr}_A G$. Then the Isaacs correspondent χ^* of χ has the same Schur index as χ.*

Proof. Let $K = [G, A]$ and $L = K'$. There is an A-invariant constituent ϑ of χ_K, and a unique A-invariant constituent φ of ϑ_L. We work in the semidirect product of A and G. Set $M = AL$ and $H = ALC$. Then $MK \trianglelefteq AG$, $\mathbf{C}_{K/L}(M) = 1$ and $H = \mathbf{N}_{AG}(M)$. Thus the theorem applies and yields a bijection between $\operatorname{Irr}(G \mid \vartheta)$ and $\operatorname{Irr}(LC \mid \varphi)$ preserving Schur indices. It maps, by the very definition of the Isaacs correspondence, χ to an character ξ which is the Isaacs correspondent of χ^*. Now induction yields the result. □

4.51 Remark. All the results remain true if "Schur index" is replaced by the Brauer equivalence class $[[\chi]]_{\mathbb{F}}$ defined by an irreducible character χ.

Chapter 5

Glauberman Correspondence

5.1 Theorem. *Let G be a finite group, $K \trianglelefteq G$ and $M \leqslant G$ with $KM \trianglelefteq G$, $(|M|, |K|) = 1$ and M solvable. Set $H = \mathbf{N}_G(M)$ and $L = H \cap K$. Then*

1. *The Glauberman correpondence defines a bijection between $\operatorname{Irr}_M K$ and $\operatorname{Irr} L$ as H-sets.*
2. *If $\vartheta \in \operatorname{Irr} K$ and $\varphi \in \operatorname{Irr} L$ correspond, then there is a magic representation for the configuration $G_\vartheta, H_\varphi, K, L, \vartheta, \varphi$ over $\mathbb{C}$.*
3. *If M is a p-group, p a prime, then there is a magic representation over the field $\mathbb{F} = \mathbb{Q}_p(\vartheta)$, where $\mathbb{Q}_p$ is the field of p-adic numbers.*

Proof. Note that $L = \mathbf{C}_K(M)$. Thus the Glauberman correspondence is a bijection $\beta(K, M)$ from $\operatorname{Irr}_M K$ onto $\operatorname{Irr} L$. It follows from the uniqueness of the Glauberman correspondence that $\vartheta\beta(K, M)^h = \vartheta^h\beta(K, M)$ for all $h \in H$. So if $\varphi = \vartheta\beta(K, M)$ where $\vartheta \in \operatorname{Irr}_M K$, then $G_\vartheta \cap H = H_\varphi$. Note that by the Frattini argument we have $G = KH$. It follows $G_\vartheta = KH_\varphi$. In the proof of 2., we assume that $G_\vartheta = G$ to simplify notation. We work by induction on $|M/\mathbf{C}_M(K)|$. Choose $P \trianglelefteq M$ such that $P/C_M(K)$ is a chief factor of H. It is known that $\beta(K, M) = \beta(K, P)\beta(\mathbf{C}_K(P), M/P)$ [27]. Let $\eta = \vartheta\beta(K, P) \in \operatorname{Irr}_M(\mathbf{C}_K(P))$. Then η is H-invariant and thus invariant in $\mathbf{N}_G(P) = \mathbf{C}_K(P)H$. We may apply induction to $\mathbf{N}_G(P), H, \mathbf{C}_K(P), L, \eta, \varphi$. If $P < M$ we also may apply induction to $G, \mathbf{N}_G(P), K, \mathbf{C}_K(P), \vartheta, \eta$ and use Proposition 2.17 to finish the proof. Thus we may assume that $M/\mathbf{C}_M(K)$ is a chief factor of H, in particular a p-group.

For the rest of the proof, let $P = M/\mathbf{C}_M(K)$ be a p-group. Let $\mathbb{F} = \mathbb{Q}_p(\vartheta) = \mathbb{Q}_p(\varphi)$. (The last equation results from the fact that the Glauberman correspondence commutes with field automorphisms.) Setting $i = e_\vartheta e_\varphi$ and $S = (i\mathbb{F}Ki)^L$, we have to show that there is a magic representation $\sigma : H/L \to S^*$.

Let R be the integral closure of $\mathbb{Z}_p$ in $\mathbb{F}$. This is an unramified extension of $\mathbb{Z}_p$ with maximal ideal Rp. Let $k = R/Rp$ and let $\widetilde{\ } : RG \to kG$ be the canonical homomorphism. Since K is a p'-group, we have e_ϑ, e_φ and $i \in RK$. We set $\Sigma = (iRKi)^L$ and $T = (\widetilde{i}kK\widetilde{i})^L = \Sigma/p\Sigma$. Note that $T \cong \mathbf{M}_n(k)$, and as R is complete we can lift matrix units and it follows that $\Sigma \cong \mathbf{M}_n(R)$ and $S \cong \mathbf{M}_n(\mathbb{F})$.

Our first goal is to show that there is a magic representation $\tau : H/L \to T^*$. Now remember that there is a connection between the Glauberman correspondence and the Brauer homomorphism: The action of the group P makes kK into a P-algebra and we have the Brauer homomorphism $\beta_P : (kK)^P \to k\,\mathbf{C}_K(P) = kL$ sending $\sum_{k\in K}\lambda_k k$ to $\sum_{k\in L}\lambda_k k$. This map is an algebra homomorphism. Moreover, $\beta_P(\widetilde{e_\vartheta}) = \widetilde{e_\varphi}$ [26, §18]. It follows that $\beta_P(\widetilde{i}) = \beta_P(\widetilde{e_\vartheta}\widetilde{e_\varphi}) = \widetilde{e_\varphi}$ and $\beta_P(T^P) \subseteq (kL\widetilde{e_\varphi})^L = k\widetilde{e_\varphi}$. Since $\beta_P(i) = \widetilde{e_\varphi} \neq 0$ and β_P is an k-algebra homorphism, equality holds, so $\beta_P(T^P) = k\widetilde{e_\varphi}$.

Note that T is a permutation P-algebra, which means that T has a basis that is permuted by P: It is a direct summand of $(kK)^L$ and the last has a P-invariant basis, namely the L-conjugacy class sums.

We will next appeal to a theorem of Dade which we explain now: First, the action of P on T defines uniquely a group homomorphism from P to T^*, as $H^2(P, k^*) = 1 = H^1(P, k^*)$. We let notation as if $P \subseteq T^*$, which is harmless even if the homomorphism is not injective. Now Dade's theorem can be stated as follows: *There is a group homomorphism $\varepsilon : \mathbf{N}_{T^*}(P) \to k^*$ extending the canonical map $(T^P)^* \to k^*$.* In this form it was proved by Puig [53]. In particular, for $\lambda \in k^*$ we have $\lambda\varepsilon = \lambda$.

The action of H on T defines a projective representation $\tau : H/L \to T^*$. As H normalizes P, we see that $\tau(H/L) \subseteq \mathbf{N}_{T^*}(P)$. So we can replace $\tau(Lh)$ by $\varepsilon(\tau(Lh))^{-1}\tau(Lh)$ to get a magic representation from H/L into T^*.

Now it follows from Theorem 3.8 that there is $\sigma : H/L \to \Sigma^*$ lifting τ. This finishes the proof. □

We should remark that this theorem was announced by Dade, at least in the case where M is a p-group [8]. He used it to prove the McKay conjecture for p-solvable groups, more precisely to construct a bijection between the p'-characters of a p-solvable group G and those of the normalizer of a Sylow p-subgroup of G. A simpler proof of the McKay-conjecture was given by Okuyama and Wajima using an argument that is now known as the "Okuyama-Wajima counting argument". This argument proves that $|\mathrm{Irr}(G \mid \vartheta)| = |\mathrm{Irr}(L \mid \varphi)|$ in the above situation by showing that $h \in H_\varphi$ is φ-good if and only if h is ϑ-good. This in turn follows from Theorem 5.1 for H/M abelian [46, §15], [38]. The proof of Dade's theorem, which we used, depends on his classification of endo-permutation modules for abelian

p-groups, a rather deep result. Recently, Turull [64] proved a stronger version of Theorem 5.1 using a strengthened version of Dade's theorem.

It is natural to ask wether a similar result holds for the Isaacs correspondence. We conclude this section with some remarks on this topic. So assume the situation of Theorem 5.1, but skip the condition that M is solvable. Then Glauberman correspondence or Isaacs correspondence are defined, and when both are defined, they are equal. It is thus unambiguous to speak of the Glauberman-Isaacs correspondence. Given $\vartheta \in \operatorname{Irr}_M K$ and its Glauberman-Isaacs correspondent φ, it is thus natural to ask wether the Clifford extensions associated with the character triples $(G_\vartheta, K, \vartheta)$ and (H_φ, L, φ) are isomorphic. For the Glauberman correspondence, this is true, as shown above. The problem was stated in an expository paper of Navarro [51] on character correspondences and coprime action, and is attributed to Lluís Puig by him. More precisely, Navarro asks wether the character triples above are isomorphic. Mark L. Lewis [45] answered this question in the positive, as he showed for the Isaacs correspondence that the character triples are isomorphic in the sense of Isaacs, and pointed out that for the Glauberman correspondence a positive answer is contained in the work of Dade [8]. But while Dade proves in fact that the associated cohomology classes are equal, Lewis proves only an isomorphism of character triples in the weaker sense of Isaacs. The slightly more general problem of proving equality of the associated cohomology classes is thus still open, as far as I can see. We contend ourself with the remark that in attempts to solve this, a situation as in Corollary 2.12 occurs (with $U \trianglelefteq G$).

Appendix A

Strongly G-Graded Algebras

A.1 G-Graded Algebras and Modules

We collect some facts about strongly G-graded algebras and their modules. This material can be found in papers of Dade [9, 12] or in some books on representation theory [3, 49]. There is also a book of A. Marcus especially on group graded algebras [47]. G will always denote a group, and R a commutative ring. We use the following conventions: module homomorphisms are usually written opposite the scalars, and, if nothing else is said, A-module means left A-module. Thus homomorphisms are mostly written on the right.

A.1 Definition. A G-graded R-algebra is an R-algebra A with a family of R-submodules A_g, such that the following assertions hold:

1. $A = \bigoplus_{x \in G} A_x$,
2. $A_x A_y \subseteq A_{xy}$.

A is strongly (or fully) G-graded, if the stronger condition

2' $A_x A_y = A_{xy}$ for all $x, y \in G$

holds.

If every g-component A_g contains a unit u_g of A, then A is called a crossed product (of G with A_1).

Some authors [49] use the term „strongly graded algebra“ for what we call crossed products.

A.2 Remark. If A is strongly graded, then G acts on the ideals of A_1 via $I^x = A_{x^{-1}} I A_x$.

Next we recall the Miyashita action:

A.3 Theorem. *[12, Theorem 2.1] Let A be a strongly G-graded algebra and M and N be left A-modules.*

1. *For every $g \in G$ and $\varphi \in \mathrm{Hom}_{A_1}(M,N)$ there is a unique $\varphi^g \in \mathrm{Hom}_{A_1}(M,N)$ such that*

$$a_g(m\varphi^g) = (a_g m)\varphi \quad \text{for all} \quad a_g \in A_g \text{ and } m \in M.$$

2. *This defines an action of G on $\mathrm{Hom}_{A_1}(M,N)$ respecting addition and composition of homomorphisms.*

3. *For $H \leqslant G$, $\mathrm{Hom}_{A_H}(M,N) = \mathrm{Hom}_{A_1}(M,N)^H$, where $A_H = \bigoplus_{h\in H} A_h$.*

Sketch of proof. There are $a_1,\dots,a_n \in A_{g^{-1}}$ and $b_1,\dots,b_n \in A_g$ with $a_1b_1 + \dots + a_nb_n = 1_A$ (since $A_{g^{-1}}A_g = A_1$). If $b_i(m\varphi^g) = (b_i m)\varphi$, then we must have $m\varphi^g = \sum_i a_i(b_i m)\varphi$. Define φ^g by the last equation and check that it has the required properties. □

A.4 Remark. Similarly, G acts on $\mathbf{C}_A(A_1)$ by $c^g = \sum_i a_i c b_i$, and this action fulfills $a_g c^g = c a_g$ for $c \in \mathbf{C}_A(A_1)$ and $a_g \in A_g$. Moreover, this action respects the grading of $\mathbf{C}_A(A_1)$, so that G acts on $Z(A_1)$. It does so in a way compatible with the action on the ideals of A_1, in the sense that $z^g A_1 = (zA_1)^g$.

A.5 Proposition. *[9, §3] Let A be a strongly G-graded algebra and M and N G-graded A-modules. Suppose that G is finite or M finitely generated. Then $\mathrm{Hom}_A(M,N)$ is G-graded by*

$$\mathrm{Hom}_A(M,N)_g = \{\alpha \in \mathrm{Hom}_A(M,N) \mid M_1\varphi \subseteq N_g\}.$$

Moreover, $\mathrm{Hom}_A(M,N)_g \cong \mathrm{Hom}_{A_1}(M_1,N_g)$. If K is another graded A-module, then

$$\mathrm{Hom}_A(M,N)_g \,\mathrm{Hom}_A(N,K)_h \subseteq \mathrm{Hom}_A(M,K)_{gh}.$$

In particular, $\mathrm{End}_A(M)$ is a G-graded ring.

A.6 Proposition. *Let M be a graded module over the strongly graded ring A. Then $(\mathrm{End}_A M)_g$ contains a unit if and only if $M_x \cong M_1$ as A_1-modules.*

A.2 Graded Morita equivalences

To study Clifford Theory in view of Schur indices and Galois action, Turull [60] introduced the concept of Clifford classes. These are certain equivalence classes of simple G-algebras. Marcus [48] introduced the viewpoint of strongly G-graded algebras. In the next sections we review these concepts and show how the results of this thesis fit into that theory. We begin to consider equivalence of strongly G-graded algebras.

We will need Morita's theory of equivalences of module categories, as explained in [40, §18] or [1, Chapter II]. Let P be a left G-graded A-module. Let $(A, B, {}_AP_B, {}_BQ_A, \alpha, \beta)$ be the Morita context associated with P [40, §18C]. That is, $B = \operatorname{End}_A P$, $Q = \operatorname{Hom}_A(P, A)$ and $\alpha\colon P \otimes_B Q \to A$ is defined by $(p \otimes q)^\alpha = pq$, and $\beta\colon Q \otimes_A P \to B$ is defined by $p_0(q \otimes p)^\beta := (p_0 q) \cdot p$. From Proposition A.5 it follows that B and Q are graded, too, and it is easy to check that α and β are homomorphisms of graded bimodules, that is $(P_x \otimes Q_y)^\alpha \subseteq A_{xy}$ and similarly for β.

A.7 Definition. A G-graded Morita context is a six-tuple $(A, B, M, N, \alpha, \beta)$ where A and B are G-graded algebras, ${}_AM_B$ and ${}_BN_A$ are graded bimodules, and $\alpha\colon {}_AM \otimes_B N_A \to A$ and $\beta\colon {}_BN \otimes_A M_B \to B$ are homomorphisms of graded bimodules, such that $(m_1 \otimes n)^\alpha m_2 = m_1(n \otimes m_2)^\beta$ (MNM-associativity) and $(n_1 \otimes m)^\beta n_2 = n_1(m \otimes n_2)^\alpha$ for all m, m_1, $m_2 \in M$ and n, n_1, $n_2 \in N$. The Morita context is called surjective if α and β are surjective.

Note that if A and B are R-algebras and if ${}_AM_B$ is an A-B-bimodule, then M has a left and a right R-module structure. It is not necessarily true that $rm = mr$ for all $m \in M$ and $r \in R$, however. But for the rest of this appendix, we tacitely assume: *All A-B-bimodules ${}_AM_B$ over R-algebras A and B are R-balanced, that is $rm = mr$ for $r \in R$ and $m \in M$.*

A.8 Definition. We say that two G-graded R-algebras A and B are graded Morita equivalent, if there is a graded surjective Morita context

$$(A, B, {}_AM_B, {}_BN_A, \alpha, \beta)$$

(where M and N are R-balanced).

A graded Morita equivalence is a special case of an ordinary Morita equivalence and so the Morita Theorems hold in this case. In particular, α and β are isomorphisms, M and N are progenerators as A- and as B-modules, and $(A, B, M, N, \alpha, \beta)$ is the Morita context associated with M. By „Morita I“, $X \mapsto X \otimes_A M$ and $Y \mapsto Y \otimes_B N$ define a category equivalence between right A-modules and right B-modules, and $U \mapsto N \otimes_A U$ and $V \mapsto M \otimes_B V$

a category equivalence between left A-modules and left B-modules. In the terminology of Bass [1, Chapter II], these are equivalences of R-categories, since we assume M and N to be R-balanced. Before going on, we note

A.9 Lemma. *Let A be strongly G-graded and M a graded left A-module. Then*

1. *${}_AM$ is finitely generated if and only if ${}_{A_1}M_1$ is.*
2. *${}_AM$ is projective if and only if ${}_{A_1}M_1$ is projective.*
3. *If ${}_{A_1}M_1$ is a generator, then ${}_AM$ is a generator.*

Proof. 1. Clear since M_1 is a direct summand of M and $M \cong A \otimes_{A_1} M_1$.

2. If ${}_AM$ is projective, it is a summand of a free A-module. But A as A_1-module is projective, so M is a direct summand of a projective A_1-module. Since M_1 is a direct summand of M as A_1-module, it follows that M_1 is projective as an A_1-module. Conversely, if M_1 is projective, then M_1 is a direct summand of a free module F, and thus $M \cong A \otimes_{A_1} M_1$ is a direct summand of the free A-module $A \otimes_{A_1} F$.

3. If ${}_{A_1}M_1$ is a generator, then there are finitely many $\varphi_i \in \operatorname{Hom}_{A_1}(M_1, A_1)$ and $m_i \in M_1$ with $\sum_i m_i\varphi_i = 1_{A_1}$. As $M \cong A \otimes_{A_1} M_1$, the φ_i define homomorphisms $\widehat{\varphi}_i \colon M \to A$ by scalar extension. Then $\sum_i m_i\widehat{\varphi}_i = 1$, so ${}_AM$ is a generator.

□

A.10 Example. Let $G = \{1, x\}$ be a cyclic group of order 2, and let $A = \mathbf{M}_2(k)$. Set

$$A_1 = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}, A_x = \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}, \quad M_1 = \left\{ \begin{pmatrix} * \\ 0 \end{pmatrix} \right\}, M_x = \left\{ \begin{pmatrix} 0 \\ * \end{pmatrix} \right\}.$$

Obviously $M = M_1 \oplus M_x$ is an A-progenerator, but M_1 is not an A_1-progenerator.

A.11 Theorem. *[47, Corollary 2.3.7] Assume that $(A, B, M, N, \alpha, \beta)$ is a graded Morita context with α and β surjective. Then the following assertions hold:*

(i) $B \cong \operatorname{End}_A M \cong \operatorname{End} N_A$ and $A \cong \operatorname{End} M_B \cong \operatorname{End}_A N$ as G-graded R-algebras.

(ii)

$$N \cong \mathrm{Hom}_A(M, A) \cong \mathrm{Hom}_B(M, B)$$
$$\text{and} \quad M \cong \mathrm{Hom}_A(N, A) \cong \mathrm{Hom}_B(N, B)$$

as graded bimodules.

(iii) Let ${}_AU$, ${}_BV$, X_A and Y_B be G-graded modules. Then

$$N \otimes_A U \cong \mathrm{Hom}_A(M, U) \qquad X \otimes_A M \cong \mathrm{Hom}_A(N, X)$$
$$M \otimes_B V \cong \mathrm{Hom}_B(N, V) \qquad Y \otimes_B N \cong \mathrm{Hom}_B(M, Y)$$

as G-graded modules over A or B, respectively.

For a subgroup $H \leqslant G$ and a G-graded algebra we write $A_H = \bigoplus_{h \in H} A_h$, and similar for G-graded modules. Moreover, given a graded Morita context as before, we write

$$\alpha_H \colon M_H \otimes_{B_H} N_H \to A_H, \qquad (m \otimes_{B_H} n)^\alpha = (m \otimes_B n)^\alpha, \qquad \text{and}$$
$$\beta_H \colon N_H \otimes_{A_H} M_H \to B_H, \qquad (n \otimes_{A_H} m)^\alpha = (n \otimes_A m)^\alpha.$$

The inclusion $A_H \hookrightarrow A$ induces the restriction functor from A-modules to A_H-modules, and similarly for B-modules. We also have induction from A_H-modules to A-modules, which sends the A_H-module to the A-module $X \otimes_{A_H} A$. If no confusion can arise, we denote the induced module simply by X^G.

A.12 Theorem. *[47, Corollary 2.3.7e] Let $(A, B, M, N, \alpha, \beta)$ be a G-graded Morita context with α and β surjective, where A and B are strongly graded. Then we have:*

1. *$(A_H, B_H, M_H, N_H, \alpha_H, \beta_H)$ is an H-graded Morita context with α_H and β_H surjective.*

2. *The resulting category equivalences commute with restriction and induction of modules: That is, for an right A-module,*

$$X \otimes_A M \cong X \otimes_{A_H} M_H$$

as right B_H-modules, and for an right A_H-module U,

$$(U \otimes_{A_H} A) \otimes_A M \cong (U \otimes_{A_H} M_H) \otimes_{B_H} B$$

as right B-modules. Similar statements are true for left modules.

3. *Let C be another ring and ${}_CU_A$ and ${}_CV_A$ be bimodules. Then*

$$\mathrm{Hom}({}_CU_{A_1}, {}_CV_{A_1}) \cong \mathrm{Hom}({}_CU \otimes_A M_{B_1}, {}_CV \otimes_A M_{B_1})$$

as G-modules (where the G-action is the Miyashita action).

4. $\mathbf{C}_A(A_1) \cong \mathbf{C}_B(B_1)$ *as G-algebras.*

A.13 Example. Let A be a G-graded R-algebra and $e \in A_1$ an idempotent. Suppose that $AeA = A$. Then $(A, eAe, Ae, eA, \alpha, \beta)$ is a graded Morita context, where α and β are induced by multiplication. This Morita context is surjective and yields that A and eAe are graded Morita equivalent. In general, eAe will not be strongly graded, even if A is. But if we assume that A is strongly graded and that $A_1eA_1 = A_1$, then eAe is strongly graded, as was observed by Dade [4, Corollary 3.3].

The next proposition is well known. It can be regarded as a special case of the last example.

A.14 Proposition. *Let A be strongly G-graded. Let e be a central primitive idempotent of A_1, and let*

$$T = \{x \in G \mid e^x = e\}$$

be its inertia group. Set $f = \mathrm{T}_T^G(e) = \sum_{g \in G:T} e^g$. Then induction defines an equivalence from the category of A_Te-modules to the category of Af-modules. For any A_Te-modules U and V,

$$\mathrm{Hom}_{A_T}(U, V) \cong \mathrm{Hom}_A(U \otimes_{A_T} A, V \otimes_{A_T} A) \quad \text{via} \quad \alpha \mapsto \alpha \otimes 1.$$

Proof. We have $AeA = Af$, where f is a central idempotent of A. Thus Af and eAe are graded Morita equivalent, by the last example. The action of G on $\mathbf{Z}(A_1)$ is such, that $eA_g = A_ge^g$. It follows that for $g \notin T$, we have $eA_ge = A_ge^ge = 0$. Thus $eAe = A_Te = eA_T$. If U is an module over $eAe = A_Te$, then

$$U \otimes_{eAe} eA = U \otimes_{A_Te} eA \cong U \otimes_{A_T} A$$

naturally, since $Ue = U$ and $U(1-e) = 0$. □

A.15 Theorem. *Suppose that A and B are G-graded R-algebras, and let Λ be another R-algebra. If A and B are graded Morita equivalent (as R-algebras), then $A \otimes_R \Lambda$ and $B \otimes_R \Lambda$ are graded Morita equivalent, too. If Λ is commutative, the equivalence is one of Λ-algebras and -categories.*

Proof. The grading of $A \otimes_R \Lambda$ is given by $A \otimes_R \Lambda = \bigoplus_{x \in G} A_x \otimes_R \Lambda$, and similar for B. Suppose the equivalence between A and B comes from the surjective G-graded Morita-context

$$(A, B, {}_AM_B, {}_BN_A, \alpha, \beta).$$

We may form the Morita context

$$(A \otimes_R \Lambda, B \otimes_R \Lambda, {}_{A\otimes_R\Lambda}M \otimes_R \Lambda_{B\otimes_R\Lambda}, {}_{B\otimes_R\Lambda}N \otimes_R \Lambda_{A\otimes_R\Lambda}, \widehat{\alpha}, \widehat{\beta}),$$

where $\widehat{\alpha}$, for example, is defined by

$$\big((m \otimes_R \lambda_1) \otimes_{B\otimes_R\Lambda} (n \otimes_R \lambda_2)\big)^{\widehat{\alpha}} = (m \otimes_B n)^\alpha \otimes_R \lambda_1\lambda_2.$$

We skip the tedious verifications that the bimodule structures and maps involved are well defined, since they are not difficult. We only remark that to define the left $A \otimes_R \Lambda$-module structure of $M \otimes_R \Lambda$, we need that M is R-balanced: This comes from the fact that $M \otimes_R \Lambda$ is defined via the *right* R-module structure of M. It is then clear that the new Morita context is G-graded and surjective, too, so that $A \otimes_R \Lambda$ and $B \otimes_R \Lambda$ are graded Morita equivalent. □

A.16 Definition. A strongly graded algebra A is called simple, if A_1 has no nonzero proper G-invariant ideals.

A.17 Theorem. *Let A be a strongly G-graded R-algebra (G a finite group) and U a simple A_1-module. Suppose that A_1 is semilocal. Set*

$$I = \bigcap_{g \in G} (\operatorname{Ann}_{A_1}(U))^g.$$

The following strongly graded algebras are simple and belong to the same equivalence class:

a) A/IA,

b) $\operatorname{End}_A(A \otimes_{A_1} V)$ *where V is an A_1-module with $IV = 0$ and $A_g \otimes_{A_1} V \cong V$ as A_1-modules for all $g \in G$.*

Proof. Let $D = \operatorname{End}_{A_1} U$. Since $A_1/J(A_1)$ is semisimple, we have that $\dim U_D$ is finite and by the density theorem, $A_1/\operatorname{Ann}_{A_1} U \cong \operatorname{End} U_D$.

That A/IA is strongly graded and simple is clear.

Now let V as in b). Then V is semisimple and its annihilator is G-invariant. Since I is the maximal G-invariant ideal contained in the maximal ideal $\operatorname{Ann}_{A_1} U$, we get $I = \operatorname{Ann}_{A_1} V$. As A_1/I is a semisimple ring, V is projective as an A_1/I-module, and a generator. Thus $A \otimes_{A_1} V$ is an A-progenerator, and thus A and $E = \operatorname{End}_A(A \otimes_{A_1} V)$ are graded Morita equivalent. Since V is G-invariant, E is a crossed product over E_1, in particular strongly graded. □

A.18 Remark. Suppose that A is a semisimple algebra over the field $\mathbb{F}$ of characteristic 0. Let V be a simple A-module and χ the character of a simple submodule of the $A \otimes_{\mathbb{F}} \overline{\mathbb{F}}$-module $V \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. Marcus [48, 3.1] defines $[[\chi]]$ to be the equivalence class of the simple strongly graded algebra $\operatorname{End}_A(A \otimes_{A_1} V)$. One can show that $\mathbf{Z}(A_1)^G = \mathbb{F}(\chi_{A_1}) = \mathbb{F}(\{\chi(a) \mid a \in A_1\})$. Marcus goes on to show that the same equivalence class is obtained if one chooses another module which is „χ-quasihomogeneous“. This follows also directly from Theorem A.17.

A.3 G-Algebras

Turull [60] introduced the concept of central simple G-acted $\mathbb{F}$-algebras and equivalence classes thereof to study Schur indices and related questions in the context of Clifford theory. We shortly review (and generalize) this concept.

A.19 Definition. An G-algebra over R or an G-acted R-algebra is an R-algebra S together with an homomorphism $G \to \operatorname{Aut}_R(S)$. We write the action of G on S as s^g.

To define an equivalence relation on the class of G-algebras, we first introduce the concept of G-acted S-module. This is an S-module X together with an action of G on X (as R-module), written exponentially, such that $(sx)^g = s^g x^g$.

A.20 Definition. An G-acted Morita context (over R) is a Morita context $(S, T, {}_SP_T, {}_TQ_S, \alpha, \beta)$, where S and T are G-algebras, P and Q are bimodules with G-action, such that $(spt)^g = s^g p^g t^g$ and $(tqs)^g = t^g q^g s^g$, and α and β commute with the G-action.

A.21 Remark. With every Morita context $(S, T, P, Q, \alpha, \beta)$ is associated the Morita ring

$$\begin{pmatrix} S & P \\ Q & T \end{pmatrix} = \left\{ \begin{pmatrix} s & p \\ q & t \end{pmatrix} \mid s \in S, p \in P, q \in Q, t \in T \right\}.$$

Multiplication is given by formal matrix multiplication. If S and T are R-algebras and P and Q are R-balanced bimodules, then the Morita ring is an

R-algebra, too. The Morita context admits an G-action if and only if the action of G on S and T can be extended to an action of G on the Morita ring (as R-algebra).

A.22 Definition. Two G-algebras S and T are called equivalent, if there is a Morita context with G-action $(S, T, P, Q, \alpha, \beta)$, such that α and β are surjective.

This is not the original definition of Turull, but can be shown to be equivalent for central simple algebras (see below).

A.23 Lemma. *If the G-algebras S and T are equivalent, then $\mathbf{Z}(S) \cong \mathbf{Z}(T)$ as G-algebras.*

Proof. A faithfully balanced S-T-bimodule induces an isomorphism ε from $\mathbf{Z}(S)$ to $\mathbf{Z}(T)$ defined by $zx = xz^{\varepsilon}$ for $x \in X$ and $z \in \mathbf{Z}(S)$. If X admits an G-action, then

$$x^g(z^{\varepsilon})^g = (xz^{\varepsilon})^g = (zx)^g = z^g x^g = x^g (z^g)^{\varepsilon}.$$

Thus ε is an isomorphism of G-algebras. □

Now we review how G-algebras and strongly G-graded algebras are related.

A.24 Definition. Let S be an G algebra. We denote by $[G]S$ the crossed product of G with S, that is the set of formal sums

$$\sum_{g \in G} g s_g, \qquad s_g \in S,$$

with multiplication defined by

$$(g s_g)(h s_h) = gh s_g^h s_h.$$

A.25 Remark. $[G]S$ is a strongly G-graded R-algebra in a natural way.

A.26 Theorem. *Let G be a finite group. The maps*

$$S \mapsto [G]S \quad \textit{and} \quad A \mapsto \operatorname{End}_{A_1} A$$

induce a bijection between the equivalence classes of G-acted R-algebras and the equivalence classes of strongly G-graded R-algebras.

If we would restrict our attention to, say, algebras finitely generated as R-modules, then the set of equivalence classes is indeed a set. All equivalence classes do not form a set.

Proof. The proof follows partly Marcus [48, 2.16]. Note that $\operatorname{End}_{A_1} A$ is indeed a G-algebra by the Miyashita action.

Step 1. If the G-algebras S and T are equivalent, then $[G]S$ and $[G]T$ are G-graded Morita equivalent.

Proof. Let $(S, T, P, Q, \alpha, \beta)$ be a Morita context inducing the equivalence between S and T. Consider the R-module

$$[G]P = \left\{ \sum_{g \in G} g p_g \;\middle|\; p_g \in P \right\}.$$

We can make $[G]P$ into a $[G]S$-$[G]T$-bimodule by defining

$$\left(\sum_{g \in G} g s_g \right) \left(\sum_{h \in G} h p_h \right) = \sum_{g,h \in G} g h s_g^h p_h$$
$$\text{and} \quad \left(\sum_{g \in G} g p_g \right) \left(\sum_{h \in G} h t_h \right) = \sum_{g,h \in G} g h p_g^h t_h.$$

Similarly, $[G]Q$ becomes a $[G]T$-$[G]S$-bimodule. Now define a map

$$[G]\alpha \colon [G]P \otimes_{[G]T} [G]Q \to [G]S$$

by

$$\left(\sum_{g \in G} g p_g \otimes \sum_{h \in G} h q_h \right)^{[G]\alpha} = \sum_{g,h} g h (p_g^h \otimes q_h)^\alpha.$$

Define $[G]\beta$ similarly. It is now routine to verify that

$$([G]S, [G]T, [G]P, [G]Q, [G]\alpha, [G]\beta)$$

is a surjective, graded Morita context. Its Morita ring is naturally isomorphic with the skew group ring

$$[G] \begin{pmatrix} S & P \\ Q & T \end{pmatrix}.$$

□

Step 2. If $[G]S$ and $[G]T$ are graded Morita equivalent, then S and T are equivalent as G-algebras.

Proof. Let

$$([G]S, [G]T, {}_{[G]S}M_{[G]T}, {}_{[G]T}N_{[G]S}, \alpha, \beta)$$

be a graded surjective Morita context. Then

$$(S, T, M_1, N_1, \alpha_1, \beta_1)$$

is a surjective Morita context by Theorem A.12 (as $S \cong ([G]S)_1$). Moreover G acts on M_1 via $m_1^g := g^{-1}1_S \cdot m_1 \cdot g1_T$ for $m_1 \in M_1$ and $g \in G$. In the same way, G acts on N_1. Easy verifications now show that $(S, T, M_1, N_1, \alpha_1, \beta_1)$ is a Morita context with G-action, and so S and T are equivalent, as claimed. □

Until now we have shown that $S \mapsto [G]S$ yields a well defined injective map from the equivalence classes of G-algebras to the equivalence classes of strongly graded algebras. The last step yields that the map is surjective, and its inverse.

Step 3. $[G](\operatorname{End}_{A_1} A)$ and A are graded Morita equivalent for any strongly G-graded algebra A.

Proof. Set $M = A \otimes_{A_1} A$ and $M_x = A_x \otimes A$. View M as graded left A-module. Forgetting about the grading, ${}_AM \cong {}_AA^{|G|}$. It follows that ${}_AM$ is an A-progenerator. Thus A and $B = \operatorname{End}_A M$ are graded Morita equivalent.

We conclude the proof by showing that $B \cong [G](\operatorname{End}_{A_1} A)$. Remember (Proposition A.5) that the grading of B is given by

$$B_g = \{b \in B \mid M_1 b \subseteq M_g\}.$$

We define elements $\varepsilon_g \in B_g$ as follows: First, choose elements $a_{i,g} \in A_g$ and $a'_{i,g} \in A_{g^{-1}}$ $(i \in I_g)$ with $\sum_i a_{i,g}a'_{i,g} = 1$. Then set

$$(a \otimes b)\varepsilon_g = \sum_i aa_{i,g} \otimes a'_{i,g}b.$$

The following statements hold:

1. ε_g is a well defined element of B_g.
2. ε_g is independent of the choice of the $a_{i,g}$ and $a'_{i,g}$.
3. $\varepsilon_g\varepsilon_h = \varepsilon_{gh}$.
4. For $\varphi \in \operatorname{End}_{A_1} A$, we have $1 \otimes \varphi^g = \varepsilon_{g^{-1}}(1 \otimes \varphi)\varepsilon_g \in B_1$.

We only show here that $(aa_1 \otimes b)\varepsilon_g = (a \otimes a_1 b)\varepsilon_g$ for $a_1 \in A_1$ and leave the other verifications to the reader. We have:

$$\begin{aligned}(aa_1 \otimes b)\varepsilon_g &= \sum_i aa_1 a_{i,g} \otimes a'_{i,g} b = \sum_i a \sum_j a_{j,g} a'_{j,g} a_1 a_{i,g} \otimes a'_{i,g} b \\ &= \sum_{i,j} aa_{j,g} \otimes a'_{j,g} a_1 a_{i,g} a'_{i,g} b = \sum_j aa_{j,g} \otimes a'_{j,g} a_1 b \\ &= (a \otimes a_1 b)\varepsilon_g.\end{aligned}$$

As $\operatorname{End}_{A_1} A \ni \varphi \mapsto 1 \otimes \varphi \in B_1$ is an isomorphism, it follows from the above properties of ε_g that

$$\sum_g g\varphi_g \mapsto \sum_g \varepsilon_g (1 \otimes \varphi_g)$$

defines an isomorphism of G-graded algebras between $[G](\operatorname{End}_{A_1} A)$ and $B = \operatorname{End}_A M = \operatorname{End}_A(A \otimes_{A_1} A)$. The proof is finished. □

A.27 Remark. Turull [60] calls a G-algebra S simple if it has no nontrivial G-invariant ideals. A G-algebra S over a field $\mathbb{F}$ is called central simple, if it is simple and $\mathbf{Z}(S)^G = \mathbb{F}$. Finally, he calls two G-algebras S and T equivalent, if there are $\mathbb{F}G$-modules U and V such that $S \otimes_{\mathbb{F}} \operatorname{End}_{\mathbb{F}}(U) \cong T \otimes_{\mathbb{F}} \operatorname{End}_{\mathbb{F}}(V)$ as G-algebras. Marcus [48] calls a strongly G-graded algebra A *simple*, if A_1 has no G-invariant ideals. He also shows that the set of equivalence classes of simple G-algebras is mapped bijectively to the equivalence classes of central simple strongly G-graded algebras by the maps of Theorem A.26. From these results it follows that Turull's definition of equivalence and ours coincide for central simple G-algebras.

A.4 The Brauer-Clifford group

It is not true that the equivalence classes of central simple strongly graded algebras form a group, but we may multiply certain equivalence classes. This is most easily described in the context of central simple G-algebras. First we describe a more general construction:

A.28 Proposition. *Suppose that S and T are two G-algebras. Let C be a commutative G-algebra and $C \to \mathbf{Z}(S)$ and $C \to \mathbf{Z}(T)$ homomorphisms of G-algebras. Then $S \otimes_C T$ is a G-algebra and its equivalence class depends only on the equivalence classes of S and T (and their C-algebra structures).*

Proof. The G-action is defined by $(s\otimes t)^g = s^g \otimes t^g$. This is well defined since $(1_S \cdot c)^g = 1_S \cdot c^g$ for $c \in C$, and similarly for T. Suppose that S is equivalent to $\widetilde{S}$ and let $(S, \widetilde{S}, X, Y, \alpha, \beta)$ be a surjective Morita context with G-action. The faithfully balanced bimodule X induces an isomorphism of G-algebras $\mathbf{Z}(S) \cong \mathbf{Z}(\widetilde{S})$, so that by composition we get a homomorphism $C \to \mathbf{Z}(\widetilde{S})$ of G-algebras. With respect to this C-algebra structure of $\widetilde{S}$, we can form the Morita context with G-action

$$(S \otimes_C T, \widetilde{S} \otimes_C T, X \otimes_C T, Y \otimes_C T, \alpha \otimes_C T, \beta \otimes_C T),$$

where $\alpha \otimes_C T$ and $\beta \otimes_C T$ are defined in the obvious way. We leave out the easy, but tedious verifications that this is well defined. (Note that the Morita ring associated with this new context is isomorphic with

$$\begin{pmatrix} S & X \\ Y & \widetilde{S} \end{pmatrix} \otimes_C T$$

as G-algebra.) It follows that $S \otimes_C T$ and $\widetilde{S} \otimes_C T$ are equivalent G-algebras, as claimed. □

Now let $\mathbb{F}$ be a field. We consider the set of equivalence classes of finite dimensional central simple G-algebras over $\mathbb{F}$. This set has in general no natural group structure. But using the last theorem, we get a partial multiplication as follows:

A.29 Proposition. *Let S and T be central simple G-algebras over the field $\mathbb{F}$. Also assume that we have an injection $\mathbf{Z}(S) \hookrightarrow \mathbf{Z}(T)$ of G-algebras. Then $S \otimes_{\mathbf{Z}(S)} T$ is a central simple G-algebra.*

Proof. It remains to show that $S \otimes_{\mathbf{Z}(S)} T$ is central simple as G-algebra. Let $1_S = e_1 + \ldots + e_r$ be a decomposition of 1_S into central primitive idempotents of S. For each i, let $e_i = f_{i1} + \ldots + f_{in}$ be a decomposition into central primitive idempotents of T. Note that since G permutes transitively the e_i's, the number n does not depend on i. We have

$$S \otimes_{\mathbf{Z}(S)} T \cong \bigoplus_i Se_i \otimes_{\mathbf{Z}(Se_i)} T \cong \bigoplus_{i,j} Se_i \otimes_{\mathbf{Z}(Se_i)} Tf_{ij}.$$

The algebra $Se_i \otimes_{\mathbf{Z}(Se_i)} Tf_{ij}$ is simple with center isomorphic with $\mathbf{Z}(Tf_{ij})$. Thus $S \otimes_{\mathbf{Z}(S)} T$ has center isomorphic with $\mathbf{Z}(T)$, the isomorphism is given by $s \otimes t \mapsto st$. This is an isomorphism of commutative G-algebras, so since T is central simple, it follows that $\mathbf{Z}(S \otimes_{\mathbf{Z}(S)} T)^G = \mathbb{F}$. Also $S \otimes_{\mathbf{Z}(S)} T$ has no nontrivial G-invariant ideals, since its block ideals $Se_i \otimes Tf_{ij}$ are permuted transitively by G. The proof is finished. □

A.30 Theorem. *Let C be a central simple, commutative G-algebra over some field $\mathbb{F}$. Then the set of equivalence classes of G-algebras with center isomorphic to C forms an abelian group, with multiplication induced by $S \otimes_C T$.*

Proof. If S and T belong to the same equivalence class of central simple G-algebras, their centers are isomorphic as G-algebras, so the definition makes sense. The multiplication is compatible with the equivalence relation by Proposition A.28. The equivalence class of C acts as identity under this multiplication. Finally, we show that $S \otimes_C S^{\mathrm{op}}$ belongs to the same equivalence class as C: As C is a direct products of fields and S is faithful as C-module, it follows from a theorem of Azumaya [40, 18.11] that S is an progenerator as C-module. The natural homomorphism $S \otimes_C S^{\mathrm{op}} \to \operatorname{End} S_C$ is injective, since its kernel is a G-invariant ideal of $S \otimes_C S^{\mathrm{op}}$, which is simple. By reasons of dimensions, it is an isomorphism. Also we have an action of G on S. Thus the Morita context associated with the C-progenerator S admits an action of G. This yields that C and $S \otimes S^{\mathrm{op}}$ are equivalent. □

A.31 Definition. We denote the group of the last theorem by $\operatorname{BrCliff}(G, C)$ and call it the Brauer-Clifford group of G over C.

This terminology is due to Turull [64] who uses this group in his recent work on the Glauberman correspondence.

A.32 Remark. Let C be a field with G-action. There is a natural homomorphism of abelian groups

$$\operatorname{BrCliff}(G, C) \to \operatorname{Br}(C) \to 0$$

defined by forgetting the G-action.

A.33 Corollary. *If $\varphi\colon C \to Z$ is an homomorphism of commutative simple G-algebras, then $S \mapsto S \otimes_C Z$ induces an homomorphism*

$$\operatorname{BrCliff}(G, C) \to \operatorname{BrCliff}(G, Z).$$

Proof. Clear from the last results and the isomorphism of G-algebras $(S \otimes_C Z) \otimes_Z (T \otimes_C Z) \cong (S \otimes_C T) \otimes_C Z$. □

By the bijections of Theorem A.26, we also get a multiplication of equivalence classes of strongly graded algebras. We do not describe how to multiply two equivalence classes of strongly graded algebras, but rather describe how to multiply a simple G-algebra with a strongly graded algebra. This will be used in our application.

A.34 Lemma. *Let A be a strongly G-graded algebra and S a simple G-algebra, with $\mathbf{Z}(S) \cong \mathbf{Z}(A_1) = C$ as G-algebras. Let $A * S$ be the following algebra: As abelian group, $A * S = A \otimes_C S$. Multiplication is defined by*

$$(a_g \otimes s_g)(a_h \otimes s_h) = a_g a_h \otimes s_g^h s_h \quad \textit{for} \quad a_g \in A_g, a_h \in A_h \quad \textit{and} \quad s_g, s_h \in S,$$

extended linearly. Then $A * S$ is graded equivalent with $[G](S \otimes_C \operatorname{End}_{A_1} A)$.

Proof. Let $B = A * S$. It is routine to verify that B is a ring with the above multiplication. Observe that B is graded by $B_g = A_g \otimes_C S$ and that $B_1 = A_1 \otimes_C S$ with the usual multiplication in the tensor product of algebras. Note also that $A \ni a = \sum_g a_g \mapsto \sum_g a_g \otimes 1$ is an injective ring homomorphism, so we can view A as a graded subring of B. Set $M = B \otimes_{A_1} A$ and $M_g = B_g \otimes_{A_1} A$. Then ${}_B M$ is finitely generated projective, since ${}_{A_1} A$ is, and ${}_B M$ is a generator, since the map $M \ni b \otimes a \mapsto ba \in B$ defines a surjective B-linear map. Thus ${}_B M$ is a progenerator, and is graded as B-module. It remains to show that $\operatorname{End}_B M \cong [G](S \otimes_C T)$, where $T = \operatorname{End}_{A_1} A$. The proof is similar to that of Theorem A.26, where we proved that $\operatorname{End}_A(A \otimes_{A_1} A) \cong [G](T)$. Thus define $\varepsilon_g \in (\operatorname{End}_B M)_g$ by

$$(b \otimes a)\varepsilon_g = \sum_i b a_{i,g} \otimes a'_{i,g} a,$$

where $a_{i,g} \in A_g$ and $a'_{i,g} \in A_{g^{-1}}$ with $\sum_i a_{i,g} a'_{i,g} = 1$ as before. Again, ε_g is well defined, independent of the choice of the $a_{i,g}$ and $\varepsilon_g \varepsilon_h = \varepsilon_{gh}$. It follows that $\operatorname{End}_B M$ is a skew group algebra over $(\operatorname{End}_B M)_1$. We finish the proof by showing $S \otimes_C T \cong (\operatorname{End}_B M)_1$ as G-acted algebras. Define $\rho \colon S \otimes_C T \to \operatorname{End}_B M$ by

$$(b \otimes a)(s \otimes t)^\rho = bs \otimes at.$$

In verifying that this is well defined, on has to observe that the C-algebra structure of T is given by $a(c \cdot t) = cat$: Although T operates on the right of A, the center of T consists of the *left* multiplications with elements of C. Taking this into account, we get

$$(b \otimes a)(s \otimes ct)^\rho = bs \otimes cat = bsc \otimes at = (b \otimes a)(sc \otimes t)^\rho.$$

The other verifications are easy, so we omit them. We need to show that

$$(s^g \otimes t^g)^\rho = \varepsilon_g^{-1}(s \otimes t)^\rho \varepsilon_g.$$

Remember that t^g is defined by

$$at^g = \sum_i a_{i,g^{-1}}\big((a'_{i,g^{-1}} a)t\big).$$

Thus we get

$$\begin{aligned}(b\otimes a)\varepsilon_{g^{-1}}(s\otimes t)^{\rho}\varepsilon_g &= \left(\sum_i ba_{i,g^{-1}}\otimes a'_{i,g^{-1}}a\right)(s\otimes t)^{\rho}\varepsilon_g\\ &= \left(\sum_i ba_{i,g^{-1}}s\otimes (a'_{i,g^{-1}}a)t\right)\varepsilon_g\\ &= \sum_{i,j} ba_{i,g^{-1}}sa_{j,g}\otimes a'_{j,g}\big((a'_{i,g^{-1}}a)t\big)\\ &= \sum_{i,j} bs^g a_{i,g^{-1}}a_{j,g}\otimes a'_{j,g}\big((a'_{i,g^{-1}})t\big)\\ &= \sum_i bs^g\otimes a_{i,g^{-1}}\big((a'_{i,g^{-1}}a)t\big)\\ &= bs^g\otimes at^g = (b\otimes a)(s^g\otimes t^g)^{\rho}.\end{aligned}$$

The kernel of ρ must be G-invariant, and since $S\otimes_C T$ is simple, it follows that ρ is injective. Let us show that ρ is surjective. Suppose $\varphi\in(\operatorname{End}_B M)_1$. For all $x\in G$ choose $b'_{i,x}\in A_{x^{-1}}$ and $b_{i,x}\in A_x$ with $\sum_{i\in I_x} b'_{i,x}b_{i,x}=1$. (Using the notation introduced earlier, we can take $b_{i,x}=a'_{i,x^{-1}}$ and $b'_{i,x}=a_{i,x^{-1}}$.) Then choose $s_{x,i,j}\in S$ and $c_{x,i,j}\in A$ with $(1\otimes b_{i,x})\varphi=\sum_j s_{x,i,j}\otimes c_{x,i,j}$. This is possible since $(1\otimes b_{i,x})\varphi\in B_1\otimes A$ and $B_1=SA_1$. Let $t_{x,i,j}\in T$ be defined by $at_{x,i,j}=a\pi_x b'_{i,x}c_{x,i,j}$, where $\pi_x\colon A\to A_x$ is the projection. We show that $\varphi=\sum_{x,i,j}(s_{x,i,j}\otimes t_{x,i,j})^{\rho}$:

$$\begin{aligned}(b\otimes a)\varphi &= \sum_x (b\otimes a\pi_x)\varphi = \sum_x (b\otimes a\pi_x\sum_i b'_{i,x}b_{i,x})\varphi\\ &= \sum_{x,i}(ba\pi_x b'_{i,x}\otimes b_{i,x})\varphi = \sum_{x,i} ba\pi_x b'_{i,x}(1\otimes b_{i,x})\varphi\\ &= \sum_{x,i,j} ba\pi_x b'_{i,x}(s_{x,i,j}\otimes c_{x,i,j}) = \sum_{x,i,j} bs_{x,i,j}a\pi_x b'_{i,x}\otimes c_{x,i,j}\\ &= \sum_{x,i,j} bs_{x,i,j}\otimes a\pi_x b'_{i,x}c_{x,i,j} = \sum_{x,i,j} bs_{x,i,j}\otimes at_{x,i,j}\end{aligned}$$

as claimed. The proof of the lemma is finished. □

Assume Hypothesis 2.22. It is well known that $\mathbb{F}G$ can be viewed as an strongly G/K-graded algebra (even a crossed product). In the same way, $\mathbb{F}H$ is H/L-graded. Let us write $X=H/L\cong G/K$, and let us view $\mathbb{F}G$ and $\mathbb{F}H$ as X-graded algebras. According to Theorem A.17, we may associate equivalence classes of strongly X-graded algebras to φ and ϑ, respectively.

We denote these equivalence classes by $[[\varphi]]_{G,\mathbb{F}}$ and $[[\vartheta]]_{G,\mathbb{F}}$, respectively. Also, we have the simple algebra $S = (i\mathbb{F}Ki)^L$ on which X acts. This defines an equivalence class of simple X-algebras, which we denote $[S]$. We have seen that under Hypothesis 2.22, the algebras $\mathbb{F}Ke$, $\mathbb{F}Lf$ and S have centers isomorphic as G-fields, all of them being isomorphic with $\mathbb{F}(\vartheta) = \mathbb{F}(\varphi)$.

A.35 Theorem. *In the Brauer-Clifford group* $\mathrm{BrCliff}(X, \mathbb{F}(\varphi))$ *we have*

$$[[\vartheta]]_{G,\mathbb{F}} = [S] \cdot [[\varphi]]_{G,\mathbb{F}}.$$

Proof. By Example A.13 we have that $\mathbb{F}Ke$ is graded Morita equivalent with $i\mathbb{F}Ki$. By Lemma A.34, we are done if we can show that $\mathbb{F}Hf * S \cong i\mathbb{F}Ki$ as X-graded algebras. Set

$$\kappa\colon \mathbb{F}Hf * S \to i\mathbb{F}Ki, \quad a_h \otimes s \mapsto a_h s \quad \text{for} \quad a_h \in \mathbb{F}Hfh, s \in S.$$

Then φ is a homomorphism since for $h \in H$ and $a_h \in \mathbb{F}Lfh$ we have $sa_h = a_h s^h$. It is a bijection from $\mathbb{F}Lf \otimes S$ onto $i\mathbb{F}Ki$ by Lemma 2.25. Since both $\mathbb{F}H * S$ and $i\mathbb{F}Ki$ are crossed products of X, it follows easily that φ is an isomorphism of graded algebras. □

This theorem is related to Theorem 2.30. One can show that an X-algebra S is equivalent to the trivial X-algebra $Z(S)$ if and only if $S \cong \mathbf{M}_n(\mathbf{Z}(S))$ and the action comes from a crossed representation $\sigma\colon X \to S$ as in Lemma 2.27, with respect to the subalgebra $S_0 = \mathbf{M}_n(\mathbf{Z}(S)^X)$.

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List of Notation

$\mathbb{N}$	$=\{0,1,2,\ldots\}$, natural numbers
$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	rational numbers
$\mathbb{C}$	complex numbers
$\subseteq$	inclusion
$\subset$	strict inclusion
$\lvert A\rvert$, $\lvert z\rvert$	cardinality of set A respective absolute value of $z \in \mathbb{C}$
$A \dot{\cup} B$	disjoint union of A and B
$N \trianglelefteq G$	N is a normal subgroup of G
$N \triangleleft G$	N is a normal subgroup of G strictly contained in G
$\mathbf{C}_A(S)$	centralizer of S in A
A^G	alternative notation for $\mathbf{C}_A(G)$, when A is a G-algebra
$\mathrm{T}_H^G(a)$	$\mathrm{T}_H^G(a) = \sum_{r\in R} a^r$ for $G = \dot{\bigcup}_{r\in R} Hr$ and $a \in A^H$
$\mathbf{N}_G(H)$	normalizer of H in G
$\mathbf{Z}(G)$	center of group or ring G
$\mathbf{M}_n(R)$	set of $n \times n$ matrices with entries from R
$\mathrm{ZF}(A, \mathbb{F})$	Central forms on A, that is maps $\tau\colon A \to \mathbb{F}$ with $\tau(ab) = \tau(ba)$
$\mathrm{tr}_{S/\mathbb{F}}$	reduced trace of central simple $\mathbb{F}$-algebra S
$\mathrm{nr}_{S/\mathbb{F}}$	reduced norm of central simple $\mathbb{F}$-algebra S
A^*	group of units of ring A
$\mathrm{ann}_A M$	annihilator of M in A
$\mathrm{Aut}\, G$	automorphism group of group or ring G
$\mathrm{Inn}\, G$	inner automorphisms of group or ring G
$\mathrm{Out}\, G$	$\mathrm{Aut}\, G / \mathrm{Inn}\, G$
$\mathrm{Aut}_{\mathbb{F}} A$	$\mathbf{C}_{\mathrm{Aut}\, A}(\mathbb{F})$
$\mathrm{Gal}(\mathbb{E}/\mathbb{F})$	Galois group of Galois extension $\mathbb{E}/\mathbb{F}$, means the same as $\mathrm{Aut}_{\mathbb{F}}\, \mathbb{E}$
$\mathrm{T}_{\mathbb{F}}^{\mathbb{E}}$	field trace for Galois extension $\mathbb{E}/\mathbb{F}$

$\mathrm{Syl}_p\, G$	set of Sylow p-subgroups of group G
$\mathrm{ord}(g)$	order of g
$\mathrm{Irr}\, G$	set of irreducible complex characters of G
$\mathrm{Lin}\, G$	set of linear complex characters of G
χ_H	restriction of class function χ to subgroup H
ξ^G	induced character
$(\chi, \psi)_G$	inner product of class functions χ, $\psi\colon G \to \mathbb{C}$.
$\mathrm{Irr}_{\mathbb{E}}\, G$	set of characters of absolutely irreducible $\mathbb{E}G$-modules, $\mathbb{E}$ being a splitting field of G
$\mathrm{Irr}_A\, G$	set of irreducible complex A-invariant characters of G, where A acts on G
$\mathrm{Irr}(\beta)$	irreducible constituents of (generalized) character β
$\mathrm{Irr}(G \mid \vartheta)$	$\mathrm{Irr}(\vartheta^G)$ for $\vartheta \in \mathrm{Irr}\, H$ with $H \leqslant G$
$\langle\ ,\ \rangle_\varphi$	Dade-Isaacs bilinear form, see Section 4.1.
γ_ϑ	$\sum_{\chi \in \mathrm{Irr}(G\mid\vartheta)} (\gamma, \chi)_G \chi$ for classfunctions γ
$\det \chi$	determinant character of χ
$\mathrm{ord}(\chi)$	order of $\det \chi$
$\mathbb{F}(\chi)$	field generated by the values of χ over $\mathbb{F}$
e_χ	central primitive idempotent belonging to irreducible character χ, if the field of values of χ has characteristic $p > 0$ it is assumed that the module of χ is also projective.
$[[\chi]]_{\mathbb{F}}$	the equivalence class of $\mathbb{F}(\chi)Ge_\chi$ in the Brauer group of $\mathbb{F}(\chi)$.
$m_{\mathbb{F}}(\chi)$	Schur index of χ over field $\mathbb{F}$

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Thesen zur Dissertation

Character Correspondences in Finite Groups

Frieder Ladisch

1. Sei G eine endliche Gruppe, $H \leqslant G$ und $K \trianglelefteq G$ mit $HK = G$, und sei $L = H \cap K$. Die Charaktere $\vartheta \in \operatorname{Irr} K$ und $\varphi \in \operatorname{Irr} L$ seien invariant in H und die Vielfachheit n von φ in ϑ_L sei nicht null. Unter welchen Bedingungen existiert eine Bijektion zwischen $\operatorname{Irr}(G \mid \vartheta)$ und $\operatorname{Irr}(H \mid \varphi)$?

2. Sei n die Vielfachheit von φ in ϑ_L. Die obige Konfiguration definiert eine zentral einfache Algebra S der Dimension n^2 über jedem Körper, der die Werte von ϑ und φ enthält. Außerdem hat man einen Homomorphismus von H/L in die Automorphismengruppe von S.

3. Falls der Homomorphismus von H/L in $\operatorname{Aut} S$ sich zu einem Homomorphismus von H/L nach S^* heben läßt, ist dadurch eine Bijektion zwischen $\operatorname{Irr}(G \mid \vartheta)$ und $\operatorname{Irr}(H \mid \varphi)$ definiert. Wir nennen eine solche Hebung eine "magische Darstellung" für die Konfiguration aus 1.

4. Eine magische Darstellung hat einen Charakter ψ. Die Charakterkorrespondenz zu einer magischen Darstellung hat neben anderen Eigenschaften die, daß für $\chi \in \operatorname{Irr}(G \mid \vartheta)$ und den korrespondierenden Charakter $\xi \in \operatorname{Irr}(H \mid \varphi)$ die Gleichung $\chi_H = \psi\xi + \Delta$ gilt, wobei $(\Delta_L, \varphi) = 0$ ist.

5. Falls $(n, |H/L|) = 1$ ist, existiert eine magische Darstellung. Die Isomorphie der Clifford-Erweiterungen zu ϑ und φ über algebraisch abgeschlossenen Körpern wurde bereits 1970 von Dade bewiesen und von Schmid (1988) verallgemeinert.

6. Sei jetzt L ein Normalteiler von K und φ vollständig verzweigt in K, das heißt, φ ist invariant in K und $n^2 = |K/L|$. Für diesen Fall exi-

stiert bereits eine umfangreiche Literatur über die Beziehungen zwischen $\mathrm{Irr}(G \mid \vartheta)$ und $\mathrm{Irr}(H \mid \varphi)$.

7. Falls K/L abelsch und von ungerader Ordnung ist, garantiert ein Satz von Isaacs die Existenz einer magischen Darstellung, sogar einer ausgezeichneten kanonischen unter diesen. Wenn G/L ungerade Ordnung hat, entspricht einem Charakter χ über ϑ gerade derjenige Charakter von H, der in χ_H mit ungerader Vielfachheit auftaucht. Die Methoden dieser Arbeit erlauben eine signifikante Vereinfachung des Beweises.

8. Außerdem verallgemeinern wir das Resultat dahingehend, daß die Bijektion unter einer zusätzlichen Voraussetzung Schur-Indices über allen Körpern erhält, auch wenn die Voraussetzung aufgegeben wird, daß ϑ und φ invariant in H sind.

9. Sei jetzt G eine endliche Gruppe und A eine Gruppe von Automorphismen von G, so daß $(|A|, |G|) = 1$ ist. Dann existiert eine Bijektion zwischen den A-invarianten Charakteren von G und den Charakteren des Zentralisators $\mathbf{C}_G(A)$ von A in G, die Glauberman-Isaacs-Korrespondenz. Für G ungerader Ordnung wurde dies von Isaacs mit Hilfe des in 7. beschriebenen Resultates gezeigt. Wir zeigen zusätzlich, daß diese Korrespondenz (für $|G|$ ungerade) Schurindizes über allen Körpern erhält.

10. Falls in der Konfiguration aus 1. L ein Normalteiler von K ist, und ein Normalteiler $M \trianglelefteq H$ existiert, so daß $(|M/L|, |K/L|) = 1$ und $\mathbf{C}_{K/L}(M) = 1$ ist, dann existiert ebenfalls eine Magische Darstellung. Dies läßt sich mit unseren Methoden recht einfach aus einem sehr viel weniger allgemeineren Satz von Dade herleiten.

11. Eine magische Darstellung existiert auch, wenn φ der Glauberman-Korrespondent von ϑ unter einer Automorphismengruppe P ist, und $H = \mathbf{N}_G(P)$. Dies geht im wesentlichen aus einem Satz von Dade hervor, und benutzt einen tiefliegenden Klassifikationssatz, der ebenfalls von Dade stammt.

Selbständigkeitserklärung

Ich versichere hiermit an Eides statt, daß ich die vorliegende Arbeit selbständig angefertigt und ohne fremde Hilfe verfaßt habe, keine außer den angegebenen Hilfsmitteln und Quellen dazu verwendet habe und die den benutzten Werken inhaltlich oder wörtlich entnommenen Stellen als solche kenntlich gemacht habe.

Rostock, den 27. Oktober 2008

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