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Contents

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Abstract. Quantum entanglement nowadays plays a fundamental role in Quantum Optics and Quantum Information Theory. Entanglement is a nonclassical correlation between the parties of a compound quantum system. This kind of correlations cannot be described by a classical joint probability distribution between the subsystems. In this work, we present new approaches for the identification, representation by quasi-probabilities, and quantification of entanglement.

For the identification and the representation by quasi-probabilities we have derived separability eigenvalue equations. From the solution of these equations we obtain all observables witnessing the entanglement of a state. On the other hand, the solution also yields an optimized quasi-probability distribution of entanglement. The negativities of this distribution allow us to conclude that no classical probability can generate the considered state in terms of factorizable ones. For the quantification of entanglement we compare well-known entanglement measures. We conclude that the Schmidt number – the number of global superpositions – has advantageous properties compared to measures based on a distance.

We generalize our method to so-called Schmidt number states and multipartite entangled states. We relate our findings to the notion of nonclassicality of radiation fields. Moreover, we transfer our new methods to this notion.

Zusammenfassung. Verschränkung spielt heutzutage eine fundamentale Rolle in der Quantenoptik und Quanteninformationstheorie. Verschränkung ist eine nichtklassische Korrelation zwischen den Parteien eines zusammengesetzten Quantensystems. Diese Art der Korrelationen kann nicht durch eine klassische gemeinsame Wahrscheinlichkeitsverteilung zwischen den Teilsystemen beschrieben werden. In dieser Arbeit präsentieren wir neue Ansätze zur Identifikation, Darstellung mit Quasi-Wahrscheinlichkeiten und Quantifizierung von Verschränkung.

Für die Identifikation und die Darstellung mit Quasi-Wahrscheinlichkeiten haben wir Separabilität-Eigenwerts-Gleichungen abgeleitet. Durch die Lösung dieser Gleichungen erhalten wir alle Observablen, die die Verschränkung von Zuständen nachweisen können. Andererseits liefert die Lösung optimierte Verschränkungs-Quasi-Wahrscheinlichkeitsverteilungen. Negativitäten in diesen Verteilungen erlauben uns zu schlussfolgern, dass keine klassische Verteilung den untersuchten Zustand durch faktorisierte Zustände erzeugen kann. Für die Quantifizierung von Verschränkung vergleichen wir verschiedene Verschränkungsmaße. Wir schlussfolgern, dass die Schmidtzahl - Anzahl von globalen Überlagerungen - vorteilhafte Eigenschaften gegenüber Maßen, die auf Abständen basieren, hat.

Wir verallgemeinern unsere Methode auf sogenannte Schmidtzahlzustände und Mehrmoden-verschränkte Zustände. Wir bringen unsere Ergebnisse in Relation mit dem Begriff der Nichtklassizität von Strahlungsfeldern. Darüber hinaus übertragen wir unsere neuen Methoden auf diesen Begriff.

Where are the Proofs? This work contains the results of eight manuscripts [I, II, III, IV, VII, V, VI, VIII]. All the needed proofs are given in these manuscripts. In this work we describe our findings by examples and figures.

Used text decoration

- An underlined word denotes a new property.
- An *emphasized* text denotes a heuristic question or notion.

List of used abbreviations

- CV – continuous variable
- PNCP – positive, but not completely positive
- LOCC – local operations and classical communication
- PT, PPT, NPT – partial transposition, positive partial transposition, negative partial transposition
- SE value/vector – separability eigenvalue/-vector

Nomenclature of variables

- We use the common abbreviation $|a\rangle \otimes |b\rangle = |a, b\rangle$.
- σ denotes a classical (coherent, separable, etc.) mixed or pure quantum state.
- ϱ denotes a nonclassical mixed or pure quantum state.
- ρ denotes a quantum state without further specification of its properties.
- Λ denotes a linear map from one quantum state to another.
- $^{\Gamma}$ denotes a linear map of Hermitian operators.
- $\hat{\delta}(x)$ denotes the multidimensional operator-value Dirac δ distribution.
- dP denotes a signed integration measure.
- dP_{cl} denotes a probability measure.
- Capital roman letters are operators, e.g. L.
- Calligraphic letters denote sets, e.g. $\mathcal{S}$.
- $\mathsf{Lin}(\mathcal{H}, \mathcal{H})$ denotes the set of linear maps with the domain and codomain $\mathcal{H}$.
- $\mathsf{Herm}(\mathcal{H})$ denotes the set of Hermitian maps with the domain and codomain $\mathcal{H}$.

- $\mathcal{S}$ denotes a special sets of quantum states. The superscript $\mathcal{S}^{(\text{pure})}$ denotes that the set contains only pure states. The following indices denote
  - $\mathcal{S}_{AB}$ separable states;
  - $\mathcal{S}_{\infty}$ all states;
  - $\mathcal{S}_r$ Schmidt number r states ($\mathcal{S}_1 = \mathcal{S}_{AB}$);
  - $\mathcal{S}_{\text{full}}$ fully separable states;
  - $\mathcal{S}_{\text{part}}$ partially separable states.
- The same subscripts are valid for the maximal expectation value for sets of quantum states given by a function f.
- The maximal Schmidt number is denoted as $r_{\max} = \min\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}$.
- The swap operator is $V = \sum_{k,l} |k,l\rangle\langle l,k|$.
- The vector $|\Phi\rangle = \sum_{k=1}^{d} |k,k\rangle$ denotes a unnormalized state with a Schmidt rank d.

1 Motivation

Quantum physics includes some of the most astonishing result in physics. The consequences of the quantumness of nature are considered to be in contrast to our everyday experiences. At the beginning of quantum mechanics the domain of this field of research was bounded to small amounts of energy and small distances. However, distant particles in a compound quantum system were well-known to include quantum effects. This non-locality is often regarded as a spooky interaction of compound quantum systems [1, 2]. Today, entanglement can be observed in distance of more than 100 km [3]. Massive mirrors in the domain of 1 kg can be prepared in a certain quantum state [4, 5].

The philosophical question of the quantumness of nature – especially the universal character of quantum physics – must be reconsidered. Therefore, it is important to create a tool box of methods for the identification and characterization of quantum effects. This is the main aim of this work. We consider correlations between subsystems of a compound quantum systems. This will be done in the framework of entanglement.

The superposition principle. The quantum superposition principle is the most striking effect of quantum physics. It explains the duality between wave and particle description. Even the non-commuting property of observables, e.g. for the Pauli matrices $[\sigma_x, \sigma_y] \neq 0$, is a consequence of the superposition principle. Namely, the eigenvectors of σ_y are superpositions of the eigenvectors of σ_x and the other way around. The direct relation between non-commuting observables and entanglement has been studied in Ref. [6]. In this work we will describe some additional consequences of the superposition principle in connection with nonclassical correlations.

Entanglement. There is an enormous growth of the fields of Quantum Information Processing, Quantum Computation, and Quantum Technology [7, 8, 9, 10]. All of these fields use the quantum property of entanglement. In general, all entangled states can be used for quantum tasks which cannot be simulated in terms of classical correlations [11, 12, 13].

Here, we use the notion of entanglement defined as the complement of separability. A quantum state σ in a bipartite quantum system is separable by definition, if it can be written as

$$\sigma = \int dP_{\mathrm{cl}}(a,b)\, |a\rangle\langle a| \otimes |b\rangle\langle b|, \tag{1.1}$$

where $P_{\mathrm{cl}}(a,b)$ denotes a classical joint probability [14]. All pure separable, or factorizable, states will be written as $|a\rangle \otimes |b\rangle = |a,b\rangle$. The pure entangled states can

be written as superpositions of local states,

$$|\psi\rangle = \sum_{k,l} \psi_{k,l} |k,l\rangle. \tag{1.2}$$

By performing the singular value decomposition of the matrix $(\psi_{k,l})_{k,l}$ we obtain the Schmidt decomposition of a pure state [7], which reads as

$$|\psi\rangle = \sum_{k=0}^{r} \lambda_k |e_k, f_k\rangle, \tag{1.3}$$

where r denotes the Schmidt rank, $\lambda_k > 0$ denote the Schmidt coefficients, and $|e_k\rangle$, $|f_{k'}\rangle$ are orthonormal vectors in the corresponding Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$, respectively. The Schmidt rank of a separable state is obviously one. For entangled states, the Schmidt rank denotes the minimal number of factorizable vectors which has to be superimposed to generate this state. Hence, the Schmidt rank relates entanglement to the quantum superposition principle.

Experimental realization of entanglement. Entanglement has been used in many experiments, such as: entanglement in semi-conductor quantum dots [15]; multipartite entanglement in Dicke states [16, 17]; entangled photon pairs in energy-time [18]; quantum teleportation with mesoscopic objects [19]; and quantum dense coding in continuous variable systems [20]. The latter one is based on the two-mode squeezed-vacuum state.

Another example for such an experimental situation is given in Fig. 1.1. Let us assume we have two squeezed light sources. In addition, let us assume, that the squeezed quadratures are orthogonal to each other. These two nonclassical light beams are the inputs of a beam splitter, cf. [21, 22]. The two output beams have correlations which cannot be explained by a classical correlated joint probability of the individual output beams. One of the output beams passes a medium which influences the quantum correlations between the subsystems. Some of these influences can cause a loss of all quantum correlations, and only classical correlations can be reconstructed by the local measurements. The question arises: *How much entanglement survives for a given medium?* This simple question includes the identification and the quantification of entanglement. Such properties of entanglement will be discussed in this work.

The eigenvalue problem. In this work we will present a method which is related to the eigenvalue problem in linear algebra. Therefore it is useful to recall some facts about the eigenvalue problem. Let us denote with $\mathcal{S}_\infty^{(\mathrm{pure})}$ the set of all pure quantum states $|\psi\rangle\langle\psi|$ with $|\psi\rangle \in \mathcal{H}$ and $\langle\psi|\psi\rangle = 1$, and $\mathcal{S}_\infty$ denotes the set of all pure and mixed quantum states. In general any quantum state ρ can be written as

$$\rho = \int_{\mathcal{S}_\infty^{(\mathrm{pure})}} dP_{\mathrm{cl}}(\psi)\, |\psi\rangle\langle\psi|, \tag{1.4}$$

with $P_{\mathrm{cl}}(\psi)$ being a classical probability distribution.

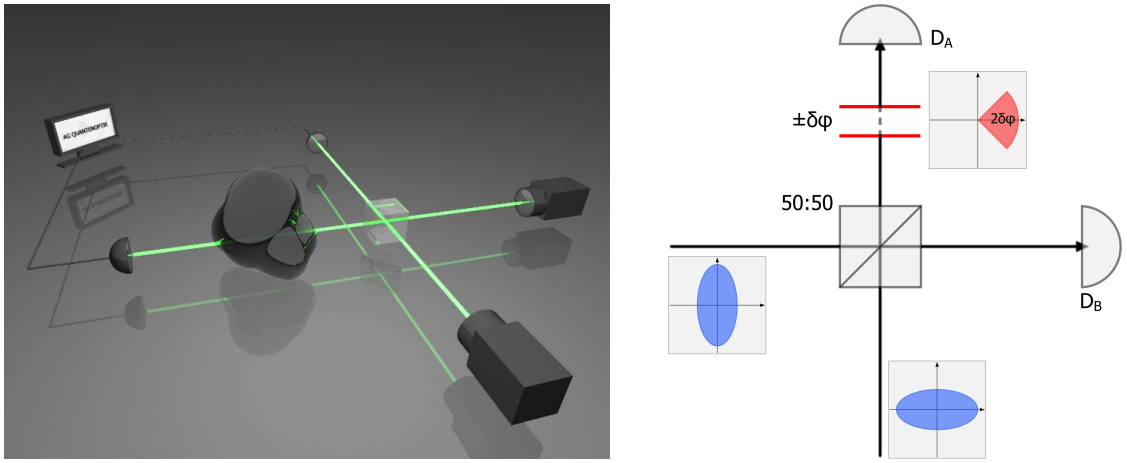

Figure 1.1: Two squeezed light sources enter a 50:50 beam splitter. One of the output beams suffers a local phase and amplitude randomization by a medium. The detectors indicate a measurement for a reconstruction of the density matrix of the output beams, e.g. by a homodyne detection scheme [23, 24].

The eigenvalue problem for an observable L and for a state ρ is given by

$$L|\phi_k\rangle = L_k|\phi_k\rangle \quad \text{and} \quad \rho|\psi_k\rangle = p_k|\psi_k\rangle. \tag{1.5}$$

We obtain that the maximal expectation value of L is given by

$$f_\infty(L) = \sup\{\langle\psi|L|\psi\rangle : |\psi\rangle \in \mathcal{S}_\infty^{(\text{pure})}\} = \max_k\{L_k\}. \tag{1.6}$$

Obviously, for all quantum states $\mathrm{Tr}(\rho L) \leq f_\infty(L)$ holds. The other way around, an operator ρ is a quantum state, iff $\mathrm{Tr}(\rho L) \leq f_\infty(L)$[1] is fulfilled for any observable L together with the normalization $\mathrm{Tr}\,\rho = 1$. This means that the maximal eigenvalue of the observable L delivers boundaries for the identification of quantum states in the set of all Hermitian operators.

In addition it is clear that the quantum state ρ has a spectral decomposition as

$$\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|, \tag{1.7}$$

with p_k being a classical probability distribution, or with the probability distribution $P_{\text{cl}}(\psi) = \sum_k p_k\delta(\psi - \psi_k)$. This means quantum state can be given in an integral form of Eq. (1.4), but it can also be decomposed in a convex manner by its eigenvectors.

A third important fact is the transformation of the eigenvalue problem. A transformed operator $L' = TLT^{-1}$ has the same eigenvalues like the initial operator L, $L'_k = L_k$, whereas the the eigenvectors transform as $|\phi'_k\rangle = T|\phi_k\rangle$.

These obvious facts deliver us an idea how to proceed, when solving an optimization problem for a state or an observable with respect to the property separability.

[1] The operator $C = f_\infty(L)\mathbb{I} - L$ is an arbitrary positive semi-definite operator. The expectation value $\mathrm{Tr}(\rho C) \geq 0$ yields the positive semi-definite property of the state ρ.

In the notion of separability, we will obtain equations called separability eigenvalue equations. They will resemble the situation of the (ordinary) eigenvalue problem with analogous implications.

2 Nonclassical Quantum States

2.1 Single-mode nonclassicality

Before going into detail with the correlations of compound quantum systems, we consider the single mode situation. Here, we restrict ourselves to the description of radiation fields. We consider known methods for the definition, the identification, and the quantification of the quantumness in a single mode. For an introduction to Quantum Optics see e.g. [25, 26].

The Glauber-Sudarshan P function. The pure classical states of the Harmonic oscillator are the coherent states $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$, where $|n\rangle$ denotes the Fock basis. In general every quantum state can be written as

$$\rho = \int_{\mathbb{C}} dP(\alpha)\, |\alpha\rangle\langle\alpha|, \tag{2.1}$$

with $P(\alpha)$ being a quasi-probability [27, 28]. If the P function is a classical probability, $P = P_{\mathrm{cl}}$, the state ρ is called classical. The definition of a nonclassical state is given by the complement, $P \neq P_{\mathrm{cl}}$. The state is nonclassical, if the P function is not a classical probability [29].

Each quantum state ρ defines exactly one P function which allows the definition of nonclassicality on this basis. But the P function can be highly singular – it may contain derivations of the δ distribution. This deficiency can be overcome by regularizing filter functions, see [30] and references therein.

Normally ordered expectation values. As we already mentioned above, the superposition principle delivers non-commuting observables. A famous example is the commutation relation between the annihilation operator $\hat{a}$ and the creation operator $\hat{a}^\dagger$:

$$[\hat{a}, \hat{a}^\dagger] = \mathbb{I}, \tag{2.2}$$

with the identity operator $\mathbb{I}$. The coherent states are eigenvalues of the annihilation operators, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. The relative variance of the photon number, $\hat{n} = \hat{a}^\dagger\hat{a}$, of a coherent state is given by

$$\frac{\Delta n}{\bar{n}} = \frac{\sqrt{\langle\alpha|\hat{n}^2|\alpha\rangle - \langle\alpha|\hat{n}|\alpha\rangle^2}}{\langle\alpha|\hat{n}|\alpha\rangle} = \frac{\sqrt{|\alpha|^4 + |\alpha|^2 - |\alpha|^4}}{|\alpha|^2} = \frac{1}{|\alpha|} \overset{\alpha\to\infty}{\to} 0, \tag{2.3}$$

representing the correspondence principle of Bohr. This relates, for large amplitudes, the coherent state to a nearly noiseless, classical oscillating wave.

One possibility for the identification of nonclassical states is given in terms of normally ordered operators [31]. This means we exchange the order of moments – powers of $\hat{a}$ and $\hat{a}^\dagger$ – without using the commutation relation. The positivity of the expectation value for coherent states is not affected by this ordering, e.g. $\langle :(\Delta\hat{n})^2: \rangle \geq 0$ for coherent states, where $: \cdot :$ denotes the normal ordering. The relation of the phase-space representations and operator ordering can be found, for example, in Ref. [32, 33, 34, 35]. In general, for any nonclassical state ϱ, there exists a normally ordered operator $:\hat{f}^\dagger \hat{f}:$ such that

$$\mathrm{Tr}(:\hat{f}^\dagger \hat{f}:\varrho) < 0. \tag{2.4}$$

The operator $:\hat{f}^\dagger \hat{f}:$ witnesses the nonclassicality of the state. It is non-negative for classical states, and might become negative for nonclassical ones. There are a number of methods identifying nonclassicality, for later purposes let us only mention the characterization in terms of matrices of moments [36].

Quantification of nonclassicality. The first quantification of nonclassical states is considered to be expressed in terms of distances [37]. The intuition is clear: *The closer a state is to the set of classical quantum state, the less nonclassicality is in this state.* At this point let us focus on a problem which is well-known for entanglement measures [38, 39]. We considered this problem in the context of nonclassicality in [VII]. Let us assume we have two nonclassical quantum states. The simple question is: *Which quantum state is more nonclassical?*

In this context two problems arise. First, we are free to choose a distance. Second, a given distances can be linearly transformed. Let us illustrate this for an example in Fig. 2.1. Depending on the choice of the distance, the first state has a smaller, a larger or an equal amount of nonclassicality in comparison with the second state. The quantification of nonclassicality in terms of distances already includes this paradoxical situation. It becomes clear that we need to find a quantification which is consistent with the currently accepted axioms of quantification, but also overcomes the problem of ambiguities of the ordering of quantum states.

In physics we are used to have properties which are somehow invariant. These properties allow us to characterize a physical system independent of the particular description, e.g. the space-time curvature is independent of the choice of coordinates. As another example let us compare the situation with thermodynamics. Water can be in one of the three states of matter – solid, liquid or gas. All states in one phase have similar properties. By cooling or heating, we observe a spontaneous phase transition between the phases, but at a certain temperature we can observe liquid water or vapor for different pressures. Hence, it is problematic to compare states by the temperature. This means we have different properties, whereas all states of the gas phases are related to each other. Moreover, we would not quantify the *solidness* of a system by the distance to the solid phase in the phase diagram of water. Otherwise we would have states with an equal *solidness* in a liquid and vapor phase, which is due to the triple point.

In addition to the quantification of nonclassicality in general, we also consider the following. Different quantum processes may need different kinds of quantum corre-

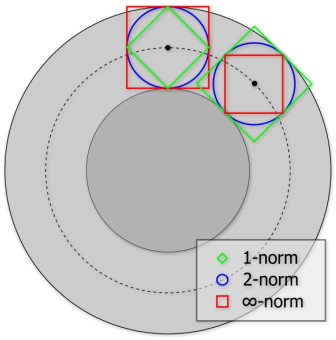

Figure 2.1: We choose three different norms: 1(green)-, 2(blue)-, and ∞(red)-norm. We take two points. In the 2-norm both points have the same distance to the convex set (dark gray area) of classical states. The distance of the upper point is in the 1-norm smaller than for the other point. The distance of the upper point is in the ∞-norm larger than for the other point.

lations. Thus, it might be useful to define an operational measure for quantifying the quantum correlation in connection with the considered operation/process.

2.2 Multi-mode quantum systems

Let us consider in detail the multi-mode description of quantum systems. In more than one mode, we have additional nonclassical correlations between these modes. One major method for the description of this quantum effects is given in terms of entanglement.

Multi-mode description. Let us consider a multi-mode quantum system. The description of this n-mode system is based on the tensor product structure of single mode vectors,

$$|\psi\rangle = |a_1\rangle \otimes \cdots \otimes |a_n\rangle = |a_1, \ldots, a_n\rangle. \tag{2.5}$$

This state is referred as fully separable. A general pure quantum state is a superposition of fully separable basis elements

$$|\psi\rangle = \sum_{k_1,\ldots,k_n} \psi_{k_1,\ldots,k_n} |k_1, \ldots, k_n\rangle \in \mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i. \tag{2.6}$$

The structure of the linear operators, $L \in \mathsf{Lin}(\mathcal{H}, \mathcal{H})$[1] , is given in the same form,

$$L = \sum_{k_1,\ldots,k_n} L_{k_1,\ldots,k_n} A_{k_1} \otimes \cdots \otimes A_{k_n}, \tag{2.7}$$

[1]The set $\mathsf{Lin}(\mathcal{V}_1, \mathcal{V}_2)$ is defined by all linear maps with the domain $\mathcal{V}_1$ and a codomain $\mathcal{V}_2$.

where $\{A_{k_i}\}$ denotes a basis of linear operators $\mathsf{Lin}(\mathcal{H}_i, \mathcal{H}_i)$ and $L_{k_1,\ldots,k_n} \in \mathbb{C}$. Thus, any operator is given by linear combinations of product operators $A_{k_1} \otimes \cdots \otimes A_{k_n}$.[2]

Space-time correlations by the P functional The single-mode P function can be expressed as the following expectation value [25]:

$$P(\alpha) = \langle{:}\hat{\delta}(\hat{a} - \alpha){:}\rangle. \tag{2.8}$$

In the case of the radiation field, $\mathrm{Re}(\alpha)$ denotes the classical field with a phase $\arg(\alpha)$ and amplitude $|\alpha|$. The conjugate momentum of the field is given by $\mathrm{Im}(\alpha)$. The operator $\hat{a}$ is the above given annihilation operator of the quantum description of the fields,

$$\hat{x} \propto \hat{a} + \hat{a}^\dagger \quad \text{and} \quad \hat{p} \propto \frac{1}{i}\left(\hat{a} - \hat{a}^\dagger\right), \tag{2.9}$$

with the field $\hat{x}$ and its canonical momentum $\hat{p}$.

In the multi-mode setting we have the following. The classical field – with a given phase and a given amplitude – corresponds to $E^{(+)}(\mathbf{r}, t)$. Each value of $(\mathbf{r}, t)$ denotes one mode at a certain time. They are represented by the quantum analogue $\hat{E}^{(+)}(\mathbf{r}, t)$. The generalized P functional is defined analogously to Eq. (2.8) by

$$P[E^{(+)}(\mathbf{r}_1, t_1), \ldots, E^{(+)}(\mathbf{r}_n, t_n)] = \langle \overset{\circ}{\circ} \prod_{k=1}^{n} \hat{\delta}\left[\hat{E}^{(+)}(\mathbf{r}_k, t_k) - E^{(+)}(\mathbf{r}_k, t_k)\right] \overset{\circ}{\circ} \rangle, \tag{2.10}$$

with the notion $\overset{\circ}{\circ}\cdot\overset{\circ}{\circ}$ for normally and time ordered expectation values and $\hat{E}^{(+)}(\mathbf{r}_k, t_k)$ being the k-th space-time component of the field [40].

The time dependence suffers additional requirements. In addition to effects like time dependent commutation relations, it requires the ordering of the unitary evolution in time,

$$\begin{aligned} \overset{\circ}{\circ}\hat{E}^{(+)}(t_2)\hat{E}^{(+)}(t_1)\overset{\circ}{\circ} &= {:}\hat{E}^{(+)}(t_1)\hat{E}^{(+)}(t_2){:} \\ \text{and } \overset{\circ}{\circ}\hat{E}^{(-)}(t_1)\hat{E}^{(-)}(t_2)\overset{\circ}{\circ} &= {:}\hat{E}^{(-)}(t_2)\hat{E}^{(-)}(t_1){:}, \end{aligned} \tag{2.11}$$

for $t_1 < t_2$, by neglecting the non-commuting property of the time dependent operators.

Apart from this general approach, we restrict ourselves to systems at equal times, $t = t_1 = \cdots = t_n$. This is useful for the description of entanglement. But let us note that a general temporal entanglement description must be considered in the future where this assumption cannot be made.

Further on, let us consider the superposition of two radiation fields A and B in relation to the quantum superposition. A classical superposition of two field is given by $\hat{E}_A^{(+)} + \hat{E}_B^{(+)}$ in the operator space. Whereas the quantum superposition is given in the state space, e.g. a two-mode odd coherent state $|\alpha\rangle_A \otimes |\alpha\rangle_B - |-\alpha\rangle_A \otimes |-\alpha\rangle_B$ [41].

[2] The multi-mode description of Hermitian operators, and therefore all quantum states, is given in the same form. The additional restriction for Hermitian operators is $A_{k_i} \in \mathsf{Herm}(\mathcal{H}_i)$ and $L_{k_1,\ldots,k_n} \in \mathbb{R}$.

Relation between entanglement and nonclassicality There are some surprising relations between entanglement and nonclassicality. Remarkably are those which are given in terms of moments for the common identification of entanglement and nonclassicality [42, 43, 44, 45]. Through this work we will compare nonclassicality in terms of coherent states with entanglement.

First let us consider an inclusion, see e.g. [II]. It is clear that a classical two-mode P function implies a separable quantum state, cf. Eq. (1.1),

$$\sigma = \int dP_{\mathrm{cl}}(\alpha, \beta)\, |\alpha, \beta\rangle\langle\alpha, \beta|. \tag{2.12}$$

A classical and entangled multi-mode coherent state cannot exist. But there are some nonclassical and separable quantum states with a negative P function, such as

$$\rho = |1\rangle\langle 1| \otimes |0\rangle\langle 0|, \tag{2.13}$$

with $|0\rangle$ representing the vacuum state and $|1\rangle$ the nonclassical one photon Fock state. In Fig. 2.2 this discussed relation between the separable and classical states is given. This implies that entanglement is a sub-phenomena of all quantum correlation in a radiation field.[3]

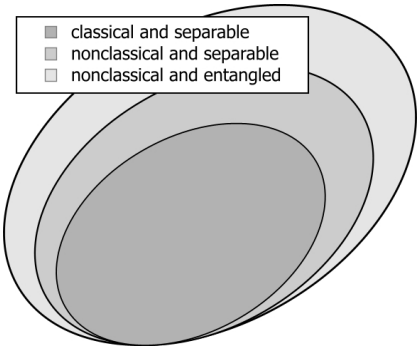

Figure 2.2: All gray areas together form the set of all quantum states. The mid gray and the dark gray area define the set of separable quantum states. The dark gray area defines the separable and classical quantum states.

Note that, beside entanglement in radiation fields, there is also entanglement in other fermion and/or boson systems, or between interacting fields and particles [46, 47]. No matter how we define nonclassicality in such a system – which is in general an open problem – entanglement resembles correlations which cannot be described by classical joint probabilities. In this sense entanglement can be found in any multipartite quantum system.

[3] Separable states are the classical reference for entanglement, and multi-mode coherent states, such as $|\alpha, \beta\rangle$, are the reference for nonclassicality. Multi-mode coherent states are separable.

Preview

In the following, we concentrate on three points in connection with entanglement. Analogously to the single mode characterization of nonclassicality we consider:

1. the identification of entanglement by witness operators similar to $\langle :\hat{f}^\dagger \hat{f}: \rangle < 0$ for nonclassical states;

2. the identification of entanglement by quasi-probabilities similar to the P function;

3. and the quantification of entanglement by the quantum superposition principle.

We will show that the first two items can be solved by the separability eigenvalue problem. We solve this problem for a few examples to demonstrate our methods. For the quantification, we will focus on the fundamental role of the superposition principle. With this ansatz we overcome the ambiguity of comparing entanglement of two quantum states by distances.

We generalize our methods – developed for bipartite entanglement – to so-called Schmidt number states and multipartite systems. Such states will be classified in the following chapter. We also address to the continuous variable entanglement as it appears in multi-mode radiation fields. A general approach for the quantification of nonclassicality in arbitrary anharmonic quantum systems, as well as a convex decomposition method in those systems will be considered. We also consider methods in relation to the so-called partial transposition, which are of some importance for quantum information theory.

3 Entanglement

A pure factorizable state in a bipartite quantum system is given by $|a, b\rangle$. A mixed separable state was defined by pure factorizable states together with a classical joint probability distribution [14]. An entangled quantum state cannot be represented in such a form. The Schmidt decomposition [7] of a pure quantum state delivered a method for the identification of pure entangled states – the Schmidt rank being greater than one.

In this chapter we investigate general mixed entangled quantum states. We consider the identification of continuous variable entanglement, Sec. 3.1 and Ref. [III]. In Sec. 3.2, we study well-known classes of entangled states in bipartite and multipartite systems. The identification of entanglement is considered in Sec. 3.3 with a new optimized approach in terms of arbitrary Hermitian test operators [I]. To strengthen the relation between nonclassicality and entanglement, we developed a representation of entangled states in terms of quasi-probabilities in Sec. 3.4 and Ref [II, IV, VIII]. The quantification of entanglement is studied in Sec. 3.5 and Ref. [V], and the quantification of general nonclassicality in Ref. [VII].

3.1 Continuous variable entanglement

For a bipartite radiation field, we have a quantum system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim(\mathcal{H}) = \dim(\mathcal{H}_A)\dim(\mathcal{H}_B) = \infty$. Such a quantum system is referred as a system of continuous variables (CV). A special class of CV states is given by Gaussian states (for a complete characterization of multimode Gaussian entanglement see [48]). Other methods also apply to CV entanglement, e.g. [42, 43], but, for example, there exist states for which the so-called PT criterion (partial transposition [49]) does not apply [50]. Thus, the general identification of CV entanglement was so far unknown.

The general mathematical description of CV operators, e.g. the density operator, is given in terms of methods described by functional analysis. These methods are more complex than finite dimensional linear algebra. It was known that entanglement in a finite dimensional subsystem delivers entanglement in CV, but: *Is entanglement in a continuous variable system always visible in finite dimensional subsystems?*

A finite dimensional system $\dim(\mathcal{H}_{A,B}) < \infty$ can be handled by a simpler toolbox of mathematical methods. We considered the general property of CV entanglement in [III]. The main finding is that all kinds of entanglement can be treated in finite dimensional subspaces. This means: *Continuous variable entanglement is always visible in a finite subsystem.* This finding was also formulated for the multipartite case of entanglement. Let us illustrate the result with two examples [III].

Example 1 *Let us consider the following Bell-like state* $|\chi_k\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle + |k,k\rangle)$ *in a CV system. Using the local projection given by* $P_k = \sum_{i=1}^{k} |i\rangle\langle i|$*, we obtain* $P_k \otimes P_k|\chi_k\rangle = |\chi_k\rangle$*. Obviously the state is entangled in a compound system of* k *dimensional subsystems, but it is not entangled in the projected subspace of* $k-1$ *dimensions,* $P_{k-1} \otimes P_{k-1}|\chi_k\rangle \propto |1,1\rangle$*.*

Now let us consider $k \to \infty$*. It follows that* $P_k \otimes P_k|\chi_\infty\rangle \propto |1,1\rangle$ *for any* $k \in \mathbb{N}$*, and therefore* $|\chi_\infty\rangle$ *is not entangled at all. However,* $|\chi_\infty\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle + |\infty,\infty\rangle)$ *seems to be entangled. The resolution of this paradoxical situation is that* $|\chi_\infty\rangle$ *is no longer a vector in* $\mathcal{H}$*. Thus, it is neither entangled nor separable, it is no quantum state. This example shows that we cannot shift the superposition property in a way that finite subsystem states are separable whereas the continuous variable state has any kind of entanglement.*

Example 2 *Now let us consider the two-mode squeezed-vacuum state, given as* $|q\rangle = \sqrt{1-|q|^2}\sum_{k=0}^{\infty} q^k|k,k\rangle$ *with* $|q| < 1$*. This state has continuous variable entanglement, which can be detected in finite subsystems. However, it cannot be completely described in finite systems, which is due to the infinite Schmidt rank.*

Let us note the following two facts. The Schmidt coefficients must decrease to zero for $k \to \infty$ *for being an element of the compound Hilbert space. From* $\langle\psi|\psi\rangle = 1$ *as an infinite but converging series follows that this cannot be changed by manipulating the Schmidt coefficients by a local transformation. The second fact is that the state can also be described by a sequence of finite states converging to* $|q\rangle$*. This means that an increasing number of dimensions implies an increasing Schmidt rank. In other words the state has a number of superpositions which exceeds any finite number.*

In both examples the entanglement could be described in terms of arbitrarily large but finite dimensional spaces. As we already said, this is of a great advantage for the mathematical treatment of entanglement. From the physical point of view, it also proves the fact that a state reconstruction with an arbitrary small error from a finite set of measurements delivers the property entangled/separable for a countable number of measurements.

Relation to nonclassicality. For nonclassicality the situation is different. The truncation of the Hilbert space to finite dimensional subsystems, e.g. in Fock basis, delivers nonclassicality for any state. For example, the truncated coherent state $|\alpha_N\rangle \propto \sum_{k=0}^{N} \frac{\alpha^k}{\sqrt{k!}}|k\rangle$ is nonclassical. This is due to the fact that $|\alpha_N\rangle$ is a pure state, but not a coherent state [51].

3.2 Classes of entangled states

The correlations between quantum systems deliver a huge number of states with different kinds of entanglement. There are a lot of classifications of such states, e.g. symmetric states [52], Werner states [14], bound and free entangled states [53], isotropic states [54], and cluster and graph states [55, 56]. Here, let us restrict to best known and – for us – most important families of states. An introduction of different kinds of entanglement can be found in Ref. [9, 10].

Relation to nonclassicality. For nonclassical states there are also families of nonclassical states, e.g. Fock states, squeezed states [57], even/odd coherent states [41], nonlinear coherent states [58], etc. Each classification of the states is given in connection with some nonclassical properties of the states, e.g. sub-Poisson photon statistics, sub-vacuum quadrature noise (both with an infinite number of superpositions of coherent states) and superpositions of two coherent states. In the case of entanglement there are also different classifications of states related to the superposition principle.

3.2.1 Schmidt number states

We already considered the Schmidt rank as the number of global superpositions. The number of the dimensions of the subspaces deliver the maximal possible Schmidt rank, $r_{\max} = \min\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}$. A pure separable state has a Schmidt rank of one, an entangled qubit (e.g. a Bell state) has a Schmidt rank two, an entangled qutrit has a Schmidt rank three, etc., and all quantum states have a Schmidt number less or equal to $r_{\max}$. For some reviews of generalized Schmidt number states see [59, 60, 61].

Pure states. Let us consider the pure states with a Schmidt rank less or equal to a given r. They are elements of the set $\mathcal{S}_r^{(\mathrm{pure})}$. Such a state is a superposition of up to r factorizable vectors. The set of such Schmidt rank states are included in each other by $\mathcal{S}_r^{(\mathrm{pure})} \subset \mathcal{S}_{r'}^{(\mathrm{pure})}$ (for $r < r'$).

Mixed states. A mixture – or convex combination – of Schmidt rank r states delivers the Schmidt number r states [62],

$$\sigma = \int_{\mathcal{S}_r^{(\mathrm{pure})}} dP_{\mathrm{cl}}(\psi)\, |\psi\rangle\langle\psi|, \tag{3.1}$$

where $P_{\mathrm{cl}}(\psi)$ is a classical probability distribution. This means it has a Schmidt number up to the Schmidt rank of the entangled vector $|\Phi\rangle = \sum_{k=1}^{r} |k,k\rangle$.

Any quantum state which cannot be written in this form has a Schmidt number larger than r. Let us note that the Schmidt number of a state ρ is exactly r, iff it is a classical mixture as in Eq. (3.1), $\rho \in \mathcal{S}_r$, but not a classical mixture of those states with $r' = r - 1$, $\rho \notin \mathcal{S}_{r-1}$. We use the common notion for the Schmidt number of a state $r_{\mathrm{S}}(\rho) = r$. The embedding of the sets can be found in Fig. 3.1.

3.2.2 Multipartite entanglement

The notion of entanglement is not so simple in the multi-mode case. Let us focus on the tree-mode situation, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Even for pure states, there are various different kinds of entanglement. For mixed states the situation becomes even more complex. However, the identification of entanglement in finite subspaces, as a necessary and sufficient condition for CV entanglement, is also concluded in Ref. [III].

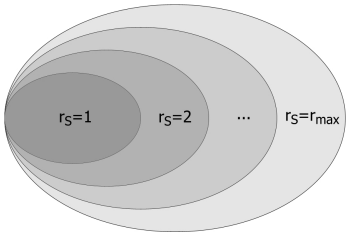

Figure 3.1: Here the embedding of the sets $\mathcal{S}_r$ is given. The convex set $\mathcal{S}_r$ includes all states with a Schmidt number less or equal to r. The set of separable states is $r_{\mathrm{S}} = 1$. The set of entangled qubits is $r_{\mathrm{S}} = 2$. The two-mode squeezed-vacuum state is an element of the set with $r_{\mathrm{S}} = \infty$. Or equivalently, this state is not element of any set with a finite Schmidt number.

Pure states. There are fully separable pure states, $|a,b,c\rangle \in \mathcal{S}_{\mathrm{full}}^{(\mathrm{pure})}$. All three modes separate for such states and they resemble the factorized structure of bipartite factorizable states. Such states are denoted as pure fully separable states [9]. Factorizable states have no entanglement properties.

Another family are the pure partially separable states, $\mathcal{S}_{\mathrm{part}}^{(\mathrm{pure})}$. They are given by states for which one subsystem can be separated, e.g. $\frac{1}{\sqrt{2}}(|0,1\rangle + |1,0\rangle) \otimes |0\rangle$. These states have entanglement properties, but only between two subsystems.

The third class of states are states, where no factorization of any subsystems is possible. One example is the $|GHZ\rangle$ state [63], having the property that a partial trace over system C delivers an separable two-mode state,

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle), \tag{3.2}$$

with $\mathrm{Tr}_C|GHZ\rangle\langle GHZ| = \frac{1}{2}|0,0\rangle\langle 0,0| + \frac{1}{2}|1,1\rangle\langle 1,1|$. Another famous example is the $|W\rangle$ state [64] which is still entangled, when tracing out system C,

$$|W\rangle = \frac{1}{\sqrt{3}}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle), \tag{3.3}$$

with $\mathrm{Tr}_C|W\rangle\langle W| = \frac{1}{3}|0,0\rangle\langle 0,0| + \frac{2}{3}\left(\frac{1}{\sqrt{2}}[|0,1\rangle + |1,0\rangle]\frac{1}{\sqrt{2}}[\langle 0,1| + \langle 1,0|]\right)$.

A realization of such states can be found in Refs. [65, 66]. Such difficulties already arises in a three-qubit system. By increasing the number of dimensions of the subsystems or the number of modes, the situation becomes more complex. There are states with a multi-mode Schmidt decomposition, e.g. the state $|GHZ\rangle$ or examples in Ref. [67], but in general there is no Schmidt decomposition in the multi-mode case [64]. This is a consequence of multi-linear algebra, where no general singular value decomposition exists.

Mixed states. The definition of fully separable states is given by

$$\sigma = \int dP_{\mathrm{cl}}(a,b,c)\, |a,b,c\rangle\langle a,b,c|, \tag{3.4}$$

with P_{cl} being a classical probability distribution. A quantum state which is not fully separable is at least partially entangled. This is a direct generalization of entanglement to multipartite systems.

Partially separable states are defined as

$$\sigma = \int_{\mathcal{S}_{\mathrm{part}}^{(\mathrm{pure})}} dP_{\mathrm{cl}}(\psi)\, |\psi\rangle\langle\psi|. \tag{3.5}$$

Thus, a partially separable state is a classical mixture of pure partially factorizable states. A quantum state which is not even partially separable is fully entangled. This property is sometimes denoted as genuine entanglement.

It is worth to note that every fully separable state is also partially separable. This delivers an analogous situation as it is given in the Schmidt number case. Let us stress that the superposition principle can be applied twice. The partially entangled states are superpositions of fully factorizable states, and the fully entangled states can be expressed in terms of superpositions of partially factorizable states. Like in the case of Schmidt number states, this superposition property delivers inclusions of convex sets, see Fig. 3.2.

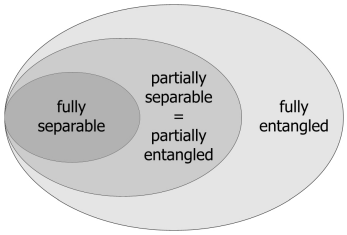

Figure 3.2: The convex set of partially separable states $\mathcal{S}_{\mathrm{part}}$ (mid gray together with dark gray area) includes the convex set of fully separable states $\mathcal{S}_{\mathrm{full}}$ (dark gray area).

3.3 Entanglement identification

One of the most important task in connection with the entanglement theory is the identification of entangled states. There are sufficient detection methods in terms of Bell inequalities [68] (Clauser-Horne-Shimony-Holt-inequalities [69]), entropic inequalities [70], and uncertainty relations [71]. In addition, there are also necessary and sufficient conditions in terms of entanglement witnesses and positive, but not completely positive maps, e.g. partial transposition. In the following we only refer to the latter ones.

3.3.1 Partial transposition

The presently most important and best developed method for the identification of CV entanglement is given in terms of the partial transposition (PT) [49]. In general, the PT condition is only necessary for the detection of entanglement, but includes a

large class of states. For example, the PT criterion applies to all pure states. This entanglement criterion belongs to the class of necessary and sufficient conditions named positive, but not completely positive (PNCP) maps [72].

A single mode quantum state ρ will be mapped by the transposition of the density matrix to another density operator, ρ^{T}. For bipartite states the situation is different. The PT condition for entanglement reads as follows. *A quantum state ϱ is entangled, if the partially transposed state $\varrho^{\mathrm{PT}} = \varrho^{\mathbb{I}\otimes\mathrm{T}}$ is not a quantum state, $\varrho^{\mathrm{PT}} \not\geq 0$.* Here, the transposition is performed in the second mode, but could be equivalently performed in the first one. To understand the notion of states with a negative partial transposition (NPT) let us consider the PT criterion in detail.

A quantum state ϱ is NPT, iff it exists a $|\psi\rangle$ such that

$$0 > \langle\psi|\varrho^{\mathrm{PT}}|\psi\rangle = \mathrm{Tr}(\varrho[|\psi\rangle\langle\psi|]^{\mathrm{PT}}). \tag{3.6}$$

All operators of the structure $[|\psi\rangle\langle\psi|]^{\mathrm{PT}}$ can be obtained by the swap operator V [14],

$$V = [|\Phi\rangle\langle\Phi|]^{\mathrm{PT}} = \sum_{k,l} |k,l\rangle\langle l,k|, \tag{3.7}$$

with the vector $|\Phi\rangle = \sum_{k=1}^{d} |k,k\rangle$, and a local map $A \otimes B|\Phi\rangle = |\psi\rangle$. The swap operator can be decomposed as follows

$$V = \mathbb{I}\otimes\mathbb{I} - 2\sum_{k>l} |\psi^-_{k,l}\rangle\langle\psi^-_{k,l}| \text{ and } |\psi^-_{k,l}\rangle = \frac{1}{\sqrt{2}}(|k,l\rangle - |l,k\rangle). \tag{3.8}$$

The PT criterion is necessary and sufficient in systems $\mathbb{C}^2\otimes\mathbb{C}^2$, $\mathbb{C}^3\otimes\mathbb{C}^2$, and $\mathbb{C}^2\otimes\mathbb{C}^3$ [72] and in the case of bipartite Gaussian states [73]. In all other systems, there exist entangled states with a positive partial transposition (PPT states), see e.g. [50]. These states refer to a class of states being bound entangled.

3.3.2 Entanglement witnesses

In this work we will focus on the detection of entanglement by the method of entanglement witnesses [72, 74]. This method is based on the Hahn-Banach Theorem, and therefore it is necessary and sufficient. An entanglement witness is a Hermitian operator W with

$$\mathrm{Tr}(\sigma W) \geq 0 \text{ for all } \sigma \text{ separable}, \tag{3.9}$$

$$\mathrm{Tr}(\varrho W) < 0 \text{ for an entangled state } \varrho. \tag{3.10}$$

For an optimized witness, there exists at least one pure separable state $|a,b\rangle$ such that $\langle a,b|W|a,b\rangle = 0$ [75, 76]. The necessary and sufficient entanglement condition is: *A quantum state ϱ is entangled, iff a witness W exists such that* $\mathrm{Tr}(W\varrho) < 0$.

Example 3 *Let us consider the swap operator V. This is an optimized entanglement witness. Therefore let us study the action of V when applying to a separable state,*

$$V|a,b\rangle = |b,a\rangle.$$

From the convex structure of separable states it follows the positivity for separable states by pure states,

$$\langle a, b|V|a, b\rangle = |\langle b|a\rangle|^2 \geq 0.$$

Whereas from the spectral decomposition in Eq. (3.8) follows the negativity for some entangled states.

It is hard to construct all entanglement witnesses from the given structure of Eqs. (3.9) and (3.10). We considered a related approach but with a simpler structure. First, let us consider the relation of entanglement witnesses with the corresponding method for the identification of nonclassicality.

Relation to Nonclassicality. Entanglement witnesses can be directly related to normally ordered operators [31], $:f^\dagger f:$. The expectation value of these operators is non-negative for all classical states, but can be negative for nonclassical ones. The normally ordered operators of the given structure also deliver necessary and sufficient conditions for the detection of nonclassicality.

3.3.3 Positive, but not completely positive maps

The best studied example of positive, but not completely positive (PNCP) maps is the partial transposition. It was shown by the Choi-Jamiołkowski isomorphism between PNCP maps and entanglement witnesses, that this method is also necessary and sufficient [72, 77, 78]. However, the PNCP condition is not very practicable, since only a few maps are known. This part is rather short since we will show with a simple argumentation the idea of this isomorphism.

A PNCP map is a linear map Γ mapping quantum state to quantum state in a single mode case, $\rho' = \rho^\Gamma/(\operatorname{Tr}\rho^\Gamma)$. This means the positivity of the operator under the map Γ is conserved. We can summarize these facts in the following equation.

$$\rho^\Gamma = \left(\sum_{k,l} \rho_{k,l}|k\rangle\langle l|\right)^\Gamma = \sum_{ij}\left(\sum_{k,l}\Gamma_{i,k,j,l}\rho_{k,l}\right)|i\rangle\langle j| \geq 0 \tag{3.11}$$

The entanglement condition by PNCP maps is [72]: *A bipartite quantum state σ is separable, iff for all PNCP maps Γ holds $\sigma^{\mathbb{I}\otimes\Gamma} \geq 0$.* This means that applying Γ only on subsystem B will always deliver a positive semi-definite operator in the case of separable states.

Mapping a single-mode Hermitian operator to a single-mode Hermitian operator delivers the condition $\Gamma_{i,k,j,l} = \Gamma^*_{j,l,i,k}$. The positivity of mapped pure states $(|x\rangle\langle x|)^\Gamma$ is sufficient to prove the positivity,

$$\langle y|\left(|x\rangle\langle x|\right)^\Gamma|y\rangle = \langle x, y|L_\Gamma|x, y\rangle \geq 0, \tag{3.12}$$

for all $|y\rangle$ and $L_\Gamma = \sum_{i,j,k,l}\Gamma_{i,k,j,l}|i\rangle\langle j| \otimes |k\rangle\langle l|$. But this is equivalent to the entanglement witness condition in Eq. (3.9). Therefore the construction of all PNCP maps is as limited as the construction of all entanglement witnesses. Or, the other way around, the construction of all entanglement witnesses delivers all PNCP maps Γ.

3.3.4 Hermitian test operators

As we pointed out above, our approach is different. However, our method is not only necessary and sufficient, it is also optimized. This method is based an optimization procedure called separability eigenvalue (SE) problem. This optimization is also relevant for other in connection with of entanglement.

Our optimized entanglement condition reads as follows [I]: *A quantum state ρ is entangled, iff there exists a Hermitian operator L with*

$$\mathrm{Tr}(\rho L) > f_{AB}(L) = \sup\{\mathrm{Tr}(\sigma L) : \sigma \in \mathcal{S}_{AB}\}. \tag{3.13}$$

The set $\mathcal{S}_{AB}$ denotes the set of all separable quantum states σ. Equivalently, we could also use

$$\mathrm{Tr}(\rho L) < \inf\{\mathrm{Tr}(\sigma L) : \sigma \in \mathcal{S}_{AB}\}. \tag{3.14}$$

The value of the function $f_{AB}(L)$ denotes the maximal expectation value of L for separable states. We will discuss how to obtain the value of this function in relation to the separability eigenvalue problem in Chapter 4.

Rewriting our condition, we obtain that all optimized entanglement witnesses W can be written as

$$W = f_{AB}(L)\mathbb{I} \otimes \mathbb{I} - L, \tag{3.15}$$

where L denotes an arbitrary Hermitian operator. In this context it was known that such a construction delivers an entanglement witness [79], but it was unclear if all witnesses have such a form. As we already pointed out, the function f_{AB} can be obtained by the SE equations. They deliver both, the optimal expectation value (SE value) and the vector (SE vector) which yields this value.

Example 4 *We consider a single test operator $L = \sum_{k,l=0}^{\infty} \mathrm{sinc}(\delta\varphi[k-l])|k,k\rangle\langle l,l|$ The function f_{AB} can be calculated as $f_{AB}(L) = 1$. The state to be tested is the following,*

$$\rho_{\delta\varphi} = \frac{1}{2\delta\varphi} \int_{-\delta\varphi}^{+\delta\varphi} d\varphi \, (e^{i\varphi\hat{n}} \otimes \mathbb{I})|\epsilon\rangle\langle\epsilon|(e^{-i\varphi\hat{n}} \otimes \mathbb{I}), \;\; \textit{with } |\epsilon\rangle = \sqrt{1-\epsilon^2} \sum_{k=0}^{\infty} \epsilon^k |k,k\rangle$$

for $0 < \epsilon < 1$ resembling the two-mode squeezing. This means that a two-mode squeezed-vacuum state $|\epsilon\rangle$ is given. The quantum channel of the subsystem A suffers a phase randomization. This randomization is assumed to be equally distributed in the interval $[-\delta\varphi, +\delta\varphi]$.

For no phase randomization, $\delta\varphi = 0$, we have entangled states $\rho_0 = |\epsilon\rangle\langle\epsilon|$ with an infinite Schmidt rank. For a total phase randomization, $\delta\varphi = \pi$, we obtain a separable quantum state $\rho_\pi = (1-\epsilon^2)\sum_{k=0}^{\infty} \epsilon^{2k}|k,k\rangle\langle k,k|$. In Fig. 3.3 it is shown that we can identify entanglement with our method for any phase diffusion below the total one with one single test operator L [VI].

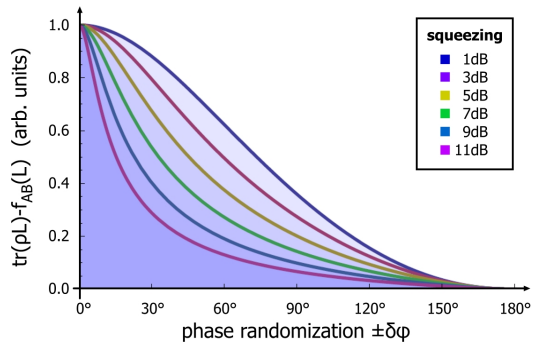

Figure 3.3: The identification of entanglement of a two-mode squeezed-vacuum state is given. One quantum channel is randomized in phase in an interval $[-\delta\varphi, +\delta\varphi]$. We identify this state to be entangled for all randomization below the full randomization, $\delta\varphi < \pi$. The functions are scaled to the maximal mean values. The widths of the functions show that an increasing squeezing delivers a higher sensitivity to dephasing.

Schmidt number states. Again the situation can be generalized by using the superposition principle. First let us consider the Schmidt number case. In the same way as described above the following condition can be found for Schmidt number r states [VI]. A quantum state ϱ has a Schmidt number larger than r, iff there exists an observable L, such that

$$\mathrm{Tr}(\varrho L) > f_r(L), \tag{3.16}$$

with the maximal expectation value for all Schmidt number r states,

$$f_r(L) = \sup\{\langle\psi|L|\psi\rangle : |\psi\rangle \in \mathcal{S}_r^{(\mathrm{pure})}\}. \tag{3.17}$$

For the new defined function f_r we will also find equations representing generalizations of the separability eigenvalue equations. The Schmidt number witnesses [80] can be obtain in a similar form of Eq. (3.15).

Multipartite entanglement. For the multipartite case such necessary, sufficient and optimized conditions follow analogously. A quantum state ϱ is partially entangled, iff there exists an observable L, such that

$$\mathrm{Tr}(\varrho L) > f_{\mathrm{full}}(L), \tag{3.18}$$

with the maximal expectation value for fully separable states,

$$f_{\mathrm{full}}(L) = \sup\{\langle\psi|L|\psi\rangle : |\psi\rangle \in \mathcal{S}_{\mathrm{full}}^{(\mathrm{pure})}\}. \tag{3.19}$$

A quantum state ϱ is fully entangled, iff there exists an observable L, such that

$$\mathrm{Tr}(\varrho L) > f_{\mathrm{part}}(L), \tag{3.20}$$

with the maximal expectation value for all partially separable states,

$$f_{\text{part}}(L) = \sup\{\langle\psi|L|\psi\rangle : |\psi\rangle \in \mathcal{S}_{\text{part}}^{(\text{pure})}\}. \tag{3.21}$$

We will also generalize the separability eigenvalue problem for this case. The multipartite entanglement witnesses [81] can be obtained in a similar form as in the bipartite case, cf. Eq. (3.15).

3.3.5 Detection scheme

For the verification of entanglement of a given state ϱ it is sufficient to find one test operator L with $\text{Tr}(\varrho L) > f_{AB}(L)$. However, *which test operator is the correct one?* For the verification of separability, we would have to check for every test operator if $\text{Tr}(\varrho L) \leq f_{AB}(L)$. Of course, performing an infinite number of tests is impossible. Therefore we considered a approximation scheme, cf. Fig. 3.4, together with an error estimation of such a scheme [I].

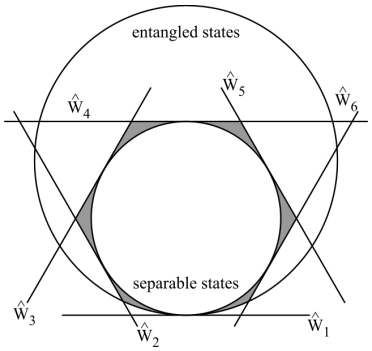

Figure 3.4: The approximation of the separable states by linear forms $w_i(\rho) = \text{Tr}\,(W_i\rho)$ is given. This illustration considers a grid of six operators. The gray areas denote entangled states which cannot be shown to be entangled for the given grid.

3.4 Quasi-probabilities of entanglement

In this section we focus on the representation of entangled mixed quantum states in terms of factorizable states. We show that that such a representation is ambiguous. A method to overcome this deficiency will be studied.

3.4.1 Representation of entangled states

It was shown that any entangled state ϱ can be written as

$$\varrho = (1+\mu)\sigma - \mu\sigma', \tag{3.22}$$

where σ and σ' are separable states and $\mu > 0$ is a real number [82, 83]. Thus, we do not have a convex, but a linear decomposition for entangled states. Let us note

that a separable state requires $\mu = 0$, this means a convex decomposition. This finding is a consequence of the fact that local Hermitian operators, $A \otimes B$, form a basis in the compound operator space of systems $\mathcal{H}_A$ and $\mathcal{H}_B$. Hence any operator, including ρ, can be represented as linear combination of factorizable operators.

There are two facts showing that the state is given by a quasi-probability. First, the value $(1 + \mu) > 1$, whereas for classical discrete probabilities it must be less or equal than one. This part might be compensated by the $(-\mu)$ term. The more important term is the negative contribution $-\mu\sigma'$ itself.

From the decomposition of the separable parts σ and σ' in terms of pure factorizable states, we obtain the general formula of the decomposition of arbitrary quantum states as

$$\rho = \int dP(a, b)\, |a, b\rangle\langle a, b|, \tag{3.23}$$

with P being a quasi-probability [II]. A quantum state is separable, if $P = P_{\text{cl}}$ is a classical probability. An entangled state requires that P is not a classical probability.

Ambiguities. Everything seems to be fine until now, but there are ambiguities in this joint quasi-probability. The Glauber-Sudarshan P function is unique for any state. This means each P function corresponds to one state and the other way around. For entanglement the situation is different. Each quantum state can be given by an infinite number of quasi-probabilities for entanglement. An ambiguous decomposition can be found in Example 5.

A separable quantum state requires only that one of all possible quasi-probabilities is non-negative. On the other hand, the entanglement of a quantum state can be verified only if all quasi-probabilities are negative. A question automatically arises: *Exists a best quasi-probability giving a "if and only if" condition for separability and entanglement?* The answer is yes [II]. Moreover, we present a decomposition scheme that delivers an optimal quasi-probability of entanglement P_{Ent}. The negativities of this quasi-probability are necessary and sufficient for the identification of entanglement. It follows that this scheme delivers a positive P_{Ent} for separable states [II, VIII].

3.4.2 Optimized quasi-probability P_{Ent}

We developed a method, delivering optimized, positive joint probabilities for separable states, and quasi-probabilities for entangled ones. Thus, it overcomes the problems of ambiguity given in the previous paragraph. This approach is again based on the separability eigenvalue problem [II].

Crucial for this representation is the fact, that separable states σ can be given as convex combination of the following form. The separability eigenvalue problem of the density matrix σ delivers solutions $|a_i, b_i\rangle$ together with the separability eigenvalues g_i. The state itself can be written in the form

$$\sigma = \sum_i p_i |a_i, b_i\rangle\langle a_i, b_i|, \tag{3.24}$$

with the convex coefficients p_i, $p_i \geq 0$ and $\sum_i p_i = 1$. It means that any separable state can be given as a convex combination of its separability eigenvectors, see Fig. 3.5 and Refs. [II, VIII].

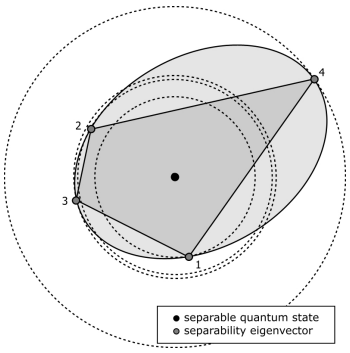

Figure 3.5: A separable state of the convex set $\mathcal{S}_{AB}$ is considered (light gray area). The distance of this element to the pure factorized states is optimized, which yields the SE values and SE vectors. The convex set of states with an optimized distance is defined (dark gray area). The figure shows that the initial quantum state is element of this new convex set.

Due to the definition of the optimal values $g_j = \langle a_j, b_j|\sigma|a_j, b_j\rangle$ we obtain the linear equations for all i, cf. Eq. (3.24),

$$g_j = \sum_i |\langle a_i, b_i|a_j, b_j\rangle|^2 p_i, \tag{3.25}$$

$$\vec{g} = \mathbf{G}\vec{p}, \tag{3.26}$$

with $\vec{g} = (g_j)_j$ the vector of separability eigenvalues, a generalized Gram-Schmidt matrix $\mathbf{G} = (|\langle a_i, b_i|a_j, b_j\rangle|^2)_{i,j}$, and a probability vector $\vec{p} = (p_i)_i$. This is a linear equation, which can be solved. Our method is designed such that a positive solution $p_i \geq 0$ will be obtained for the separable state σ'[II].

If the state is not separable, then our method does deliver negativities. This means at least one of the elements p_i is negative. However, this shows that the state cannot be represented by a non-negative solution. This yields the entanglement property of the state in terms of optimized quasi-probabilities, $P_{\text{Ent}}(a, b)$.

We conclude the following. For finding the optimal joint quasi-probability of entanglement P_{Ent}, we have to solve the separability eigenvalue problem of the density matrix of the quantum state ρ. We define the vector $\vec{g}$ (by the separability eigenvalues), the matrix $\mathbf{G}$ by the separability eigenvectors and solve the linear equation $\vec{g} = \mathbf{G}\vec{p}$. We obtain the quasi-probability P_{Ent} and the property entanglement as

$$P_{\text{Ent}}(a, b) = \sum_i p_i \delta(a - a_i)\delta(b - b_i), \tag{3.27}$$

$$\rho \text{ separable } \Leftrightarrow P_{\text{Ent}} \geq 0 \ (\forall i: \ p_i \geq 0), \tag{3.28}$$

$$\rho \text{ entangled } \Leftrightarrow P_{\text{Ent}} \ngeq 0 \ (\exists i: \ p_i < 0). \tag{3.29}$$

Example 5 *Let us consider the following quantum state*

$$\begin{aligned}\sigma =&\frac{1}{4}|0,0\rangle\langle 0,0| + \frac{1}{4}|1,0\rangle\langle 1,0| + \frac{1}{4}|0,1\rangle\langle 0,1| + \frac{1}{4}|1,1\rangle\langle 1,1| + \frac{1}{2}|\mathfrak{s}_0,\mathfrak{s}_0\rangle\langle\mathfrak{s}_0,\mathfrak{s}_0|\\ &- \frac{1}{8}|\mathfrak{s}_3,\mathfrak{s}_3\rangle\langle\mathfrak{s}_3,\mathfrak{s}_3| - \frac{1}{8}|\mathfrak{s}_1,\mathfrak{s}_3\rangle\langle\mathfrak{s}_1,\mathfrak{s}_3| - \frac{1}{8}|\mathfrak{s}_3,\mathfrak{s}_1\rangle\langle\mathfrak{s}_3,\mathfrak{s}_1| - \frac{1}{8}|\mathfrak{s}_1,\mathfrak{s}_1\rangle\langle\mathfrak{s}_1,\mathfrak{s}_1|,\end{aligned}$$

with the single mode state $|\mathfrak{s}_n\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i^n|1\rangle\right)$*. This is a decomposition of the quantum state in terms of separable states. The not optimized negativities of the state are clearly given. However, we obtain a positive decomposition after applying the optimization method,*

$$\sigma = \frac{5}{8}|\mathfrak{s}_0,\mathfrak{s}_0\rangle\langle\mathfrak{s}_0,\mathfrak{s}_0| + \frac{1}{8}|\mathfrak{s}_2,\mathfrak{s}_0\rangle\langle\mathfrak{s}_2,\mathfrak{s}_0| + \frac{1}{8}|\mathfrak{s}_0,\mathfrak{s}_2\rangle\langle\mathfrak{s}_0,\mathfrak{s}_2| + \frac{1}{8}|\mathfrak{s}_2,\mathfrak{s}_2\rangle\langle\mathfrak{s}_2,\mathfrak{s}_2|.$$

This example shows two facts. The representation of mixed states by separable states is ambiguous, and the optimization delivers a positive joint probability for this separable state. The optimized classical probability distribution is visualized in the left part of Fig 3.6.

Example 6 *Now let us consider the Bell state*

$$\varrho = |\Phi^+\rangle\langle\Phi^+|, \text{ with } |\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0,0\rangle + |1,1\rangle\right).$$

This state is entangled. We obtain an optimal decomposition by applying our method as

$$\begin{aligned}\varrho =&\frac{1}{2}\left(|0,0\rangle\langle 0,0| + |1,1\rangle\langle 1,1|\right)\\ &+ \frac{1}{4}\left(|\mathfrak{s}_0,\mathfrak{s}_0\rangle\langle\mathfrak{s}_0,\mathfrak{s}_0| + |\mathfrak{s}_1,\mathfrak{s}_3\rangle\langle\mathfrak{s}_1,\mathfrak{s}_3|\right) + \frac{1}{4}\left(|\mathfrak{s}_2,\mathfrak{s}_2\rangle\langle\mathfrak{s}_2,\mathfrak{s}_2| + |\mathfrak{s}_3,\mathfrak{s}_1\rangle\langle\mathfrak{s}_3,\mathfrak{s}_1|\right)\\ &- \frac{1}{4}\left(|\mathfrak{s}_0,\mathfrak{s}_2\rangle\langle\mathfrak{s}_0,\mathfrak{s}_2| + |\mathfrak{s}_1,\mathfrak{s}_1\rangle\langle\mathfrak{s}_1,\mathfrak{s}_1|\right) - \frac{1}{4}\left(|\mathfrak{s}_2,\mathfrak{s}_0\rangle\langle\mathfrak{s}_2,\mathfrak{s}_0| + |\mathfrak{s}_3,\mathfrak{s}_3\rangle\langle\mathfrak{s}_3,\mathfrak{s}_3|\right).\end{aligned}$$

The optimized nonclassical joint quasi-probability distribution for entanglement is visualized in the right hand side of Fig 3.6.

Relation to nonclassicality. We have seen that the Glauber-Sudarshan P function can be negative for separable states. In addition we have seen that the (linear) decomposition of states in terms of factorizable states is ambiguous. We have shown that the optimized P_{Ent} quasi-probabilities overcomes this problems. It exceeds the properties of a classical joint probability, if and only if the quantum state is entangled. For separability it plays the same role as the P function for classical states.

Relation to the eigenvalue problem. *The separability eigenvalue problem, again?* It seem that the separability eigenvalue problem is somehow universal for the property entanglement. The identification of entanglement by Hermitian test operators and its representation by P_{Ent} can be obtained by solving the SE problem of the density matrix of the state. In the introduction, we have seen that the eigenvalue problem delivers the same properties in the general case.

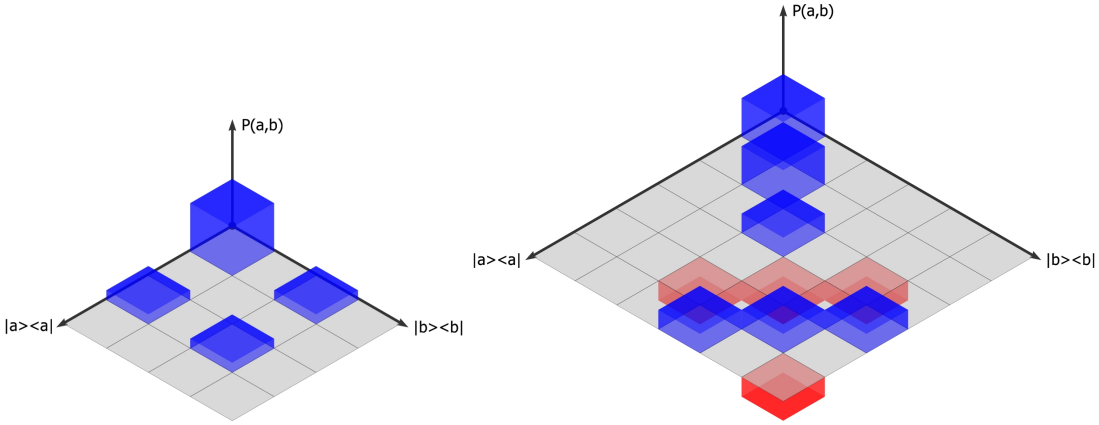

Figure 3.6: On the left hand side a classical joint probability of a mixed separable quantum state is given. The axis limiting the gray area enumerate different vectors $|a\rangle \in \mathcal{H}_A$ and $|b\rangle \in \mathcal{H}_B$ for the decomposition of the state in terms of separable states. On the right hand side we find the example of an Bell state, with P_{Ent} having negativities.

Generalizations. Again, generalizations are possible for Schmidt number r states and for multipartite systems (fully and partially separable). We follow the same arguments, see [VIII], to construct optimized quasi-probabilities for Schmidt number r states and partially and fully separable multipartite quantum states. The resulting Schmidt number r quasi-probability $P_{\text{Ent},r}$ (or $P_{\text{Ent,fully}}$, $P_{\text{Ent,part}}$) are negative, iff the state under consideration is not a Schmidt number r (or fully/partially separable) quantum state. In Ref. [VIII] it is shown, that for any finite dimensional Hilbert space and a convex subset a linear equation in the form of Eq. (3.26) can be obtained by the corresponding Schmidt number or multipartite eigenvalue problem.

3.5 Quantification

In addition to the general property of being entangled or not, it is interesting to know: *How much entanglement has a quantum state?* This means the aim is to relate the entanglement of different quantum states. In the introduction Sec. 2.1 and Ref. [VII], we have already seen that distance measures have the problem that the choice of the distance dramatically influences the amount of nonclassicality. For the quantification of entanglement, we start with different kinds of operations that do not effect the separability of a state.

3.5.1 Local operations and classical communications

Now we consider different classes of local operations and classical communication (LOCC). We distinguish between pure and mixed operations, and deterministic and non-deterministic ones. A review about such classification can be found in [9].

The general classification of linear operations Λ mapping a quantum state to another has been first considered in [85] for a single mode case. A deterministic

operation Λ is an operation preserving the trace of a quantum state, $1 = \mathrm{Tr}\,\rho = \mathrm{Tr}\,\Lambda(\rho)$. On the other hand for a non-deterministic operation holds $1 = \mathrm{Tr}\,\rho \geq \mathrm{Tr}\,\Lambda(\rho)$.

Local filtering operations. These are operations mapping

$$|a\rangle\langle a| \otimes |b\rangle\langle b| \mapsto A|a\rangle\langle a|A^\dagger \otimes B|b\rangle\langle b|B^\dagger, \tag{3.30}$$

with arbitrary $A \otimes B \in \mathsf{Lin}(\mathcal{H}, \mathcal{H})$ [86, 87]. The general local filter operations are divided in several substructures. Local filtering operations cannot generate a mixed state from a pure state.

Local invertible operations are operations of the form $S \otimes T$, with S and T being invertible. They map a factorizable vector to a factorizable vector in a one to one correspondence. In general, they affect the Schmidt coefficients of a pure entangled state, but the Schmidt rank remains invariant. These operations are used for many applications and results, including: *A quantum state σ is separable, iff $(S\otimes T)\sigma(S^\dagger \otimes T^\dagger)$ is separable* [88, V]. Such operations have been studied in Ref. [V]. It follows that any pure Schmidt rank r state can be obtained by any other pure Schmidt rank r state.

A special subclass of these local invertible operations are local unitary operations $U_A \otimes U_B$, preserving the inner product of two vectors, $U_{A,B}^{-1} = U_{A,B}^\dagger$. The Schmidt coefficients and the rank remain invariant under these operations. Moreover, they are the only deterministic local filtering operations.

Local projections are projections of the form $P \otimes Q$. We used such operations to solve the CV entanglement problem in finite dimensions [III]. In general they affect both: they may lower the Schmidt rank and change the Schmidt coefficients.

Stochastic separable operations. In addition to local filter operations, which do not affect the purity of a pure state, we consider classical mixtures. We already discussed that the mixture of separable states is always separable. However, there are also examples of mixing entangled states, which deliver a separable one. Thus, a mixing procedure can only convert an entangled state to a separable one. The most general form is given in terms of stochastic separable operations [89, 90],

$$|a\rangle\langle a| \otimes |b\rangle\langle b| \mapsto \Lambda_{AB}^{\mathrm{sep}}(|a\rangle\langle a| \otimes |b\rangle\langle b|) = \sum_k A_k|a\rangle\langle a|A_k^\dagger \otimes B_k|b\rangle\langle b|B_k^\dagger, \tag{3.31}$$

where $A_k \otimes B_k$ denotes a local filtering operation for each k.

These operations can be further specified in the background of classical communication. First we can have a situation without communication, just noise in each quantum channel. Such deterministic operations are called local operations,

$$|a\rangle\langle a| \otimes |b\rangle\langle b| \mapsto \Lambda_A(|a\rangle\langle a|) \otimes \Lambda_B(|b\rangle\langle b|) = \sum_k A_k|a\rangle\langle a|A_k^\dagger \otimes \sum_l B_l|b\rangle\langle b|B_l^\dagger. \tag{3.32}$$

A factorizable state $\rho_A \otimes \rho_B$ will be mapped by a local operation to another factorizable state.

The next step would be a one-way forward classical communication,

$$|a\rangle\langle a| \otimes |b\rangle\langle b| \mapsto \Lambda^{\rightarrow}_{AB}(|a\rangle\langle a| \otimes |b\rangle\langle b|) = \sum_k A_k|b\rangle\langle b|A_k^\dagger \otimes \Lambda_{B,k}(|a\rangle\langle a|). \tag{3.33}$$

For example, the measurement in system A (given by the operators A_k) influences the local operation $\Lambda_{B,k}$ in system B (forward) by sending the classical information about the measurement outcome of B_k. In the same manner the one-way backward classical communication is defined by interchanging the subsystems. The two-way classical communication is given by forward and backward operations, which is only slightly different from the set of stochastic separable operations. All classical communication operations are considered to be deterministic.

Example 7 *Let us study an example of a global map – the swap operator V,*

$$V|a,b\rangle = |b,a\rangle,$$

being an interesting example. It maps in a deterministic manner a separable state to a separable one. The exchange of the systems for arbitrary $|a,b\rangle$, e.g. by an instantaneous two-way teleportation scheme, requires a non-local quantum protocol. This means we need a quantum channel between the subsystems to exchange $A \leftrightarrow B$.

However, there are local unitary maps $U_A \otimes U_B$ for a particular choice $|a_0,b_0\rangle$ which deliver the same result

$$U_A \otimes U_B|a_0,b_0\rangle = |b_0,a_0\rangle, \text{ with } U_A = \sum_k |b_k\rangle\langle a_k| = U_B^\dagger,$$

and $|a_k\rangle$, $|b_{k'}\rangle$ being orthonormal. These are deterministic local unitary operations doing the same as the swap operator, $V|a_0,b_0\rangle = |b_0,a_0\rangle$. In contrast to the specific choice of $|a_0,b_0\rangle$, the swap operator leads to an exchange for any choice $|a,b\rangle$.

Let us consider an entangled state $|\psi\rangle = \sum_{k=1}^r \lambda_k|k,k\rangle$. This particular state transforms as $V|\psi\rangle = \sum_{k=1}^r \lambda_k|k,k\rangle$. The swap operator does neither change the Schmidt rank nor the Schmidt coefficients. For states with a Schmidt decomposition in a different basis, both can be changed (the Schmidt rank can only decrease, cf. Eq. (3.8)).

Example 8 *Let us consider losses in one channel by a non-Hermitian Hamilton operator, $H = -i\hbar\gamma\sum_k k|k\rangle\langle k|$. This situation is given for the evolution in cavity QED before a quantum jump [91]. The evolution of the state $|\psi\rangle = \sum_k \lambda_k|k,k\rangle$ is given by*

$$|\psi(t)\rangle = e^{-iHt/\hbar} \otimes \mathbb{I}|\psi\rangle = \sum_{k=0}^{\infty} e^{-\gamma tk}\lambda_k|k,k\rangle.$$

We choose an initial Schmidt rank r state with the following coefficients,

$$\lambda_k = \begin{cases} e^{\gamma t_0 k} & \text{for } k = 0,\ldots,r-1 \\ 0 & \text{for } k \geq r. \end{cases}$$

The initial quantum state has no equally distributed Schmidt coefficient, whereas for $t = t_0$ we have equally distributed Schmidt coefficients. In comparison, the Schmidt rank is r for all times. This operation is a non-deterministic local invertible operation.

Now, what is an LOCC operation? To be honest, I cannot answer this question strictly. In different publications, the authors use different families of operations. All families of considered operations have the property, that they map a separable state to a separable one. However, the most common notion of LOCC is given by deterministic two-way classical communication. The set of stochastic separable operations includes these operations, but it also includes more than these operations, for instance local invertible operations. We see – even though we restricted ourselves – that there is a large number of families referred to be LOCC operations.

3.5.2 Analysis

Convex cone construction. We have seen that there are operations Λ which are non-deterministic, or not trace preserving. First, let us consider the case of a given map Λ, for which exist a state with $\operatorname{Tr}\Lambda(\rho) > 1$. This is not even a non-deterministic operation, but we can change it to one by

$$\Lambda'(\rho) = \frac{\Lambda(\rho)}{\sup\{\Lambda(\tilde{\rho}) : \tilde{\rho} \in \mathcal{S}_\infty\}}, \tag{3.34}$$

with $\operatorname{Tr}\Lambda'(\rho) \leq 1$ for all ρ.

The factorizable structure of a pure state $|a\rangle \otimes |b\rangle$ does not depend on the normalization of this state. In fact, for any positive semi-definite (trace-class) operator $L \in \mathsf{Herm}(\mathcal{H})$ we can ask if this operator is separable. Therefore, we may define the state

$$\rho = \frac{L}{\operatorname{Tr} L}. \tag{3.35}$$

This means that the normalization constant does not deliver any information about the fact, whether a quantum operator is separable or not. This resembles a construction of a cone of separable state $\mathcal{S}_{AB}^{(\mathrm{cone})}$,

$$\mathcal{S}_{AB}^{(\mathrm{cone})} = \{c\sigma : c \geq 0 \wedge \sigma \in \mathcal{S}_{AB}\}. \tag{3.36}$$

This is a typical mathematical procedure, when dealing with optimization on convex sets. The detection and the representation of entanglement is not affected by this scaling structure, e.g. one can apply Eq. (3.35). For the question of the amount of entanglement of a given state we can neglect the classification of operations into deterministic and non-deterministic ones [VII].

Generating mixed states by stochastic separable operations. Any quantum state can be rewritten in terms of stochastic separable operations acting on a pure state [V]. Let r be the Schmidt number of ρ, $r_{\mathrm{S}}(\rho) = r$, and $|\Phi\rangle = \sum_{k=1}^{r} |k,k\rangle$, now we obtain

$$\rho = \sum_l A_l \otimes B_l |\Phi\rangle\langle\Phi| (A_l \otimes B_l)^\dagger. \tag{3.37}$$

In the case $r_{\mathrm{S}}(\rho) < r$, we can first perform a local projection to obtain a smaller Schmidt rank of $|\Phi\rangle$. Thus, all elements of the set $\mathcal{S}_r$ can be obtained by a single

element of $\mathcal{S}_r^{(\text{pure})}$ and one stochastic separable operation. On the other hand, no element of $\mathcal{S}_{r+1} \setminus \mathcal{S}_r$ can be constructed in such a way by an element of $\mathcal{S}_r^{(\text{pure})}$. The number of global superpositions to generate ρ can only decrease when applying a stochastic separable operation.

3.5.3 General quantification

The quantification of entanglement is given by a function E mapping a quantum state to its amount of entanglement with the properties [92, 93]

$$E(\sigma) = 0 \Leftrightarrow \sigma \in \mathcal{S}_{AB} \quad \text{and} \quad \forall \Lambda \text{ LOCC}: \; E(\rho) \geq E(\Lambda(\rho)). \tag{3.38}$$

Sometimes it is useful to shift the first condition to: *The measure is minimal* $E(\sigma) = E_{\min}$ *only for separable states.* This delivers an equivalent definition. It is also obvious, that a strictly monotonically increasing function h delivers also an entanglement measure $E_h(\rho) = h(E(\rho))$. However the definition of an entanglement measure suffers from some problems. First, let us start with a trivial example.

Example 9 *We define the measure E as*

$$E_\epsilon(\sigma) = E_{\min} \text{ for } \sigma \text{ separable and } E_\epsilon(\varrho) = E_{\min} + \epsilon \text{ for } \varrho \text{ entangled and } \epsilon > 0.$$

This is a valid entanglement measure, but does not deliver much insight into the structure of entangled states.

The definition of a measure depends on the choice of the family of LOCC operations [V]. Different choices of LOCC deliver different amounts of entanglement. For example, local invertible operations are usually not considered to be LOCC operations, but they are used for entanglement distillation protocols [94, 95]. This automatically influences the notion of a maximally entangled state [V]. On the other hand, the LOCC as deterministic two-way classical communication does only refer to a quantum communication task, and it does not apply to the situation of entanglement as a resource of quantum computation [96].

Entropic measures. Distance or entropic measures of entanglement are directly related to the ambiguous relation of the amount of entanglement. We have seen that different choices of distances deliver different relations of the amount of entanglement for entangled quantum states. The same can be formulated in terms of entropic measures, which are only monotonic functions of distances.

Example 10 *Let us illustrate this statement with the relative entropy of entanglement [97]*

$$E(\rho) = \inf_{\sigma \in \mathcal{S}_{AB}} \operatorname{Tr} \rho |\log \rho - \log \sigma|.$$

The infimum is taken over all separable states σ for a fixed quantum state ρ. Let us recall the fact the function $\|L\| = \operatorname{Tr}|L|$ is a norm for $L \in \mathsf{Herm}(\mathcal{H})$. The operator

function $|L|$ is defined for the spectral decomposition as $|L| = \sum_k |L_k|\,|\psi_k\rangle\langle\psi_k|$[1]. The norm can be converted to another one – in general into pseudo-norm – by a metric defined by the quantum state $\|L\|_\rho = \mathrm{Tr}\rho|L|$. We note that the operator function $\log L$ is a strictly monotonic increasing function. Moreover, for positive semi-definite operators it is invertible, e^L. Now we replace L with $L = \log\rho - \log\sigma$ and use the definition of distance entanglement measures to obtain

$$E(\rho) = \inf_{\sigma\in\mathcal{S}_{AB}} \|\log\rho - \log\sigma\|_\rho.$$

This means we have revealed the relative entropy as a monotonic operator function, $\log(L)$, of distance measure $\|L\|_\rho$.

This example shows that distance measures and entropic measures are closely related. Thus it is obvious that entropic measures which are defined in terms of distances suffer from the same ambiguity as distance measures. The questions arises if there exists a better quantifier of entanglement. It should have the following properties:

1. It should satisfy the definition of an entanglement measure in Eq. (3.38);
2. The desired entanglement measure should be defined for a preferably large class of LOCC operations;
3. It should give insights into the structure of entanglement (which is not the case in Example 9);
4. It should relate the entanglement between **all** quantum states in an **unambiguous** way;
5. It should have a clear physical interpretation;
6. Last but not least the measure must be accessible in experiments.

The Schmidt number. Such a measure exists and it is well known. We already defined the Schmidt number of a quantum state [62]. It fulfills all the requirements:

1. This number is known to be an entanglement measure;
2. The operations for its definition are all deterministic and non-deterministic stochastic separable operations;
3. It is a non-trivial measure;
4. All quantum states are element of exactly one set $\mathcal{S}_r \setminus \mathcal{S}_{r-1}$. It delivers for any quantum state the largest Schmidt rank of entangled qudits in this state, which can be compared with any other state [VI];

[1]Here, an operator function $F(L)$ is given in terms of the spectral decomposition of $L = \sum_k L_k|\psi_k\rangle\langle\psi_k|$ and a real function F. It delivers $F(L) = \sum_k F(L_k)\,|\psi_k\rangle\langle\psi_k|$.

5. Its physical interpretation as the number of global quantum superpositions has been discussed above, see [V];

6. We derived necessary and sufficient, optimized conditions in terms of measurements for the identification of the Schmidt number [VI].

In Ref. [V] we have shown that all entanglement measures E which use LOCC operations including local projections are monotones of the Schmidt number. This means that a decreasing Schmidt number does not allow an increasing entanglement given in terms of E. In addition, we showed that all entanglement measures E which use LOCC operations including local invertibles have the property that the pure states with a maximal Schmidt rank are maximally entangled. This means that the Schmidt coefficients do not influence the amount of entanglement given in terms of such an universal entanglement measure E [V]. It is also clear that the Schmidt number is not the only measure with the desired property. For example, if we also take the purity of a quantum state into account, this can further specify the amount of entanglement.

Sometimes there are more requirements for an entanglement measure. The monotonicity axiom $E(\rho) \geq E(\Lambda(\rho))$ is sometimes replaced by a stronger one. Namely, the measure does not increase on average. We showed that this is also fulfilled for the Schmidt number [V]. This is related to the convexity of this measure, which is another requirement that can be postulated. Obviously the Schmidt number is convex, due to the definition of the convex sets $\mathcal{S}_r$. The last additional requirement studied here, will be given in the following example.

Example 11 *In quantum information processing it is useful to consider copies of states [98]. This means we have at the same time N copies of the state, $\rho \mapsto \rho \otimes \cdots \otimes \rho = \rho^{\otimes N}$. The entanglement measure is considered to obey $E(\rho^{\otimes N}) = N \cdot E(\rho)$, which is fulfilled for the measure used in Example 10.*

Let us consider a pure Schmidt rank r state $|\Phi\rangle = \sum_{k=1}^{r} |k\rangle_A \otimes |k\rangle_B$. Now let us take N copies

$$|\Phi\rangle^{\otimes N} = \sum_{k_1,\ldots,k_N=1}^{r} |k_1,\ldots,k_N\rangle_A \otimes |k_1,\ldots,k_N\rangle_B.$$

Obviously we have $r_{\mathrm{S}}(|\Phi\rangle\langle\Phi|) = r$ and $r_{\mathrm{S}}([|\Phi\rangle\langle\Phi|]^{\otimes N}) = r^N \neq Nr$. However, a monotonic increasing function, $r_{\mathrm{S,log}} = \log r_{\mathrm{S}}$, yields the desired property,

$$r_{\mathrm{S,log}}([|\Phi\rangle\langle\Phi|]^{\otimes N}) = N \log r = N \cdot r_{\mathrm{S,log}}(|\Phi\rangle\langle\Phi|).$$

Sometimes also a so-called ancilla state is considered $\rho \mapsto \rho \otimes \rho_{\mathrm{anc}}$. Also in this case it follows $r_{\mathrm{S,log}}(\rho \otimes \rho_{\mathrm{anc}}) = r_{\mathrm{S,log}}(\rho) + r_{\mathrm{S,log}}(\rho_{\mathrm{anc}})$.

Nonclassicality Measures. Using the superposition principle to quantify the entanglement of a quantum state can be generalized to define nonclassicality measures for single-mode systems. In Ref. [VII], we have shown that the convex ordering procedure and the quantification of nonclassicality is possible by the number of coherent superpositions. The number of superpositions of coherent states, needed to

generate a quantum state, quantifies the amount of nonclassicality. The superposition principle is the only physically allowed procedure for leaving the convex set of classical quantum states.

Example 12 *The coherent state $|\alpha\rangle$ has no quantum superpositions of classical states, since it is the classical analogue. It has zero coherent quantum superpositions. The odd coherent states [41], $[2(1-\exp(-2|\alpha|^2))]^{-1/2}(|\alpha\rangle - |-\alpha\rangle)$, have a nonclassicality measure of one. One quantum superposition of coherent states is necessary to describe this state completely. There are also states with an infinite amount of nonclassicality. For example, any Fock state $|n\rangle$ $(n>0)$ – representing the particle properties of the radiation field – needs infinitely many superpositions to be obtained by coherent states,*

$$|n\rangle = \frac{\sqrt{n!}}{2\pi} e^{-1/2} \int_0^{2\pi} d\varphi e^{-in\varphi} |e^{i\varphi}\rangle,$$

where $|e^{i\varphi}\rangle$ are the coherent states on the unit circle in phase space.

The squeezed state has a similar property. The Wigner function of a pure squeezed state is non-negative. However, the squeezed state does not fulfill the correspondence principle of Bohr[2], which is fundamental for the analogy of coherent states to classical mechanics. This nonclassical property is given by the negative P function and can be quantified by an infinite number of superpositions of coherent states to obtain the squeezed state.

Here, the correspondence to the local unitary operations is the displacement operator, local invertibles analogs are the coherent noise free amplifications, and mixtures of arbitrary states can only decrease the number of quantum superpositions, e.g.

$$\rho = \sum_{n=0}^{\infty} e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} |n\rangle\langle n|, \tag{3.39}$$

with ρ being a phase randomized coherent state and being a mixture of strongly nonclassical Fock states.

Multipartite entanglement. The number of superpositions also apply in the case of partial or full entanglement, cf. [VII]. The definition would be the same as in the Schmidt number case for bipartite entanglement. But a more detailed measure might also allow to distinguish, for example, between a $|W\rangle = \frac{1}{\sqrt{3}}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle)$ state with three superpositions and a GHZ state also consisting of three superpositions, $|GHZ\rangle = \frac{1}{\sqrt{3}}(|1,1,1\rangle + |2,2,2\rangle + |3,3,3\rangle)$. This problem has to be studied in future.

[2]The violation of the correspondence principle of Bohr can be obtained by selecting a sequence of squeezed states, with an anti-squeezed quadrature proportional to $|\alpha|$ and a displacement of α with respect to the vacuum state. For $|\alpha| \to \infty$, this family of squeezed states has a non-zero relative anti-squeezed quadrature.

3.5.4 Generalizations for pseudo and operational measures.

We have generalized the method to pseudo and operational measures [V]. We relax the axioms of the quantification in two ways. We restrict to the set of operations which can be performed in a specific quantum protocol, and the useful amount of entanglement for a certain quantum task can be zero even if the state is entangled.

Operational measures. As we pointed out, the LOCC in terms of deterministic two-way classical communication protocols are suitable for the quantum communication scenario. For the quantification of entanglement we used the most general family of stochastic separable operations. This yields the definition of a operational measure. For such a measure, we only use a sub-semigroup of all stochastic separable operations [V], e.g. LOCC. We relax the monotonicity condition as follows

$$\begin{aligned} \Lambda \text{ stochastic separable operation}: &\ E(\rho) \geq E(\Lambda(\rho)) \\ \text{relax to } \rightarrow\ \Lambda \text{ LOCC}: &\ E(\rho) \geq E(\Lambda(\rho)). \end{aligned} \tag{3.40}$$

This yields the initial monotonicity condition using two-way classical communication, see Eq. (3.38), as an operational measure. The desired quantum task of communication can perform only some stochastic separable operations.

Pseudo measures. On the other hand, we can also relax the first condition in Eq. (3.38). A pseudo measure can be minimal for entangled states as well. This is useful in the case when the kind of entanglement in the state is not useful for the considered scenario [V]. An example could be a NPT-measure for the task of distillation. All entangled PPT states have a zero amount of useful entanglement for distillation,

$$\begin{aligned} \sigma \text{ separable} &\quad \Leftrightarrow \quad E(\sigma) = 0 \\ \text{relax to } \rightarrow\ \sigma \text{ separable} &\quad \Rightarrow \quad E(\sigma) = 0, \end{aligned} \tag{3.41}$$

here the direction "$\Leftarrow$" would be given for PPT bound entangled states.

Example 13 *In Ref [V] we considered an example of a pseudo and operational measure. Our set of LOCC is given by only some stochastic separable operations* $\Lambda_1, \ldots, \Lambda_K$*. These are the operations which are considered in an experimental setup. In addition, we have a single observable* L*. We may define*

$$E(\rho) = \max\left(\left\{ \frac{\mathrm{Tr}\left(\left[\frac{\Lambda_k(\rho)}{\mathrm{Tr}\Lambda_k(\rho)} \right] L \right) - f_{AB}(L)}{f_\infty(L) - f_{AB}(L)} : k = 1, \ldots, K \right\} \cup \{0\} \right).$$

This quantifies the usable amount of entanglement for the possible operations Λ_k *and the observable* L*. This measure is an operational one, since not all stochastic separable operations can be included in this particular experiment. In addition it is a pseudo measure. Besides all separable states, there may be states which are entangled and have no usable entanglement. The useful amount is a value between 0 and 1. In Ref [V] we studied the case* $L = -V$ *(*$f_{AB}(-V) = 0$ *and* $f_\infty(-V) = -1$*) for PPT states.*

4 The Separability Eigenvalue Problem

The separability eigenvalue (SE) problem is an algebraic approach to an analytical optimization problem: *What is the optimal expectation value of a measurement of an observable for separable quantum states?* It applies to arbitrary dimensional Hilbert spaces, even for continuous variable systems. Together with the partial transposition, it can be used to detect all states which are positive under partial transposition and entangled, so-called bound entanglement.

Some interesting properties can be found when we solve this problem. First of all, we obtain the Schmidt decomposition [I] of any pure quantum state by this method, whereas the initial question has nothing to do with the Schmidt decomposition. The solution of the SE problem for Hermitian operators delivers all entanglement witnesses [I]. Moreover, the unambiguous representation of entangled states by quasi-probabilities is given by the solutions of the SE problem for the mixed quantum state itself [II]. It delivers some invariant properties which are axioms of the quantification of entanglement, leading to surprising new results [V].

This very general method sheds new light on the phenomena of entanglement. Even though it was designed for solving a mathematical problem, it delivers a manifold of insights to the physics of entangled states. It includes the representation, identification and quantification of entanglement. Moreover, there might be some presently undiscovered features of the SE equations.

In this chapter we define the separability eigenvalue equations. We study some properties of the solutions. Moreover, we generalize this problem to Schmidt number r states and multipartite systems. We also consider some strategies for the solution of this new equations.

4.1 The separability eigenvalue problem

The SE problem arises when an optimization of the following form is needed,

$$\langle a,b|L|a,b\rangle \rightarrow \text{optimum, with } \langle a,b|a,b\rangle \equiv 1. \tag{4.1}$$

This is an optimization problem which can be solved with the method of Lagrange multipliers. Note that the separable states $|a,b\rangle$ are the extremal points of the convex set $\mathcal{S}_{AB}$. This means each element of the convex set can be given as a convex combination of the extremal points, and the extremal points cannot be given as a convex combination of any other elements of the convex set. Starting from this optimization procedure, we obtain the separability eigenvalue problem [I].

Definition 1 *The separability eigenvalue (SE) problem for an operator L in its first form is given by*

$$L_b|a\rangle = g|a\rangle \text{ and } L_a|b\rangle = g|b\rangle,$$

with $L_b = \mathrm{Tr}_B([\mathbb{I} \otimes |b\rangle\langle b|]L)$ and $L_a = \mathrm{Tr}_A([|a\rangle\langle a| \otimes \mathbb{I}]L)$, $g = \langle a, b|L|a, b\rangle$ being the SE value, and $|a, b\rangle$ being the SE vector.

The separability eigenvalue (SE) problem for an operator L in its second form is given by

$$L|a, b\rangle = g|a, b\rangle + |\chi\rangle,$$

with a bi-orthogonal perturbation $|\chi\rangle$: $0 = \langle a|\chi\rangle \in \mathcal{H}_A$ (projection to $|a\rangle$) and $0 = \langle b|\chi\rangle \in \mathcal{H}_B$ (projection to $|b\rangle$).

In the first form of the SE problem we have two single-mode eigenvalue equations. This explains the name of our method. The operators L_a and L_b deliver single-mode eigenvalue equations which depend on the solution of the other one. In this form the SE equations deliver a coupled set of eigenvalue equations.

In the second form we have an perturbed eigenvalue problem in the compound space, $\mathcal{H}_A \otimes \mathcal{H}_B$. The perturbation $|\chi\rangle$ is orthogonal to any projection in a space $\mathsf{span}\{|a\rangle\} \otimes \mathcal{H}_B$ and $\mathcal{H}_A \otimes \mathsf{span}\{|b\rangle\}$. Also from this form we immediately see a relation to the eigenvalue problem. Namely, an eigenvalue with a eigenvector $|a, b\rangle$ is also a separability eigenvalue, $|\chi\rangle = 0$[1].

4.2 Entanglement properties

As we have seen above the SE problem has some general properties in connection to entanglement. For the identification by Hermitian test operators L and the representation by entanglement quasi-probabilities, P_{Ent}, the SE equations has to be solved. Even the quantification of entanglement is affected by the SE problem.

Determination of entanglement. We have seen that the function $f_{AB}(L)$ – the maximal expectation value of L for separable states – is needed to give a general identification of all entangled and separable states [I] and Sec. 3.3.4. Thus we need to find all optimal (maximal) values of L. Afterward we obtain

$$f_{AB}(L) = \max\{g : \ g \text{ SE value of } L\}, \tag{4.2}$$

cf. Eq. (1.6). Let us note that the test operator L can detect entanglement, iff $f_{AB}(L) < f_\infty(L)$, with $f_\infty(L) = \sup\{\mathrm{Tr}(\rho L) : \rho \in \mathcal{S}_\infty\}$. In the case $f_{AB}(L) = f_\infty(L)$ the entanglement condition $\mathrm{Tr}(\varrho L) > f_{AB}(L)$ cannot be fulfilled for any entangled state ϱ. This is equivalent to the fact that the eigenspace of the largest eigenvalue contains a factorizable state, see [VI].

[1] A factorizable eigenvector fulfills $L|a, b\rangle = g|a, b\rangle + 0$.

Quasi-probabilities of Entanglement The second application of the SE problem is related to the unambiguous representation of entangled states by separable states through a quasi-probability. Here, we need the solutions of the SE equations of the state ρ to obtain the vector $\vec{g}$ and the matrix $\mathbf{G}$ delivering P_{Ent} of the state, see Ref. [II] and Sec. 3.4.2. The vector $\vec{g} = (g_i)_i$ is the vector of SE values, and the matrix $\mathbf{G}$ is the generalized Gram-Schmidt matrix $(\mathbf{G})_{i,j} = |\langle a_i, b_i | a_j, b_j \rangle|^2$ in the operator space $\mathsf{Herm}(\mathcal{H})$.

As we have shown above this is also a method for detecting entanglement by negativities of a quasi-probability distributions. The negativities turned out to be a result of the global quantum superposition principle.

The SE problem and the Schmidt decomposition. Let us consider an example presented in Ref. [I]. We aim to solve the SE problem for projections of the form $L = |\psi\rangle\langle\psi|$. Surprisingly we find a relation to the Schmidt decomposition of the state $|\psi\rangle$.

Example 14 *Now let us solve the SE equations for a special case. Let us consider the projection L with a Schmidt decomposition $|\psi\rangle = \sum_{k=1}^{d} \lambda_k |e_k, f_k\rangle$. The solutions of the SE problem is given by*

$$g = 0,\ |e_k, f_l\rangle \quad \textit{for } k \neq l$$
$$g = |\lambda_k|^2,\ |e_k, f_k\rangle \quad \textit{for } k = l.$$

For example, we can apply this to the two-mode squeezed-vacuum state $|q\rangle$. It follows that the non-zero SE values are $(1 - |q^2|)|q|^{2k}$ ($|q| < 1$, $k \in \mathbb{N}$).

For this pure projection $|\psi\rangle\langle\psi|$, the SE solutions and the Schmidt decomposition of $|\psi\rangle$ are identical. However, our initial definition of the SE equations had no relation to the Schmidt decompositions. This simple example delivers a strong relation between the property entanglement and the SE problem.

The SE Problem under partial transposition. Let a given solution of the SE problem of L be g and $|a, b\rangle$. It follows that g, $|a, b^*\rangle$ is a solution of the SE problem of L^{PT} [I]. The PT is performed in mode B, and the state $|b^*\rangle$ is given by

$$|b\rangle = \sum_k b_k |k\rangle \ \Leftrightarrow\ |b^*\rangle = \sum_k b_k^* |k\rangle.$$

The consequences of this PT-property are surprising. It does not affect the SE value, if we solve the SE problem for L or L^{PT}. Together with the general identification of entanglement via Hermitian test operators, we can identify bound entangled states.

Example 15 *Let us consider the swap operator $V = [|\Phi\rangle\langle\Phi|]^{\text{PT}}$. Since we already solved the SE problem for pure states, the solution of the PT follows immediately as*

$$f_{AB}(V) = 1 \textit{ and } f_{AB}(-V) = -\inf\{\text{Tr}(\sigma V) : \sigma \in \mathcal{S}_{AB}\} = 0.$$

The latter equation displays the entanglement witness character of V. It has negative eigenvalues, but the lowest possible expectation value under separable states is zero.

The SE problem under translations and scaling. It is worth to note that the SE problem has some more interesting properties. Let us consider the translated and scaled operator ($\lambda \neq 0$)

$$L' = \lambda L + \kappa \mathbb{I} \otimes \mathbb{I}. \tag{4.3}$$

The SE vectors of L' are the same as for L and the eigenvalues change as $g' = \lambda g + \kappa$. This delivers the possibility to restrict the detection of entanglement to positive semi-definite operators (shift), $L' \geq 0$, with an operator norm one (scaling), $\|L'\| = 1$. Another possible conclusion of the translation and scaling property is that we can assume an invertible operator L.

Local transformations. We have also seen that the quantification of entangled states strongly depends on the role of local-invertible maps $S \otimes T$, with the inverse $(S \otimes T)^{-1}$ [V, VI]. It is surprising how they affect the SE problem. Iff g is the SE value with the SE vector $|a, b\rangle$ of L, then

$$L' = (S \otimes T)L(S \otimes T)^{-1} \text{ locally transformed operator,} \tag{4.4}$$

$$|a', b'\rangle = (S \otimes T)|a, b\rangle \text{ locally transformed SE vector of } L', \tag{4.5}$$

$$g' = g \text{ locally transformed SE value.} \tag{4.6}$$

This has a direct relation to the (ordinary) eigenvalue problem under transformations, cf. Chapter 1. However, it also includes the factorizable structure of pure separable quantum states. Moreover, the close relation between entanglement and the SE problem gives us the possibility to explain why local invertible operations play such an important role for the quantification of entanglement. It also justifies the exceptional treatment of the Schmidt number as an entanglement measure.

The dual SE problem. The dual problem is related to interchanging domain and co-domain of an operator. The dual SE problem reads as

$$\tilde{L}(|a, b\rangle + |\tilde{\chi}\rangle) = \tilde{g}|a, b\rangle. \tag{4.7}$$

This can be easily understood when we assume an invertible operator L with the SE problem

$$L|a, b\rangle = g|a, b\rangle + |\chi\rangle. \tag{4.8}$$

An exchange of domain and co-domain can be done by considering L instead of L^{-1}. Multiplying Eq. (4.8) with the operator $\frac{1}{g}L^{-1}$ we obtain the dual formulation

$$\frac{1}{g}|a, b\rangle = L^{-1}\left(|a, b\rangle + \frac{1}{g}|\chi\rangle\right). \tag{4.9}$$

The identification of the operator $\tilde{L} = L^{-1}$, the SE value $\tilde{g} = g^{-1}$, and the bi-orthogonal perturbation $|\tilde{\chi}\rangle = \frac{1}{g}|\chi\rangle$ delivers the dual SE problem in Eq. (4.7).

This direct identification can be used to obtain solutions of the SE equations. Take an arbitrary vector $|\psi\rangle$ and apply L^{-1}. Iff the outcome is a separable vector $|a, b\rangle$ of the Schmidt decomposition of $|\psi\rangle$ then $|a, b\rangle$ is a solution of the SE equation of L.

4.3 Generalizations

In this section we generalize the powerful tool of the SE equations. One generalization is given in terms of Schmidt number states (number of global superpositions). This generalization is of importance for the quantification of entanglement. It allows us to decide whether the entangled state contains an entangled qudit or not. The second generalization is given in terms of multipartite entangled states. Entanglement in multipartite systems contains a large number of different kinds of entanglement, e.g. $|W\rangle$ states or $|GHZ\rangle$ states (the entanglement survives/does not survive when tracing out a subsystem) and partially and fully entangled states. But even in this complicated situation a SE problem can be found.

4.3.1 Schmidt number r states

We have generalized the separability eigenvalue problem for the case of Schmidt number states [VI]. Again this method can be used to obtain Schmidt number witnesses and Schmidt number quasi-probabilities. Here, the main task is the following. What is the maximal expectation value of a Hermitian operator L for pure states $|\psi_{\leq r}\rangle \in \mathcal{S}_r^{(\text{pure})}$ with a Schmidt number less or equal to r,

$$\langle\psi_{\leq r}|L|\psi_{\leq r}\rangle \rightarrow \text{optimum}. \tag{4.10}$$

In this case a major problem was the fact that we have a lot of conditions in the Schmidt decomposition of $|\psi_{\leq r}\rangle$. We have positive Schmidt coefficients $\lambda_k > 0$, orthogonal and normalized vectors $|e_k\rangle$ and $|f_{k'}\rangle$ in two modes, and the restriction of the Schmidt number $r(\psi_{\leq r}) \leq r$. Altogether the Schmidt decomposition is too strict. We overcome this problem by using a weaker decomposition [VI],

$$|\psi_{\leq r}\rangle = \sum_{k=1}^{r} |x_k, y_k\rangle, \tag{4.11}$$

with arbitrary elements $|x_k\rangle \in \mathcal{H}_A$ and $|y_{k'}\rangle \in \mathcal{H}_B$ which may be linearly dependent or even zero. Hence, the Schmidt rank is less or equal to r. The only remaining condition is the normalization of the compound state $\langle\psi_{\leq r}|\psi_{\leq r}\rangle = 1$.

We used this weaker decomposition and obtained the SE equations of Schmidt number r states. The result can be formulated in the following form,

$$L|\psi\rangle = g|\psi\rangle + |\chi\rangle, \tag{4.12}$$

where $|\psi\rangle$ is a vector with a Schmidt rank $r(\psi) \leq r$ ($\lambda_k > 0$ and $|e_k\rangle$, $|f_{k'}\rangle$ orthonormal for $k, k' = 1, \ldots, r$), the optimal expectation value g and a bi-orthogonal perturbation $|\chi\rangle = \sum_{k=r+1}^{d} \tilde{\lambda}_k |e_k, f_k\rangle$.

The form generalizes the second form of the SE equations, wherein now the factorizable vector is replaced by a vector with r superpositions of factorizable vectors (Schmidt rank r). Here, the surprising fact is that in the case $r \rightarrow r_{\max}$, the Schmidt number eigenvalue problem delivers the ordinary eigenvalue problem, $L|\psi\rangle = g|\psi\rangle$.

Example 16 *Let us consider the Schmidt number r equations for a projection* $L = |\psi\rangle\langle\psi|$ *with the usual Schmidt decomposition of* $|\psi\rangle$*, cf. [VI]. The Schmidt number r eigenvalues g are given by*

$$g = \lambda_{i_1}^2 + \cdots + \lambda_{i_{r'}}^2,$$

with $0 \leq r' \leq r$ *and Schmidt coefficients* λ_i *for different indices* $i = i_1, \ldots, i_{r'}$*.*

The SE problem has led to SE values with only one $|\lambda_i|^2$*. Here, a general operator L can have a larger overlap with the the set of pure states with a Schmidt number less or equal to r,* $\mathcal{S}_1^{(\text{pure})} \subset \mathcal{S}_r^{(\text{pure})}$*. Hence, in the considered case of a projection L, the maximal expectation value is given in terms of the r largest Schmidt coefficients.*

Example 17 *Now let us check if the swap operator is suitable to detect whether a state has a Schmidt number greater than two or not. The condition for a Schmidt number greater than two for a state* ϱ *in terms of the swap operator reads as* $\mathrm{Tr}(\varrho V) > f_2(V)$ *(maximal expectation value of V for a Schmidt number two) or* $\mathrm{Tr}(\varrho[-V]) < f_2(-V)$ *(minimal expectation value of V of a Schmidt number two). Using the decomposition of the swap operator in Eq. (3.8), we find that the spectral decomposition is already given in terms of rank two states.*

It follows that the maximal/minimal expectation value for all quantum states f_∞ *is the same as for Schmidt number two states,* $f_\infty(\pm V) = f_2(\pm V) = \pm 1$*. Hence, the swap operator cannot detect if a quantum state has a Schmidt number larger than two or not. This is of some importance to relate PPT bound entanglement with local-hidden-variables models in bipartite systems [99].*

4.3.2 The multipartite case

Fully separable states. Until now, we only considered the bipartite case. The study of quantum correlations in terms of entanglement requires also a method for multipartite entanglement. For deriving the SE problem for the multipartite case for n subsystems, we follow the bipartite scenario. We start with a fully factorizable state

$$|\psi\rangle = |a_1, \ldots, a_n\rangle. \tag{4.13}$$

Further on, we consider an optimization problem for an observable L under the normalization condition,

$$G = \langle a_1, \ldots, a_n | L | a_1, \ldots, a_n\rangle \rightarrow \text{optimum}, \tag{4.14}$$

$$C = \langle a_1, \ldots, a_n | a_1, \ldots, a_n\rangle - 1 \equiv 0. \tag{4.15}$$

Using the method of Lagrangian multipliers g we obtain the desired optimization condition as

$$0 = \frac{\partial G}{\partial \langle a_k|} - g\frac{\partial C}{\partial \langle a_k|}. \tag{4.16}$$

In addition we assume the normalization $\langle a_l | a_l\rangle = 1$. The optimization condition can be reformulated as the SE for the n-partite case ($k = 1, \ldots, n$) as

$$L_{a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n} | a_k\rangle = g | a_k\rangle, \tag{4.17}$$

with

$$L_{a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n} = \mathrm{Tr}_{1,\ldots,k-1,k+1,\ldots,n}\left(\left[\otimes_{l=1}^{k-1}|a_l\rangle\langle a_l| \otimes \mathbb{I} \otimes_{l=k+1}^{n} |a_l\rangle\langle a_l|\right] L\right),$$
$$g = \langle a_1,\ldots,a_n|L|a_1,\ldots,a_n\rangle. \tag{4.18}$$

It is also of advantage to consider Eq. (4.17) together with the action of L,

$$L|\psi\rangle = g|\psi\rangle + |\chi\rangle, \tag{4.19}$$
$$\forall k = 1,\ldots,n\ \forall |x\rangle \in \mathcal{H}_k :\ 0 = \langle a_1,\ldots,a_{k-1},x,a_{k+1},\ldots,a_n|\chi\rangle, \tag{4.20}$$

where $|\chi\rangle$ is a n-orthogonal perturbation. In the case of $n = 2$ multipartite SE equations are the usual bipartite SE equations. It is also obvious, that the two-mode findings can also be applied for this n-mode case.

Example 18 *Let us consider the generalized SE problem for the Greenberger-Horne-Zeilinger state* $|GHZ\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)$ *and* $L = |GHZ\rangle\langle GHZ|$. *From symmetry reasons we only consider the third component. The SE problem reads as* $L_{a,b}|c\rangle = g|c\rangle$. *We obtain*

$$L_{a,b} = \frac{1}{2}(a_0^* b_0^*|0\rangle + a_1^* b_1^*|1\rangle)(a_0 b_0\langle 0| + a_1 b_1\langle 1|).$$

The non-trivial solutions, $g \neq 0$, *delivers* $|c\rangle = N(a_0^* b_0^*|0\rangle + a_1^* b_1^*|1\rangle)$ *with a normalization constant* N. *After some algebra, it follows that the components of the optimal vector are either* 0 *or* 1, *and the maximal* g *is*

$$g_{\max} = \langle a,b,c|L|a,b,c\rangle = \frac{1}{2}, \tag{4.21}$$

whereas the maximal eigenvalue of L *is obviously* 1.

Example 19 *Now, let us consider the generalized SE problem for the W state,* $|W\rangle = \frac{1}{\sqrt{3}}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle)$ *and* $L = |W\rangle\langle W|$. *The SE problem reads as* $L_{a,b}|c\rangle = g|c\rangle$. *We obtain*

$$L_{a,b} = \frac{1}{3}(a_1^* b_0^*|0\rangle + a_0^* b_1^*|0\rangle + a_0^* b_0^*|1\rangle)(a_1 b_0\langle 0| + a_0 b_1\langle 0| + a_0 b_0\langle 1|).$$

The non-trivial solutions, $g \neq 0$, *delivers* $|c\rangle = N[(a_1^* b_0^* + a_0^* b_1^*)|0\rangle + a_0^* b_0^*|1\rangle]$ *with a normalization constant* N. *Like in the previous example, it follows that the components of the optimal vector are either* 0 *or* 1, *and the maximal* g *is*

$$g_{\max} = \langle a,b,c|L|a,b,c\rangle = \frac{1}{3}. \tag{4.22}$$

The maximal eigenvalue of L *for such a projection operator is obviously* $f_\infty(L) = 1$. *The state* $|W\rangle$ *is a superposition of three local elements which explains the smaller value* $g_{\max}$ *compared to the projection to the* $|GHZ\rangle$ *state.*

Partially separable states. Until now, we only considered fully separable states as classical. For some purposes, it is also possible to consider partially separable states as classical, e.g. $\frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle) \otimes |0\rangle$. In this case we have to study both: the SE problem in the bipartite case and the SE problem for fully separable n-partite case, e.g. $(\mathcal{H}_A) \otimes (\mathcal{H}_B \otimes \mathcal{H}_C)$ (and permutations) and $(\mathcal{H}_A) \otimes (\mathcal{H}_B) \otimes (\mathcal{H}_C)$, respectively.

For simplicity let us assume that one of the n systems can be separated from the others. In this case the expectation value of L reads as

$$(\langle a| \otimes \langle \psi|)L(|a\rangle \otimes |\psi\rangle), \tag{4.23}$$

together with all permutations of the systems, and $|\psi\rangle$ being entangled or separable in $n-1$ modes. The permutation denotes the fact that it does not matter which subsystem can be separated to identify partially entangled states. Now we obtain that the maximal expectation value can be calculated as the maximum over all bipartite cases $\mathcal{H}_A = \mathcal{H}_k$ and $\mathcal{H}_B = \otimes_{l=1}^{k-1}\mathcal{H}_l \otimes_{l=k+1}^{n} \mathcal{H}_l$.

Example 20 *Again, we consider the tripartite case with the states $|\phi\rangle = |GHZ\rangle$ or $|\phi\rangle = |W\rangle$, and the operator $L = |\phi\rangle\langle\phi|$. We consider mode C as the separated party with the state $|c\rangle$, and the other modes are given by the state $|\psi_{AB}\rangle$. Due to the symmetry we will have the same expectation values for the other separations.*

Let us start with the GHZ state. The SE multipartite operator is given by

$$L_{\psi_{AB}} = \frac{1}{2}(\psi^*_{0,0}|0\rangle + \psi^*_{1,1}|1\rangle)(\psi^*_{0,0}\langle 0| + \psi^*_{1,1}\langle 1|). \tag{4.24}$$

From the bipartite SE problem follows that the maximal expectation value is $g_{\max} = \frac{1}{2}$.

For the W state, the situation is more sophisticated. The reduced operator is

$$L_{\psi_{AB}} = \frac{1}{3}([\psi^*_{0,1} + \psi^*_{1,0}]|0\rangle + \psi^*_{0,0}|1\rangle)([\psi_{0,1} + \psi_{1,0}]\langle 0| + \psi_{0,0}\langle 1|).$$

The vector $|c\rangle \in \mathcal{H}_C$ is a solution of the SE problem, if

$$|c\rangle = N([\psi^*_{0,1} + \psi^*_{1,0}]|0\rangle + \psi^*_{0,0}|1\rangle),$$

with the normalization constant $N = (|\psi^*_{0,1} + \psi^*_{1,0}|^2 + |\psi_{0,0}|^2)^{-1/2}$. It follows $g = \langle \psi_{AB}, c|L|\psi_{AB}, c\rangle = \frac{1}{3}(|\psi^*_{0,1} + \psi^*_{1,0}|^2 + |\psi_{0,0}|^2)$, which is obviously maximal for $\psi_{1,1} = \psi_{0,0} = 0$ and $\psi_{0,1} = \psi_{1,0}$. The maximal value is $g_{\max} = \frac{2}{3}$.

In comparison with Examples 18 and 19, we see that for the GHZ state the expectation value of partially separable and fully separable states is invariant. However, for the W state the partially separable states deliver a two times larger expectation value than the fully separable ones.

We see that a straight-forward generalization of the SE problem to multipartite systems already delivers interesting results. The examples yield the following entanglement conditions: *The state ϱ is at least partially entangled, if*

$$\langle GHZ|\varrho|GHZ\rangle > f_{\text{full}}(|GHZ\rangle\langle GHZ|) = \frac{1}{2}, \tag{4.25}$$

$$\langle W|\varrho|W\rangle > f_{\text{full}}(|W\rangle\langle W|) = \frac{1}{3}, \tag{4.26}$$

and fully entangled if

$$\langle W|\varrho|W\rangle > f_{\text{part}}(|W\rangle\langle W|) = \frac{2}{3}. \tag{4.27}$$

The test operator $L = |GHZ\rangle\langle GHZ|$ cannot make a difference between partially and fully entangled states, $f_{\text{full}}(L) = f_{\text{part}}(L)$, which is possible for a test operator based on the W state. The boundary delivered by the partially separable states for the GHZ operator is smaller than for the W operator.

4.4 Solutions

So far we have solved the SE problem in different cases. We considered the pure state projections and obtained the Schmidt decomposition and entanglement conditions. We considered the swap operator as a prominent example of an entanglement witness and solved the SE problem for this operator as well. In addition, we solved the SE problem in the case of generalizations for Schmidt number r states and multipartite entanglement. In this section let us consider more generally the SE problem and methods for finding solutions of the important SE equations.

Example 21 *Let us now see what properties a bipartite Hermitian matrix L has, if a certain solution of the SE problem is given. We consider a two-qutrit system,* $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = 3$ *with basis* $\{|1\rangle, |2\rangle, |3\rangle\}$*. Let* $|a, b\rangle = |1, 1\rangle$ *be a separability eigenvector for a SE value g. The block structure of L reads as*

$$L = \left[\begin{array}{ccc} \left[\begin{array}{ccc} g & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{array}\right] & \left[\begin{array}{ccc} 0 & * & * \\ * & * & * \\ * & * & * \end{array}\right] & \left[\begin{array}{ccc} 0 & * & * \\ * & * & * \\ * & * & * \end{array}\right] \\ \left[\begin{array}{ccc} 0 & * & * \\ * & * & * \\ * & * & * \end{array}\right] & \left[\begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] & \left[\begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] \\ \left[\begin{array}{ccc} 0 & * & * \\ * & * & * \\ * & * & * \end{array}\right] & \left[\begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] & \left[\begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] \end{array}\right],$$

where $*$ denotes a not further specified expression. For this operator L_1 reads as

$$L_1 = \text{Tr}_B([\mathbb{I} \otimes |1\rangle\langle 1|]L) = \left[\begin{array}{ccc} g & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{array}\right],$$

and has an eigenvector $|1\rangle$ with an eigenvalue g. The same holds for the operator with a traced out system A. Let us note that $|1, 1\rangle$ is also an eigenvector if all elements in the first column and row of L contain only zeros (except the eigenvalue g). This visualizes the fact that a factorizable (ordinary) eigenvector is also a SE vector.

While solving the SE problem for different cases of test operators L and quantum states ρ, it turns out that there are two important matrix decompositions. Both deliver solutions of the SE problem.

Operator decomposition 1. For simplicity let us consider a calculus, which is useful for our considerations. This decomposition turns out to be useful especially for the detection of Schmidt number r state [VI]. Again, it relates the eigenvalue problem and the Schmidt decomposition and the SE problem.

Therefore let us consider an arbitrary state $|\psi\rangle$, an associated map M_ψ, and a state $|\Phi\rangle$,

$$|\psi\rangle = \sum_{i,j} \psi_{i,j}|i,j\rangle,\ M_\psi = \sum_{i,j} \psi_{i,j}|i\rangle\langle j|,\ \text{and}\ |\Phi\rangle = \sum_{k=1}^{r_{\max}} |k,k\rangle. \tag{4.28}$$

We obtain the following equalities,

$$|\psi\rangle = M_\psi \otimes \mathbb{I}|\Phi\rangle = \mathbb{I} \otimes M_\psi^{\mathrm{T}}|\Phi\rangle \tag{4.29}$$

$$M \otimes N|\Phi\rangle = MN^{\mathrm{T}} \otimes \mathbb{I}|\Phi\rangle = \mathbb{I} \otimes NM^{\mathrm{T}}|\Phi\rangle \tag{4.30}$$

$$\langle\psi|\psi'\rangle = \mathrm{Tr}(M_\psi^\dagger M_{\psi'}). \tag{4.31}$$

In addition, let us note: The rank of M_ψ is the Schmidt rank of $|\psi\rangle$; the singular values of M_ψ are the Schmidt coefficients of $|\psi\rangle$ [7]. Any Hermitian operator L can be given in a decomposition as

$$L = \sum_k L_k (M_k \otimes \mathbb{I})|\Phi\rangle\langle\Phi|(M_k^\dagger \otimes \mathbb{I}), \tag{4.32}$$

with $L_k \in \mathbb{R}$. Let us illustrate this representation of operators with an example.

Example 22 *Let us assume diagonal matrices $M_k = \sum_l d_{k,l}|l\rangle\langle l|$. In this case we obtain*

$$L = \sum_k L_k \sum_l d_{k,l}|l,l\rangle \sum_{l'} d_{k,l}^*\langle l',l'| = \sum_{l,l'} \left(\sum_k L_k d_{k,l} d_{k,l'}^* \right) |l,l\rangle\langle l',l'|.$$

From the second form of the SE equations, we obtain from the solutions [VI]. The trivial ones are $g = 0$ and all $|k,l\rangle$ $(k \neq l)$. The non-trivial solutions have a SE value $g_l = \left(\sum_k L_k d_{k,l} d_{k,l}^*\right)$ and a SE vector $|a_l, b_l\rangle = |l,l\rangle$.

Operator decomposition 2. Now, let us consider different decomposition of the operator. Therefore let us consider a Hermitian operator L,

$$L = \sum_{i,j} L_{ij} \otimes |i\rangle\langle j|, \tag{4.33}$$

with single-mode operators $L_{ij} = L_{ji}^\dagger$. This decomposition can be done for all operators, and it represents the block-matrix structure of operators acting on a compound Hilbert space. Again, we illustrate this representation with an analytically solvable example.

Example 23 *Let the operators L_{ij} in subsystem A have a diagonal form $L_{ij} = \sum_k d_{ijk}|k\rangle\langle k|$, which delivers*

$$L = \sum_k |k\rangle\langle k| \otimes \sum_{i,j} d_{ijk}|i\rangle\langle j|,$$

and $d_{ijk} = d^*_{jik}$ delivering $L \in \mathsf{Herm}(\mathcal{H})$. For each k follows that the matrix $(d_{ijk})_{i,j}$ is also Hermitian with a spectral decomposition as $d_{ijk} = \sum_l g_{k,l}\phi_{k,l,i}\phi^*_{k,l,j}$ (with the orthonormal eigenvectors $|\phi_{k,l}\rangle = \sum_i \phi_{k,l,i}|i\rangle$ and $\langle\phi_{k,l}|\phi_{k,l'}\rangle = \delta_{l,l'}$),

$$L = \sum_{k,l} g_{k,l}|k\rangle\langle k| \otimes \sum_i \phi_{k,l,i}|i\rangle \sum_j \phi^*_{k,l,j}\langle j| = \sum_{k,l} g_{k,l}|k\rangle\langle k| \otimes |\phi_{k,l}\rangle\langle\phi_{k,l}|.$$

Now the solutions of the SE problem in the first form can be easily obtained. The SE values are $g = g_{k,l}$ for the SE vectors $|a,b\rangle = |k\rangle \otimes |\phi_{k,l}\rangle$. Let us note, if we consider a quantum state $L = \rho$, this solution of the SE problem proves its separability.

When solving the SE problems in the first form, this decomposition seems to be more suitable, for the second form the previous decomposition is preferable. However, since both forms are equivalent, it depends on the operator L which decomposition should be chosen. The major number of examples presented in this work were related to the solution of the SE problem of test operators and/or quantum states. We obtained for large classes of operators analytical solutions of the SE problem in finite and infinite dimensions, in bipartite and multipartite systems, and superposition (Schmidt number) states.

Relation to the eigenvalue problem. It turned out to be of some advantage to consider the well-known eigenvalue problem, when solving the SE problem and for analyzing of the solutions of the SE problem. All relations and properties of the SE solutions can be obtained by studying the eigenvalue problem in a related case. Afterward we translate the situation to separable states by assuming the same properties in the SE case. In our examples, the SE values are combinations of the eigenvalues and the Schmidt coefficients.

As a remark let us note that the computational complexity of the SE problem is at least the same as for the eigenvalue problem. It seems not likely that we can obtain all SE solutions for all operators L. If the contrary was true, then the eigenvalue problem of all single mode operators L could be solved by solving the SE problem of $L \otimes \mathbb{I}$. This is obviously not possible and can be understood by the well known fact that the general separability problem cannot be solved in polynomial time, see Ref. [100, 101, 102] for the non-polynomial-hardness of the separability problem.

However, in the single-mode case there are numerical methods to obtain the solutions of eigenvalue problem. Besides analytical solutions, such numerical methods must be developed in future for the SE problem. Especially the maximal SE value is needed together with an numerical error estimation.

5 Conclusions, Summary, and Outlook

In my diploma thesis [103], I derived the separability eigenvalue (SE) equations and applied it to simple situations. Afterward, we continued the research as it is presented in this work. A lot of new properties and relations to other problems in mathematical physics have been found. In Ref. [I], we have formulated properties of the solution of the SE equations, solved them for rank-one projection operators, considered a construction scheme for a grid of operators, and studied the identification of PPT bound entangled states. Another example of continuing my diploma thesis in another direction is the optimal decomposition of entangled quantum states in terms of separable ones by solving the SE equations [II]. Let us summarize all discussed findings and let us make an outlook.

5.1 Characterization of correlations

Families of entangled states. At the beginning of our treatment we analyzed different families of entangled states. We considered the well-known Schmidt number states, NPT and PPT states, partially and fully separable states. We studied a certain class of pure states with certain properties, and defined classical statistical mixtures of these pure states, cf. Fig. 5.1. This delivered the convex structure of the set of states with a classical correlation. We have shown that the quantum superpositions can deliver nonclassical/quantum correlations in this notion [IV, VII, VIII].

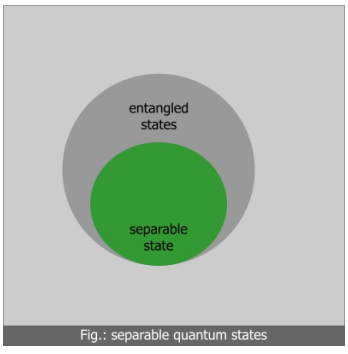

Figure 5.1: The figure shows the (green) set of separable quantum states embedded in the set of all quantum states (gray and green areas). A convex combination of separable states is separable, whereas a convex combination of entangled states is not necessarily entangled.

Identification. A characterization of the quantum correlations can be given in terms of the detection method. This means that the type of the detection characterizes the correlations. A well-known example is given by the classification of NPT and PPT states.

Beyond PT entanglement we considered the identification of entanglement for any quantum state. We could prove that CV entanglement can always be treated in finite dimensions [III]. We studied entanglement witnesses to obtain optimized, necessary and sufficient conditions for entanglement in terms of general Hermitian operators [I, VI]. The optimization method yields the SE equations. An approximate approach for the detection of continuous variable entanglement is given in Ref. [I] and visualized in Fig. 5.2.

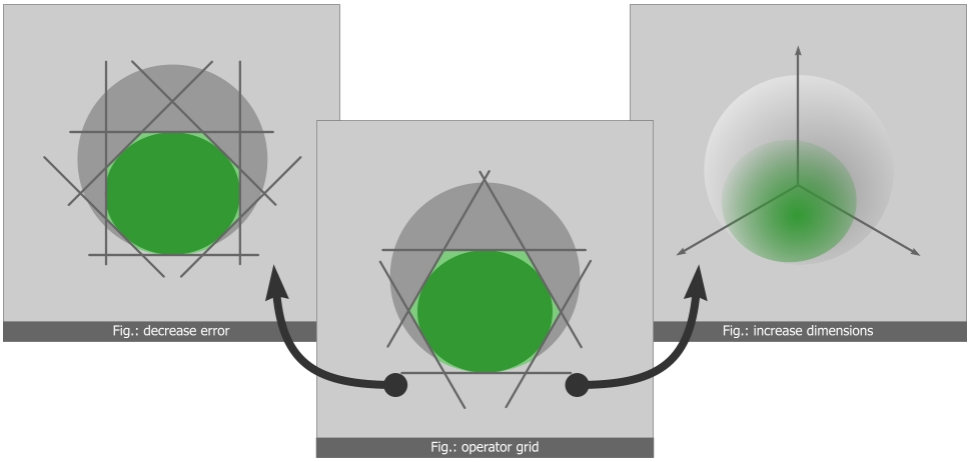

Figure 5.2: A grid of operators can be chosen for the detection of entanglement with test operators (central part of the picture). If the considered state is not detected, we have two possibilities. The right pictures shows that the dimension might be increased, or on the left part of the picture the operator grid can be chosen to be narrower.

Quasi-probability. Another method for the characterization of quantumness is given in terms of negativities in the corresponding quasi-probabilities. The most prominent example is the Glauber-Sudarshan P function. Negativities in this function display nonclassical features of the corresponding state. The quantum superposition allowed us to break the classical statistical mixtures to obtain the negativities.

It turned out that deriving quasi-probabilities for entanglement is problematic. Starting from the known situation, that any operator can be written as a difference of two separable operators, a quasi-probability for entanglement can be constructed. The problem was that the initial representation is ambiguous. A related problem is to find the convex decomposition of a separable state, to prove its separability.

We overcome the ambiguity by an optimized quasi-probability of entanglement P_{Ent} [II, VIII]. This led again to the SE equations for the state. The optimized P_{Ent} is negative, iff the state is entangled. It also delivers the positive decomposition of mixed separable quantum state.

Measures. A third approach for the characterization of correlations is the quantification of the nonclassicality. The well-know axiomatic quantification of entanglement delivers a method for the quantification of these correlations. This approach is based on the so-called LOCC operations.

The most prominent quantification of entanglement is related to distance measures. We have shown that the choice of the metric of the distance delivers different ordering of entangled states or nonclassical quantum states [V, VII]. We overcome this problem by a rigorous mathematical treatment of convex ordering procedures. The physical aspect we found is that the number of superpositions delivers a quantification for both nonclassical correlations and entanglement. This result was already of some importance when formulating the detection and representation by quasi-probabilities.

Let us note that our method is not in contrast to the previous definition of entanglement measures. The number of superpositions is a nonclassicality measure, but it also satisfies additional constrains. In the case of entanglement measures the additional constrain include local invertible operations.

We generalized the quantification of entanglement method to pseudo-measures and operational measures [V]. These generalizations consider only specific kinds of correlations like NPT entanglement. Or they apply to certain experimental situations.

Relation to nonclassicality. We also pointed out the relation between nonclassicality (convex combinations of coherent states) and entanglement, see e.g. [IV]. We have shown that there are general approaches for the representation of entanglement and nonclassicality in terms of quasi-probabilities [II, VIII]. For systems beyond the harmonic oscillators our ordering procedure [VII] and the convex decomposition of classical correlated states [VIII] also apply.

5.2 Methods for entanglement

Continuous variable entanglement. We have shown that entanglement in continuous variable systems can be treated in finite dimensions [III]. This enables the possibility to use methods for finite vector spaces only. The treatment of entanglement in terms of functional analysis is not needed which simplifies the mathematical description of this property.

Classical operations. We carefully considered the role of local filter operations and stochastic separable operations. By the convex cone construction we showed that the normalization of the quantum state is not so important to characterize its properties. Especially the role of local invertibles has been studied [V]. We have seen that they do not influence the SE values of operators. By applying them to entanglement measures, we found that they deliver measures which are invariant under the choice of a metric.

Optimization methods. Above we already explained the approximation for the identification of entanglement by a grid of operators [I]. The second optimization scheme we presented for quasi-probabilities of entanglement, P_{Ent} [II]. Here, we solve the SE problem for the quantum state. With the solutions we obtain a linear problem $\mathbf{G}\vec{p} = \vec{g}$ which has to be solved. The solutions delivered the P_{Ent} quasi-distribution of any separable or entangled, mixed or pure quantum state.

Separability eigenvalue problem. At several points we obtained an optimization problem which we named separability eigenvalue problem. We solved this problem analytically for a few examples in this work. We also pointed out the relation between the (ordinary) eigenvalue problem and the SE problem at several points. Let us recall some catchwords to remember its importance:

1. optimized identification of bipartite entanglement;
2. optimized quasi-probabilities of entanglement;
3. Schmidt decomposition of pure states;
4. generalization to Schmidt number r states;
5. generalization to multipartite systems (fully and partially entangled states);
6. SE solutions under partial transposition;
7. identification of PPT bound entanglement;
8. SE solutions under local invertible transformations;
9. examples: pure states including: all bipartite pure states, GHZ state, W state and all pure Schmidt number r states;
10. examples: mixed states in continuous variable systems (phase randomized two-mode squeezed-vacuum);
11. examples: different mixed separable states;
12. examples: the swap operator.

Quantum superposition principle. From the beginning we considered the quantum superposition principle. It has a major importance for the characterization of correlations by entanglement and nonclassicality of radiation fields [V, VII]. All nonclassical effects only occur as a consequence of some form of quantum superpositions. In comparison with classical statistical physics, it delivers the additional quantum correlations. Another important feature is the reduction of the wave function after a measurement. This property has not been studied in this work, but it should be included in future.

5.3 Outlook

Detection. For the detection of entanglement in the present form, it is necessary to test all operators L. It would be nice if we could reduce the number of tests. Due to the properties of local invertible transformations, $T \otimes S$, and the possibility of defining a locally transformed metric by $T^\dagger T \otimes S^\dagger S$, it seems likely that this could deliver a method to translate the properties of the state to properties of a suitable test operator. Negativities of a witness $W = f_{AB}(L) - L$ do not depend on the choice of a metric, and the separability property cannot be changed under local transformations. The equivalence of test operators and positive, but not completely positive maps also delivers methods to classify entanglement beyond negativity of partial transposition.

Quasi-probabilities. We have seen that entanglement is only one nonclassical property of radiation fields. From the bare fact that the P function includes all the quantum information of the state follows that it also includes the entanglement information. However, which kinds of negativities of the P function directly display the negativities of the P_{Ent} function is not known so far. A possible answer could be given in terms of matrices of moments which already delivered interesting similarities between nonclassicality and negativity under partial transposition.

Quantification. We started to compare topological (distance based ones) with algebraic properties of measures. Especially in the multipartite case the question of a measure is of some importance. It is clear that the number of superpositions is a measure. But it is not clear how to include specific properties of the GHZ (now as three superpositions $|GHZ\rangle = \frac{1}{\sqrt{3}}(|1,1,1\rangle + |2,2,2\rangle + |3,3,3\rangle)$) in relation to the W states to refine the measure. The separability eigenvalue problem in the multipartite case should give hints which invariant properties are useful for this quantification.

Separability eigenvalue problem. As we have seen, the most important question in connection with entanglement in this work is given in terms of the separability eigenvalue problem. Analytical solutions of wider families of operators are needed. Numerical methods must be developed to obtain the cases where an analytical solution is missing. More properties of these equations should be revealed for a fundamental understanding of quantum correlations in terms of entanglement.

This can be done in several ways. It turned out that it is useful to consider the ordinary eigenvalue problem to learn more about the separability eigenvalue problem. Methods of multi-linear algebra seem to be useful, especially in the case of multipartite entanglement, to further classify the structure of entangled states.

Temporal correlations in the notion of entanglement. For the nonclassicality of a state, it was shown how to define space-time correlations. In the case of entanglement, it is not clear how to define entanglement in time. Some approaches concerning the invariance of entanglement under Lorentz transformations have been investigated, but the general case remains open. Also the evolution of entanglement

has been studied in different systems, in relation to "birth" and "death" of entanglement. However, there is no definite definition of entanglement between systems separated in different times. Using time-ordered operators might deliver an insight into this open question.

Nonclassicality. All the methods presented in this work can help to understand not only entanglement, but also nonclassicality of radiation fields. We used approaches known in the context of nonclassicality to translate them to the phenomenon entanglement. The other way around, derived methods for entanglement could also apply to nonclassicality. We have done some first approaches in the context of number of superpositions of coherent states to generalize the Schmidt number to a nonclassicality measure.

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Appendix
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The Convex Decomposition Problem of Quantum states [Draft]

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We consider the convex decomposition of an element of a convex set. We show that elements of the convex set with a stationary distance are sufficient for the decomposition. The convex decomposition problem is departed into two well understood problems. First finding maxima and minima of a given function, and second a linear problem. Our method can be applied to various problems in mathematical physics. Here we consider an example in quantum entanglement: The convex decomposition of a given quantum state into pure quantum states with a fixed Schmidt rank.

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I. INTRODUCTION

A set $\mathcal{C}$ is convex by definition, if

$$c_k \in \mathcal{C} \Rightarrow \sum_{k=1}^{n} \lambda_k c_k \in \mathcal{C}, \tag{1}$$

with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$ ($n \in \mathbb{N}$). Due to these properties, the positive factors can be interpreted as a probability for the corresponding element. Beyond quantum physics we consider convex sets in various systems in mathematical physics, e.g. statistical physics. But which elements of a convex set are needed for the convex decomposition of an arbitrary element?

The problem of the convex decomposition of mixed quantum states occures in Quantum Information Theory at a very early point. Namely, the decomposition of a given quantum state in terms of factorizeable states. The definition of separability by Werner [1] is: A quantum state σ is not entangled, if and only if it can be written as a convex combination of factorizable states $|a_k, b_k\rangle\langle a_k, b_k|$. Thus, poving separability means that we need to find factorizable states together with the convex coefficients for the decomposition of the state.

Finding this decomposition is not easy. Which pure factorizeable states $|a_k, b_k\rangle\langle a_k, b_k|$ can be choosen for a convex decomposition of a separable quantum states σ? What are the values of the convex coefficients λ_k? Is the state σ separable at all? The decompositon of quantum states in terms of factorizable states is ambiguous [2]. There are examples with a non-optimal decomposition which is not-convex, but an optimal decomposition is convex. We have overcome this problem by obtaining an optimal decomposition, which is non-negative iff the considered state is separable [2]. The method is based on the so called separability eigenvalue equations [3]. Solving these equations, we obtain a quasi-probability distibution with negativities for entangled states, and a convex decomposition for separable ones.

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The problem of the convex decomposition of an element can be generalized to Schmidt number states. Each pure state can be written as a superposition of factorizeable states. The Schmidt rank r of a pure bipartite state denotes the minimal number of nonlocal superpositions which is needed to generate the state [4]. The statistical mixture of pure states with a Schmidt rank less or equal to r denotes the convex set of states with a Schmidt number less or eual to r [5–8]. For $r = 1$ we obtain separable quantum states; for $r = 2$ we obtain separable states, and mixtures of Schmidt rank two states like Bell states. For example, a states with a Schmidt number greater than 2 cannot be written as a convex combination of pure Bell-like states. Let us note, that for the identification of entanglement it is sufficient to consider finite spaces only [9].

In the present article, we consider the general decompositon of an element of a convex subset of a Banach space. We define stationary points – points with a local maximum or minimum in the distance to the element under study. We prove that the considered element is within the convex set generated by these stationary points. The number of elements needed for the convex decomposition can be further reduced. We apply our method to finite dimensional Hilbert spaces. This example can be used for the convex decomposition of general Schmidt number quantum states and quasi-probabilities.

II. CONVEX DECOMPOSITION WITH STAIONARY POINTS

Let us now formulate the general problem. Let us consider a convex set $\mathcal{C}$ and an element $g \in \mathcal{C}$. We aim to find a convex combination of g by the extremal points of $\mathcal{C}$. The extremal points of $\mathcal{C}$ are the elements of a set $\mathcal{E}$. These elements of $\mathcal{E}$ cannot be written as a convex combination of other elements of $\mathcal{C}$. In addition, all possible convex combinations of elements of $\mathcal{E}$ deliver the complete convex set, $\mathcal{C} = \overline{\mathsf{conv}\,\mathcal{E}}$. Now we want to rewrite g in terms of elements of $x \in \mathcal{E}$ with a stationary distance $\|g - x\| =$ optimal. This means that all elements $y \in \mathcal{E}$ close to x have either a larger or a smaler distance to g. For simplicity let us shift the problem, such that $g = 0$.

We consider a Banach space $(\mathcal{X}, \|\cdot\|)$ and its dual space $\mathcal{X}'$; $\mathcal{C}$ is a compact, convex subset of $\mathcal{X}$, and $0 \in \mathcal{C}$; the closed set of extremal points of $\mathcal{C}$ is denoted as $\overline{\mathcal{E}}$; and the abbreviations for a sphere is $\mathcal{B}_r(x) = \{y \in \mathcal{X} : \|x - y\| \leq r\}$ and a metric is denoted as $d(x) = \|x\|$.

Definition 1 *An element $x \in \mathcal{M}$ is called* **stationary** *point of $\mathcal{M}$, if*

$$\exists \epsilon > 0 \ \forall y \in \mathcal{B}_\epsilon(x) \cap \mathcal{M} : d(x) \geq d(y)$$
(stationary maximum)

or

$$\exists \epsilon > 0 \ \forall y \in \mathcal{B}_\epsilon(x) \cap \mathcal{M} : d(x) \leq d(y)$$
(stationary minimum).

These points have an optimal (minimal or maximal) distance to the element $g = 0$ in comparison with their neighbours of a given arbitrary set $\mathcal{M}$. Note that $\overline{\mathcal{E}} \subset \mathcal{C}$, and therefore $\overline{\mathcal{E}}$ is compact. Due to compactness of $\overline{\mathcal{E}}$, it has at least one stationary maximum and minimum, e.g. the global maximum and minimum of the metric $d|\overline{\mathcal{E}}$ restricted to extreme lements. Further on, it is clear that an isolated point of $\mathcal{M}$ is a stationary point of $\mathcal{M}$.

Lemma 1

$$\forall p \in \mathcal{X}' \ \exists x_1, x_2 \text{ stationary points of } \overline{\mathcal{E}} : \\ p(x_1) \geq 0 \wedge p(x_2) \leq 0.$$

Proof. From $0 \in \overline{\mathcal{E}}$ follows that 0 is a stationary minimum of $\overline{\mathcal{E}}$ and the conjecture follows. Now assume $0 \notin \overline{\mathcal{E}}$ and $\exists p \in \mathcal{X}'$ $\forall x$ stationaty points of $\overline{\mathcal{E}} : p(x) < 0$. $0 \in \mathcal{C} \Rightarrow \exists y \in \overline{\mathcal{E}} : p(y) > 0$. It follows that $\mathcal{M}_p = \overline{\{y \in \overline{\mathcal{E}} : p(y) \geq 0\}} \neq \emptyset$ which is suffciently departed from the stationary points of $\overline{\mathcal{E}}$. Let $x' \in \mathcal{M}_p$ denote a point with $d(x') = \sup_{y \in \mathcal{M}_p} d(y)$ – the existence follows from the compactness of $\mathcal{M}_p$. It follows from Definition 1, that x' is a stationary point of $\overline{\mathcal{E}}$ with $p(x') \geq 0$. From the contradiction to the assumption follows the conjecture. ■

Lemma 1 states that for any hyperplane $p(x) = 0$ in $\mathcal{X}$ exists stationary points above (or on) and under (or on) the hyperplane. The importance of this finding is connected with the following identity:

$$g = 0 \Leftrightarrow \forall p \in \mathcal{X}' : p(g) = 0. \tag{2}$$

For a convex decomposition of $g = 0$ it is necessary to have points above (or on) and under (or on) all hyperplanes given by p.

The question is, if the condition of points above (or on) and under (or on) is sufficient to generate the element $g = 0$ as a convex combination. Therefore, let us considier a compact subset $\mathcal{D}$ of $\mathcal{X}$ with this property

$$\forall p \in \mathcal{X}' \ \exists x_1, x_2 \in \mathcal{D} : \ p(x_1) \geq 0 \wedge p(x_2) \leq 0. \tag{3}$$

We find

Lemma 2 *For a set $\mathcal{D}$ with the property given in Eq. (3) holds $0 \in \overline{\mathsf{conv}\ \mathcal{D}}$.*

Proof. Assume $0 \notin \Delta = \overline{\mathsf{conv}\ \mathcal{D}}$. From the theorem of Hahn-Bannach follows $\exists p' \in \mathcal{X}' \ \forall y \in \Delta : p'(y) < 0$. This is a contradiction to the given property of $\Delta \supset \mathcal{D}$: for any $p \in \mathcal{X}'$ exists $x_1, x_2 \in \Delta : p(x_1) \geq 0 \wedge p(x_2) \leq 0$. ■

Now we can conclud from Lemma 1 and Lemma 2 that $0 \in \mathcal{C}$ can be written as a convex combination of stationay points. By performing the shift $x \mapsto x+g$, with an arbitrary $g \in \mathcal{C}$ and an arbitrary compact and convec set $\mathcal{C} \subset \mathcal{X}$, we can formulate the following theorem.

Theorem 1 (Convex decomposition)

1. *Any element $g \in \mathcal{C}$ can be written as a convex combination of its stationary points $x \in \overline{\mathcal{E}} : \|g - x\| =$* optimal.
2. *Any element $g \in \mathcal{C}$ can be written as a convex combination of a subset $\mathcal{D}$ of its stationary points, with the additional property: $\forall p \in \mathcal{X}' \ \exists x_1, x_2 \in \mathcal{D} : p(x_1 - g) \geq 0 \wedge p(x_2 - g) \leq 0$.* ■

This is the first main finding of our work. For the convex decomposition of g, we have to find the stationary points of $\overline{\mathcal{E}}$ with respect to g. We delete some elements such that Eq. (3) is fulfilled. The resulting elements deliver a convex combination of g.

From the Carathéodory's theorem and the Krein–Milman theorem we can conclude that $\dim(\mathsf{span}\,\mathcal{C}) + 1$ elements of $\overline{\mathcal{E}}$ are sufficient to generate any arbitrary element $g \in \mathcal{C}$. But it is not clear which elements must be choosen for this decomposition. Usually Theorem 1 uses more than these minimal number of elements, but it is clear which elements necessarily deliver a convex decomposition.

Moreover, if we use the negative version of Theorem 1, we get the following corollary.

Corollary 1

If $g \in \mathcal{X}$ cannot be decomposed as a convex combination as given in Theorem 1, then follows $g \notin \mathcal{C}$. ■

The stationary points of $\overline{\mathcal{E}}$ with respect to $g \in \mathcal{X}$ exist independently from $g \in \mathcal{C}$ or $g \notin \mathcal{C}$. But a convex decomposition with stationary poins fails only in the latter one.

III. APPLICATION TO FINITE DIMENSIONAL HILBERT SPACES

In the following let us consider the restriction $n = \dim \mathcal{X} < \infty$. For our application for entanglement in Quantum Information Theory this is not a restriction, as we discussed above [2]. Let us briefly recall that the dual space $\mathcal{X}'$ has the same (finite) dimension like $\mathcal{X}$. All norms in finite complex spaces are equivalent. We may also consider $\mathcal{X}$ to be a Hilbert space $\mathcal{H} = \mathcal{X}$ (by using the 2-norm, inner product $\langle\cdot|\cdot\rangle$, and $\mathcal{X}' \cong \mathcal{H}$), and we replace $d(x) = \|x\|_2{}^2$ to generate a differentiable function d.

Up to now, it is not clear how to obtain the stationary points $x_i \in \overline{\mathcal{E}}$ with respect to a given $g \in \mathcal{C}$, and the question how to obtain the convex coefficients $p_i \geq 0$ ($\sum_i p_i = 1$) for

$$g = \sum_i p_i x_i. \tag{4}$$

We already observed above, that the isolated points of $\overline{\mathcal{E}}$ are stationary ones. Let us assume that all other elements are piecewise given as differentiable manifolds, e.g. given by the map Φ. Obviously we obtain the stationary points by

$$d(g - \Phi(t)) \to \text{optimum} \text{ or } \partial_t d(g - \Phi(t)) = 0. \tag{5}$$

In general, elements of the boundary can also be optimal points.

As we have seen in Theorem 1 not all optimal points are needed for a convex decomposition. It is sufficient to find all statonary points until the property given in Eq. (3) is fullfilled. Let us denote such a set of stationary points as $\mathcal{D}$. For example, this is possible if every orthant with the origin in g – n-dimensional generalization of quadrants ($n = 2$) and octants ($n = 3$) – has et least one element of $\mathcal{D}$. This can be also done by choosing stationary points x_i with $\{x_i - g : i = 1, \ldots, \dim \mathsf{span}\mathcal{C}$ linear independent, and an additional $x_0 \in \mathcal{D}$ with linear decomposition $x_0 - g = \sum_i q_i(x_i - g)$ for $q_i \leq 0$. In this case we obtain the minimal number of elements for the convex decompositions. Thus, $\mathcal{D}$ can be chosen to have a finite number of elements.

Now, our element can be written as

$$g = \sum_{x \in \mathcal{D}} p_x x. \tag{6}$$

By using the projection $d_y = \langle y|g\rangle$ we find

$$\forall d \in \mathcal{D} : d_y = \sum_{x \in \mathcal{D}} p_x \langle y|x\rangle. \tag{7}$$

The resulting linear system must have a solution, since $g \in \mathsf{span}\,\mathcal{D}$ as a conclusion of Theorem 1.

Theorem 2 *We define* $\vec{d} = (d_y)_{y \in \mathcal{D}}$, $\vec{p} = (p_x)_{x \in \mathcal{D}}$, *and* $\mathsf{G} = (\langle y|x\rangle)_{x,y \in \mathcal{D}}$. *The linear system*

$$\mathsf{G}\vec{p} = \vec{d}$$

has a positive solution (for all x*:* $p_x \geq 0$*) and* $g = \sum_{x \in \mathcal{D}} p_x x$, *if and only if* $g \in \mathcal{C}$. ■

Note that G is a symetric, positive semidefinite operator. Thus, a solution of this system exists. In the case $g \in \mathsf{span}\,\mathcal{D} \setminus \mathsf{conv}\,\mathcal{D}$ the solution of $\mathsf{G}\vec{p} = \vec{d}$ must have negative elements, $p_{x_0} < 0$. Whereas in the case $g \notin \mathsf{span}\,\mathcal{D}$ the given solution does not resemble the point $g \neq \sum_{x \in \mathcal{D}} p_x x$.

Theorem 2 is the second main finding. The previously convex decomposition problem has been splitted into to a linear problem and a general optimization problem, see Eq. (5). Solutions and numerical approaches are well known for each sub-problem.

IV. RECONSTRUCTION SCHEME FOR SCHMIDT NUMBER r STATES

In the following let us apply the methods for Schmidt number r states. The Schmidt decomposition of a pure state $|\psi\rangle\langle\psi|$ with the Schmidt rank r reads as [4],

$$|\psi\rangle = \sum_{k=1}^{r} \lambda_k |e_k\rangle \otimes |f_k\rangle, \tag{8}$$

with orthomormal $|e_k\rangle$ and $|f_k\rangle$, and Schmidt coefficients $\lambda_k > 0$. Let us denote with $\mathcal{S}_{r,0}$ the set of all pure quantum states with a Schmidt rank less or equal than r. Elements of the set $\mathcal{S}_r$ are Schmidt number r states. These are all quantum states, which can be generated as a convex combination pure states in $\mathcal{S}_{r,0}$, $S_r = \mathsf{conv}\, S_{r,0}$. Let us note that the case $r = 1$ – separable quantum states – has been considered in an analogous way in [2]. For the other trivial case $r = \min\{\dim \mathcal{H}_A, \dim \mathcal{H}_B\} = R$ follows that $\mathcal{S}_R$ is the set of all quantum states, cf. [8].

We identify the following: $\mathcal{H} = \mathsf{Herm}(\mathcal{H}_A \otimes \mathcal{H}_B)$ the set of linear Hermitian operators together with the inner product $\mathrm{tr}\left(L_1^\dagger L_2\right)$; the convex set $\mathcal{C} = \mathcal{S}_r$. Let us briefly recall that $\overline{\mathcal{E}} = \mathcal{S}_{r,0}$. A pure state cannot be written as a convex combination of other pure states, and all quantum states can be written as a mixture of pure states. This is also true for a subset of quantum states.

The function which must be optimised for ρ reads as

$$\mathrm{tr}\,(\rho - |\psi\rangle\langle\psi|)^2 = 2\langle\psi|\rho|\psi\rangle + C \to \text{optimum}, \tag{9}$$

with $|\psi\rangle\langle\psi| \in \mathcal{S}_{r,0}$, $C = 1 + \mathrm{tr}\ \rho^2 = \text{const.}$ (Normalization: $\langle\psi|\psi\rangle = 1$). This means

$$\langle\psi|\rho|\psi\rangle \to \text{optimum}, \tag{10}$$

and obtain solutions $|\psi_k\rangle$. This kind of optimization procedure has been considered in [10]. The optimization delivers a set of algebraic equations, called r-Separability-Eigenvalue-Equations.

Concluding our findings, we can sketch a reconstruction scheme for Schmidt number r states.

1. Solve the r-Separability-Eigenvalue-Equations for the given quantum state ρ [10], and obtain solutions $|\psi_k\rangle$.
2. Again we define

$$\vec{d} = (\langle\psi_k|\rho|\psi_k\rangle)_k,\ \vec{p} = (p_k), \tag{11}$$

$$\mathsf{G} = (\mathrm{tr}\,[|\psi_l\rangle\langle\psi_l|\psi_k\rangle\langle\psi_k|])_{k,l} = (\langle\psi_k|\psi_l\rangle^2)_{k,l}. \tag{12}$$

3. Now, we have to solve $\vec{d} = \mathsf{G}\vec{p}$.
4. It follows $\rho \in \mathcal{S}_r$, if and only if

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|, \tag{13}$$

for $p_k \geq 0$, and $\sum_k p_k = 1$.

We obtain a convex decomposition of ρ in terms of pure Schmidt number r states, if and only if ρ is a Schmidt number r state. Note that this decomposition procedure delivers in the case $r = \min\{\dim \mathcal{H}_A, \dim \mathcal{H}_B\}$ the spectral decomposition of the quantum state.

In addition to the convex decomposition of a Schmidt number r state, let us cansider the consequences what happens in the case $\varrho \notin \mathcal{S}_r$. The existence of the set of stationary points $\mathcal{D}$ as described above can also be done. Therefore the failure of the convex decomposition can be due to (i) some coefficients are negative, $p_{k_0} < 0$, or (ii) $\rho \neq \sum_k p_k |\psi_k\rangle\langle\psi_k|$.

It has been shown in [11, 12], that any quantum state for finite Hilbert spaces can be given as

$$\hat{\rho} = (1+\mu)\hat{\sigma} - \mu\hat{\sigma}', \tag{14}$$

with $\hat{\sigma}$ and $\hat{\sigma}'$ being separable states and $\mu \geq 0$. This can be easyly generalized to Schmidt number r states. This is equivalent to thee fact, that ϱ is a linear (and not a convex) combination of stationary states, $\varrho \in \mathsf{span}\,\mathcal{S}_{r,0}$. Thus, we can conclude that a decomposition of ϱ is possible and option (ii) cannot happen. Whereas option (i) is always fulfilled for states with a Schmidt number larger than r.

In such a situation, we define a quasi-probability $P_{r-\mathrm{Ent}}$ as

$$P_{r-\mathrm{Ent}}(\psi) = \sum_k p_k \delta_{\psi_k}(\psi), \tag{15}$$

with the Dirac-δ-distribution. In general this is a signed measure with the property

$$1 = \int_{\mathcal{S}_r} dP_{r-\mathrm{Ent}}(\psi), \tag{16}$$

$$\varrho = \int_{\mathcal{S}_r} dP_{r-\mathrm{Ent}}(\psi)|\psi\rangle\langle\psi|. \tag{17}$$

Moreover we can conclude the following corollary.

Corollary 2 *A quantum state σ is a Schmidt number r state, if and only if P_{r-Ent} is a probability ($P_{r-Ent} \geq 0$).* ■

Corollary 2 is a non-trivial result. Usually the linear decomposition of a quantum state in terms of Schmidt number r states is ambiguous. This means in some cases a linear decomposition of an Schmidt number r state has negativities, whereas a convex decomposition exists. For example, this can occures if a the set $\mathcal{D}$ does not fulfill the property given in Eq. (3). In such a case, a linear decomposition (negative coefficients are allowed) is possible but not a convex one (non-negative coefficients only). Due to Theorem 1 and Theorem 2 this cannot happen for $P_{r-\mathrm{Ent}}$ as it is defined by the reconstruction scheme and Eq. (17).

V. SUMMARY AND CONCLUSIONS

We considered the convex decomposition of an element of a convex set. We showed that the elements of the convex set with a maximal or minimal distance to the considered element are sufficient for its convex decomposition. We could reduce the number of elements for a convex decomposition. The problem of convex decomposition has been splitted into two individual problems. A linear problem and an optimization problem for a distance function.

The method has been applied to finite dimensional Hilbert spaces. Here we especially considered the decomposition of quantum states in terms of Schmid tnumber r states. We were able to define a quasi-probability distribution, which is negative if and only if the quantum state has a larger Schmdit number than r. This delivers a method for the identification of Schmidt number r states. The optimization problem in this case delivers a set of algebraic equations.

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