The Morse complex for reaction-diffusion equations

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CHAPTER 1

Introduction and main results

Let M be a compact smooth Riemannian manifold and $f : M \to \mathbb{R}$ a Morse-Smale function. The Morse-Smale-Witten chain complex $(C_q, \partial_q)_{q\in\mathbb{Z}}$ [**2**] is defined as follows: C_q is the free abelian group generated by the critical points $\mathrm{Cr}_q(f)$ of f which have index q, and

$$\partial_{q+1}(x) = \sum_{y\in C_q(f)} n(x,y)y \tag{1.1}$$

for every $x \in \mathrm{Cr}_{q+1}(f)$. Here, $n(x, y)$ is a finite sum over the the set of orbits connecting x and y where each orbit is given a sign reflecting its orientations. The singular homology of M equals the homology of the Morse-Smale-Witten chain complex associated with M. This has been shown in [**20**] using the Conley index. Essentially, the proof relies on two facts:

- The connecting homomorphism (in Conley index theory) associated with a single, isolated connecting orbit is an isomorphism, and
- using a suitable choice of an orientation on the set of orbits connecting two critical points yields that the boundary operator in (1.1) agrees with a connecting homomorphism in Conley index theory.

The first result has been shown by McCord [**16**]. Floer [**10**] gives a proof for both claims. In this work, we will generalize the results sketched above. Roughly speaking, we will work with semiflows generated by reaction-diffusion equations (or more general semilinear parabolic equations) under assumptions which are weak variants of the Morse-Smale property of f. In fact, though the term equilibrium is used, none of the proofs rely directly on a gradient or gradient-like structure of the semiflow. In what follows, we will describe the structure of this work and its result divided into three parts. The first part is a generalization of McCord's proof to reaction diffusion equations, the second one uses a new approach to tackle the orientation of connecting orbits. Finally, one can define a chain complex, replacing the critical points by hyperbolic equlibria, such that the homology of this chain complex is isomorphic to the singular homology of the Conley index. It is likely that the chain complex agrees with the Morse-Smale-Witten chain complex, but we do not give a proof here. Moreover, in the case of attractors, there seems to be a generalization stating that even the singular homology of an attractor (as opposed to an isolating neighborhood thereof) equals the homology of the chain complex.

1. The Homotopy Conley Index along heteroclinic trajectories

It is well-known that hyperbolic equilibria of reaction-diffusion equations have the homotopy Conley index of a pointed sphere, the dimension of which is the Morse index of the equilibrium. A similar result concerning the homotopy Conley index along heteroclinic solutions of ordinary differential equations under the assumption that the respective stable and unstable manifolds intersect transversally, is due to McCord (see [**16**, Theorem 3.1]). This result has recently been generalized by Dancer [**7**] to some reaction-diffusion equations by using finite-dimensional approximations. Roughly speaking, the homotopy Conley index is calculated in $L^2(\Omega)$ under remarkably weak smoothness assumptions on the non-linearity. As Dancer remarks [**7**, Remark 2.2], his result also covers the (Čech) cohomology in $L^p(\Omega)$, $1 < p < \infty$.

Unfortunately, the proof of [**16**, Theorem 3.1] contains an error and, as such, is incomplete. To see this, consider the following ordinary differential equation on $\mathbb{R}^2$:

$$\begin{aligned}\dot{x} &= 1 - x^2\\ \dot{y} &= x^2 y.\end{aligned}$$

$(-1, 0)$ and $(1, 0)$ are hyperbolic critical points and there is a solution $(u(t), 0)$ connecting $(-1, 0)$ and $(1, 0)$. It is easy to see that $\{(x, y) \in \mathbb{R}^2 : \ x < 1\}$ is the unstable manifold of $(-1, 0)$ and $\{(x, 0) \in \mathbb{R}^2 : \ x > -1\}$ is the stable manifold of $(1, 0)$. The tangential spaces of both manifolds intersect transversally in every point $(u(t), 0)$, $t \in \mathbb{R}$. According to the proof of [**16**, Theorem 3.1], there is a continuation to

$$\begin{aligned}\dot{x} &= 1 - x^2\\ \dot{y} &= 0^2 y.\end{aligned}$$

Evidently, $[-1, 1] \times \{0\}$ is not even an isolated invariant set relative to this flow. One might conjecture that this problem could be resolved by an arbitrarily small perturbation. However, there are also examples which show that this is generally not possible. The proof (of [**16**, Theorem 3.1]) relies on the assumption that 0 is an isolated rest point with respect to $\dot{y} = A(x)y$ for every $x \in [-1, 1]$. Now let $\varepsilon > 0$ and consider the following perturbation of our original equation

$$\begin{aligned}\dot{x} &= 1 - x^2\\ \dot{y} &= (x^2 - \varepsilon^2)y =: A(x)y.\end{aligned}$$

This means that 0 is not isolated with respect to $\dot{y} = A(\pm\varepsilon)y$ and every sufficiently small perturbation will retain these problematic points. Furthermore, the homotopy index of 0 relative to $\dot{y} = -\varepsilon^2 y$ is not Σ^1 as stated but Σ^0.

Dancer notes in [**7**] that "it should be possible to give a more natural direct proof [...] at least in the C^1 case". In this paper we provide a genuinely infinite-dimensional proof for a theorem which is closely related to Dancer's result in the C^1 case. It is possible to compute the homotopy Conley index in $L^p(\Omega)$ (not only the cohomology) directly, provided the solution is sufficiently regular. We face several technical difficulties due to the inifinite-dimensional situation, which, fortunately, are all overcome.

We are now in a position to state the first result. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $\partial\Omega$ be of class C^2. Let $2 \le p < \infty$ and $f : \ \Omega \times \mathbb{R} \to \mathbb{R}$. Suppose that for almost all $x \in \Omega$ there is a partial derivative $f_u(x, u)$ which is continuous in u and that $\operatorname{ess\,sup}_{x\in\Omega} \sup_{|u|\le r} |f_u(x, u)| < \infty$ for all $r \in \mathbb{R}^+$. Assume further that f and $(x, u) \mapsto f_u(x, u)$ are Carathéodory functions.

We consider the problem

$$\begin{array}{rcll} u_t(x,t) & = & \Delta u(x,t) + f(x,u(x,t)) & t>0,\ x\in\Omega \\ u(x,t) & = & 0 & t>0,\ x\in\partial\Omega. \end{array} \tag{1.2}$$

Let A_p denote the closure of $-\Delta:\ \{u\in C^2(\Omega):\ u_{|\partial\Omega}=0\}\to L^p(\Omega)=:X$ in $W^{2,p}(\Omega)$ and define the Nemitskii (superposition) operator $\hat{f}\in C^1(C(\bar{\Omega}),L^p(\Omega))$ by

$$(\hat{f}(u))(x) := f(x,u(x)) \quad x\in\Omega$$

so that $(D\hat{f}(\xi)\eta)(x) = f_u(x,\xi(x))\eta(x)$ a.e..
For k sufficiently large, A_p+kI is a positive operator having compact resolvent. Letting $\xi\in X^\alpha$, it follows that all eigenvalues of $A-D\hat{f}(\xi)$ are real.
Let $p\geq\max\{2,N\}$, $A:=A_p$, and $v:\ \mathbb{R}\to X^\alpha$ be a heteroclinic mild solution of

$$\dot{x}+Ax=\hat{f}(x) \tag{1.3}$$

and suppose that $v(t)\to e^\pm$ as $t\to\pm\infty$ in X^α (resp. $C(\bar{\Omega})$). It follows that $v\in C^1(\mathbb{R},L^p(\Omega))$. Choosing $0<\alpha<1$ large enough, we can further assume that there is a continuous inclusion $X^\alpha\subset C(\bar{\Omega})$ (see [**13**, Theorem 1.6.1]).
In the following theorem we will replace transversality by weaker assumptions, which have the advantage of not relying on the existence of global stable manifolds (see also [**7**]).

THEOREM 1.1. *Let u be a heteroclinic mild solution of* (1.3) *with $u(t)\to e^\pm$ as $t\to\infty$ in X^α (resp. $C(\bar{\Omega})$) and suppose that*

(1) *e^+,e^- are hyperbolic equilibria,*
(2) *the Morse indices satisfy $m(e^-)=m(e^+)+1$,*
(3) *all eigenvalues of $A-Df(e^\pm)$ are simple,*
(4) *$e^{\lambda t}(u(t)-e^+)\not\to 0$ for some $\lambda\in\mathbb{R}$, and*
(5) *every full bounded (in $C(\bar{\Omega})$) mild solution of*

$$\dot{y}+Ay=D\hat{f}(u(t))y$$

is a multiple of $\dot{u}$.

Then the homotopy Conley index $h(\pi,\bar{u})$ of $\bar{u}:=\mathrm{cl}\{u(t):\ t\in\mathbb{R}\}$ is well-defined and trivial, that is, $h(\pi,\bar{u})=\bar{0}$, where π denotes the semiflow which is induced by mild solutions of (1.3).

Conditions ensuring that the assumptions of Theorem 1.1 hold for every heteroclinic mild solution of (1.3) are discussed in the following section. In view of the growth condition in Theorem 1.1, it should be noted that in [**17**] Meshkov gives an example of an equation

$$\Delta u = q(x,t)u$$

on the three-dimensional torus which has a non-trivial solution $u(x,t)$ with $|u(x,t)|\leq Ce^{-ct^2}$ for some real constants c,C.
Theorem 1.1 is proved by reducing the general problem subsequently to a special case, the homotopy index of which can be calculated.
It follows from Theorem 3.2 that $u(t)$ satisfies the hypotheses of Proposition 4.4. Therefore, we can apply Theorem 5.12, which is the main result of Section 5 and states that the

homotopy index of $\bar{u}$ relative to π equals the homotopy index of a suitable linear skew product semiflow.
The structure of a certain class of linear skew product semiflows, which are defined on a trivial bundle, is discussed in Section 6. Theorem 6.8 is the main result of this section and completes the proof of Theorem 1.1.

2. An orientation for connecting orbits

We continue working with the semiflow π introduced in the previous section. Suppose that u is a solution of π for which the assumptions of Theorem 1.1 hold. For each of the equilibria e^- and e^+, there are $A - Df(e^-)$-invariant (resp. $A - Df(e^+)$) subspaces $E^-(e^-)$ (resp. $E^-(e^+)$) associated with $\{\operatorname{Re}\sigma(A - Df(e^-)) < 0\}$ (resp. $\{\operatorname{Re}\sigma(A - Df(e^+)) < 0\}$).
By $E = E_1 \oplus E_2$, we mean that E_1 and E_2 are closed linear subspaces of a normed space E with $E_1 \cap E_2 = \{0\}$ and $E = E_1 + E_2$. The canonical projection $P : E_1 \oplus E_2 \to E_1$ is given by $P(e_1 \oplus e_2) := e_1$.
Provided that the assumptions of Theorem 1.1 hold, we obtain that $\dim E^-(e^-) = \dim E^-(e^+) + 1 =: n+1$ for some $n \in \mathbb{N}$. Let $\{x_1, \ldots, x_{n+1}\}$ be a basis for $E^-(e^-)$ consisting of eigenvectors of $A - Df(e^-)$ and let $\{y_1, \ldots y_n\}$ denote an (arbitrary) basis for $E^-(e^+)$. These bases define toplinear isomorphisms $\Phi_{-1} : \mathbb{R}^{n+1} \to E^-(e^-)$, $\hat{\Phi}_{-1} : \mathbb{R}^n \to \operatorname{span}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}\}$, and $\Phi_1 : \mathbb{R}^n \to E^-(e^+)$, where we set

$$\begin{aligned} \Phi_{-1}(\mu_1, \ldots, \mu_{n+1}) &:= \sum_{k=1}^{n+1} \mu_k x_k \\ \hat{\Phi}(\mu_1, \ldots, \mu_n) &:= \Phi_{-1}(\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_{n+1}) \\ \Phi_1(\mu_1, \ldots, \mu_n) &:= \sum_{k=1}^{n} \mu_k y_k. \end{aligned}$$

Φ_{-1} and Φ_1 induce orientations, that is, they induce isomorphisms of connected simple systems, $\langle o_{-1} \rangle : \mathcal{S}^{n+1} \to \mathcal{C}(\pi, \{e^-\})$ (resp. $\langle o_1 \rangle : \mathcal{S}^n \to \mathcal{C}(\pi, \{e^+\})$).
Under the assumptions of Theorem 1.1, it holds that $\|u(t) - e^-\|_\alpha^{-1}(u(t) - e^-)$ converges to an eigenvector $\pm x_i$ of $A - Df(e^-)$ as $t \to -\infty$. We can further assume that there is an eigenvector η of $A - Df(e^+)$ with $\|u(t) - e^+\|_\alpha^{-1}(u(t) - e^+) \to \eta$ as $t \to \infty$. η belongs to an eigenvalue $\lambda > 0$. If there is an $A - Df(e^+)$ invariant subspace F of X such that $X = E^-(e^+) \oplus \operatorname{span}\{\eta\} \oplus F$, then, for large $t \in \mathbb{R}$, there is a decomposition of X, which defines a family $P(t) : E^-(e^+) \oplus \operatorname{span}\{\dot{u}(t)\} \oplus F \to E^-(e^+)$ of canonical projections. Furthermore, let Π_t denote the semigroup associated with the semiflow π, that is, $\Pi_t(x) = x\pi t$, $t \in \mathbb{R}^+$. It follows from our assumptions that, for every $t \in \mathbb{R}^+$, Π_t is continuously differentiable.
We now consider a linear operator $\mathbb{R}^n \to \mathbb{R}^n$:

$$C(t, \Delta, u) := \Phi_1^{-1} \circ P(t + \Delta) \circ D\Pi_\Delta(u(t)) \circ \hat{\Phi}.$$

It describes the geometrical connection from $E^-(e^-)$ to $E^-(e^+)$ given by linearization of π along u. Let $\delta(u) := \lim_{(t,t+\Delta)\to(-\infty,\infty)} \operatorname{sgn}\det C(t, \Delta, u)$.

Theorem 1.2. *Suppose that e^- and e^+ are hyperbolic equilibria. Then, for every heteroclinic solution $u(t)$ which satisfies*

(1) *$u(t) \to e^\pm$ as $t \to \pm\infty$,*

(2) $\|u(t)\|_{\alpha}^{-1}u(t) \to \nu x_i$ *as* $t \to -\infty$, $\nu \in \{-1,1\}$, *and*
(3) *the assumptions of Theorem 1.1*

it holds that $\delta(u)$ *is well defined and*

$$\partial_q \circ H_q\langle o_{-1}\rangle \circ \mu_q = \nu \cdot (-1)^{1+i} \cdot \delta(u) \cdot H_{q-1}\langle o_1\rangle \circ \mu_{q-1}.$$

Here, $\partial_q : H_q\langle \pi, \{e^-\}\rangle \to H_{q-1}\langle \pi, \{e^+\}\rangle$ *denotes the* q*-th connecting homomorphism associated with* u*, which is the connecting homomorphism associated with* $(u(\mathbb{R})\cup\{e^-,e^+\}, \{e^+\}, \{e^-\})$.

The theorem follows immediately from Theorem 7.42.

3. Morse homology

Let K be an isolated π-invariant set admitting a strongly admissible isolating neighborhood. Then K contains finitely many equiblibria. Suppose that for every full solution $u : \mathbb{R} \to K$ one of the following alternatives holds:

(1) $u \equiv e$, e is a hyperbolic equilibrium
(2) there are equilibria $e^{\pm}$ such that the Morse indices satisfy $m(e^+) < m(e^-) - 1$ and $u(t) \to e^{\pm}$ as $t \to \pm\infty$
(3) the assumptions of Theorem 1.1 are satisfied for u

Equivalently, we could assume that the assumptions at the beginning of Section 2 in Chapter 8 hold.

For every equilibrium e, choose an (ordered) basis $\mathcal{B}(e)$ of eigenvectors if possible[1]. Let there be given a connecting orbit[2] Γ, and let u be a solution of π whose trace is the connecting orbit. Let $\operatorname{sgn} u := \nu(-1)^{1+i}\delta(u)$ where we use the notation of Theorem 1.2. Different choices of u for the same connecting orbit yield the same number, so we can define $\operatorname{sgn}(\Gamma) := \operatorname{sgn}(u)$. Note that $\operatorname{sgn}(u)$ depends on the bases $\mathcal{B}(e)$ chosen above.

Now, let $e, f \in K$ be equilibria, and let $C(e,f)$ denote the set of all orbits connecting them. Given two equilibria $e, f \in K$, the set of orbits connecting them is finite (Lemma 8.12). Hence, the definition $n(x,y) := \sum_{\Gamma \in C(x,y)} \operatorname{sgn}(\Gamma)$ makes sense. We can now proceed as in the definition of the Morse-Smale-Witten chain complex, which is presented at the beginning of this chapter: let C_q, $q \in \mathbb{Z}$, denote the free abelian group generated by the set of equlibria with Morse index q, and let

$$\partial_{q+1}(x) = \sum_{y \in C_q} n(x,y)y.$$

Proposition 8.7 and Proposition 8.18 imply that the chain complex WN defined in chapter 8 is isomorphic to $(C_q, \partial_q)_{q\in\mathbb{Z}}$. We may thus apply Proposition 8.15 to obtain that the homology of WN is isomorphic to the singular homology of the Conley index of K. To sum it up:

Theorem 1.3. *For every* $q \in \mathbb{Z}$*, there is an isomorphism*

$$H_q((C_{\tilde q}, \partial_{\tilde q})_{\tilde q \in \mathbb{Z}}) \approx H_q\langle \pi, K\rangle.$$

[1] Our assumptions guarantee the existence of such a basis for an equilibrium e only if there is a heteroclinic solution $u : \mathbb{R} \to K$ with $u(t) \to e$ as $t \to \infty$ or $t \to -\infty$.
[2] see Definition 8.11

CHAPTER 2

Preliminaries

1. The nonlinearity

For the sake of completeness, we will give proofs for the properties of the Nemitskii operator used above.

LEMMA 2.1. *Let* $1 \leq p < \infty$. $\hat{f} \in C^1(C(\bar{\Omega}), L^p(\Omega))$ *with*

$$(D\hat{f}(u)v)(x) = f_u(u(x))v(x) \tag{2.1}$$

whenever $f_u(u(x))$ *is defined.*

PROOF. It is clear that $\hat{f}$ is well-defined since Ω is bounded. Let $u, \Delta \in C(\bar{\Omega})$ and $h \in [0,1]$. It follows that $M := \sup_{h\in[0,1]} \|u + h\Delta\| < \infty$ and by the assumptions of this lemma

$$C := \sup_{x\in\Omega\ |u|\leq M} |f_u(x)| < \infty.$$

For every $n \in \mathbb{N}$ and almost every $x \in \Omega$, there is a $\theta = \theta(x,n) \in [0,1]$ such that

$$h^{-1}(f(x, u(x) + h\Delta(x)) - f(x, u(x))) = f_u(x, u(x) + \theta h\Delta)\Delta(x) \to f_u(x, u(x))\Delta(x)$$

as $h \to 0$.
Moreover, we have $\left|h^{-1}(\hat{f}(u(x) + h\Delta(x)) - \hat{f}(u(x)))\right| \leq C\|\Delta\|_{C(\bar{\Omega})}$ for almost all $x \in \Omega$ so that Lebesgue's theorem of dominated convergence implies that $h^{-1}(\hat{f}(u + h\Delta) - \hat{f}(u)) \to D\hat{f}(u)\Delta$, where $D\hat{f}$ is given by (2.1). For every $u \in C(\bar{\Omega})$, it holds that $D\hat{f}(u)v \in \mathcal{L}(C(\bar{\Omega}), L^p(\Omega))$ with

$$\left\|D\hat{f}(u)\right\| \leq C.$$

For $v, u_1, u_2 \in C(\bar{\Omega})$ with $\|v\| \leq 1$, we have

$$\begin{aligned}\left\|D\hat{f}(u_1)v - D\hat{f}(u_2)v\right\| &= \left(\int_\Omega |(f_u(u_1(x)) - f_u(u_2(x)))v(x)|^p\, dx\right)^{1/p} \\ &\leq \left(\int_\Omega |f_u(u_1(x)) - f_u(u_2(x))|^p\, dx\right)^{1/p}.\end{aligned}$$

Now let $\tilde{u}_n \to u$ in $C(\bar{\Omega})$. It follows as before that there is a real constant C' such that $|f_u(\tilde{u}_n(x))| < C'$ for all $n \in \mathbb{N}$ and almost every $x \in \Omega$. Hence, $\int_\Omega |f_u(\tilde{u}_n(x)) - f_u(u(x))|\, dx \to 0$ as $n \to \infty$ by dominated convergence. This shows that $D\hat{f}(\tilde{u}_n) \to D\hat{f}(u)$, proving the continuity of $u \mapsto D\hat{f}(u)$. $\square$

LEMMA 2.2. *In addition to the assumptions of Lemma 2.1, suppose that for every $r \in \mathbb{R}$ there exist constants $\delta > 0$ and $C \in \mathbb{R}^+$ such that*

$$\text{ess sup}_{x\in\Omega} \sup_{|u_1|,|u_2|\le r} |f_u(x,u_1) - f_u(x,u_2)| \le C|u_1-u_2|^\delta.$$

Then for every $1 \le q \le \infty$, $D\hat{f} : C(\bar{\Omega}) \to \mathcal{L}(L^q(\Omega), L^q(\Omega))$ is locally Hölder continuous.

Here, we simply extend $D\hat{f}$ to $L^q(\Omega)$ by setting

$$(D\hat{f}(u)v)(x) := f_u(x,u(x))v(x) \quad x \in \Omega \quad u \in C(\bar{\Omega}) \quad v \in L^q(\Omega).$$

PROOF. For every $u \in C(\bar{\Omega})$, we have

$$\left\| D\hat{f}(u)v \right\|_{L^q} \le \sup_{x\in\Omega} |f_u(x,u(x))| \|v\|_{L^q}$$

for all $v \in L^q(\Omega)$. Let $r > \|u\|_{C(\bar{\Omega})}$ and $\tilde{u} \in C(\bar{\Omega})$ with $\|\tilde{u}\| < r$. By our assumptions, there are $\delta > 0$ and $C \in \mathbb{R}^+$ such that for every $x \in \Omega$,

$$|f_u(x,u(x)) - f_u(x,\tilde{u}(x))| \le C|u(x) - \tilde{u}(x)|^\delta.$$

It follows that

$$\left\| (D\hat{f}(u) - D\hat{f}(\tilde{u}))v \right\|_{L^q} \le C\|u-\tilde{u}\|^\delta_{C(\bar{\Omega})} \|v\|_{L^q}.$$

□

2. Notation

Although most of the notation is more or less standard, a couple of symbols should at least be mentioned. $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) denotes the set of all non-negative (resp. non-negative) real numbers. W^u and W^s denote unstable respectively stable manifolds, the precise meaning is given when they are used. σ is used to designate the spectrum of an operator. The open (resp. closed) ball with radius r and center x is denoted by $B_r(x)$ (resp. $B_r[x]$). If X is a set, then $\#X$ denotes the cardinality of X.

We will frequently deal with trivial vector bundles. They are considered as continuous families $U(x)$, $x \in [a,b]$, of vector space homomorphisms. When no confusion can arise, we will identify U with its image, just like the notation of the topology is usually suppressed. A more detailed exposition of this terminology can be found in the appendix.

Given normed spaces X and Y, and a continuous linear operator $F \in \mathcal{L}(X,Y)$, $\|F\|_{X,Y}$ is used sometimes to make the norm unambiguous. $\text{ISO}(X,Y)$ denotes the set of all $F \in \mathcal{L}(X,Y)$ which are toplinear isomorphisms. The notion of fractional power spaces follows **[13]**. If $F \in \mathcal{L}(X^\alpha, X^\beta)$, then $\|F\|_{\alpha,\beta}$ denotes the operator norm.

Finally, if X, Y are topological spaces, $f : X \to Y$ is a homeomorphism, and π is a (local) semiflow on X, then $f[\pi]$ is the semiflow on Y which is obtained by conjugacy, that is, u is a solution of π if and only if $f \circ u$ is a solution of $f[\pi]$.

3. Exponential decay

Suppose that the assumptions of Lemma 2.2 hold so that $D\hat{f} : C(\bar{\Omega}) \to \mathcal{L}(L^2(\Omega), L^2(\Omega))$ is locally Hölder continuous.

Let $u(t)$ be a mild solution of (1.3) defined for all $t \in \mathbb{R}^+$ with $u(0) \ne e^+$ and $u(t) \to e^+$ in $C(\bar{\Omega})$ as $t \to \infty$. $u(t)$ has a continuous derivative $\dot{u} : \mathbb{R}^+ \to X = L^p(\Omega)$. Suppose that

$\lambda(u) := \sup\{\mu \in \mathbb{R}^+ : e^{\mu t}\|u(t) - e^+\|_\alpha \to 0 \text{ as } t \to \infty\} = \infty$, that is, $e^{\lambda t}\|u(t) - e^+\|_\alpha \to 0$ as $t \to \infty$ for all $\lambda \in \mathbb{R}^+$.

Define $B(t) \in C(\mathbb{R}^+, \mathcal{L}(L^2(\Omega), L^2(\Omega)))$ by $(B(t)y)(x) := f_u(x, u(x))y(x)$ and $B(\infty) \in \mathcal{L}(L^2(\Omega), L^2(\Omega))$ by $B(\infty)y(x) := f_u(x, e^+(x))y(x)$. Due to the Hölder-continuity of $D\hat{f}$, there is a real constants $\tilde{C}$ with $\|B(t) - B(\infty)\| \leq \tilde{C}e^{-t}$ for all $t \in \mathbb{R}^+$.

Now, $\dot{u}(t)$ is a mild solution of

$$\dot{y} + A_2 y = B(t)y,$$

where we take $X := H := L^2(\Omega)$, and $\alpha = 0$.

Using the continuity of the inclusion $L^p(\Omega) \subset L^2(\Omega)$ and Lemma 3.6, it follows that $e^{\lambda t}\|\dot{u}(t)\|_2 \to 0$ as $t \to \infty$ for all $\lambda \in \mathbb{R}^+$.

We can apply Proposition 3.13 and obtain an $\varepsilon > 0$ such that $\dot{u}(t) = 0$ for all $t \in \mathbb{R}^+$ with

$$\sup_{s \geq t} \|B(s) - B(\infty)\| \leq \varepsilon^2.$$

Let

$$t_0 := \inf\{t \in \mathbb{R}^+ : \dot{u}(t) = 0\}$$

and assume that $0 < t_0$. For all $\tilde{t} \geq t_0$, it follows that $u(\tilde{t}) = e^+$ and $B(\tilde{t}) = B(\infty)$, so, by the continuity of $u(t)$, there is an $0 \leq \tilde{t}_0 < t_0$ with $\sup_{s \geq \tilde{t}_0} \|B(s) - B(\infty)\| \leq \varepsilon^2$. We thus have $\dot{u}(t) = 0$ for all $t \geq \tilde{t}_0$, a contradiction to the minimality of t_0.

Lemma 3.6 now implies that $\lambda(u) = \lambda(\dot{u}) < \infty$.

4. Hyperbolicity, transversality, and simple eigenvalues

It has been shown in [**3**] that generically (with respect to the nonlinearity) all equilibria are hyperbolic, the eigenvalues of their linearizations are simple, and their respective stable and unstable manifolds intersect transversally.

As already noted in [**7**], it is not necessary to assume the existence of global stable manifolds. Indeed, a sufficient condition can be formulated solely in terms of the linear equation.

To show that the assumptions of Theorem 1.1 hold in the case of transversality, let $e^+, e^- \in X^\alpha$ be hyperbolic equlibria with Morse indices $m(e^+) = n$ and $m(e^-) = n + 1$ for some $n \in \mathbb{N}$, and let $u(t)$ be a mild solution of (1.3) with $u(t) \to e^\pm$ as $t \to \pm\infty$.

The tangential spaces are characterized in [**3**, Lemma 4.b.1]. Translated to our notation (see Definition 5.5), we have

$$T_{u(t)}W^s(e^+) = B^+(T\pi, u(t))$$
$$T_{u(t)}W^u(e^-) = B^-(T\pi, u(t)).$$

Since $\operatorname{codim} T_{u(t)}W^s(e^+) = \dim T_{u(t)}W^u(e^-) - 1$ (using the Morse indices of $e^\pm$), one has $\dim(B^+(T\pi, u(t)) \cap B^-(T\pi, u(t))) = 1$, that is, every full bounded (in X^α) mild solution of

$$\dot{y} + Ay = D\hat{f}(u(t))y \tag{2.2}$$

is a multiple of $\dot{u}$ as stated in the assumptions of Theorem 1.1. Of course, if $v : \mathbb{R} \to X = L^p(\Omega)$ is a mild solution of (2.2), then $v(t) \in X^\alpha$ for all $t \in \mathbb{R}$ and $\sup_{t \in \mathbb{R}} \|v(t)\|_\alpha < \infty$.

5. Conley Index

The purpose of this section is to give a short overview over the most important concepts of Conley index theory for semiflows on metric spaces. A more detailed exposition can be found in [**4**] and [**18**].

Let B be a topological space and $A \subset B$. Let $(\tilde{A}, \tilde{B}) := (A, B)$ if $B \neq \emptyset$ and $(\tilde{A}, \tilde{B}) := (A \cup \{*\}, \{*\})$ (endowed with the sum topology) otherwise. Here, we assume that $* \notin A$. Now let A/B denote the set of equivalence classes in $\tilde{A}$ where $a, \tilde{a} \in \tilde{A}$ are related if they are equal or $\{a, \tilde{a}\} \subset \tilde{B}$. A/B is equipped with the quotient topology.

Let π be a local semiflow defined on a metric space X. A subset $S \subset X$ is called *invariant* if for every $x \in S$ there exists a full solution $u : \mathbb{R} \to S$ of π through x that is, $u(0) = x$.

Let $Y \subset X$, $(x_n)_n$ a sequence in Y, and $(t_n)_n$ a sequence in $\mathbb{R}^+$ such that $t_n \to \infty$ and $x_n\pi[0, t_n] \subset Y$. Y is called *admissible* if the sequence of endpoints $x_n\pi t_n$ is precompact for every such pair of sequences. We say that π *does not explode in* Y if for every $x \in X$ either $x\pi t$ is defined for all $t \in \mathbb{R}^+$ or there is a $t_0 \in \mathbb{R}^+$ such that $x\pi[0, t_0]$ is defined and $x\pi t_0 \notin Y$. Y is called *strongly π-admissible* if it is admissible and π does not explode in Y.

Now let $Z \subset Y \subset X$. Z is called *Y-positively invariant* if it holds that $x\pi[0, t] \subset Y$ whenever $x\pi[0, t]$ is defined and $x\pi[0, t] \subset Z$.

Z is called an *exit ramp* for Y if for every $x \in Y$ with $x\pi[0, t]$ defined and $\not\subset Z$, there is a $t_0 \in [0, t_0]$ such that $x\pi[0, t_0] \subset Y$ and $x\pi t_0 \in Z$.

Definition 2.3 (Definition 2.4 in [**4**]). A pair (N_1, N_2) is called an *FM-index pair* for (π, S) if:

(1) N_1 and N_2 are closed subsets of X with $N_2 \subset N_1$ and N_2 is N_1-positively invariant;
(2) N_2 is an exit ramp for N_1;
(3) S is closed, $S \subset \mathrm{int}_X(N_1 \setminus N_2)$ and S is the largest invariant set in $\mathrm{cl}_X(N_1 \setminus N_2)$.

Assume that there exists a strongly π-admissible isolating neighborhood N for S, that is, $N \subset X$ is a closed and strongly π-admissible neighborhood of S such that S is the largest invariant set in N. Then the homotopy Conley index $h(\pi, S)$ is defined to be the homotopy type of $(N_1/N_2, \{[N_2]\})$ where (N_1, N_2) is an FM-index pair for (π, S) such that $\mathrm{cl}_X(N_1 \setminus N_2)$ is strongly π-admissible.

Let $u(t)$ satisfy the assumptions of Theorem 1.1 and let π denote the semiflow on X^α induced by mild solutions of (1.3). Then $S := \bar{u}$ is an isolated invariant set admitting a strongly π-admissible isolating neighborhood. In particular, the homotopy Conley index $h(\pi, \bar{u})$ is well-defined under these assumptions.

Furthermore, $\{\pi, \bar{u}, e^+, e^-)$ is an attractor-repeller decomposition of $\bar{u}$. Suppose we are given an arbitrary attractor-repeller decomposition (π, S, A, A^*). A triple (N_1, N_2, N_3) is an *FM-index triple* for $(\pi, \bar{u}, A, A^*)$ if (N_1, N_3) is an FM-index pair for $(\pi, \bar{u})$ and if (N_2, N_2) is an FM-index pair for e^+. As a consequence, the sequence

$$\Delta(N_2/N_3)/\Delta\{[N_3]\} \xrightarrow{i} \Delta(N_1/N_3)/\Delta\{[N_3]\} \xrightarrow{p} \Delta(N_1/N_2)/\Delta\{[N_2]\} \tag{2.3}$$

is weakly exact. Here, Δ denotes the singular chain functor, which passes a topological space to its singular chain complex. Generally, a sequence of chain maps

$$C_1 \xrightarrow{i} C_2 \xrightarrow{p} C_3$$

is called weakly exact if $p \circ i = 0$, $\ker i = 0$, and $[x] \mapsto p(x)$ induces an isomorphism $H_q(C_2/\operatorname{im} i) \to H_q(C_3)$. There is a covariant functor which passes weakly exact sequences of chain maps to long exact sequences in singular homology. Applying this functor to (2.3), one obtains a long exact sequence

$$\longrightarrow H_{q+1}(N_1/N_2, \{[N_2]\}) \xrightarrow{\partial_{q+1}} H_q(N_2/N_3, \{[N_3]\}) \longrightarrow H_q(N_1/N_3, \{[N_3]\}) \longrightarrow .$$

Since these sequences are rather lengthy, we will abbreviate them sometimes by

$$\longrightarrow H_{q+1}[N_1/N_2] \xrightarrow{\partial_{q+1}} H_q[N_2/N_3] \longrightarrow H_q[N_1/N_3] \longrightarrow .$$

The boundary operator $(\partial_q)_{q\in\mathbb{Z}}$ is called the connecting homomorphism associated with the weakly exact sequence or, if appropriate, the attractor-repeller decomposition. In the context of a heteroclinic solution u, the connecting homomorphism associated with u will denote the connecting homomorphism of $\bar{u} = \operatorname{cl}\{u(t) : \ t \in \mathbb{R}\}$.

We will frequently use the notion of $\mathcal{S}$-continuity. It has been defined in [**18**, Definition I.12.1]. Let Λ be a metric space and $(\pi_\lambda, K_\lambda)_{\lambda\in\Lambda}$ be a family for which the following holds:

(1) π_λ is a local semiflow on X;
(2) there is a strongly π_λ-admissible isolating neighborhood N_λ for K_λ relative to π_λ;
(3) whenever $\lambda_n \to \lambda$ in Λ, then $\pi_{\lambda_n} \to \pi_\lambda$, N_λ is a strongly π_{λ_n}-admissible isolating neighborhood for K_{λ_n} relative to π_{λ_n}, and N_λ is $(\pi_{\lambda_n})_n$-admissible.

These conditions are equivalent to the original definition.

CHAPTER 3

Abstract semilinear parabolic equations

Let H be a real Hilbert space, and let $A_H : D(A_H) \subset H \to H$ be a sectorial operator such that

(1) A_H has compact resolvent;
(2) A_H is densely defined;
(3) $\operatorname{Re}\lambda > 0$ for all $\lambda \in \sigma(A_H)$.

Let X be a real Banach space with continuous inclusion $X \subset H$, and let

$$A : D(A) \subset X \to X$$

be a sectorial operator such that

(1) A is densely defined;
(2) A has compact resolvent;
(3) $Ax = A_H x$ for all $x \in \mathcal{D}(A)$.

Fix $\alpha \in [0,1[$, let X^α denote the α-th fractional power space (see [**13**]), and let $f \in C^1(\mathcal{U}, X^0)$ where $\mathcal{U} \subset X^\alpha$ is open.
We consider mild solutions of the Cauchy problem

$$\begin{aligned} \dot{x}(t) + Ax(t) &= f(x(t)) \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

which induce a local semiflow on X^α ([**13**, Theorem 3.3.3], [**1**, Theorem A.3]). This semiflow is denoted by π_f, respectively π whenever the meaning is clear.

Definition 3.1. For $u : [0,\infty[\to X^\alpha$, let

$$\lambda(u) := \sup\{\gamma \in \mathbb{R} : \ e^{\gamma t}\|u(t)\|_\alpha \to 0 \text{ as } t \to \infty\}.$$

Theorem 3.2. *Let $u : \mathbb{R} \to X^\alpha$ be a heteroclinic solution of* (3.1) *with*

$$\begin{aligned} u(t) &\to e^- & t &\to -\infty \\ u(t) &\to e^+ & t &\to \infty. \end{aligned}$$

For each $e \in \{e^-, e^+\}$, assume that $A - Df(e)$ is hyperbolic and that the spectrum $\sigma(A - Df(e))$ consists of isolated simple eigenvalues, all of which are real. Assume further that $\lambda(u - e^+) < \infty$ (Definition 3.1).
Letting $\rho^+(v,t) := \int_t^\infty \|\dot{v}(s)\|_\alpha \, ds$, the following holds:

(1) *There is a $0 < \lambda^+ \in \sigma(A - Df(e^+))$ and an associated eigenvector η^+ such that $\rho^+(u,t)^{-1}u(t) \to \eta^+$ as $t \to \infty$*

and there is another solution v^+ of (3.1) *defined for all $t \geq 0$ such that*

(2) $\rho^+(v^+,t)^{-1}v^+(t) \to -\eta^+$ *in* X^α *as* $t \to \infty$.

Moreover, with $\rho^-(v,t) := \int\limits_{-\infty}^{t} \|\dot{v}(s)\|_\alpha \, ds$,

(3) *there is a* $0 < \lambda^- \in \sigma(A - Df(e^-))$ *and an associated eigenvector* η^- *such that* $\rho^-(u,t)^{-1}u(t) \to \eta^-$ *in* X^α *as* $t \to -\infty$

and there is another solution v^- *of* (3.1) *defined for all* $t \le 0$ *such that*

(4) $\rho^-(v^-,t)^{-1}v^-(t) \to \eta^-$ *in* X^α *as* $t \to -\infty$.

Proof. Let $L^+ := A - Df(e^+)$, $g^+(x) := f(x) - Df(e^+)x$ and $u^+(t) := u(t) - e^+$. $u^+(t)$ is a solution of

$$\dot{x}(t) + L^+x(t) = g^+(x(t) + e^+) - g^+(e^+).$$

It follows from Lemma 3.6 that $\lambda(\dot{u}) < \infty$ and from Lemma 3.5 that $\|\dot{u}^+(t)\|_\alpha^{-1}\dot{u}^+(t)$ converges to an eigenvector η of L^+. Therefore, claim (1) is a consequence of Lemma 3.7. v^+ is obtained from Proposition 3.8; we have $\|v^+(t)\|_\alpha^{-1}v^+(t) - \eta \to 0$ as $t \to \infty$. It follows from Lemma 3.5 that there is an eigenvalue $\tilde{\eta}$ of L^+ such that $\|\dot{v}^+(t)\|_\alpha^{-1}\dot{v}^+(t) \to \tilde{\eta}$ as $t \to \infty$. Using Lemma 3.7, we conclude $\tilde{\eta} = \eta$, which proves (2).
Analogously, $u^-(t) := u(t) - e^-$ is a solution of

$$\dot{x}(t) + L^-x(t) = g^-(x(t) + e^-) - g^-(e^-)$$

with $L^- := A - Df(e^-)$ and $g^-(x) := f(x) - Df(e^-)x$.
The convergence in (3) now follows from Corollary 3.11 and the existence of v^- follows from Proposition 3.12 and Corollary 3.11. □

1. Estimates

Assume that $f(0) = 0$ and let $u : \ [0,\infty[\to X^\alpha$ be a solution of (3.1) with $u(t) \to 0 \in X^\alpha$ as $t \to \infty$. Set $L := A - Df(0)$ and $g(x) := f(x) - Df(0)x$, $x \in X^\alpha$. Then L is a sectorial operator, $g(0) = 0$, and $u(t)$ is also a solution of

$$\dot{x}(t) + Lx(t) = g(x(t)) \tag{3.2}$$

with $u(t) \to 0$ as $t \to \infty$ and we have

$$\mathcal{L}(X^\alpha, X) \ni Dg(0) = 0. \tag{3.3}$$

Assume that $\sigma(L)$ consists of a sequence of simple eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \to \infty$ as $n \to \infty$.
For each $\gamma \in \mathbb{R} \setminus \operatorname{Re}\sigma(L)$ there are linear projections $P^\pm(\gamma) : \ X \to X$ such that $P^+(\gamma)x + P^-(\gamma)x = x$ for all $x \in X$ and for some real constant $M > 0$ we have

$$\begin{aligned}
&\left\|e^{-Lt}P^-(\gamma)x\right\|_\alpha \le Me^{-\gamma t}\|x\|_0 && t \le 0\\
&\left\|e^{-Lt}P^+(\gamma)x\right\|_\alpha \le Me^{-\gamma t}\|x\|_\alpha && t \ge 0\\
&\left\|e^{-Lt}P^+(\gamma)x\right\|_\alpha \le Me^{-\gamma t}t^{-\alpha}\|x\| && t > 0
\end{aligned} \tag{3.4}$$

(see [**13**, Theorem 1.5.3]).
By [**13**, Lemma 3.3.2], u is differentiable in all $t > 0$ and $\dot{u} : \ ]0,\infty[\to X^0$ is continuous with $\dot{u}(t) \in X^\alpha$ for all $0 < t \in \mathbb{R}^+$.

Let $C_\mu(\mathbb{R}^+, X^\alpha)$ denote the set of all $f \in C(\mathbb{R}^+, X^\alpha)$ with $\lambda(f) > \mu$, which is equipped with the norm $\|f\|_{C_\mu} := \sup_{s\in\mathbb{R}^+} \|e^{\mu s} f(s)\|_\alpha$.

LEMMA 3.3. *Let $\mu \in \mathbb{R}^+ \setminus \operatorname{Re}\sigma(L)$ and*

$$P^-(\mu)K_\mu(x_0, f)(t) := -\int_t^\infty e^{-L(t-s)} P^-(\mu) f(s)\, ds$$

$$P^+(\mu)K_\mu(x_0, f)(t) := e^{-Lt} P^+(\mu)x_0 + \int_0^t e^{-L(t-s)} P^+(\mu) f(s)\, ds.$$

Then $K_\mu \in \mathcal{L}(X^\alpha \times C_\mu(\mathbb{R}^+, X^0), C_\mu(\mathbb{R}^+, X^\alpha))$.

PROOF. Let $0 < \delta \in \mathbb{R}^+$ such that $[\mu - \delta, \mu + \delta] \subset \mathbb{R} \setminus \operatorname{Re}\sigma(L)$. We then have $P^-(\mu - \delta) = P^-(\mu)$ and $P^+(\mu+\delta) = P^+(\mu)$, so by (3.4) there is an $M > 0$ such that for all $s, t \in \mathbb{R}^+$

$$\begin{aligned}
\left\|e^{-L(t-s)} P^-(\mu) f(s)\right\|_\alpha &\le M e^{-(\mu-\delta)(t-s)} \|f(s)\|_0 && t - s \le 0\\
\left\|e^{-L(t-s)} P^+(\mu) f(s)\right\|_\alpha &\le M e^{-(\mu+\delta)(t-s)} (t-s)^{-\alpha} \|f(s)\|_0 && t - s > 0\\
\left\|e^{-Lt} P^+(\mu) x_0\right\|_\alpha &\le M e^{-\mu t} \|x_0\|_\alpha.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left\|e^{-L(t-s)} P^-(\mu) f(s)\right\|_\alpha &\le M e^{-\mu t} e^{\delta(t-s)} \|f\|_{C_\mu} && t - s \le 0\\
\left\|e^{-L(t-s)} P^+(\mu) f(s)\right\|_\alpha &\le M e^{-\mu t} (t-s)^{-\alpha} e^{-\delta(t-s)} \|f\|_{C_\mu} && t - s > 0
\end{aligned}$$

showing that K_μ is well-defined and

$$\|K_\mu(x_0, f)\|_{C_\mu} \le M\|x_0\|_\alpha + \left(M \int_0^\infty e^{-\delta s}\, ds + M \int_0^\infty s^{-\alpha} e^{-\delta s}\, ds \right) \|f\|_{C_\mu}.$$

□

LEMMA 3.4. *Let $0 \ne x \in X^0$. Then there exists a $\mu \in \mathbb{R} \setminus \operatorname{Re}\sigma(L)$ with $P^-(\mu)x \ne 0$.*

PROOF. Let $(\eta_i)_{i\in\mathbb{N}}$ denote an orthonormal basis for H and let $(\lambda_i)_{i\in\mathbb{N}}$ denote the associated eigenvalues. Then there is an eigenvector η_i with $\langle x, \eta_i\rangle \ne 0$. Letting $x(t) := e^{-Lt}x$, $t \in \mathbb{R}^+$, and $\mu \in \mathbb{R} \setminus \operatorname{Re}\sigma(L)$ with $\mu > \lambda_i$, it follows that $e^{\mu t}\|x(t)\|_H \not\to 0$, so by the continuity of the inclusion $X \subset H$, one has $e^{\mu t}\|x(t)\|_{X^0} \not\to 0$ as $t \to \infty$. We have $x(t) = P^-(\mu)x(t) + P^+(\mu)x(t)$ with $e^{\mu t}\|P^+(\mu)x(t)\|_{X^0} \to 0$ as $t \to \infty$. This shows that $P^-(\mu)x(0) = P^-(\mu)x \ne 0$ whenever $\mu > \lambda_i$. □

2. Exponential decay

LEMMA 3.5. *Assume that $\sigma(L) \subset \mathbb{R}$ and let $v \in \{u, \dot u\}$ with $0 \le \lambda(v) < \infty$. Then*

(1) $\lambda(v) \in \operatorname{Re}\sigma(L) = \sigma(L)$;

(2) *there is an eigenvector η of L which belongs to the eigenvalue $\lambda(v)$ (that is $\eta \in \mathcal{D}(L)$ and $L\eta = \lambda(v)\eta$) such that*

$$\left\| \frac{v(t)}{\|v(t)\|_\alpha} - \eta \right\|_\alpha \to 0 \text{ as } t \to \infty.$$

PROOF. Following [**1**, A.3.2], let $B(t) := \int_0^1 Dg(su(t))\,ds$ if $v = u$ and $B(t) := Dg(u(t))$ if $v = \dot{u}$. In either case we have $B(t) \to 0$ in $\mathcal{L}(X^\alpha, X)$ as $t \to 0$, and v is a mild solution of

$$\dot{x}(t) + Lx(t) = B(t)x(t).$$

Now, claim (1) follows from [**1**, Theorem A.10]. The second claim is a consquence of [**1**, Corollary A.11] and the assumptions on $\sigma(L)$. □

If L is hyperbolic, then a particular consequence of Lemma 3.5 is that $u(t) \in W_{\text{loc}}$ (that is $\lambda(u) > 0$) for all t large enough, where W_{loc} denotes the local stable manifold given by [**1**, Theorem A.12]. Until further notice, we will assume that L is hyperbolic.

LEMMA 3.6. $\lambda(\dot{u}) = \lambda(u)$ *and for all* $t \in \mathbb{R}^+$

$$u(t) = -\int_t^\infty \dot{u}(s)\,ds$$

PROOF. We start with the integral expression. Letting $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, one has

$$u(t_2) = u(t_1) + \int_{t_1}^{t_2} \dot{u}(s)\,ds.$$

Taking $t_2 \to \infty$, we obtain all $t \in \mathbb{R}$

$$u(t) = -\int_t^\infty \dot{u}(s)\,ds. \tag{3.5}$$

The right side is integrable since by our assumptions there are $M \in \mathbb{R}$ and $0 < \mu \in \mathbb{R}$ such that $\|\dot{u}(t)\|_\alpha \leq Me^{-\mu t}$ for all $t \geq 0$.
It follows from [**1**, Theorem A.12 d)] that $\lambda(\dot{u}) \geq \lambda(u)$. Conversely, let $0 < \mu \in \mathbb{R}$, such that $e^{\mu t}\|\dot{u}(t)\|_\alpha \to 0$ as $t \to \infty$. Letting $C := \sup_{s\in\mathbb{R}^+} e^{\mu s}\|\dot{u}(s)\|_\alpha < \infty$, it follows that

$$\|u(t)\|_\alpha \leq \int_t^\infty \|\dot{u}(s)\|\,ds \leq C\int_t^\infty e^{-\mu s}\,ds \leq C\mu^{-1}e^{-\mu t}$$

showing that $\lambda(u) \geq \lambda(\dot{u})$. □

LEMMA 3.7. *Assume that* $\|\dot{u}(t)\|_\alpha^{-1}\dot{u}(t) \to -x_0$ *in* X^α *as* $t \to \infty$. *Then* $\rho(t)^{-1}u(t) \to x_0$ *in* X^α *as* $t \to \infty$, *where* $\rho(t) := \int_t^\infty \|\dot{u}(s)\|_\alpha\,ds$.
Moreover, $\rho(t)^{-1}\|u(t)\|_\alpha \to 1$ *as* $t \to \infty$.

PROOF. By Lemma 3.6, we have for all $t \in \mathbb{R}^+$

$$u(t) = -\int_t^\infty \dot{u}(s)\,ds$$

and thus

$$u(t) = \int_t^\infty \|\dot u(s)\|_\alpha x_0 \, ds - \int_t^\infty \|\dot u(s)\| x_0 + \dot u(s) \, ds,$$

where

$$\left\| \int_t^\infty \|\dot u(s)\|_\alpha x_0 + \dot u(s) \, ds \right\|_\alpha \le \sup_{s\ge t} \left\| \frac{\dot u(s)}{\|\dot u(s)\|_\alpha} + x_0 \right\|_\alpha \int_t^\infty \|\dot u(s)\|_\alpha \, ds$$
$$= \rho(t) \sup_{s \ge t} \left\| \frac{\dot u(s)}{\|\dot u(s)\|_\alpha} + x_0 \right\|_\alpha$$

showing that

$$\left\| \frac{u(t)}{\rho(t)} - x_0 \right\|_\alpha \to 0 \text{ as } t \to \infty.$$

Our assumptions imply that $\|x_0\|_\alpha = 1$, so $\rho(t)^{-1}\|u(t)\|_\alpha \to \|x_0\|_\alpha = 1$ as $t \to \infty$, completing the proof. □

3. Convergence as $t \to \infty$

Let the assumptions on f at the beginning of Section 1 hold, and let $u : \mathbb{R}^+ \to X^\alpha$ be a mild solution of (3.2) with $u(t) \to 0$ as $t \to \infty$. Assume that the spectrum of $L = A - Df(0)$ consists of simple, real, and isolated eigenvalues $(\lambda_i)_i \in I$ with $0 \ne \lambda_i$ for all $i \in I$.
We have already mentioned that the angle $\frac{u(t)}{\|u(t)\|_\alpha}$ converges. The inverse question is whether there exists a solution v which converges to a given eigenvector of L.
The proof primarily refines a part of [**1**, Theorem A.12]. However, we need more control over the constants involved. In the case of ordinary differential equations in finite dimensions and under slightly more restrictive assumptions on the nonlinearity, Proposition 3.8 can also be deduced from [**6**, Theorem 13.4.5].

Proposition 3.8. *Let $0 < \lambda$ be an eigenvalue of L and let η denote an associated eigenvector with $\|\eta\|_\alpha = 1$. Then there is a solution $u : [0, \infty[\to X^\alpha$ of (3.1) with*

$$\left\| \|u(t)\|_\alpha^{-1} u(t) - \eta \right\|_\alpha \to 0 \text{ as } t \to \infty. \tag{3.6}$$

Let $B(t) \in C([0,\infty[, \mathcal{L}(X^\alpha, X^0))$ and consider the following perturbation of (3.2)

$$\dot u(t) + Lu(t) = g(u(t)) + B(t)u(t), \tag{3.7}$$

which can also be written as

$$\dot u + Lu = \hat g(u) \tag{3.8}$$

with $\hat g : C(\mathbb{R}^+, X^\alpha) \to C(\mathbb{R}^+, X^0)$, $\hat g(u)(t) := g(u(t)) + B(t)u(t)$. The purpose of introducing B is to cover two variants of the following lemma simultaneously.

Lemma 3.9. *Let $\mu \in \mathbb{R} \setminus \operatorname{Re}\sigma(L)$ and let K_μ be given by Lemma 3.3. Let $M = M(\mu) := \max\{2\|K_\mu\|, 1\}$ and $0 < \rho \le \infty$.*
Provided that

$$\kappa(\rho) := \sup_{\|x\|_\alpha \le \rho,\, \|y\|_\alpha \le \rho} \frac{\|g(x) - g(y)\|}{\|x - y\|_\alpha} \le \frac{1}{2M} \tag{3.9}$$

and

$$\sup_{t \in \mathbb{R}^+} \|B(t)\|_{\alpha,0} \le \frac{1}{2M}, \tag{3.10}$$

the following holds:

(1) *If* $u : \mathbb{R}^+ \to X^\alpha$ *is a solution of* (3.8) *with* $\lambda(u) \geq \mu$, *then* $\lambda(u) > \mu$ *and* $u = K_\mu(P^+(\mu)u(0), \hat{g}(u))$.
(2) *If* $u \in C_\mu(\mathbb{R}^+, X^\alpha)$ *is a solution of*
$$u = K_\mu(P^+(\mu)u(0), \hat{g}(u)), \tag{3.11}$$
then u *is a mild solution of* (3.7).
(3) *If* $u_1, u_2 \in C_\mu(\mathbb{R}^+, X^\alpha)$ *are solutions of* (3.11) *with* $\sup_{t\in\mathbb{R}^+} \|u_i(t)\|_\alpha \leq \rho$ *for* $i \in \{1,2\}$, *then* $\|u_1 - u_2\|_{C_\mu} \leq M\|P^-(\mu)(u_1(0) - u_2(0))\|_\alpha$.
(4) *There exists a continuous map* $S = S_\mu : B_{\frac{\rho}{M}}[0] \subset X^\alpha \to C_\mu(\mathbb{R}^+, X^\alpha)$ *such that for all* $x \in \mathcal{D}(S)$ *one has* $S(x) = K_\mu(x, \hat{g}(S(x))) = K_\mu(P^+(\mu)x, \hat{g}(S(x)))$ *and* $P^+(\mu)S(x)(0) = P^+(\mu)x$.

Remark 1. *Since* $Dg(0) = 0$ *is the Fréchet-derivative, there always exists a* ρ *such that* (3.9) *holds.*

Proof. Letting $C_\mu := C_\mu(\mathbb{R}^+, X^\alpha)$, we have
$$\|K_\mu(x_1, \hat{g}(u)) - K_\mu(x_2, \hat{g}(v))\|_{C_\mu} \leq \frac{M}{2}(\|x_1 - x_2\|_\alpha + \kappa(\rho)\|u - v\|_{C_\mu}) + \frac{1}{4}\|u - v\|_{C_\mu}$$
for all $x_1, x_2 \in X^\alpha$ and for all $u, v \in C(\mathbb{R}^+, X^\alpha)$ with $\|u\|_{C(\mathbb{R}^+,X^\alpha)} \leq \rho$ and $\|v\|_{C(\mathbb{R}^+,X^\alpha)} \leq \rho$. In view of (3.9),
$$\|K_\mu(x_1, \hat{g}(u)) - K_\mu(x_2, \hat{g}(v))\|_{C_\mu} \leq \frac{M}{2}\|x_1 - x_2\|_\alpha + \frac{1}{2}\|u - v\|_{C_\mu} \tag{3.12}$$
for all $x_1, x_2 \in X^\alpha$, and all $u, v \in C_\mu(\mathbb{R}^+, X^\alpha)$ with $\|u\|_{C(\mathbb{R}^+,X^\alpha)} \leq \rho$ and $\|v\|_{C(\mathbb{R}^+,X^\alpha)} \leq \rho$.

(1) Let u be a solution of (3.7) with $\lambda(u) \geq \mu$. By Lemma 3.5, we have $\lambda(u) > \mu$, so for all $t \geq r \geq 0$,
$$e^{-L(-t)}P^-(\mu)u(t) = e^{Lr}P^-(\mu)u(r) + \int_r^t e^{-L(-s)}P^-(\mu)\hat{g}(u)(s)\, ds \to 0$$
as $t \to \infty$ because for $(-t) < 0$ we have $\left\|e^{-L(-t)}P^-(\mu)u(t)\right\|_\alpha \leq Me^{\mu t}\|u(t)\|_\alpha \to 0$ as $t \to \infty$. This shows that u is a solution of (3.11).
(4) Let $Y := B_\rho[0] \subset C_\mu(\mathbb{R}^+, X^\alpha)$ and let $x_0 \in P^+(\mu)X^\alpha$ with $\|x_0\|_\alpha \leq \frac{\rho}{M}$. $\tilde{K}y := K_\mu(x_0, \hat{g}(y))$ defines a contraction mapping on Y since by (3.12)
$$\left\|\tilde{K}y\right\|_{C_\mu} \leq \frac{\rho}{2} + \frac{1}{2}\|y\|_{C_\mu} \leq \rho.$$
Hence, there is a unique fixed point for every x_0.
(3) By (3.12), we have
$$\begin{aligned}\|u_1 - u_2\|_{C_\mu} &= \|K_\mu(x_0, \hat{g}(u_1)) - K_\mu(x_0, \hat{g}(u_2))\|_{C_\mu} \\ &\leq \frac{M}{2}\left\|P^+(\mu)(u_1(0) - u_2(0))\right\|_\alpha + \frac{1}{2}\|u_1 - u_2\|_{C_\mu},\end{aligned}$$
so
$$\|u_1 - u_2\|_{C_\mu} \leq M\left\|P^+(\mu)(u_1(0) - u_2(0))\right\|_\alpha.$$

(2) u is a mild solution of (3.7) since for all $t_1, t_2 \in \mathbb{R}^+$ with $t_1 \leq t_2$,

$$\begin{aligned} P^-(\mu)u(t_2) - P^-(\mu)e^{-L(t_2-t_1)}u(t_1) &= -\int_{t_2}^{\infty} e^{-L(t_2-s)}P^-(\mu)\hat{g}(u)(s)\,ds \\ &\quad + e^{-L(t_2-t_1)}\int_{t_1}^{\infty} e^{-L(t_1-s)}P^-(\mu)\hat{g}(u(s))\,ds \\ &= \int_{t_1}^{t_2} e^{-L(t_2-s)}P^-(\mu)\hat{g}(u(s))\,ds. \end{aligned}$$

□

Proof of Proposition 3.8. Let $\mu_1 < \lambda < \mu_2$ be real numbers such that

$$[\mu_1, \mu_2] \cap \sigma(L) = \{\lambda\},$$

let $1 \leq M(\mu_i)$, $i \in \{1, 2\}$ be given by Lemma 3.9, let $M := \max\{M(\mu_1), M(\mu_2)\}$, and choose $\rho > 0$ small enough that $\kappa(\rho) < \frac{1}{2M}$.
Let $0 < \varepsilon \leq \frac{\rho}{M^2}$, and let u denote the unique solution of

$$u = K_{\mu_1}(\varepsilon\eta, g \circ u).$$

It follows that $\sup_{t \in \mathbb{R}^+} \|u(t)\|_\alpha \leq M\|\varepsilon\eta\|_\alpha \leq \frac{\rho}{M}$.
Suppose that $\lambda(u) > \lambda$, so by Lemma 3.5, $\lambda(u) \geq \mu_2$, which implies that u is a solution of

$$u = K_{\mu_2}(P^+(\mu_2)\varepsilon\eta, g \circ u).$$

It follows that $\|u\|_{C_\mu} \leq M\|P^+(\mu_2)\varepsilon\eta\|_\alpha = 0$, a contradiction to $P^+(\mu_1)u(0) = \varepsilon\eta$, implying that $\lambda(u) = \lambda$.
It is another consequence of Lemma 3.5 that either $\|u(t)\|_\alpha^{-1}u(t) \to \eta$ or $\|u(t)\|_\alpha^{-1}u(t) \to -\eta$ as $t \to \infty$, so in either case it holds that $\|u(t)\|_\alpha^{-1}P^+(\mu_2)u(t) \to 0$ as $t \to \infty$.
Suppose that $\|u(t)\|_\alpha^{-1}u(t) \to -\eta$ as $t \to \infty$ and let $w : \mathbb{R}^+ \to X^\alpha$ be given by $w(t) := S_{\mu_2}(P^+(\mu_2)u(t))(0) = S_{\mu_2}(u(t))(0)$. We then have

- $w(0) = 0$ since $P^+(\mu_2)u(0) = 0$,
- $\|u(t)\|_\alpha^{-1}\|w(t)\|_\alpha \leq M\|u(t)\|_\alpha^{-1}\|P^+(\mu_2)u(t)\|_\alpha \to 0$ as $t \to \infty$ by the boundedness of S,
- $P^+(\mu_2)w(t) = P^+(\mu_2)u(t)$ for all $t \in \mathbb{R}^+$.

It now follows that $\|u(t)\|_\alpha^{-1}(u(t) - w(t)) \to -\eta$ as $t \to \infty$. By the intermediate value theorem, there exists a $t_0 \in \mathbb{R}^+$ such that

$$(P^+(\mu_1) - P^+(\mu_2))(u(t_0) - w(t_0)) = 0,$$

and so $P^+(\mu_1)u(t_0) = P^+(\mu_1)w(t_0)$.
$v := S_{\mu_2}(P^+(\mu_2)w(t_0)) = S_{\mu_2}(w(t_0))$ is a solution of

$$v = K_{\mu_1}(P^+(\mu_1)w(t_0), g \circ v), \tag{3.13}$$

and it holds that $\sup_{t \in \mathbb{R}^+} \|v(t)\|_\alpha \leq M\|u(t_0)\|_\alpha \leq \rho$. There is another solution of (3.13), namely

$$u_{t_0} = K_{\mu_1}(P^+(\mu_1)w(t_0), g \circ u_{t_0}),$$

where $u_{t_0}(t) := u(t_0 + t)$, $t \in \mathbb{R}^+$, denotes the time-t_0-shifted solution.
It follows that $v = u_{t_0}$, and so $\lambda(u) = \lambda(v) \geq \mu_2 > \lambda$, a contradiction. □

4. Local unstable manifold

PROPOSITION 3.10. *Let $0 < \nu$, $\sigma(L) = \sigma_1 \cup \sigma_2$ with $\sigma_1 := \{\lambda \in \sigma(L) : \lambda < -\nu < 0\}$ and $\sigma_2 := \{\lambda \in \sigma(L) : \lambda > \nu > 0\}$, and σ_1 a finite set.*
Then there exists a submanifold W^u_{loc} of X^α such that

(1) *$v(t) \in W^u_{\mathrm{loc}}$ for all $t \in \mathbb{R}^-$ sufficiently small whenever $v : \mathbb{R}^- \to X^\alpha$ is a mild solution of (3.1) with $v(t) \to 0$ as $t \to -\infty$;*
(2) *the restriction*
$$P : W^u_{\mathrm{loc}} \to P^-(0)W^u_{\mathrm{loc}}$$
of $P^-(0) : X^\alpha \to P^-(0)X^\alpha$ is a homeomorphism (between subspaces of X^α) and its inverse P^{-1} has a continuous Fréchet-derivative DP^{-1};
(3) *$DP^{-1}(0)y = y$ for all $y \in P^-(0)X^\alpha$.*

This is an application of Theorem 71.1 (Saddle Point Property) in [**21**]. Another reference is Theorem 5.1.2 in [**13**]. The differentiability of P^{-1} can be found in [**19**, Theorem 3.3].

5. Convergence as $t \to -\infty$

For large $t \in \mathbb{R}$, $u(-t)$ can be described by an ordinary differential equation in finite dimensions (see [**21**, Theorem 71.1], [**13**, Theorem 5.1.2], [**19**, Theorem 3.3]). We can then reverse the time and obtain analogous statements for $t \to -\infty$.
Let the assumptions on f at the beginning of Section 3.2 hold, and let $u : \mathbb{R}^- \to X^\alpha$ be a solution of (3.1) with $u(t) \to 0 \in X^\alpha$ as $t \to -\infty$.
Set $L := A - Df(0)$ and $g := f - Df(0)$. Then L is a sectorial operator and $u(t)$ is also a solution of
$$\dot{x}(t) + Lx(t) = g(x(t)) \tag{3.14}$$
with $u(t) \to 0$ as $t \to \infty$ and we have
$$\mathcal{L}(X^\alpha, X) \ni Dg(0) = 0. \tag{3.15}$$
By [**13**, Lemma 3.3.2], u is differentiable in all $t > 0$ and $\dot{u} : \,]-\infty, 0[\, \to X^0$ is continuous with $\dot{u}(t) \in X^\alpha$ for all $0 > t \in \mathbb{R}^-$ and $\alpha \in [0, 1[$.
Assume that $u(0) \in W^u_{\mathrm{loc}}$ and let $P = P^-(0)$ denote the projection. Define $\tilde{L}^- \in \mathcal{L}(PX^\alpha, PX^\alpha)$ by
$$\tilde{L}^- x := -PLx \tag{3.16}$$
and
$$\tilde{g}^-(x) := -Pg(P^{-1}(x)). \tag{3.17}$$
Setting $v(t) := Pu(-t)$, $t \in \mathbb{R}^+$, we have for all $t > 0$
$$\begin{aligned}\dot{v}(t) &= -\dot{u}(-t)\\ &= -(-PLPu(-t) + Pg(u(-t))\\ &= -\tilde{L}^- v(t) + \tilde{g}^-(v(t)),\end{aligned} \tag{3.18}$$

that is, $v(t)$ is a solution of the (ordinary) differential equation

$$\dot{v}(t) + \tilde{L}^{-} v(t) = \tilde{g}^{-}(v(t)).$$

Since σ_1 is finite, it is clear that $\lambda(v) < \infty$ (see [**6**, Theorem 13.4.3]).

COROLLARY 3.11. *Let $\rho^{-}(t) := \int_{-\infty}^{t} \|\dot{u}(s)\|_\alpha \, ds$ (see also Lemma 3.7). There is an eigenvector η of L, which belongs to the eigenvalue λ, such that in X^α*

$$(\rho^{-}(t))^{-1} u(t) \to \eta \tag{3.19}$$

$$\dot{\rho}^{-}(t)^{-1} \dot{u}(t) = \|\dot{u}(t)\|_\alpha^{-1} \dot{u}(t) \to \eta \tag{3.20}$$

as $t \to -\infty$.

PROOF. By Lemma 3.5, there exists an eigenvalue $\eta \in X^1$ with $\|\dot{v}(t)\|_\alpha^{-1} \dot{v}(t) \to -\eta$ as $t \to \infty$.
Proposition 3.10 implies that

$$\dot{u}(-t) = -\dot{v}(t) + P^{+}(0)\dot{u}(-t) = -\dot{v}(t) - P^{+}(0) D P^{-1}(u(-t))\dot{v}(t)$$

and so $\|\dot{v}(t)\|_\alpha^{-1}(P^{+}(0)\dot{u}(t)) \to 0$ in X^α as $t \to \infty$.
Hence,

$$\begin{aligned} \|\dot{v}(t)\|_\alpha^{-1} \dot{u}(-t) &= \|\dot{v}(t)\|_\alpha^{-1}(P^{-}(0)\dot{u}(-t) + P^{+}(0)\dot{u}(-t)) \\ &= \|\dot{v}(t)\|_\alpha^{-1}(-\dot{v}(t) + P^{+}(0)\dot{u}(-t)) \to \eta + 0 \end{aligned}$$

and particularly $\|\dot{v}(t)\|_\alpha^{-1} \|\dot{u}(-t)\|_\alpha \to 1$ as $t \to \infty$.
Moreover, we have $\dot{\rho}^{-}(t) = \|\dot{u}(t)\|_\alpha$, which shows (3.20).
Finally, it follows as in the proof of Lemma 3.7 (note the different sign before the integral) that

$$u(t) = \int_{-\infty}^{t} \dot{u}(s) \, ds \quad t \leq 0$$

and consequently $(\rho^{-}(t))^{-1} u(t) \to \eta$. $\square$

PROPOSITION 3.12. *Let $\lambda < 0$ be an eigenvalue of L and let η denote an associated eigenvector with $\|\eta\|_\alpha = 1$. Then there is a mild solution $u : \,]-\infty, 0] \to X^\alpha$ of (3.1) with*

$$\left\| \|u(t)\|_\alpha^{-1} u(t) - \eta \right\|_\alpha \to 0 \text{ as } t \to -\infty. \tag{3.21}$$

PROOF. It follows from Proposition 3.8 that there exists a solution $v(t)$ of (3.18), defined for all $t \in \mathbb{R}^{+}$, such that $\|v(t)\|_\alpha^{-1} v(t) \to \eta$ in X^α. Hence, $u(-t) := P^{-1}(v(t))$ has the desired properties since

$$\|v(t)\|_\alpha^{-1} u(-t) = \|v(t)\|_\alpha^{-1} P^{-1}(v(t)) \to DP^{-1}(0)\eta = \eta \quad \text{in } X^\alpha$$

and thus

$$\|u(t)\|_\alpha^{-1} u(t) = \frac{\|v(t)\|_\alpha}{\|u(t)\|_\alpha} \frac{u(t)}{\|v(t)\|_\alpha} \to \|\eta\|_\alpha^{-1} \eta \quad \text{in } X^\alpha$$

as $t \to \infty$. $\square$

6. A sufficient condition for an exponential decay rate

PROPOSITION 3.13. *Let $\delta > 0$, $B \in C([0,\infty], \mathcal{L}(H,H))$ symmetric with $e^{2\delta t}(B(t)-B(\infty)) \to 0 \in \mathcal{L}(H,H)$ as $t \to \infty$, $(\eta_i)_{i\in\mathbb{N}}$ an orthonormal basis for H which consists of eigenvalues of $L := A_H - B(\infty)$, A_H be symmetric, $(\eta_i)_{i\in\mathbb{N}}$ an orthonormal basis for H which consists of eigenvectors of $L := A_H - B(\infty)$, and let $u : \mathbb{R}^+ \to X^\alpha$ be a mild solution of*

$$\dot{u}(t) + A_H u(t) = B(t)u(t) \tag{3.22}$$

with $\lambda(u) = \infty$.
Then there is an $\varepsilon > 0$ such that $u(t) = 0$ for all $t \in \mathbb{R}^+$ with $\sup_{s\geq t} \|B(s) - B(\infty)\|_{H,H} \leq \varepsilon^2$.

LEMMA 3.14. *Let the assumptions of Proposition 3.13 hold and let $K_\mu \in \mathcal{L}(H \times C_{\mu+\delta}(\mathbb{R}^+, H), C_\mu(\mathbb{R}^+, H))$ be defined as in Lemma 3.3.*
Then K_μ is well-defined and $C_K := \sup_{\mu\in\mathbb{R}^+\setminus\sigma(L)} \|K_\mu\| < \infty$. Moreover, for all $x \in H$ one has $\|P^+(\mu)x\|_H \to 0$ as $\mu \to \infty$.

C_μ is defined as before but with respect to $X = H$ and $\alpha = 0$, that is, the norm on H is considered.

PROOF. For each $i \in \mathbb{N}$, let λ_i denote the eigenvalue associated with η_i. The eigenvalues are (due to the symmetry of $A_H - B(\infty)$) real. We thus have

$$\langle e^{-Lt}x, \eta_i\rangle = e^{-\lambda_i t}\langle x, \eta_i\rangle \quad x \in H \quad i \in \mathbb{N}. \tag{3.23}$$

Every $x \in H$ may now be written as $x = \sum_{i\in\mathbb{N}} \langle x, \eta_i\rangle\eta_i$ and one has

$$\|x\|_H^2 = \sum_{i\in\mathbb{N}} \langle x, \eta_i\rangle^2. \tag{3.24}$$

Since for every $\mu \in \mathbb{R}\setminus\sigma(L)$ the projections $P^-(\mu)$ and $P^+(\mu)$ are the orthogonal projections in H, that is,

$$\begin{aligned} P^-(\mu)x &= \sum_{i\in\mathbb{N}:\ \lambda_i<\mu} \langle x, \eta_i\rangle\eta_i \text{ and} \\ P^+(\mu)x &= \sum_{i\in\mathbb{N}:\ \lambda_i>\mu} \langle x, \eta_i\rangle\eta_i, \end{aligned}$$

it follows that $P^+(\mu)x \to 0$ in $\mathcal{L}(H,H)$ as $\mu \to \infty$.
Furthermore, for every $i \in \mathbb{N}$ we have $e^{-Lt}\eta_i = e^{-\lambda_i t}\eta_i$, which shows that for all $\mu \in \mathbb{R}\setminus\sigma(L)$

$$\begin{aligned} \left\|e^{-Lt}P^-(\mu)x\right\|_H &\leq e^{-\mu t}\|x\|_H && t \leq 0 \\ \left\|e^{-Lt}P^+(\mu)x\right\|_H &\leq e^{-\mu t}\|x\|_H && t > 0. \end{aligned}$$

It follows that K_μ is well-defined and

$$\|K_\mu(x_0, f)\|_{C_\mu} \leq \|x_0\|_H + 2\int_0^\infty e^{-\delta s}\, ds \|f\|_{C_{\mu+\delta}}.$$

□

PROOF OF PROPOSITION 3.13. Let C_K be given by Lemma 3.14, suppose that

$$\|B(t) - B(\infty)\|_{H,H} \leq e^{-2\delta t}M,$$

and choose $\varepsilon := \frac{1}{2C_K}$ and $\tau \in \mathbb{R}^+$ such that

$$\|B(t) - B(\infty)\|_{H,H} \leq \frac{\varepsilon^2}{M}$$

for all $t \geq \tau$. We now have

$$\|B(t) - B(\infty)\|_{H,H}^2 \leq \varepsilon^2 e^{-2\delta t}.$$

Let $0 < \mu \in \mathbb{R}^+ \setminus \sigma(L)$ be arbitrary. $v := u(\tau + t)$ is a mild solution of

$$\dot{x} + Lx = \tilde{B}(t)x := (B(t+\tau) - B(\infty))x \tag{3.25}$$

with $\lambda(u) = \lambda(v) = \infty$.
It follows from Lemma 3.9 that $v = K_\mu(P^+(\mu)v(0), \hat{B}v)$, where we set $(\hat{B}u)(t) := \tilde{B}(t)u(t)$. We thus have

$$\begin{aligned}
\|v\|_{C_\mu} &\leq C_K \left(\left\|P^+(\mu)v(0)\right\|_H + \left\|\hat{B}v\right\|_{C_{\mu+\delta}} \right) \\
&\leq C_K \left\|P^+(\mu)v(0)\right\|_H + C_K \varepsilon \|v\|_{C_\mu} \\
&\leq C_K \left\|P^+(\mu)v(0)\right\|_H + \frac{1}{2}\|v\|_{C_\mu}
\end{aligned}$$

and consequently

$$\|v\|_{C_\mu} \leq 2C_K \left\|P^+(\mu)v(0)\right\|_H.$$

This estimate holds for arbitrary $\mu \in \mathbb{R} \setminus \sigma(L)$, that is,

$$\|u(\tau)\|_H \leq 2C_K \left\|P^+(\mu)u(\tau)\right\|_H \to 0 \text{ as } \mu \to \infty,$$

proving that $u(\tau) = 0$. $\square$

CHAPTER 4

Construction of the diffeomorphism

Recall the assumptions at the beginning of chapter 3. We consider the semiflow induced by mild solutions of

$$\dot{u}(t) + Au(t) = f(u(t)). \tag{4.1}$$

In particular, we assume that $f \in C^1(\mathcal{U}, X)$, where $\mathcal{U}$ is an open set in X^α. Fix an eigenvalue $\eta \in X^1$ of A, let $F := \operatorname{span}\{\eta\}$, and let $E \subset X$ be another subspace with $X = F \oplus E$. For $\alpha \in [0,1]$, let $E^\alpha := E \cap X^\alpha$ be endowed with $\|.\|_\alpha$.
Using $L := A$, let the projections P^- and P^+ be defined as in chapter 3.
Suppose that u is a heteroclinic full solution and $\bar{u} := \operatorname{cl}\{u(t) : t \in \mathbb{R}\}$ is an isolated invariant set. In order to calculate its homotopy index it is helpful to assume that $\bar{u}$ lies entirely in a one-dimensional subspace of the considered phase space X^α. Therefore, we construct a diffeomorphism which maps the image of $\bar{u}$ into a one-dimensional subspace.
There is a simple "prototypical" situation where the construction is obvious, namely if one assumes that u has a "main direction", that is, there is a one-dimensional subspace and an associated projection such that the image of $\dot{u}$ under this projection does not vanish for any $t \in \mathbb{R}$. In this case, one could consider a mapping $(t, e) \mapsto u(t) + e$, $e \in E$, where E denotes the complementary subspace. The following theorem is a generalization of this basic idea.
Obviously, the smoothness of such a mapping is – at least in the direction of t – limited by the smoothness of u. There are other problems which have not been considered in this informal introduction: the diffeomorphism should be defined in a neighborhood of $\bar{u}$ and the semiflow obtained by applying the diffeomorphism should still be induced by mild solutions of a semilinear parabolic equation like (4.1).
Theorems of this kind are often referred to as *tubular neighborhood theorems*, but (as far as known to the author) they are either stated in a finite-dimensional setting or they require more smoothness than C^1 and would thus impose additional restrictions on the non-linearity f in (4.1).

THEOREM 4.1. *Let $\gamma \in C^1([0,1], X^0)$ such that $0 \neq \dot{\gamma}(t)$ for all $t \in [0,1]$, $\Xi \subset [0,1]$ be finite, and $\xi \in [0,1]$ with $\dot{\gamma}(\xi) \notin E$.*
Then there exist a neighborhood U of $[0,1] \times \{0\}$ in $[0,1] \times E$ and a diffeomorphism $\varphi :\ U \to \varphi(U) \subset X$ such that

(1) *there exists a $\mu \in \mathbb{R} \setminus \operatorname{Re}(\sigma(A))$ such that $\varphi(x,y) = \gamma(x) + \Phi(x)P^-(\mu)y + P^+(\mu)y$, where $\Phi :\ [0,1] \to \mathcal{L}(P^-(\mu)E, P^-(\mu)X)$ is continuous;*
(2) $\Phi(\xi) = \mathrm{id}$*;*
(3) $\Phi(x)$ *is locally constant in a neighborhood of* Ξ*.*
(4) *for all (x_0, y_0) in U there are continuous (Fréchet-)derivatives $D_y D\varphi(x_0, y_0)$ and $D_y(D\varphi(x_0, y_0)^{-1})$.*

LEMMA 4.2. *Let the assumptions of Theorem 4.1 hold. Then there exists a* $\mu \in \mathbb{R} \setminus \sigma(A)$ *with* $P^-(\mu)\dot\gamma(t) \neq 0$ *for all* $t \in [0,1]$.

PROOF. It follows from Lemma 3.4 that for every $t \in [0,1]$ there is a $\mu_t \in \mathbb{R} \setminus \sigma(A)$ with $P^-(\mu_t)\dot\gamma(t) \neq 0$. The continuity of $\dot\gamma(t)$ implies that there is an open neighborhood U_t of t such that $P^-(\mu_t)\dot\gamma(s) \neq 0$ for all $s \in U_t$. $\{U_t\}_{t\in[0,1]}$ is an open covering of $[0,1]$, hence by compactness, there is a finite subcovering $\{U_{t_k}\}_{k\in\{1,\ldots,n\}}$, $n \in \mathbb{N}$. Let $\mu := \max\{\mu_{t_k} : k \in \{1,\ldots,n\}\}$. We then have for all $k \in \{1,\ldots,n\}$ and all $s \in U_{t_k}$ $P^-(\mu_t)P^-(\mu)\dot\gamma(s) = P^-(\mu_t)\dot\gamma(s) \neq 0$ so that $P^-(\mu)\dot\gamma(s) \neq 0$ for all $s \in [0,1]$. □

LEMMA 4.3. *Let* $k \in \mathbb{N}$, $\Xi \subset [0,1]$ *finite,* $\xi \in \Xi$, *and* $\Phi \in C([0,1], \mathrm{ISO}(\mathbb{R}^k,\mathbb{R}^k)) \cap C^1([0,1], \mathcal{L}(\mathbb{R}^k,\mathbb{R}^k))$. *Then there is a sequence* $\Phi_n \in C([0,1], \mathrm{ISO}(\mathbb{R}^k,\mathbb{R}^k)) \cap C^1([0,1], \mathcal{L}(\mathbb{R}^k,\mathbb{R}^k))$ *such that*

(1) $\Phi_n \to \Phi$ *in* $C([0,1], \mathrm{ISO}(\mathbb{R}^k,\mathbb{R}^k))$*;*
(2) Φ_n *is locally constant in a neighborhood of* Ξ*;*
(3) $\Phi_n(\xi) = \Phi(\xi)$ *for all* $n \in \mathbb{N}$.

PROOF. Using the differentiability of Φ, we can write

$$\Phi(x) = \Phi(\xi) + \int_\xi^x F(s)\,ds$$

with $F \in C([0,1], \mathcal{L}(\mathbb{R}^k,\mathbb{R}^k))$.
Define F_n in $L^\infty([0,1], \mathcal{L}(\mathbb{R}^k,\mathbb{R}^k))$ by

$$F_n(x) := \begin{cases} 0 & \mathrm{dist}(x,\Xi) \leq \frac{1}{2n} \\ F(x) & \text{otherwise.} \end{cases}$$

It follows that F_n is well-defined and that $\|F_n - F\|_\infty \leq \|F\|_\infty < \infty$.
Finally, choose $\tilde F_n \in C([0,1], \mathrm{ISO}(\mathbb{R}^k,\mathbb{R}^k)) \cap C^1([0,1], \mathcal{L}(\mathbb{R}^k,\mathbb{R}^k))$ with $\left\|\tilde F_n - F_n\right\|_\infty \leq 1/n$, and let Φ_n be defined by

$$\Phi_n(x) = \Phi(\xi) + \int_\xi^x F_n(s)\,ds.$$

We have

$$\|\Phi_n - \Phi\|_\infty \leq \|F\|_\infty \frac{\#\Xi}{n} + \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Φ_n is an isomorphism for all n sufficiently large. □

PROOF OF THEOREM 4.1. Let μ_0 be given by Lemma 4.2, $\mu_0 \leq \mu \in \mathbb{R} \setminus \sigma(A)$, $P := P^-(\mu)$, and $E_0 := PX \subset X^1$ ($\dim E_0 < \infty$). By choosing μ large enough, we can assume that $\eta \in E_0$ (η is the eigenvector defining F).
$P\dot\gamma : [0,1] \to E_0$ induces a monomorphism $U : [0,1] \times F \to [0,1] \times E_0$ of bundles in the sense of Appendix A, where $U(t)(r\eta) := rP\dot\gamma(t)$. By the assumption that $P\eta = \eta$, one has $E_0 = F \oplus PE$. By Corollary A.11, there exists an isomorphism $\Phi_0 = (U \oplus S_0) \in C([0,1], \mathcal{L}(E_0, E_0))$ such that $S_0(\xi)y = y$ and $\Phi_0(t)\eta = P\dot\gamma(t)$ for all $t \in [0,1]$.
By the Weierstrass approximation theorem, there is another sequence $(\Phi_n = (U \oplus S_n))_{n\in\mathbb{N}}$ in $C([0,1], \mathcal{L}(E_0, E_0)$ such that for each $n \in \mathbb{N}$, S_n is continuously Fréchet-differentiable, $S_n(\xi) = \mathrm{id}$, and $\Phi_n \to \Phi_0$ uniformly in t with respect to the norm in $\mathcal{L}(E_0, E_0)$. Using

Lemma 4.3, we can assume that Φ_n is locally constant in a neighborhood (depending on n) of Ξ for all $n \in \mathbb{N}$.
Let $t \in [0,1]$ and define $H_{n,t}$ by

$$\begin{aligned}\Phi_0(t)^{-1}\Phi_n(t) &= \Phi_0(t)^{-1}(\Phi_0(t) + (\Phi_n(t) - \Phi_0(t)))\\ &= 1 + \Phi_0(t)^{-1}(\Phi_n(t) - \Phi_0(t))\\ &= 1 + H_{n,t}.\end{aligned}$$

Using the Neumann series, there exists an inverse of $\Phi_0(t)^{-1}\Phi_n(t)$ whenever $\|H_{n,t}\| < 1$. We have

$$\|H_{n,t}\| \leq \left\|\Phi_0^{-1}(t)\right\| \|\Phi_n(t) - \Phi_0(t)\| \leq \sup_{t\in[0,1]} \left\|\Phi_0^{-1}(t)\right\| \sup_{t\in[0,1]} \|\Phi_n(t) - \Phi_0(t)\|$$

for all $t \in [0,1]$, where $\sup_{t\in[0,1]} \left\|\Phi_0^{-1}(t)\right\| < \infty$ by Corollary A.4 and $\sup_{t\in[0,1]} \|\Phi_n(t) - \Phi_0(t)\| \to 0$ as $n \to \infty$ by the uniform approximation. Hence, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $t \in [0,1]$, $\Phi_n(t) = (\Phi_0 \circ \Phi_0^{-1} \circ \Phi_n)(t)$ is an isomorphism in $\mathcal{L}(E_0, E_0)$, and particularly a homeomorphism by Corollary A.4.
Let $\Phi := \Phi_{n_0}$ and define $\varphi :\ [0,1] \times E \to X$ by

$$\varphi(t,y) := \gamma(t) + \Phi(t)Py + (1-P)y.$$

Let $(t_0, y_0) \in [0,1] \times E$, let $(t,y) \in \mathbb{R} \times E$ and let $h \in \mathbb{R}^+$. We have for h small enough

$$\begin{aligned}&\frac{1}{h}\left(\varphi(t_0 + ht, y_0 + hy) - \varphi(t_0, y_0)\right)\\ &= \frac{1}{h}\left(\gamma(t_0 + ht) - \gamma(t_0) + \Phi(t_0 + ht)(Py_0 + hPy) - \Phi(t_0)Py_0\right) + (1-P)y\\ &= \frac{1}{h}\left(\gamma(t_0 + ht) - \gamma(t_0) + \Phi(t_0 + ht)hPy + \Phi(t_0 + ht)Py_0 - \Phi(t_0)Py_0\right) + (1-P)y\\ &\to t\dot{\gamma}(t_0) + \Phi(t_0)Py + (D\Phi(t_0)t)Py_0 + (1-P)y \text{ as } h \to 0.\end{aligned}$$

In particular, $(t_0, y_0) \mapsto (D\Phi(t_0)1)Py_0$ is continuous, so there is a continuous Fréchet derivative, namely

$$D\varphi(t_0, y_0)(t,y) = (D\gamma(t_0) + (D\Phi(t_0)1)Py_0)t + \Phi(t_0)Py + (1-P)y. \tag{4.2}$$

We have for $(t,y) \in \mathbb{R} \times E$ and $t_0 \in [0,1]$

$$\begin{aligned}PD\varphi(t_0,0)(t,y) &= P\dot{\gamma}(t_0)t + P\Phi(t_0)Py\\ &= \Phi(t_0)(\eta t + Py),\end{aligned}$$

showing that $PD\varphi(t_0, 0) :\ \mathbb{R} \times PE \to PX = E_0$ is an isomorphism for all $t_0 \in [0,1]$. Therefore, it follows that

$$D\varphi(t_0,0)(t,y) = P\Phi(t_0)(\eta t + Py) + (1-P)(\dot{\gamma}(t_0)t + y)$$

is an isomorphism, the inverse is given by

$$\begin{aligned}(t, y_1) &= (PD\varphi(t_0,0))^{-1}Py\\ D\varphi(t_0,0)^{-1}y &= (t, y_1 + (1-P)(y - \dot{\gamma}(t_0)t)).\end{aligned}$$

The inverse mapping theorem now implies that φ is a local diffeomorphism.

Suppose that there does not exist an open neighborhood U of $[0,1]\times\{0\}$ in $[0,1]\times E$ such that $\varphi_{|U}$ is injective. Then there are sequences $(t_n,y_n)\to(t_0,0)$ in $[0,1]\times E$ and $(\tilde{t}_n,\tilde{y}_n)\to(\tilde{t}_0,0)$ in $[0,1]\times E$ (by the compactness of $[0,1]$) such that $t_n\neq\tilde{t}_n$ and $\varphi(t_n,y_n)=\varphi(\tilde{t}_n,\tilde{y}_n)$ for all $n\in\mathbb{N}$. It follows from the continuity of φ that $\gamma(t_0)=\varphi(t_0,0)=\varphi(\tilde{t}_0,0)=\gamma(\tilde{t}_0)$ and since γ is injective, we have $t_0=\tilde{t}_0$. This is a contradiction since φ is a local homeomorphism. We have shown that there exists an open neighborhood U of $[0,1]\times E$ such that $\varphi_{|U}:\ U\to\varphi(U)$ is a homeomorphism.
Finally, we have $D_yD\varphi(x_0,y_0)y=(D\Phi(x_0)1)Py\cdot p_x$, where $p_x:\ \mathbb{R}\times E\to\mathbb{R}$, $p_x(x,y)=x$ for all $(x,y)\in\mathbb{R}\times E$. Hence $D_yD\varphi(x_0,y_0)^{-1}$ exist and is given by

$$D_yD\varphi(x_0,y_0)^{-1}y=-D\varphi(x_0,y_0)^{-1}\circ D_yD\varphi(x_0,y_0)y\circ D\varphi(x_0,y_0)^{-1}.$$

□

PROPOSITION 4.4. *Let u be a solution of (4.1) with $u(t)\to 0=:e^+$, $u(-t)\to e^-$, and $\|u(t)\|_\alpha^{-1}u(t)\to\eta$ as $t\to\infty$.*
Then there exist an open neighborhood $U\subset[0,1]\times E^\alpha$ of $[0,1]\times\{0\}$, a neighborhood V of $K:=\operatorname{cl}\{u(t):\ t\in\mathbb{R}\}$, and a diffeomorphism $\varphi:U\to V$ such that

(1) *$\varphi(x,y)=\gamma(x)+\Phi(x)P^-(\mu)y+P^+(\mu)y$, where $\mu\in\mathbb{R}\backslash\operatorname{Re}(\sigma(A))$, $\gamma\in C^1([0,1],X^\alpha)$, and $\Phi\in C([0,1],\mathcal{L}(P^-(\mu)E,P^-(\mu)X)$ is locally constant in a neighborhood of $\varphi^{-1}(\{e^-,e^+\})$;*
(2) *$\Phi(\gamma^{-1}(e^+))=\mathrm{id}_E$;*
(3) *for all (x_0,y_0) in U there are continuous (Fréchet-)derivatives $D_yD\varphi(x_0,y_0)$ and $D_y(D\varphi(x_0,y_0)^{-1})$;*
(4) *$\varphi(\mathbb{R}\times\{0\}\cap U)$ is invariant under the restriction of π to V and we have $K\subset\gamma(]0,1[)$;*
(5) *$x\mapsto A\varphi(x,0)$ is continuous.*

LEMMA 4.5. *Let $u,v^+:\mathbb{R}\to X^\alpha$ be given by Theorem 3.2. Then there is a closed neighborhood $[a,b]$ of 0 and a homeomorphism $p^+:\ [a,b]\to\{u(t):\ t\in[0,\infty[\}\cup\{v^+(t):\ t\in[0,\infty[\}\cup\{e^+\}\subset X^\alpha$ such that*

(1) *$p^+\in C^1([a,b],X^\alpha)$;*
(2) *$\dot{p}^+(t)\neq 0$ for all $t\in[a,b]$;*
(3) *$(p^+,\dot{p}^+)(a)=(u(0),\|\dot{u}(0)\|_\alpha^{-1}\dot{u}(0))$, $p^+(0)=e^+$, $p^+(b)=v^+(0)$;*
(4) *Ap^+ is continuous.*

PROOF. Let λ^+, η^+ and $\rho(t):=\rho(u,t)$ be given by Theorem 3.2. Let $\rho^{-1}(u,.)$ denote the inverse mapping, that is,

$$\int_{\rho^{-1}(u,t)}^{\infty}\|\dot{u}(s)\|_\alpha\,ds=t.$$

Define further

$$p^+(t):=\begin{cases}u(\rho^{-1}(u,-t)) & t\in[-\rho(u,0),0[\\ e^+ & t=0\\ v^+(\rho^{-1}(v^+,t)) & t\in\,]0,\rho(v^+,0)]\,.\end{cases}$$

We now have

$$\begin{aligned}
\dot p^+(t) &= -\left(\frac{d}{dt}\rho^{-1}(-t)\right)\dot u(\rho^{-1}(-t)) \\
&= -\frac{1}{\dot\rho(\rho^{-1}(-t))}\dot u(\rho^{-1}(-t)) \\
&= \frac{1}{\|\dot u(\rho^{-1}(-t))\|_\alpha}\dot u(\rho^{-1}(-t))
\end{aligned}$$

and substituting $t = \rho(s)$ one obtains $\dot p^+(\rho(s)) = -\|\dot u(s)\|_\alpha^{-1}\dot u(s)$ for all $s \in [0,\infty[$. We have $\dot p^+(t) \to \eta^+$ as $t \to 0$ and $Ap^+(\rho(t)) = Au(t) = f(u(t)) - \dot u(t)$. The last term is continuous in t and it holds that $f(u(t)) - \dot u(t) \to f(e^+) = Ap^+(0)$ as $t \to \infty$.
The second branch of p^+, that is, the case $t > 0$, can be treated analogously. □

There is an equivalent for negative times to the previous lemma; its proof is omitted.

LEMMA 4.6. *Let $u, v^- : \mathbb{R} \to X^\alpha$ be given by Theorem 3.2. Then there is a closed neighborhood $[a,b]$ of 0 and a homeomorphism $p^- : [a,b] \to \{u(t) : t \in\,]-\infty,0]\} \cup \{v^-(t) : t \in\,]-\infty,0]\} \cup \{e^-\} \subset X^\alpha$ such that*

(1) *$p^- \in C^1([a,b],X^\alpha)$;*
(2) *$\dot p^-(t) \neq 0$ for all $t \in [a,b]$;*
(3) *$(p^-,\dot p^-)(a) = (u(0), \|\dot u(0)\|_\alpha^{-1}\dot u(0))$, $p^-(0) = e^-$, $p^-(b) = v^-(0)$;*
(4) *Ap^- is continuous.*

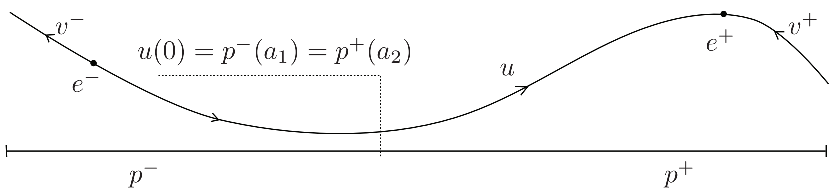

FIGURE 4.1. Construction of γ

PROOF OF PROPOSITION 4.4. Let $p^- : [a_1,b_1] \to X^\alpha$ and $p^+ : [a_2,b_2] \to X^\alpha$ be given by Lemma 4.5 and Lemma 4.6 and let

$$\tilde\gamma(x) := \begin{cases} p^-(a_1 - x) & x \in [a_1 - b_1, 0] \\ p^+(a_2 + x) & x \in [0, b_2 - a_2]. \end{cases}$$

In view of Lemma 4.5 and Lemma 4.6, we have particularly $p^-(a_1 - 0) = p^+(a_2 + 0) = u(0)$ and $\dot p^-(a_1) = \dot p^+(a_2) = \|\dot u(0)\|_\alpha^{-1}\dot u(0)$. Therefore, $\gamma : [0,1] \to X^\alpha$,

$$\gamma(t) := \tilde\gamma(t(b_2 - a_2) + (1-t)(a_1 - b_1)),$$

is well-defined and continuously differentiable.
Since $\|u(t)\|_\alpha^{-1}u(t) \to \eta$ as $t \to \infty$, it is clear that $\dot\gamma(\gamma^{-1}(e^+)) = \eta$. Hence, we can apply Theorem 4.1 to γ and obtain a mapping φ for which (1), (2), and (3) hold.
(4) and (5) follow from the choice of γ (see also Figure 4) and the two lemmas: Lemma 4.5 and Lemma 4.6. □

CHAPTER 5

Isolation and homotopy equivalence

For a hyperbolic equilibrium (a stationary solution), it is a usual technique to compute its homology index by computing the homology index of its linearization. Given two equilibria and an orbit connecting them, the assumption that the respective stable and unstable manifolds intersect transversally is a substitute for the hyperbolicity assumption in the zero-dimensional case of a single equilibrium. However, it is not immediately clear what linearization shall mean. Simply passing to the tangential space is not possible, since it is a one-dimensional subbundle of the full tangential space, which corresponds to the given orbit, that is, given a heteroclinic solution u of a differentiable semiflow π, the pair $(u, \lambda\dot{u})$ is a solution of $T\pi$ (using Definition 5.2) for every $\lambda \in \mathbb{R}$. Hence, $K := \mathrm{cl}\{(u(t), 0) : t \in \mathbb{R}\}$ is not an isolated invariant set, and $\tilde{K} := \mathrm{cl}\{(u(t), \lambda\dot{u}(t)) : \lambda, t \in \mathbb{R}\}$ is not compact, which means that there does not exist a $T\pi$-admissible isolating neighborhood of $\tilde{K}$.

1. Linear skew product semiflows

Sell and You use in **[21]** the notion of *linear skew product semiflows*. We will borrow the concept since it is a suitable abstraction for our Conley index calculations.

Definition 5.1. Let F be a Banach space and let $a < b$ be real numbers. A *linear skew product semiflow* on $(]a, b[, F)$ is a semiflow $\pi = (\xi, \Phi)$ on $]a, b[\times F$, where

$$(x, y)\pi t = (x\xi t, \Phi(x, t)y) \quad \forall (t, x, y) \in D(\pi).$$

Here, ξ is a flow on $]a, b[$ and for every $(x, t) \in D(\xi)$ we have $\Phi(x, t) \in \mathcal{L}(F, F)$.
Let $\mathrm{SK}(]a, b[, F)$ denote the set of all linear skew product semiflows on $(]a, b[, F)$ and let $\pi \in \mathrm{SK} := \mathrm{SK}([a, b], F) \subset \mathrm{SK}(]a, b[, F)$ if there exists an $\varepsilon > 0$ and a $\tilde{\pi} \in \mathrm{SK}(]a-\varepsilon, b+\varepsilon[, F)$ with $(x, y)\pi t = (x, y)\tilde{\pi} t$ whenever the left side is defined.
Given a decomposition $F = F_1 \oplus F_2$ into closed subspaces and semiflows $\pi_1 = (\xi, \Phi_1) \in \mathrm{SK}([a, b], F_1)$, $\pi_2 = (\xi, \Phi_2) \in \mathrm{SK}(\xi, \Phi_2)$, define $\pi_1 \oplus \pi_2 \in \mathrm{SK}([a, b], E)$ by $\pi_1 \oplus \pi_2 = (\xi, \Phi_1 \oplus \Phi_2)$, where $(\Phi_1 \oplus \Phi_2)(t, x)(y_1 \oplus y_2) = \Phi_1(t, x)y_1 \oplus \Phi_2(t, x)y_2$.

Let M be an open subset of a Banach space F and let $\gamma : [0, 1] \to \Gamma$ be a diffeomorphism. One may regard TM as $M \times F$, and $[0, 1] \times F$ is diffeomorphic to $\Gamma \times X^\alpha$. $U : [0, 1] \times \mathbb{R} \to TM$, $U(x)y := \dot{\gamma}(x) \cdot y$ is a subbundle in the sense of Appendix A. In particular, it follows from Corollary A.11 that $TM/T\Gamma$ is a metric space (the definition according to Appendix A and the definition below conincide).

Definition 5.2. Let M be an open subset of a Banach space F and let Γ be a C^1-submanifold of M. For $x \in \Gamma$ define

$$T_xM/T_x\Gamma := \{\{\eta + \eta' : \eta' \in T_x\Gamma\} : \eta \in T_xM\}$$

and

$$TM/T\Gamma := \{(x,\eta) : \ x \in \Gamma \text{ and } \eta \in T_xM/T_x\Gamma\}.$$

Let π be a C^1-semiflow on M and let Γ be invariant under π. Then π induces a natural semiflow $T\pi$ on TM which is defined by

$$T\pi(t,(x,\eta)) := (x\pi t, D(\pi(t,.))(x)\eta)\,.$$

By the linearized semiflow $\pi'(\Gamma)$ along Γ we mean the linear skew product semiflow on $TM/T\Gamma$ which is defined by

$$\pi'(t,(x,\eta)) := p\,(T\pi(t,(x,\eta)))$$

where $p : \{(x,\eta) \in TM : \ x \in \Gamma\} =: TM(\Gamma) \to TM/T\Gamma$ denotes the canonical projection that is, $p(x,\eta) = (x,[\eta])$.
Let $TM/T\Gamma$ be equipped with the quotient topology and let each fiber be equipped with the norm $\|[y]\|_\xi := \|[y]\|_{T_\xi M/T_\xi\Gamma} := \inf\{\|y-y'\| : \ y' \in T\Gamma\}$, $\xi \in \Gamma$.

Lemma 5.3. *Let M and Γ satisfy the assumptions of Definition 5.2. Then $\pi' := \pi'(\Gamma)$ is a semiflow.*

Proof. First, one has to show that π' is well-defined. Since $T\pi$ is a linear skew product semiflow, it may be decomposed into its components: let $T\pi = (\xi,\Phi)$. Now, let $y_1, y_2 \in F$ with $[y_1]_{F/T_x\Gamma} = [y_2]_{F/T_x\Gamma}$, let $x \in [a,b]$ and let $t \in \mathbb{R}^+$ such that $\Phi(t,x)$ is defined. We then have $y_1 - y_2 \in T\Gamma$ so that $\Phi(t,x)y_1 - \Phi(t,x)y_2 = \Phi(t,x)(y_1-y_2) \in T\Gamma$ due to the invariance of $T\Gamma$, implying that $[\Phi(t,x)y_1]_{F/T_{x\xi t}\Gamma} = [\Phi(t,x)y_2]_{F/T_{x\xi t}\Gamma}$.
Now, π' inherits its properties from $T\pi$. In particular, it is continuous due to the choice of the quotient topology and

$$\begin{aligned}
[x,y]\pi'(t_1+t_2) &= [(x\xi(t_1+t_2), \Phi(x,t_1+t_2)y] \\
&= [(x\xi t_1)\xi t_2, \Phi(x\xi t_1,t_2)\Phi(x,t_1)y] \\
&= ([x,y]\pi' t_1)\pi' t_2 \quad (t_1+t_2,(x,y)) \in \mathcal{D}(T\pi).
\end{aligned}$$

□

Lemma 5.4. *Let M and Γ satisfy the assumptions of Definition 5.2 and $\dim\Gamma = n \in \mathbb{N}$. Further, let $T\pi = (\xi,\Phi)$ and suppose that $\Phi(t,x)y \neq 0$ for all $(t,x) \in \mathcal{D}(\Phi)$ and all $0 \neq y \in T_x\Gamma$.*
Finally, let $[u(t),v(t)]$ be a solution of π' which is defined for all $t \in [-t_0,0]$. Then there is a unique solution $(u(t),\tilde{v}(t))$ of $T\pi$ satisfying $[u(t),v(t)] = [u(t),\tilde{v}(t)]$ and $v(0) = \tilde{v}(0)$.

Proof. We have $[\Phi(t_0,u(-t_0))v(-t_0)] = [v(0)]$, so there is a solution $(u(t),w(t))$ of $T\pi$ with $w(t) - v(t) \in T_{u(0)}\Gamma$ for all $t \in [-t_0,0]$.
Moreover, the restriction $\Phi(t_0,u(-t_0)) : \ T_{u(-t_0)}\Gamma \to T_{u(0)}\Gamma$ is an isomorphism since it is injective, and so there exists a unique $\eta \in T_{u(-t_0)}\Gamma$ with $\Phi(t_0,u(-t_0))\eta = v(0) - w(0)$.
The linearity of $\Phi(t_0,u(-t_0))$ now implies that $\Phi(t_0,u(-t_0))(v(-t_0)+\eta) = w(0)+\Phi(t_0,u(-t_0))\eta = v(0)$. By the invariance of $T\Gamma$, we have $[u(t),\tilde{v}(t)]\pi' t = [u(t),w(t)]$ for all $t \in [-t_0,0]$, where we set $\tilde{v}(t) := w(t) + \Phi(t+t_0)\eta$. □

Definition 5.5. Let M and π satisfy the assumptions of Definition 5.2.

For every $x \in \Gamma$ let $y \in B^-(T\pi, x)$ iff there is a solution $(u,v):\ \mathbb{R}^- \to TM$ of $T\pi$ such that $(u(0), v(0)) = (x, y)$ and $\sup_{t\in\mathbb{R}^-} \|v(t)\| < \infty$; and let $y \in B^+(T\pi, x)$ iff there exists a solution $(u,v):\ \mathbb{R}^+ \to TM$ of $T\pi$ with $(u(0), v(0)) = (x, y)$ and $\sup_{t\in\mathbb{R}^+} \|v(t)\| < \infty$.

The above notion of a bounded solution can be translated to $TM/T\Gamma$:

DEFINITION 5.6. Let M, Γ, and π satisfy the assumptions of Definition 5.2.
For every $x \in \Gamma$ let $y \in B^-(\pi', x)$ iff there is a solution $(u,v):\ \mathbb{R}^- \to TM/T\Gamma$ of π' such that $(u(0), v(0)) = (x, y)$ and $\sup_{t\in\mathbb{R}^-} \|v(t)\|_{T_{u(t)}M/T_{u(t)}\Gamma} < \infty$; and let $y \in B^+(\pi', x)$ iff there exists a solution $(u,v):\ \mathbb{R}^+ \to TM/T\Gamma$ of π' with $(u(0), v(0)) = (x, y)$ and $\sup_{t\in\mathbb{R}^+} \|v(t)\|_{T_{u(t)}M/T_{u(t)}\Gamma} < \infty$.

The transversal intersection of the respective stable and unstable manifolds (or weaker, of the respective local stable manifold and the unstable manifold) has one implication concerning $T\pi$ which is crucial (and sufficient) for the following linearization procedure, namely:

DEFINITION 5.7. Let M be an open subset of a Banach space F, and let π be a semiflow on M. Let $e^+, e^- \in M$ be hyperbolic equilibria, and let $u(t)$ be a heteroclinic solution with $u(t) \to e^-$ as $t \to -\infty$ and $u(t) \to e^+$ as $t \to \infty$ (not necessarily $e^- \neq e^+$).
u is said to be *normal* if for all $t \in \mathbb{R}$

$$\dim(B^-(T\pi, u(t)) \cap B^+(T\pi, u(t))) = 1. \tag{5.1}$$

2. Isolation

Recall the assumptions we made in chapter 3. In particular, the semiflow π is induced by mild solutions of

$$\dot u(t) + Au(t) = f(u(t)), \tag{5.2}$$

where $f \in C^1(\mathcal{U}, X)$, $\mathcal{U} \subset X^\alpha$ is open, and A has compact resolvent. We will use $F = X^\alpha$ and $T\pi$ is the semiflow induced by mild solutions of

$$\begin{aligned} \dot u(t) + Au(t) &= f(u(t)) \\ \dot v(t) + Av(t) &= Df(u(t))v(t). \end{aligned}$$

Let $u(t)$ be a solution of (5.2) such that Proposition 4.4 can be applied, and let $\varphi:\ U \to V$ and E be given by that proposition. Then the assumptions in Definition 5.2 are satisfied for $F = X^\alpha$, $M = V$, and $\Gamma = \varphi(]0,1[\times \{0\})$.
If the equilibria e^-, e^+ are hyperbolic,

$$B^+(T\pi, u(t)) + B^-(T\pi, u(t)) = T_{u(t)}M \text{ for all } t \in \mathbb{R}, \text{ and} \tag{5.3}$$

$$\dim B^-(T\pi, u(t)) = \operatorname{codim} B^+(T\pi, u(t)) + 1, \tag{5.4}$$

then (5.1) holds.
We are now in a position to state the main result of this section. Recall that $K := \mathrm{cl}\{u(t) : t \in \mathbb{R}\}$.

PROPOSITION 5.8. *Suppose that u is normal. $K_0 := [K \times \{0\}]_{TM/T\Gamma}$ is an isolated invariant set relative to π', that is, there exists an isolating neighborhood N of K_0 in $TM/T\Gamma$.*

The proof of Proposition 5.8 relies on

LEMMA 5.9. *The following holds for all $x_0 \in K_0$*

$$B^+(\pi', x_0) \subset [B^+(T\pi, x_0)]$$
$$B^-(\pi', x_0) \subset [B^-(T\pi, x_0)]$$

PROOF. We can assume w.l.o.g. that $u(0) = x_0$ or $x_0 \in \{e^-, e^+\}$.
Let $[y] \in B^+(\pi', x_0)$ and let $(u, v):\ \mathbb{R}^+ \to TM$ be a solution of $T\pi$ with $v(0) = y$.
We have $u(t) \to e \in \{e^+, e^-\}$ and $\|u(t)\|_\alpha^{-1} u(t) \to \eta$ as $t \to \infty$, where η is an eigenvector of $L := A - Df(e)$. Let $0 < \lambda$ be the associated eigenvalue, and let P_η denote the projection onto the eigenspace spanned by η that is, $P_\eta = \lim_{0 \neq \delta \to 0} P^-(\lambda + \delta) - P^-(\lambda - \delta)$.
By Lemma A.12, there exists a neighborhood V_0 of e in Γ such that for all $x \in V_0$ the canonical projection $Q(x):\ (1 - P_\eta)X^\alpha \to X^\alpha / T_x\Gamma$ lies in $\mathrm{ISO}((1 - P_\eta)X^\alpha, X^\alpha / T_x\Gamma)$ and there are constants $0 \neq m, M \in \mathbb{R}^+$ such that

$$m\|x\|_\alpha \leq \|x\|_{X^\alpha / T_x\Gamma} \leq M\|x\|_\alpha \quad \forall x \in V_0. \tag{5.5}$$

Let $t_0 \in \mathbb{R}^+$ such that $u(t) \in V_0$ for all $t \geq t_0$, and set $w(t) := Q^{-1}(u(t))([v(t)])$, $t \geq t_0$. Since $\sup_{t \in \mathbb{R}^+} \|[v(t)]\|_{X^\alpha / T_{u(t)}\Gamma} < \infty$, (5.5) implies that

$$\sup_{t \in \mathbb{R}^+} \|w(t)\|_\alpha < \infty. \tag{5.6}$$

Moreover, it holds for all $t \geq t_0$ that $[w(t) - v(t)]_{X^\alpha / T_{u(t)}\Gamma} = 0$ and so $v(t) - w(t) \in T_{u(t)}\Gamma$. Lemma A.7 implies that there is a neighborhood $V_1 \subset V_0$ of e such that $P(x) := P_\eta \in \mathrm{ISO}(T_{u(t)}\Gamma, P_\eta X^\alpha)$ for all $x \in V_1$. There is a $t_1 \in \mathbb{R}^+$ such that $t_1 \geq t_0$ and $u(t) \in V_1$ for all $t \geq t_1$. Letting $F(t) := Df(u(t)) - Df(e)$, we have

$$\begin{aligned} P_\eta F(t) v(t) &= P_\eta F(t)(v(t) - w(t)) + P_\eta F(t) w(t) \\ &= \underbrace{P_\eta F(t) P(u(t))^{-1}}_{\to 0 \text{ as } t \to \infty} \underbrace{P_\eta v(t)}_{P_\eta(v(t) - w(t))} + \underbrace{P_\eta F(t) w(t)}_{\to 0 \text{ as } t \to \infty}. \end{aligned}$$

Thus, $P_\eta(v(t) - w(t)) = P_\eta v(t)$, $t \geq t_1$, is a solution of an ordinary differential equation (in one dimension)

$$\dot{x} + \underbrace{P_\eta L x}_{\lambda x} = G(x, t)$$

We can apply [**6**, Theorem 13.3.1], which states that $P_\eta(v(t) - w(t))$ is governed by the eigenvalue $0 < \lambda$, that is, $\sup_{t \in \mathbb{R}^+} \|P_\eta(v(t) - w(t))\|_\alpha < \infty$. It follows that

$$\sup_{t \in \mathbb{R}^+} \|v(t)\|_\alpha \leq \sup_{t \in \mathbb{R}^+} \|w(t)\|_\alpha + \sup_{t \in \mathbb{R}^+} \left\|P(u(t))^{-1}\right\|_{\alpha,\alpha} \|P_\eta(v(t) - w(t))\|_\alpha < \infty,$$

and therefore $y \in B^+(T\pi, x_0)$, implying that $B^+(\pi', x_0) \subset [B^+(T\pi, x_0)]$.
Analogously, let $[y] \in B^-(\pi', x_0)$ and let $(u, v):\ \mathbb{R}^- \to TM$ be a solution of $T\pi$ with $v(0) = y$ (its existence follows from Lemma 5.4).
We have $u(t) \to e^-$ and $\|u(t)\|_\alpha^{-1} u(t) \to \tilde{\eta}$ as $t \to -\infty$, where $\tilde{\eta}$ is an eigenvector of $\tilde{L} := A - Df(e^-)$. Let $0 < \tilde{\lambda}$ be the associated eigenvalue and let $P_{\tilde{\eta}}$ denote the projection onto the eigenspace spanned by $\tilde{\eta}$.

As before, there exists a neighborhood $\tilde{V} \subset \Gamma$ of e^- such that $\tilde{P}(x) := P_{\tilde{\eta}} \in \mathrm{ISO}(T_x\Gamma, P_\eta X^\alpha)$ and the canonical projection $\tilde{Q}(x) \in \mathrm{ISO}((1-P_\eta)X^\alpha, X^\alpha/T_x\Gamma)$ for all $x \in \tilde{V}$. Now, there is a $\tilde{t}_0 \in \mathbb{R}^-$ with $u(t) \in \tilde{V}$ for all $t \le \tilde{t}_0$. Letting $\tilde{w}(t) := \tilde{Q}^{-1}(u(t))([v(t)])$, we have $\sup_{t \le \tilde{t}_0} \|\tilde{w}(t)\|_\alpha < \infty$.
Therefore, it follows as before that $P_{\tilde{\eta}}(v(t) - \tilde{w}(t))$, $t \le \tilde{t}_0$, is a solution of an ordinary differential equation, namely

$$\dot{x} + P_{\tilde{\eta}}\tilde{L}x = P_{\tilde{\eta}}((\tilde{G} \circ \tilde{P}^{-1})(u(t))x) + P_{\tilde{\eta}}(\tilde{G}(u(t))\tilde{w}(t))$$

where $\tilde{G}(x)y := Df(x)y - Df(e^-)y$, implying that $\sup_{t \le \tilde{t}_0} \|P_{\tilde{\eta}}v(t)\|_\alpha < \infty$ and consequently $\sup_{t \in \mathbb{R}^-} \|v(t)\|_\alpha < \infty$. This shows that $y \in B^-(T\pi, x_0)$ and that $B^-(\pi', x_0) \subset [B^-(T\pi, x_0)]$. $\square$

Proof of Proposition 5.8. Let N_0 be an isolating neighborhood for K relative to the restriction of π to Γ and define

$$N := \{[x, y] \in TM/T\Gamma : \ x \in N_0 \text{ and } y \in E^\alpha \text{ with } \|[y]\|_{X^\alpha/T_{u(t)}\Gamma} \le 1\}.$$

Further, let $(\tilde{u}, \tilde{v}) : \ \mathbb{R} \to N$ be a full solution of π'. It follows from Lemma 5.4 that there exists a full solution $(\tilde{u}, v)$ of $T\pi$ such that $(\tilde{u}, [v]) = (\tilde{u}, \tilde{v})$. v is bounded, that is $\sup_{s \in \mathbb{R}} \|v(s)\|_\alpha < \infty$ by Lemma 5.9.
Now, there are two cases: either $\tilde{u}(t) \in \{e^-, e^+\}$ for all $t \in \mathbb{R}$, impyling that $v \equiv 0$ by the hyperbolicity assumption, or $\tilde{u}(t) = u(t + \tau)$ for some $\tau \in \mathbb{R}$. We may assume w.l.o.g. that $\tau = 0$.
In the second case, we have for all $t \in \mathbb{R}$ $v(t) \in T\Gamma = B^+(T\pi, u(t)) \cap B^-(T\pi, u(t))$, which is equivalent to $\tilde{v}(t) = 0$ and so $\tilde{v} \equiv 0$. $\square$

3. Linearization along a solution

As in the previous section, we are given a linear subspace $E \subset X$. It is convenient to assume that $AE^1 = A(E \cap \mathcal{D}(A)) \subset E$. Let $\varphi : \ U \to V$, and $\mu \in \mathbb{R}$ be given by Proposition 4.4, and let $\varphi(x(t), y(t))$ be a solution of (5.2) which is defined on $[0, T[$. Then for all $t \in]0, T[$ $\varphi(x(t), y(t)) \in X^1$, $(x(t), y(t))$ is differentiable, and

$$D\varphi(x(t), y(t))(\dot{x}(t), \dot{y}(t)) + A\varphi(x(t), y(t)) = f \circ \varphi(x(t), y(t)). \tag{5.7}$$

Letting $P := P^-(\mu)$ and $Q := P^+(\mu)$, we can split (5.7) into an equation on PX and another one on QX. We will omit the notation of t in order to improve the readability.
On PX, we have

$$\begin{aligned} PD\varphi(x, y)(\dot{x}, \dot{y}) &= Pf \circ \varphi(x, y) - PA\varphi(x, y) \\ &=: P\tilde{f}(\varphi(x, y)), \end{aligned}$$

where the right side is again continuously Fréchet-differentiable since $PX \subset X^1$ is finite-dimensional.

On QX, one obtains

$$QD\varphi(x,y)(\dot{x},\dot{y}) + A \underbrace{Qy}_{Q(\varphi(x,y)-\varphi(x,0))} = Qf(\varphi(x,y)) - AQ\varphi(x,0)$$
$$=: Q\tilde{f}(\varphi(x,y)).$$

$\tilde{f}$ is well-defined, continuous, and $\tilde{f}\circ\varphi$ has a continuous Fréchet-derivative $D_y\tilde{f}$. Furthermore, $(x(t),y(t))$ is a solution of

$$\dot{x}(t) = g_1(x(t),y(t))$$
$$\dot{y}(t) + \tilde{A}y(t) = g_2(x(t),y(t)),$$

where we set

$$g(x,y) := (g_1,g_2)(x,y) := D\varphi(x,y)^{-1}\circ\tilde{f}\circ\varphi(x,y)$$

and $\tilde{A} := AQ$, which is again a sectorial operator since for all $y\in X^1$ we have $Ay-\tilde{A}y = APy$ with $AP\in\mathcal{L}(X^\alpha,X^0)$. The sectoriality now follows from [**13**, Corollary 1.4.5]. Moreover, by [**13**, Theorem 1.4.6], the norms induced by A and $\tilde{A}$ are equivalent.
Using Proposition 4.4, one can show

LEMMA 5.10. *$g_2 : U\cap E^\alpha \to E^0$ is continuously Fréchet-differentiable in y (with $D_yg_2\in\mathcal{L}(E^\alpha,E^0)$).*

PROOF. Indeed, we will show that $g = (g_1,g_2)$ is continuously Fréchet-differentiable in y. Letting $(x_0,y_0)\in U$, $y\in E^\alpha$ and $h\in\mathbb{R}^+$, we have for h small enough

$$\begin{aligned}
&h^{-1}(g(x_0,y_0+hy)-g(x_0,y_0))\\
&= D\varphi(x_0,y_0)^{-1}h^{-1}(\tilde{f}\circ\varphi(x_0,y_0+hy)-\tilde{f}\circ\varphi(x_0,y_0))\\
&+ h^{-1}\left(D\varphi(x_0,y_0+hy)^{-1}-D\varphi(x_0,y_0)^{-1}\right)(\tilde{f}\circ\varphi(x_0,y_0+hy))\\
&\to D\varphi(x_0,y_0)^{-1}D(\tilde{f}\circ\varphi(x_0,y_0))y + \left(D_yD\varphi^{-1}(x_0,y_0)y\right)(\tilde{f}\circ\varphi)(x_0,y_0)
\end{aligned}$$

as $h\to 0$. The limit depends continuously on (x_0,y_0) and thus

$$D_yg(x_0,y_0)y = D\varphi(x_0,y_0)^{-1}D(f\circ\varphi)(x_0,y_0)y + \left(D_yD\varphi(x_0,y_0)^{-1}y\right)(\tilde{f}\circ\varphi)(x_0,y_0)$$

with $y\in E^\alpha$. □

Let the family of semiflows $(\pi_\lambda)_{\lambda\in[0,1]}$ on $\mathbb{R}\times E^\alpha$ ($E^\alpha = E\cap X^\alpha$) be defined as follows:

DEFINITION 5.11. $(x(t),y(t))$ is a solution of π_λ if $\varphi(x(t),\lambda y(t))$ is a mild solution of (5.2) and $y(t)$ is a mild solution of

$$\dot{y}(t) + \tilde{A}y(t) = \tilde{g}_\lambda(x(t),y(t)), \tag{5.8}$$

where we set

$$\tilde{g}_\lambda(x,y) := \begin{cases}\lambda^{-1}g_2(x,\lambda y) & \lambda>0\\ D_yg_2(x,0)y & \lambda=0.\end{cases}$$

Given $\lambda\in\,]0,1]$ and a solution $\varphi(x(t),\lambda y(t))$ of (5.2), it follows that (5.8) holds, that is, $y(t)$ is a solution of (5.8).
The the theorem below is the main result of this section.

THEOREM 5.12. *Let the assumptions at the beginning of section 2 hold, and suppose that u is normal.*
Then

(1) $K := \varphi^{-1}(\operatorname{cl} u(\mathbb{R}))$ *is an isolated invariant set relative to* π_λ *for all* $\lambda \in [0,1]$*;*
(2) $h(\pi_1, K) = h(\pi_0, K)$.

In order to prove the theorem, we can make the following additional assumptions w.l.o.g.:

(1) $U \cap (\mathbb{R} \times \{0\}) =]0,1[\times \{0\}$;
(2) $\|y\|_\alpha \le 1$ for all $(x,y) \in U$;
(3) U is convex in y, that is, for all $\xi \in [0,1]$ one has $(x, \xi y_1 + (1-\xi) y_2) \in U$ whenever (x, y_1) and $(x, y_2) \in U$;
(4) $\sup_{(x,y)\in U} \|g_2(x,y)\|_\alpha < \infty$;
(5) $\sup_{(x,y)\in U} \|D_y g_2(x,y)\|_{\alpha,0} < \infty$.

LEMMA 5.13. *There exists a constant $L \in \mathbb{R}^+$ such that*

$$\|\tilde{g}_\lambda(x,y_1) - \tilde{g}_\lambda(x,y_2)\|_0 \le L \|y_1 - y_2\|_\alpha$$

for all $(x,y_1),(x,y_2) \in U$ and all $\lambda \in [0,1]$.

PROOF. Let $\lambda \in [0,1]$ and $(x,y_1),(x,y_2) \in U$. We have for all $\xi \in [0,1]$

$$\begin{aligned}\|\tilde{g}_\lambda(x,y_1) - \tilde{g}_\lambda(x,y_2)\|_0 &\le \sup_{\xi\in[0,1]} \|D_y \tilde{g}_\lambda(x, \xi y_1 + (1-\xi)y_2)\|_{\alpha,0} \|y_1 - y_2\|_\alpha \\ &\le \sup_{(x,y)\in U} \|D_y g_2(x,y)\|_{\alpha,0} \|y_1 - y_2\|_\alpha.\end{aligned}$$

□

LEMMA 5.14. *Let $\lambda_n \to 0$ in $[0,1]$, $T \in \mathbb{R}^+$, $\gamma_n \to \gamma_0$ in $C([0,T],]0,1[)$, and $h_n(t,y) := \tilde{g}_{\lambda_n}(\gamma_n(t), y)$ for $n \in \mathbb{N} \cup \{0\}$.*
Then $h_n(t,y)$ is continuous in (t,y) for all $n \in \mathbb{N} \cup \{0\}$ and for every $0 < \rho \in \mathbb{R}^+$ one has

$$\sup\{\|h_n(t,y) - h_0(t,y)\|_0 : \ t \in [0,T] \ y \in E^\alpha \ \|y\|_\alpha \le \rho\} \to 0$$

as $n \to \infty$.

PROOF. We have for all $(x_1,y),(x_2,y) \in U$

$$\begin{aligned}\|\tilde{g}_{\lambda_n}(x_1,y) - \tilde{g}_0(x_2,y)\|_0 &\le \|(\tilde{g}_{\lambda_n}(x_1,y) - \tilde{g}_{\lambda_n}(x_1,0)) - (\tilde{g}_0(x_2,y) - \tilde{g}_0(x_2,0))\|_0 \\ &\quad + \|\tilde{g}_{\lambda_n}(x_1,0) - \tilde{g}_0(x_2,0)\|_0 \\ &\le \sup_{\xi\in[0,1]} \|D_y g_2(x_1, \xi\lambda_n y) - D_y g_2(x_2, \xi\lambda_n y)\|_{\alpha,0} \|y\|_\alpha \\ &\quad + \|g_2(x_1,0) - g_2(x_2,0)\|_0.\end{aligned}$$

Suppose that our claim is not true for some $\rho \in \mathbb{R}^+$. Then there are sequences $t_n \to t_0$ in $[0,T]$, y_n in E^α, $k(n) \to \infty$ in $\mathbb{N}$ and an $\varepsilon > 0$ such that $\left\|h_{k(n)}(t_n, y_n) - h_0(t_n, y_n)\right\| > \varepsilon$ for all $n \in \mathbb{N}$. In view of the above calculation, we have for $x_n := \gamma_{k(n)}(t_n)$ and $\tilde{x}_n := \gamma_0(t_n)$

$$\begin{aligned}\left\|h_{k(n)}(t_n, y_n) - h_0(t_n, y_n)\right\|_0 &\leq \sup_{\xi\in[0,1]} \|D_y g_2(x_n, \xi\lambda_n y) - D_y g_2(\tilde{x}_n, \xi\lambda_n y)\|_{\alpha,0}\rho \\ &+ \|g_2(x_n, 0) - g_2(\tilde{x}_n, 0)\|_0 \\ &\to 0 \text{ as } n \to \infty,\end{aligned}$$

a contradiction. □

Using the previous lemmas, we are now able to prove

PROPOSITION 5.15. *Let $[a, b] \subset V$ such that $K := [a, b] \times \{0\}$ is an isolated invariant set relative to π_0.*
*Then $(\pi_\lambda)_{\lambda\in[0,1]}$ is an $\mathcal{S}$-continuous family of semiflows in the sense of [**18**, Definition 12.1], that is, for every $\lambda \in [0, 1]$, K is an isolated invariant set relative to π_λ and there is a neighborhood W of λ in $[0, 1]$ and a closed set $N \subset V$ such that*

1. *for every $\lambda \in W$, N is a strongly π_λ-admissible isolating neighborhood of K_λ relative to π_λ;*
2. *whenever $\lambda_n \to \lambda_0$ in $[0, 1]$, then $x_n\pi_{\lambda_n}t_n \to x_0\pi_{\lambda_0}t_0$ as $n \to \infty$ for every sequence $((x_n, y_n), t_n) \to ((x_0, y_0), t_0)$ in $U \times \mathbb{R}^+$, and N is $(\pi_{\lambda_n})_n$-admissible.*

PROOF. Let $\lambda_n \to \lambda_0$ in $[0, 1]$. We have to show that $\pi_n := \pi_{\lambda_n} \to \pi_{\lambda_0} := \pi_0$. For every $n \in \mathbb{N}$ let $(u_n(t), v_n(t))$, $0 \leq t \leq t_n$ denote the solution of π for which $(u_n(0), v_n(0)) = (x_n, \lambda_n y_n)$.
Suppose that $\lambda_0 \neq 0$. It follows that $(x_n, \lambda_n y_n) \to (x_0, \lambda_0 y_0)$, so by the continuity of π_1 there is a solution $(u_0(t), v_0(t))$ of π with $(u_0(0), v_0(0)) = (x_0, y_0)$ which is defined for all $t \in [0, t_0]$ and we have $(u_n(t_n), v_n(t_n)) \to (u_0(t_0), v_0(t_0))$ as $n \to \infty$. Therefore, $(x_n, y_n)\pi_n t_n = (u_n(t_n), \lambda_n^{-1} v_n(t_n)) \to (u_0(t_0), \lambda_0^{-1} v_0(t_0)) = (x_0, y_0)\pi_0 t_0$ that is, $\pi_n \to \pi_0$.
Now suppose that $\lambda_0 = 0$. By the continuity of π_1, there is a solution $(u_0(t), 0)$ of π_1 defined for all $t \in [0, t_0]$ with $(u_0(t), 0) = \lim_{n\to\infty}(u_n(t), \lambda_n v_n(t))$ for all $t \in [0, t_0]$.
For every $\tau \in [0, t_0]$, we have $\sup_{t\in[0,\tau]} |u_n(t) - u_0(t)| \to 0$ as $n \to \infty$. It follows from Lemma 5.13 that for all $n \in \mathbb{N}$

$$G_n(t, y) := \tilde{g}_{\lambda_n}(u_n(t), \lambda_n y) \quad t \in [0, \tau]$$

is Lipschitz continuous in y and from Lemma 5.14 that

$$\sup_{\|y\|_\alpha \leq \rho \quad t\in[0,\tau]} \|G_n(t, y) - G_0(t, y)\| \to 0$$

as $n \to \infty$, provided that $\rho > 0$ is sufficiently small.
Moreover, for each $n \in \mathbb{N}$, $v_n(t)$ is a mild solution of

$$\dot{y} + \tilde{A}y = G_n(t, y).$$

Let $v_0(t)$ denote the maximally defined mild solution of

$$\dot{y} + \tilde{A}y = G_0(t, y)$$

with $v_0(0) = y_0$. [**21**, Theorem 47.5] implies that $v_n(t) \to v_0(t)$ uniformly on $[0, \tau]$ provided that $v_0(t)$ is defined on $[0, \tau]$. Because $v_n(t) \in U$, we have $\|v_n(t)\|_\alpha \leq 1$ for all $t \in [0, t_n]$ and all $n \in \mathbb{N}$ so it follows from Lemma 5.13 and [**21**, Lemma 47.4] that $v_0(t)$ is defined for all $t \in [0, t_0]$. This shows again that $\pi_n \to \pi_0$.

In order to verify the strong admissibility, let $0 < \varepsilon \in \mathbb{R}^+$, let $N_0 \subset U$ be an isolating neighborhood for K with respect to π_1 and define

$$N := N(\varepsilon) := \{(x, y) \in [0, 1] \times E^\alpha : (x, 0) \in N_0 \text{ and } y \in E^\alpha \text{ with } \|y\|_\alpha \leq \varepsilon\}.$$

By choosing ε_0 small enough, $N(\varepsilon_0) \subset U$, and Lemma 5.13 and [**21**, Lemma 47.4] imply that π_λ does not explode in $N(\varepsilon)$ for all $\varepsilon \in [0, \varepsilon_0[$ and all $\lambda \in [0, 1]$.

Now let there be given sequences (x_n, y_n) in N, $\lambda_n \to \lambda_0$ in $[0, 1]$ and $t_n \to \infty$ in $\mathbb{R}^+$ such that for every $n \in \mathbb{N}$ and for all $s \in [0, t_n]$ $x_n \pi_n s \in N$, where we set $\pi_n := \pi_{\lambda_n}$. We may assume that $x_n \to x_0$. Let $(u_n(s), v_n(s)) := (x_n, y_n)\pi_n s$, $s \in [0, t_n]$. $v_n(t)$ is a mild solution of (5.8). Hence, it follows exactly as in the proof of [**18**, Theorem I.4.3] that given $\beta \in]\alpha, 1[$ there is a constant $b \in \mathbb{R}^+$ such that $\|v_n(t_n)\|_\beta \leq b$ for all $n \in \mathbb{N}$ sufficiently large. By [**13**, Theorem 1.4.8] (A has compact resolvent), the inclusion $X^\beta \subset X^\alpha$ is compact, so there exists a convergent (in X^α) subsequence of $v_n(t_n)$. This proves the claims concerning the admissibility properties.

Suppose that $N(\varepsilon_0)$ is an isolating neighborhood for (K, π_0) (this can always be achieved by choosing ε_0 small enough) and that there does not exist an $\varepsilon \in]0, \varepsilon_0[$ such that for all $\lambda \in [0, 1]$, $N(\varepsilon)$ is an isolating neighborhood for (K, π_λ). Then there is a sequence $\lambda_n \in [0, 1]$ and for every $n \in \mathbb{N}$ a full solution $(u_n(t), v_n(t))$ of $\pi_n := \pi_{\lambda_n}$ with $0 < c_n := \sup_{t \in \mathbb{R}} \|v_n(t)\|_\alpha \to 0$ as $n \to \infty$.

It follows that that $(u_n(t), c_n^{-1} v_n(t))$ is a solution of $\pi_{\lambda_n c_n}$. We may assume that $2\|v_n(0)\|_\alpha \geq c_n$ and by admissibility that $(u_n(0), v_n(0)) \to (x_0, y_0)$. We have

$$\frac{\|v_n(0)\|_\alpha}{c_n} \geq \frac{\|v_n(0)\|_\alpha}{2\|v_n(0)\|_\alpha} = \frac{1}{2},$$

showing that $y_0 \neq 0$. By [**18**, Theorem I.4.5] and since $c_n \lambda_n \leq c_n \to 0$ as $n \to \infty$, $(x_0, y_0) \in \mathrm{Inv}_{\pi_0}(N) = K$, a contradiction to $y_0 \neq 0$. We have shown that there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ and all $\lambda \in [0, 1]$ $N(\varepsilon)$ is an isolating neighborhood for K relative to π_λ. □

Lemma 5.16.

$$p_2 \circ D\Pi_{1,t}(x_0, 0) = \Phi(x_0, t) \circ p_2 \quad (x_0, 0) \in]0, 1[\cap U,$$

where $p_2 : \mathbb{R} \times E \to E$, $p_2(x, y) := y$, *denotes the canonical projection,* $\Pi_{\lambda,t} x := x\pi_\lambda t$ *and* $\pi_0 = (\xi, \Phi) \in \mathrm{SK}([0, 1], E)$.

PROOF. According to Definition 5.11 and Proposition 5.15, one has for all $x_0 \in]0,1[\times \{0\} \cap U$ and $(x,y) \in \mathbb{R} \times E^\alpha$

$$\begin{aligned}
\Phi(x_0,t) \circ p_2(x,y) &= \Phi(x_0,t)y \\
&= \lim_{\lambda \to 0+} p_2 \circ \Pi_{\lambda,t}(x_0,y) \\
&= \lim_{\lambda \to 0+} p_2(\lambda^{-1}(\Pi_{1,t}(x_0,\lambda y) - \Pi_{1,t}(x_0,0))) \\
&= p_2 \circ D\Pi_{1,t}(x_0,0)(0,y) \\
&= \underbrace{p_2 \circ D\Pi_{1,t}(x_0,0)(x,0)}_{0} + p_2 \circ D\Pi_{1,t}(x_0,0)(0,y) \\
&= p_2 D\Pi_{1,t}(x,0)(x,y),
\end{aligned}$$

where we have used the invariance of Γ under π (resp. $]0,1[\cap U$ under π_1). $\square$

PROOF OF THEOREM 5.12. Our claims follow from Proposition 5.15 and [**18**, Theorem I.12.2] if we show that $K = \varphi^{-1}(\mathrm{cl}\{u(t) : t \in \mathbb{R}\})$ is isolated relative to π_0.
Let $\tilde{M} =]0,1[\times E^\alpha$ and $\tilde{\Gamma} =]0,1[\times \{0\}$.

$$\begin{tikzcd}
T\tilde{M} \times \mathbb{R}^+ \arrow[r, "T\pi_1"] \arrow[d, "\mathrm{id}\times p_2 \times \mathrm{id}"] & T\tilde{M} \arrow[d, "\mathrm{id}\times p_2"] \\
]0,1[\times E^\alpha \times \mathbb{R}^+ \arrow[r, "\pi_0"] &]0,1[\times E^\alpha
\end{tikzcd}$$

is commutative by Lemma 5.16 and

$$\begin{tikzcd}
T\tilde{M} \times \mathbb{R}^+ \arrow[r, "T\pi_1"] \arrow[d, "\mathrm{id}\times p_2 \times \mathrm{id}"] & T\tilde{M} \arrow[d, "\mathrm{id}\times p_2"] \\
]0,1[\times E^\alpha \times \mathbb{R}^+ \arrow[d, "k \times \mathrm{id}"] &]0,1[\times E^\alpha \arrow[d, "k"] \\
T\tilde{M}/T\tilde{\Gamma} \times \mathbb{R}^+ \arrow[r, "\pi_1'"] & T\tilde{M}/T\tilde{\Gamma},
\end{tikzcd}$$

where we set $k(x,y) := [x,(0,y)]$, by the definition of π_1'. Combining the previous two diagrams (p_2 is an epimorphism) shows that

$$\begin{tikzcd}
]0,1[\times E^\alpha \times \mathbb{R}^+ \arrow[r, "\pi_0"] \arrow[d, "k \times \mathrm{id}"] &]0,1[\times E^\alpha \arrow[d, "k"] \\
T\tilde{M}/T\tilde{\Gamma} \times \mathbb{R}^+ \arrow[r, "\pi_1'"] & T\tilde{M}/T\tilde{\Gamma},
\end{tikzcd}$$

commutes.
By Proposition 5.8, $[K \times \{(0,0)\}]$ is an isolated invariant set relative to π_1'. $k :]0,1[\times E^\alpha \to T\tilde{M}/T\tilde{\Gamma}$ is a homeomorphism (a continuous bijection; the continuity of the inverse $[x,(y_1,y_2)] \mapsto (x,(0,y_2))$ follows from the choice of the quotient topology on $T\tilde{M}/T\tilde{\Gamma}$). Hence, K is isolated relative to π_0. $\square$

CHAPTER 6

Homotopy index of linear skew product semiflows

This chapter is concerned with the homotopy index of linear skew product semiflows obtained in the previous chapter. We consider linear skew product semiflows which are generated by semilinear parabolic equations and are normalized on the zero-section, that is, the semiflow $\pi = \pi(A, F) \in \mathrm{SK}([-2, 2], X^\alpha)$ is induced by mild solutions of

$$\begin{aligned} \dot{x} &= 1 - x^2 \\ \dot{y} + Ay &= F(x)y. \end{aligned} \tag{6.1}$$

Unfortunately, the right side of the above equation is not necessarily locally Lipschitz continuous if one assumes only that F is a continuous family of linear operators. Therefore, the term *mild solution* is used as follows: $(u(t), v(t))$ is called a mild solution of (6.1) if $u(t)$ is a solution of the first equation, that is, $\dot{u}(t) = 1 - u(t)^2$, and $v(t)$ is a mild solution of $\dot{y} + Ay = F(u(t))y$.
Let $[a, b]$ be an arbitrary interval and let $\tilde{\pi} \in \mathrm{SK}([a, b], X^\alpha)$ be induced by mild solutions of

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{y} + Ay &= F(x)y \end{aligned}$$

such that there exists a homeomorphism $\varphi :\ [a, b] \to [-2, 2]$ such that $\varphi \circ u(t)$ is a solution of $\dot{x} = 1 - x^2$ whenever $(u(t), v(t))$ is a solution of $\tilde{\pi}$. Then $(\varphi \circ u(t), v(t))$ is a mild solution of

$$\begin{aligned} \dot{x} &= 1 - x^2 \\ \dot{y} + Ay &= F(\varphi^{-1}(x))y \end{aligned}$$

and $(\tilde{F}(x) := F(\varphi^{-1}(x)))_{x \in [a,b]}$ is a again a continuous family of semiflows. This justifies the restriction to semiflows given by (6.1).

1. Existence, continuous dependence of solutions, and admissibility

Suppose that

- X is a Banach space;
- A is sectorial linear operator which is densely defined on X and has compact resolvent;
- X^α denotes the α-th fractional power space (see **[13]**);

and

(1) $F :\ [-2, 2] \to \mathcal{L}(X^\alpha, X^0)$ is *sufficiently continuous*, that is, there are $-2 = x_0 \leq \cdots \leq x_n = 2 \in [-2, 2]$ such that for every interval $[x_i, x_{i+1}]$, $i \in \{0, \ldots, n-1\}$, there is an $\tilde{F} \in C([x_i, x_{i+1}], \mathcal{L}(X^\alpha, X^0))$ such that $F(x) = \tilde{F}(x)$ for every $x \in]x_i, x_{i+1}[$.

(2) $-1, 1 \notin \{x_0, \ldots, x_n\}$.

LEMMA 6.1. *Let $F_n \in L^\infty([0,\tau], \mathcal{L}(X^\alpha, X))$, $n \in \mathbb{N} \cup \{0\}$, and suppose that $F_n(t) \to F_0(t)$ a.e. in $[0,\tau]$.*
Let there further exist an $M \in \mathbb{R}^+$ with

$$2\|F_n\|_\infty \leq M$$

for all $n \in \mathbb{N} \cup \{0\}$.
Then,

$$K_n v(t) = \int_0^t e^{-A(t-s)}(F_n - F_0)(s)v(s)\,ds \quad t \in [0,\tau]$$

defines a sequence of operators in $\mathcal{L}(C([0,\tau], X^\alpha), C([0,\tau], X^\alpha))$ with $\|K_n\| \to 0$ as $n \to \infty$.

PROOF. We have

$$K_n v(t) = \int_0^t e^{-A(t-s)}(F_n - F_0)(s)v(s)\,ds \quad t \in [0,\tau].$$

Using standard estimates (see [**13**]), there exist $1 \leq \tilde{M}, \tilde{\tilde{M}} \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ such that $\operatorname{Re}\sigma(A) > \mu$ and

$$\begin{aligned} \left\|e^{-At}\right\|_{\alpha,0} &\leq \tilde{\tilde{M}} t^{-\alpha} e^{-\mu t} \leq t^{-\alpha}\tilde{M} && t \in \,]0,\tau] \\ \left\|e^{-At}\right\|_{0,0} &\leq \tilde{\tilde{M}} e^{-\mu t} \leq \tilde{M} && t \in [0,\tau]\,. \end{aligned}$$

Let $\varepsilon > 0$ and $v \in C([0,\tau], X^\alpha)$. There exists a $\delta = \delta(\varepsilon) > 0$ with

$$\tilde{M} \int_0^t s^{-\alpha}\,ds < \varepsilon \text{ for all } t \in [0,\delta].$$

Consequently, we obtain that

$$\|K_n v(t)\|_\alpha = \left\| \int_0^t e^{-A(t-s)}(F_n - F_0)(s)v(s)\,ds \right\|_\alpha \leq M\varepsilon \|v\|_{C([0,\tau],X^\alpha)} \tag{6.2}$$

for all $t \in [0,\delta]$.
By Egorov's theorem (see [**9**]), there exists a measurable set $C \subset [\delta,\tau]$ with Lebesgue measure $\lambda(C) \leq \varepsilon$ and $\|F_n(t) - F_0(t)\|_{\alpha,0} \to 0$ uniformly on $[\delta,\tau] \setminus C$.

For every $t \in [\delta, \tau]$, we have

$$\begin{aligned}
\|K_n v(t)\|_\alpha &\leq \left\| \int\limits_{[\delta,\tau]\setminus C} e^{-A(t-s)}(F_n - F_0)(s)v(s)\,ds \right\|_\alpha \\
&+ \left\| \int\limits_C e^{-A(t-s)}(F_n - F_0)(s)v(s)\,ds \right\|_\alpha \\
&+ \left\| e^{-A(t-\delta)} \int\limits_0^\delta e^{-A(\delta-s)}(F_n - F_0)(s)v(s)\,ds \right\|_\alpha \\
&\leq \delta^{-\alpha}\tilde{M} \sup_{s\in[\delta,\tau]\setminus C} \|F_n(s) - F_0(s)\|_{\alpha,0}\|v\|_{C([0,\tau],X^\alpha)} \\
&+ \varepsilon\tilde{M}\operatorname{ess\,sup}_{s\in C}\|F_n(s) - F_0(s)\|_{\alpha,0}\|v\|_{C([0,\tau],X^\alpha)} + \varepsilon\tilde{M}M\|v\|_{C([0,\tau],X^\alpha)}.
\end{aligned}$$

Let $N = N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N$

$$\sup_{s\in[\delta,\tau]\setminus C} \|F_n(s) - F_0(s)\|_{\alpha,0} \leq \varepsilon\delta^\alpha.$$

In conjunction with (6.2), we have shown that for all $t \in [0,\tau]$ and all $n \geq N(\varepsilon)$,

$$\|K_n v(t)\|_\alpha \leq \tilde{M}(1 + 2M)\varepsilon\|v\|_{C([0,\tau],X^\alpha)},$$

where $\varepsilon > 0$ was arbitrary. □

Proposition 6.2. *For every $(x_0, y_0) \in\,]-2, 2[\, \times X^\alpha$, there is a unique, maximally defined mild solution $(u(t), v(t))$ of (6.1), which is defined on $J \subset \mathbb{R}^+$ and satisfies $(u(0), v(0)) = (x_0, y_0)$.*
Moreover, if $J \neq \mathbb{R}^+$, then there is a $t_0 \in \mathbb{R}^+$ with $u(t) \to -2$ as $t \to t_0-$.

Proof. Let $u(t)$, $t \in [0, T[$ be the maximally defined solution of

$$\dot{x} = 1 - x^2 \quad x \in\,]-2, 2[\,. \tag{6.3}$$

It follows from [**21**, Theorem 44.1] that there is a unique solution of

$$\dot{v} + Av = F(u(t))v \quad t \in [0, T[$$

with $v(0) = y_0$. □

Proposition 6.3. *Let $F_n \to F_0 \in L^\infty([-2,2], \mathcal{L}(X^\alpha, X))$, $n \in \mathbb{N}$, and suppose that F_n, $n \in \mathbb{N} \cup \{0\}$, are sufficiently continuous.*
Further, let $(x_n, y_n) \to (x_0, y_0) \in\,]-2,2[\times X^\alpha$ be sequences, and $(u_n, v_n) :\ [0, T_n[\, \to\,]-2, 2[\times X^\alpha$, $n \in \mathbb{N} \cup \{0\}$, the maximally defined mild solutions of $\pi(A, F_n)$ with $(u_n(0), v_n(0)) = (x_0, y_0)$.
Then $T_n \to T_0$ and $\sup_{s\in[0,t]} \|v_n(s) - v_0(s)\|_\alpha \to 0$ as $n \to \infty$ whenever $t \in [0, T_0[$.

Proof. It follows from Proposition 6.2 that $T_n \to T_0$ since the maximal domain of (u_n, v_n) depends only on u_n, which is a solution of (6.3).

In order to show the convergence, it is sufficient to consider small times t. Assume that

$$\left\|e^{-At}x\right\|_\alpha \le M\|x\|_\alpha$$
$$\left\|e^{-At}x\right\|_\alpha \le Mt^{-\alpha}\|x\|_0$$

for some $M \in \mathbb{R}^+$ and for all $t \in [0,1]$. Assume further that $\tilde{\tau} \in \,]0,1]$ is small enough that

$$M\int_0^t (t-s)^{-\alpha}\|F_n(u(s))\|_{\alpha,0}\,ds \le \frac{1}{2}$$

for all $t \in [0,\tilde{\tau}]$.
Provided that $[0,\tau] \subset [0,T_n[\,\cap\,[0,T_0[\,\cap\,[0,\tilde{\tau}]$, we now have for all $t \in [0,\tau]$

$$\begin{aligned} v_n(t) - v_0(t) &= e^{-At}(v_n(0) - v_0(0)) \\ &\quad + \int_0^t e^{-A(t-s)}F_n(u(s)(v_n(s) - v_0(s)) + (F_n(u(s)) - F_0(u(s)))v_0(s)\,ds, \end{aligned}$$

and thus

$$\|v_n(t) - v_0(t)\|_\alpha \le 2M\|v_n(0) - v_0(0)\|_\alpha + 2\|K_n v_0\|$$

for some $M \in \mathbb{R}^+$ where K_n is given by Lemma 6.1. Hence, the convergence follows if we show that

$$\|F_n(u_n(t)) - F_0(u_0(t))\|_{\alpha,0} \to 0 \text{ as } n \to \infty \text{ a.e. on } [0,T_0[\,. \tag{6.4}$$

For each $n \in \mathbb{N}$, we have either $u_n(t) \in \{-1,1\}$ or $u_n(t) \notin \{-1,1\}$ for all $t \in \mathbb{R}$. It is thus sufficient to assume that either $u_n(t) \notin \{-1,1\}$ for all $n \in \mathbb{N}$ and all t or $u_n(t) \in \{-1,1\}$ for all $n \in \mathbb{N}$ and all t.
In the first case, let $0 = t_0 \le t_1 \le \cdots \le t_l = T_0$ such that $F_0 \circ u_0$ is continuous on each of the subintervals $]t_k, t_{k+1}[$. For every $k \in \{0,\ldots,l-1\}$, every $n \in \mathbb{N}$ large enough, and almost every $s \in \,]t_k, t_{k+1}[$, it holds that

$$\begin{aligned} \|F_n(u_n(s)) - F_0(u_0(s))\|_{\alpha,0} \le& \|F_n(u_n(s)) - F_0(u_n(s))\|_{\alpha,0} \\ &+ \|F_0(u_n(s)) - F_0(u_0(s))\|_{\alpha,0} \to 0 \end{aligned}$$

as $n \to \infty$.
In the second case, $x_0 := u_n(0)$ is independent of n. Each F_n is continuous in a small neighborhood of x_0, so there exists a sequence $x_n' \in \,]-2,2[$ with $|x_n' - x_0| \to 0$, $\|F_n(x_0) - F_n(x_n')\|_{\alpha,0} \to 0$, and $\|F_n(x_n') - F_0(x_n')\|_{\alpha,0} \to 0$ as $n \to \infty$. We have

$$\begin{aligned} \|F_n(x_0) - F_0(x_0)\|_{\alpha,0} \le& \|F_n(x_0) - F_n(x_n')\|_{\alpha,0} \\ &+ \|F_n(x_n') - F_0(x_n')\|_{\alpha,0} + \|F_0(x_n') - F_0(x_0)\|_{\alpha,0} \to 0 \end{aligned}$$

as $n \to \infty$. $\square$

Corollary 6.4. *Let the assumptions of Propositon 6.3 hold.*
Then

(1) $\pi(A,F_n)$ *is a semiflow for all* $n \in \mathbb{N} \cup \{0\}$*;*
(2) $\pi(A,F_n) \to \pi(A,F)$ *and*

(3) *every closed set* $N \subset\,]-2, 2[\, \times X^\alpha$ *which is bounded with respect to* $\|.\|_{\mathbb{R}\times X^0}$ *is strongly* $\pi(A, F_n)$*-admissible.*

PROOF. The first two claims are a restatement of Proposition 6.3. In particular, it follows from Proposition 6.2 that for every $n \in \mathbb{N}$, $\pi_n := \pi(A, F_n)$ does not explode in N. Admissibility now follows as in the proof of [**18**, Theorem I.4.3] (which is stated only for solutions in the sense of [**13**]). □

2. The classes SK_i, $i \in \{-1, 0, 1, 2\}$

For the rest of this chapter, let us make the following assumptions in addition to those of the previous section:

(1) $F : [-2, 2] \to \mathcal{L}(X^\alpha, X^0)$ is sufficiently continuous;
(2) A and $A - F(1)$ are hyperbolic and have simple eigenvalues, all of which are real; let $E^\pm(\pi, e) := E^\pm(e) := P_e^\pm(0)X$, $e \in \{-1, 1\}$ denote the associated subspaces of X, where $P_{\pi,e}^\pm(0) := P_e^\pm(0) := P^\pm(0)$ is the projection onto the subspaces which belong to the positive respectively negative part of the spectrum of $L := A - F(e)$, where $\pi = \pi(A, F)$ (see section 3).

(6.1) implies that there are exactly two equilibria, namely $(-1, 0)$ and $(1, 0)$, all of which are hyperbolic.

DEFINITION 6.5. Let $\mathrm{SK}_0 := \mathrm{SK}_0(X, A) \subset \mathrm{SK}([-2, 2], X^\alpha)$ denote the set of linear skew product semiflows which is given by $\pi \in \mathrm{SK}_0$ iff

(1) π is induced by mild solutions of (6.1), which satisfies the assumptions above;
(2) $K := [-1, 1] \times \{0\}$ is an isolated invariant set relative to π;
(3) $\dim E^-(1) = \dim E^-(-1) < \infty$.

DEFINITION 6.6. Let $[-1, 1] \subset [a, b] \subset [-2, 2]$ and let $h : [a, b] \to [-2, 2]$ be a homeomorphism such that $h(-1) = -1$, $h(1) = 1$. Let $\mathrm{SK}_{-1} = \mathrm{SK}_{-1}(X, A) \subset \mathrm{SK}([a, b], X^\alpha)$ denote the set of all semiflows π for which there exists an h with the above properties and a $\tilde{\pi} \in \mathrm{SK}_0$ such that $(h \circ u(t), v(t))$ is a solution of $\tilde{\pi}$ whenever $(u(t), v(t))$ is a solution of π.

DEFINITION 6.7. Let $\pi_0, \pi_1 \in \mathrm{SK}_0$. Then $\pi_0 \sim \pi_1$ iff there exists a homotopy, that is, an $\mathcal{S}$-continuous family $(\pi_\lambda, [-1, 1] \times \{0\})_{\lambda\in[0,1]}$ such that for all $\lambda \in [0, 1]$

(1) $\pi_\lambda \in \mathrm{SK}_0$, and
(2) $E^-(\pi_\lambda, -1)$ and $E^-(\pi_\lambda, 1)$ are constant.

The main result of this chapter is stated in the theorem below. What follows are several normalization steps, either isomorphisms of bundles as defined in Appendix A or equivalences in the sense of Definition 6.7.

THEOREM 6.8. $h(\pi, [-1, 1] \times \{0\}) = \bar{0}$ *for all* $\pi \in \mathrm{SK}_0$.

Here, h denotes the homotopy index as defined in [**18**].

PROOF. Lemma 6.11 and Lemma 6.12 show that the theorem holds if and only if it holds for all $\pi \in \mathrm{SK}_2$ (which is defined below). The result now follows from Corollary 6.26. □

3. Local constancy of $F(x)$

According to our assumptions in the previous section, we have $F \in L^\infty([-2,2], \mathcal{L}(X^\alpha, X^0))$ (in particular, the assumption of sufficient continuity is stronger). Let $\|F\| := \|F\|_\infty := \|F\|_{L^\infty([-2,2],\mathcal{L}(X^\alpha,X^\alpha))} := \operatorname{ess\,sup}_{x\in[-2,2]} \|F(x)\|_{\mathcal{L}(X^\alpha,X^0)}$.

LEMMA 6.9. *Suppose that:*

(1) $\pi = \pi(A,F)$ *is induced by mild solutions of* (6.1)*; A and F satisfy the assumptions at the beginning of Section 2;*
(2) $\dim E^-(1) = \dim E^-(-1) < \infty$.

Then $K := [-1,1] \times \{0\}$ is an isolated invariant set relative to π if and only if the following holds:
Whenever $(x(t), y(t))$ is a full bounded solution of π with $x(0) = 0$, then $y(t) \equiv 0$.

For every solution $(x(t), y(t))$, $|x(t)|$ is a priori bounded. Hence, a solution $(x(t), y(t))$ is bounded if and only if it is bounded in y that is, $\sup_t \|y(t)\|_\alpha < \infty$ where the supremum is taken over all $t \in \mathbb{R}$ for which $(x(t), y(t))$ is defined.

PROOF. Suppose that every full bounded solution $(x(t), y(t))$ with $x(0) = 0$ satisfies $y(t) \equiv 0$. Let

$$N := [-3/2, 3/2] \times B_1[0] \subset]-2,2[\times X^\alpha, \tag{6.5}$$

$\lambda \in [0,1]$, and $(x(t), y(t))$ be a full solution with $(x(t), y(t)) \in N$ for all $t \in \mathbb{R}$. $y(t)$ is bounded, that is, $\sup_{t\in\mathbb{R}} \|y(t)\|_\alpha < \infty$. Since $\dot{x}(t) = 1 - x^2(t)$, we have $x(t) \in [-1,1]$ for all $t \in \mathbb{R}$. Either $x(t) \in]-1,1[$ for all $t \in \mathbb{R}$, in which case we have $y(t) \equiv 0$ by the assumption above, or $x(t) \in \{-1,1\}$ for all $t \in \mathbb{R}$, in which case $y(t) \equiv 0$ by the hyperbolicity of $A - F(\pm 1)$. Therefore, we have $(x(t), y(t)) \in K$ for all $t \in \mathbb{R}$, showing that N is an isolating neighborhood for (π, K).
Now, suppose that K is an isolated invariant set, and let N be an isolating neighborhood for K. Setting $\varepsilon := \inf\{\|y\|_\alpha : x \in [-1,1] \text{ and } (x,y) \in N\}$, it is clear that $\varepsilon > 0$. Let $(x(t), y(t))$ be a full bounded solution of $\pi = (\xi, \Phi)$. Due to the linearity of Φ, $(x(t), \mu y(t))$ is again a solution of π. Choosing $0 < \mu$ small enough, it holds that $\|\mu y(t)\|_\alpha \le \varepsilon$ for all $t \in \mathbb{R}$ that is, $(x(t), \mu y(t)) \in N$. It follows that $\mu y(t) \equiv 0$ and so $y(t) \equiv 0$. □

LEMMA 6.10. *For $\lambda \in [0,1]$, let $\pi_\lambda = \pi(A, F_\lambda)$ satisfy the assumptions of Lemma 6.9, and assume that $\lambda \mapsto F_\lambda$ is continuous.*
If it holds for every $\lambda \in [0,1]$ and for every full bounded solution $(x(t), y(t))$ of π_λ with $x(0) = 0$ that $y(t) \equiv 0$, then $\pi_0 \sim \pi_1$.

PROOF. We have to show that the family (π_λ, K) is $\mathcal{S}$- continuous. Let N be given by (6.5). It follows from Lemma 6.9 that N is an isolating neighborhood for $[-1,1] \times \{0\}$ relative to π_λ for all $\lambda \in [0,1]$.
The continuity and admissibility properties are a consequence of Corollary 6.4. □

Let $\mathrm{SK}_1 \subset \mathrm{SK}_0$ denote the subset of all semiflows $\pi(A,F)$ where F is locally constant in a neigborhood of $\{-1,1\}$, that is, there exists a $\delta > 0$ such that for all $x \in]-1-\delta, -1+\delta[$ we have $F(x) = F(-1)$ and for all $x \in]1-\delta, 1+\delta[$ $F(x) = F(1)$.

LEMMA 6.11. *For every* $\pi(A,F) \in \mathrm{SK}_0$ *there is a* $\lambda_0 \in [0,1]$ *such that* $\pi(A,F) \sim \pi(A,F_\lambda) \in \mathrm{SK}_1$ *for all* $\lambda \in [0,\lambda_0]$*, where we set*

$$F_\lambda(x) := \begin{cases} F(-1) & x \in [-1-\lambda, -1+\lambda] \\ F(1) & x \in [1-\lambda, 1+\lambda] \\ F(x) & \textit{otherwise.} \end{cases}$$

PROOF. We have $\|F_\lambda - F_0\|_\infty \to 0$ because F is continuous in a neighborhood of $\{-1,1\}$. Thus it follows from Corollary 6.4 that the assumptions of [**18**, Theorem I.4.5] hold. Let $\pi_\lambda := \pi(A,F_\lambda)$ and note that $F_\lambda(1)$ and $F_\lambda(-1)$ are constant in λ so that the hyperbolicity at each of the equilibria and the subspaces $E^-(\pm 1)$ are preserved.
Suppose that for every $\delta \in\,]0,1]$ there is a $\lambda =: \lambda(\delta) \in [0,\delta]$ and a full bounded solution $(x(t),y(t))$ of π_λ with $x(0) = 0$ and $\|y(0)\|_\alpha = 1$. By [**18**, Theorem I.4.5] there is a full bounded solution of π_0 with $x(0) = 0$ and $y(0) \neq 0$, which cannot exist in view of Lemma 6.9 since $K = [-1,1] \times \{0\}$ is isolated relative to π_0.
Hence, Lemma 6.10 implies that there exists a $\lambda_0 \in [0,1]$ such that $\pi_0 \sim \pi_\lambda$ for all $\lambda \in [0,1]$. □

Let $\mathrm{SK}_2 \subset \mathrm{SK}_1$ denote the subset of all those semiflows which satisfy the following stronger restriction (compared to the definition of SK_1): There exists a $\delta > 0$ such that $F(x) = F(-1)$ for all $x \in [-2,-1+\delta[$ and $F(x) = F(1)$ for all $x \in\,]1-\delta,2]$.

LEMMA 6.12. *For every* $\pi(A,F) \in \mathrm{SK}_1$*, it holds that* $\pi(A,F) \sim \pi(A,\tilde{F}) \in \mathrm{SK}_2$*, where we set*

$$\tilde{F}(x) := \begin{cases} F(-1) & -2 \le x \le -1 \\ F(x) & -1 < x < 1 \\ F(1) & 1 \le x \le 2. \end{cases}$$

PROOF. Let F_λ be given by

$$F_\lambda(x) := \lambda\tilde{F}(x) + (1-\lambda)F(x).$$

Let $\lambda \in [0,1]$ and let $(x(t),y(t))$ be a full bounded solution of $\pi_\lambda := \pi(A,F_\lambda)$ with $x(0) = 0$. We have $x(t) \in\,]-1,1[$ for all $t \in \mathbb{R}$, showing that $(x(t),y(t))$ is also a solution of π_0. Therefore, $y(t) \equiv 0$.
Now, the claim follows from Lemma 6.10. □

4. Decomposition into "unstable" and "stable" subbundles

Let $\pi_0 = (\xi,\Phi) \in \mathrm{SK}_2$, that is, $\pi_0 = \pi(A,F)$ and there is a $\delta \in\,]0,1[$ such that $F(x) = F(-1)$ for all $x \in [-2,-1+\delta]$ and $F(x) = F(1)$ for all $x \in [1-\delta,2]$. The goal of this section is to define a subbundle U in the sense of A.5 such that every solution $(x(t),y(t))$ defined for $t \in \mathbb{R}^-$ with $\sup_{t\in\mathbb{R}^-} \|y(t)\|_\alpha < \infty$ satisfies $(x(0),y(0)) \in U$. As a consequence, π continues to a direct sum of two linear skew product semiflows, which arise from restrictions of π_0 to U respectively an approporiate complementary subbundle (later denoted by S).
Let $E^- := E^-(\pi_0,-1)$ and define $U(x) \in \mathcal{L}(E^-,X^\alpha)$ by

$$U(x)y := y \quad x \in [-2,-1+\delta] \quad y \in E^-.$$

We continue along $[-2,2]$ by following the semiflow, that is,

$$U(x) := U(-1+\delta)\Phi(-1+\delta, t_x) \quad x \in [-1+\delta, 1-\delta]$$

where $(-1+\delta)\xi t_x = x$ defines t_x.

LEMMA 6.13. *$U(x)$ is well-defined and $U \in C([-2,1-\delta], \mathcal{L}(E^-, X^\alpha))$. Moreover, $U(x)$ is injective for all $x \in [-2, 1-\delta]$.*

PROOF. The linearity of $U(x)$ follows from the linearity of $\Phi(-1+\delta, t)$. Let τ be given by $(-1+\delta)\xi\tau = 1-\delta$. It follows that $[-1+\delta, 1-\delta] = \xi(\{-1+\delta\}\times[0,\tau])$ and the restriction of ξ to $\{-1+\delta\}\times[0,\tau]$ is a homeomorphism. Hence t_x is well-defined for all $x \in [-1+\delta, 1-\delta]$ and we have $t_x \to t_{x_0}$ whenever $x \to x_0$ and also $\Phi(-1+\delta, t_x)y \to \Phi(-1+\delta, t_{x_0})y$ for all $y \in E^-$. It is clear that $U(x)$ is bounded for every $x \in [-2, 1-\delta]$ since $\dim E^- < \infty$.
Let $x \in [-1+\delta, 1-\delta]$ and $y \in E^-$ with $U(x)y = 0$. Then there is a full solution $(u(t), v(t))$ of π_0 with $u(0) = -1-\delta$, $v(0) = y \in E^-$ and $v(t_{1-\delta}) = 0$. We have $\sup_{s\le 0} \|v(s)\|_\alpha < \infty$ since $v(0) \in E^-$ and $\sup_{s\ge 0}\|v(s)\|_\alpha \le \sup_{s\in[0,t_{1-\delta}]}\|v(s)\|_\alpha < \infty$ since $v(t_{1-\delta}) = 0$. It follows from Lemma 6.9 that $v(0) = y = 0$. □

LEMMA 6.14. *$P_1^-(0)\circ U(1-\delta)$ is a bijection.*

PROOF. Let $y \in E^-(-1)$ with $P_1^-(0)\circ U(1-\delta)y = 0$. It follows that $U(1-\delta)y = \Phi(-1+\delta, t_{1-\delta})y \in E^+(1,0)$, so $\sup_{s\ge 0}\|\Phi(-1+\delta, s)y\|_\alpha < \infty$. As in the previous proof, it follows from the isolation of $[-1,1]\times\{0\}$ that $y = 0$, showing the injectivity of $P_1^-(0)\circ U(1-\delta)$. Surjectivity holds since $\dim E^-(-1) = \dim E^-(1)$. □

Therefore, given $y_0 \in E^-(1)$, there is a $w \in E^-(-1)$ with $P_1^-(0)\circ U(1-\delta)w = y_0$. Choose a basis $\{\eta_i : \ i = 1\ldots\dim E^-(1)\}$ for $E^-(1)$ such that each η_i is an eigenvector of $L := A - F(1)$.
Further, let $\lambda_i < 0$ denote the real eigenvalue λ_i which corresponds to η_i, that is, $e^{-Lt}\eta_i = e^{-\lambda_i t}\eta_i$. For each $i \in \{1, \ldots, \dim E^-\}$, there is an $\eta_i^+ \in E^+(1)$ with $\eta_i + \eta_i^+ \in U(1-\delta)E^-(-1)$. Let $y_i \in E^-$ be given by $U(1-\delta)y_i = \eta_i + \eta_i^+$ and define

$$U(x)y_i := \eta_i + e^{-(L-\lambda_i)(t_x - t_{1-\delta})}\eta_i^+ \quad x \in [1-\delta, 1[\quad i = 1\ldots\dim E^-.$$

Finally, let

$$U(x)y := \lim_{\tilde{x}\to 1} U(\tilde{x})y \quad x \in [1,2] \quad y \in E^-.$$

REMARK 2. *Using the construction above, one has $U(1)E^-(-1) = E^-(1)$.*

Reading U as a morphism in the sense of Appendix A, we say that U is π_0 invariant if $\{(x, U(x)y) : \ (x,y) \in\ ]-2,2[\ \times E^-\}$ is π_0-invariant.

LEMMA 6.15. *$U(x) \in C([-2,2], \mathcal{L}(E^-, X^\alpha))$ is well-defined and π_0-invariant.*

PROOF. Let x_n be a sequence in $[-2,2]$ with $x_n \to 1-$. We have $t_n := t_{1-\delta+x_n} - t_{1-\delta} \to \infty$ as $n \to \infty$ and thus $U(x_n)y_i - \eta_i = e^{\lambda_i t_n}e^{-Lt_n}\eta_i^+ \to 0$ as $n \to \infty$ (recall that $\lambda_i < 0$) showing that

$$U(x) \to P_1^-(0)\circ U(1-\delta) \text{ as } x \to 1. \tag{6.6}$$

It is sufficient to prove the invariance for each basis vector y_i. Letting $i \in \{1, \ldots, \dim E^-\}$ and $r, t \geq 0$, we have

$$\begin{aligned}\Phi((1-\delta)\xi r, t)U((1-\delta)\xi r)y_i &= \Phi((1-\delta)\xi r, t)(\eta_i + e^{-(L-\lambda_i)r}\eta_i^+)\\ &= e^{-Lt}(\eta_i + e^{-(L-\lambda_i)r}\eta_i^+)\\ &= e^{-\lambda_i t}(\eta_i + e^{\lambda_i(t+r)}e^{-L(t+r)}\eta_i^+)\\ &= e^{-\lambda_i t}U((1-\delta)\xi(t+r))y_i\end{aligned}$$

showing that $\Phi((1-\delta)\xi r, t)U((1-\delta)\xi r)y_i \in U((1-\delta)\xi t)$.
The invariance for $x \geq 1$ follows from (6.6) since $P_1^-(0)$ is exactly the projection onto the e^{-Lt}-invariant subspace $E^-(1)$. □

So far, we have shown that U is a subbundle of $[-2,2] \times X^\alpha$ (Lemma 6.13 and Lemma 6.15), which is π_0-invariant (Lemma 6.15).

LEMMA 6.16. *There exist morphisms (of bundles)* $S^\beta \in C([-2,2], \mathcal{L}(E^+ \cap X^\beta, X^\beta))$, $\beta \in [0,1]$, *such that for all* $\beta \in [0,1]$

$$S^\beta(x)y = S^0(x)y \quad x \in [-2,2] \quad y \in X^\beta$$

and

$$U(x) \oplus S^\beta(x) \in \mathrm{ISO}(X^\beta, X^\beta) \quad x \in [-2,2].$$

PROOF. First, we show that there is a $\mu \in \mathbb{R} \setminus \sigma(A - F(-1))$ such that $P_{-1}^-(\mu)U(x)$ is injective for all $x \in [-2,2]$. Suppose that this is not true. Then there are sequences $\mu_n \to \infty$ in $\mathbb{R}$, $x_n \to x_0$ in $[-2,2]$, and $y_n \to y_0 \neq 0$ in E^- such that $P_{-1}^-(\mu_n)U(x_n)y_n = 0$ for all $n \in \mathbb{N}$. We can assume w.l.o.g. that $(\mu_n)_n$ is monotone increasing.
Let $k \in \mathbb{N}$ be arbitrary but fixed. We have

$$\begin{aligned}P_{-1}^-(\mu_k)U(x_0)y_0 &= \lim_{n\to\infty} P_{-1}^-(\mu_k)U(x_n)y_n\\ &= \lim_{n\to\infty\ n\geq k} 0\end{aligned}$$

since $\mu_n \geq \mu_k$ implies that $P_{-1}^-(\mu_k)U(x_n)y_n = 0$.
Now, it follows from Lemma 3.4 that $U(x_0)y_0 = 0$, a contradiction to the injectivity of $U(x_0)$.
Let $E_0 := P_{-1}^-(\mu)X$. By Lemma A.8, there is a complementary subbundle $\tilde{S} \in C([-2,2], \mathcal{L}(E_0, E_0))$ for $P_{-1}^-(\mu)U$ in E_0, which is continuous regardless of the norm on E_0.
We can now define

$$S^\beta(x)y := \tilde{S}(x)P_{-1}^-(\mu)y + P_{-1}^+(\mu)y \quad x \in [-2,2] \quad y \in X^\beta.$$

One has $U(x)y^- + S^\beta(x)y^+ = z$ if and only if

$$\begin{aligned}P_{-1}^-(\mu)(U(x)y_1 + \tilde{S}(x)y_2) &= P_{-1}^-(\mu)z\\ P_{-1}^+(\mu)(U(x)y_1 + y_3) &= P_{-1}^+(\mu)z,\end{aligned}$$

where $y_1 + y_2 \in P^-_{-1}(\mu)X \subset X^1$ and $y_3 \in P^+_{-1}(\mu)X^\beta$. The first equation has a continuous inverse regardless of the norm considered, and the second equation yields

$$y_3 = P^+_{-1}(\mu)z - P^+_{-1}(\mu)U(x)y_1,$$

which is again continuous with respect to $\|.\|_\beta$. $\square$

From Lemma 6.16, we obtain a complementary subbundle S^α (complementary to U in X^α), which is canonically homeomorphic to the quotient bundle $([-2,2] \times X^\alpha)/U$, that is, $(x,y) \mapsto (x, [S^\alpha(x)y])$ defines a homeomorphism $E^+ \cap X^\alpha \to ([-2,2] \times X^\alpha)/U$.
Define $\pi_U := (\xi, \Phi_U) \in \mathrm{SK}([-2,2] \times E^-)$ by

$$U(x\xi t)\Phi_U(x,t)y = \Phi(x,t)U(x)y \quad y \in E^- \tag{6.7}$$

and $\pi_S = (\xi, \Phi_S) \in \mathrm{SK}([-2,2] \times (E^+ \cap X^\alpha))$ by

$$[S^\alpha(x\xi t)\Phi_S(x,t)y]_{X^\alpha/U(x\xi t)} = [\Phi(x,t)S^\alpha(x)y]_{X^\alpha/U(x\xi t)} \quad y \in E^+ \cap X^\alpha. \tag{6.8}$$

PROPOSITION 6.17. *$U \oplus S^\alpha$ is an isomorphism of bundles and $(U \oplus S^\alpha)[\pi_U \oplus \pi_S] \sim \pi_0$ (see Definition 5.1 for the direct sum of the semiflows).*

In order to prove Proposition 6.17, we need the following two lemmas.

LEMMA 6.18. *Let $e \in \{-1, 1\}$. Then there exist a neighborhood V of e in $[-2,2]$, a local isomorphism $\phi_e \in C(V, \mathcal{L}(E^+(e) \cap X^\alpha, E^+(e) \cap X^\alpha)$, and a $B_e \in \mathcal{L}(E^+(e) \cap X^\alpha, E^+(e) \cap X^0)$ with $\mathrm{Re}\,\sigma(A - B_e) > 0$ such that $\phi_e(u(t))v(t)$ is a mild solution of*

$$\dot{x} + (A - B_e)x = 0 \quad B_e \in \mathcal{L}(E^+(e) \cap X^\alpha, E^+(e) \cap X^0) \tag{6.9}$$

whenever $(u(t), v(t))$, $t \in [0,T]$, is a solution of π_S with $u(t) \in V$ for all $t \in [0,T]$.

PROOF. Letting $B_e = F(e)$, we have $P^+_e(0)(A - B_e) = (A - B_e)P^+_e(0)$ due to the choice of the projection $P^+_e(0)$. Now, let V be given by Lemma A.12 such that the projection $p\colon\ V \times (E^+(e) \cap X^\alpha) \to p(V \times E^+(e) \cap X^\alpha) \subset (V \times X^\alpha)/U$ ($U(e) = E^-(e)$ by Remark 2) which is given by $p(x,y) := (x, [y]_{X^\alpha/U(x)})$, is a homeomorphism.
By shrinking V if necessary, we may assume that $F(x) = B_e$ for all $x \in V$. Let $(u(t), v(t))$, $t \in [0,T]$, be a solution of (6.9) and let $(u(t), w(t))$, $t \in [0,T]$, be a solution of π_S with $[v(0)] = [S(u(0))w(0)]$. Then, by (6.8),

$$\begin{aligned} [v(t)] &= [\Phi(u(0),t)v(0)] \\ &= [\Phi(u(0),t)S(u(0))w(0)] \\ &= [S(u(t))\Phi_S(u(0),t)w(0)] = [S(u(t))w(t)], \end{aligned}$$

so $(u(t), v(t)) = p^{-1}(u(t), [S(u(t))w(t)])$, that is, we can choose $\phi_e(x,y) = p^{-1}([x, S(x)y])$. $\square$

LEMMA 6.19. *Let $(u(t), v(t))$ be a bounded solution of π_S which is defined for all $t \in \mathbb{R}^-$. Then $v(t) \equiv 0$.*

PROOF. There is an $e \in \{-1, 1\}$ such that $u(t) \to e$ as $t \to -\infty$. Let ϕ_e be given by Lemma 6.18 and assume that that $\phi_e(u(t))$ is defined for all $t \le t_0$.
$(u(t), w(t)) := (u(t), \phi_e(u(t))v(t))$, $t \le t_0$, is a mild solution of (6.9), and $\mathrm{Re}\,\sigma(A - B_e) > 0$ implies that $w(t) \equiv 0$. This implies that $v(t) = 0$ for all $t \le t_0$, showing that $v(t) \equiv 0$. $\square$

Proof of Proposition 6.17. It is stated in Lemma 6.16 that $U \oplus S^\alpha$ is an isomorphism of bundles, that is, particularly a homeomorphism.
For every $\beta \in [0,1]$, the direct sum $E^-(-1) \oplus (E^+(-1) \cap X^\beta) = X^\beta$ defines continuous projections onto each of the components. Applying $U \oplus S^\beta$, we obtain morphisms of bundles $P^\beta, Q^\beta \in C([-2,2], \mathcal{L}(X^\beta, X^\beta)$ such that for every $x \in [-2,2]$ it holds that

- $P^\beta(x)$ is a projection onto $U(x) = U(x)E^-$,
- $Q^\beta(x)$ is a projection onto $S^\beta(x) = S(x)(E^+ \cap X^\beta)$, and
- $P^\beta(x) + Q^\beta(x) = \mathrm{id}_{X^\beta}$.

Suppose that $\pi_0 = \pi(A, F)$, and let $\pi_\lambda := \pi(A, F_\lambda)$ where we set

$$F_\lambda(x)y = P^0(x)F(P^\alpha(x)y + (1-\lambda)Q^\alpha(x)y) + Q^0(x)F(y). \tag{6.10}$$

Let $(u(t), v(t))$ be a full bounded solution of π_λ. It follows that there is a full bounded solution $(u(t), w(t))$ of π_S with

$$[(u(t), S^\alpha(u(t))w(t))]_{([-2,2]\times X^\alpha)/U} = [u(t), v(t)]_{([-2,2]\times X^\alpha)/U}.$$

Hence, $w(t) \equiv 0$ by Lemma 6.19, showing that $v(t) \in U(u(t))$ for all $t \in \mathbb{R}$. The semiflow on U is not changed by λ since $Q^\alpha(x)U(x) = 0$ for all $x \in [-2,2]$, and so it follows that $v(t) \equiv 0$. Lemma 6.10 finally implies that $\pi_0 \sim \pi_1$.
Moreover, letting $\pi_1 = (\xi, \Phi_1)$, it follows from (6.10) that $P^\alpha(x\xi t)\Phi_1(x,t)(U(x)y_1 + S^\alpha(x)y_2) = \Phi_1(x,t)U(x)y_1 = \Phi(x,t)U(x)y_1$ for all $(y_1, y_2) \in E^-(-1) \times (E^+(-1) \cap X^\alpha)$. We thus have

$$P^\alpha(x\xi t)\Phi_1(x,t)(U(x)y_1 + S^\alpha(x)y_2) = U(x\xi t)\Phi_U(x,t)y_1,$$

and

$$Q^\alpha(x\xi t)\Phi_1(x,t)(U(x)y_1 + S^\alpha(x)y_2) = S^\alpha(x\xi t)\Phi_S(x,t)y_2$$

follows immediately from the invariance of U. This shows that $(U \oplus S^\alpha)[\pi_U \oplus \pi_S] = \pi_1$. □

We continue by discussing π_U and π_S independently of each other. Until further notice, let $\pi = (\xi, \Phi)$ denote $(U \oplus S^\alpha)[\pi_U \oplus \pi_S]$, and $E^\pm = E^\pm(-1)$.

4.1. The situation on S^α.

Lemma 6.20. *There exists a strongly π_S-admissible isolating neighborhood for $(\pi_S, [-1,1] \times \{0\})$.*

Proof. Let $N \subset \,]-2,2[\, \times X^\alpha$ be a strongly π-admissible isolating neighborhood for $[-1,1] \times \{0\}$. We have

$$\Phi(x,t)S^\alpha(x)y = S^\alpha(x\xi t)\Phi_S(x,t)y \quad \forall y \in E^+,$$

and $S^\alpha([-2,2] \times E^+) \cap N$ is an isolating neighborhood for the restriction of π to $S^\alpha(]-2,2[\, \times (E^+ \cap X^\alpha))$. It follows that $(S^\alpha)^{-1}(N) = \{(x,y) \in \,]-2,2[\, \times (E^+ \cap X^\alpha) : \; (x, S^\alpha(x)y) \in U\}$ is a strongly π_S-admissible isolating neighborhood for $[-1,1] \times \{0\}$. □

Lemma 6.21. *There exist an isolating neighborhood $N_0 = [a,b] \subset \,]-2,2[$ for $[-1,1]$ relative to ξ and a constant $M \in \mathbb{R}^+$ such that $\|\Phi_S(x,t)y\|_\alpha \le M\|y\|_\alpha$ whenever $y \in E^+ \cap X^\alpha$, $x\xi[0,t]$ is defined and $x\xi[0,t] \subset N_0$.*

PROOF. Let N be given by Lemma 6.20 and choose N_0 small enough that $N_0 \times \{0\} \subset N$. Then every closed set $\tilde{N} \subset N_0 \times (E^+ \cap X^\alpha)$ with $[-1,1] \times \{0\} \subset \operatorname{int} \tilde{N}$ and $\sup_{(x,y)\in\tilde{N}} \|y\|_\alpha < \infty$ is a strongly admissible isolating neighborhood for π_S since we can choose $\varepsilon > 0$ small enough that $\{(x, \varepsilon y) : (x,y) \in \tilde{N}\} \subset N$.
Suppose that the lemma is not true. Then there are sequences $x_n \to x_0$ in N_0 and $y_n \in E^+ \cap X^\alpha$ with $\|y_n\|_\alpha = 1$ and $\tilde{t}_n$ in $\mathbb{R}^+$ such that

$$q_n := \sup_{s\in[0,\tilde{t}_n]} \|\Phi_S(x_n, s)y_n\|_\alpha \to \infty.$$

For every $n \in \mathbb{N}$, there exists a $t_n \in [0, \tilde{t}_n]$ with $\|\Phi_S(x_n, t_n)y_n\| = q_n$.
Assume that $t_n \not\to \infty$, that is, by choosing subsequences we may assume that $t_n \to t_0$, implying that $1 = \left\|\Phi_S(x_n, t_n)q_n^{-1}y_n\right\|_\alpha \to \|\Phi_S(x_0, t_0)0\|_\alpha = 0$, a contradiction, showing that $t_n \to \infty$.
By admissibility, we may further assume that $(x_n, q_n^{-1}y_n)\pi_S t_n \to (x_0, y_0) \in [-1,1] \times (E^+ \cap X^\alpha)$ with $0 \neq y_0$ and $(x_0, y_0) \in \operatorname{Inv}^-(N)$. Lemma 6.19 now implies that $y_0 = 0$, a contradiction. □

4.2. The situation on U. In this section, we will simplify the semiflow on U by constructing a suitable isomorphism.

LEMMA 6.22. *Let $e \in \{-1, 1\}$. Then there exist a neighborhood V of e in $[-2,2]$, a local isomorphism of bundles $\phi_e \in C(V, \mathcal{L}(E^-(e), E^-(e))$, and a $B_e \in \mathcal{L}(E^-(e), E^-(e))$ with $\operatorname{Re}\sigma(A - B_e) < 0$ such that $\phi_e(u(t))v(t)$ is a solution of (the ordinary differential equation in finite dimensions)*

$$\dot{x} + P_e^-(0)(A - B_e)x = 0 \quad x \in E^-(e) \tag{6.11}$$

whenever $(u(t), v(t))$, $t \in [0,T]$, is a solution of π_U with $u(t) \in V$ for all $t \in [0,T]$.

PROOF. Let $B_e := F(e)$ and let $P := P_e^-(0)$ be given by the spectral decomposition of $A - B_e$ (see also section 3). By Lemma A.7, there exists a neighborhood V of e (by possibly shrinking V we may assume that $F(x) = B_e$ for all $e \in V$) such that $p : U(V) \to V \times E^-(e)$, $p(x,y) := (x, Py)$, is a homeomorphism.
Let $(u(t), w(t))$, $t \in [0,T]$, be a solution of (6.11) and let $(u(t), v(t))$, $t \in [0,T]$, be a solution of π_U with $(u(0), w(0)) = p(u(0), U(u(0))v(0))$. Then by (6.7)

$$\begin{aligned}(u(t), w(t)) &= p(u(t), \Phi(u(0), t)w(0)) \\ &= (u(t), PU(u(t))\Phi_U(u(0), t)v(0)) \\ &= (u(t), PU(u(t))v(t)),\end{aligned}$$

so $w(t) = PU(u(t))v(t)$.
Therefore, we can choose $\phi_e(x) := PU(x)$, $x \in V$. □

PROPOSITION 6.23. *There exists a strongly π_U-admissible isolating neighborhood for $(\pi_U, [-1,1] \times \{0\})$.*

PROOF. Let $N \subset]-2,2[\times X^\alpha$ be a strongly π-admissible isolating neighborhood for $[-1,1] \times \{0\}$. We have

$$\Phi(x,t)U(x)y = U(x\xi t)\Phi_U(x,t)y \quad \forall y \in E^-,$$

and $U(]-2,2[\times E^-)\cap N$ is an isolating neighborhood for the restriction of π to $U(]-2,2[\times E^-)$. It follows that $U^{-1}(N)=\{(x,y)\in\,]-2,2[\times E^- : (x,U(x)y)\in N\}$ is a strongly π_U-admissible isolating neighborhood for $[-1,1]\times\{0\}$. □

Recall that $F(x)$ is constant on each of the intervals $[e-\delta,e+\delta]$, $e\in\{-1,1\}$, and let $a_e<b_e$ such that $[a_e,b_e]\subset[e-\delta,e+\delta]$. Further, let $\tau\in\mathbb{R}^+$ such that $b_{-1}\xi\tau=a_1$, and define $V_1 : [-2,a_1]\times E^-(-1)\to U([-2,a_1])$ by

$$V_1(x)y:=\Phi_U(x\xi(-\tau),\tau)\phi_{-1}(x\xi(-\tau))^{-1}y$$

and $V_2 : [a_1,2]\times E^-(1)\to U([a_1,2])$ by

$$V_2(x)y:=\phi_1(x)^{-1}y,$$

where ϕ_e, $e\in\{-1,1\}$, is given by Lemma 6.22.
We can now define $V\in C([-2,2],\mathcal{L}(E^-,E^-))$ (note that $E^-=E^-(-1)$ by definition) by

$$V(x)y:=\begin{cases}V_1(x)y & x\in[-2,a_1]\\ V_2(x)V_2(a_1)^{-1}V_1(a_1)y & x\in[a_1,2].\end{cases}$$

Note that $\operatorname{im}V(x)E^-\subset\operatorname{im}U(x)E^-$ for every $x\in[-2,2]$.
For every $t\in\mathbb{R}^+$ with $x\xi[0,t]\subset[-2,a_1]$ and every $y\in E^-$, we have

$$\begin{aligned}\Phi_U(x,t)V(x)y&=\Phi_U(x,t)V_1(x)y\\&=\Phi_U(x,t)\Phi_U(x\xi(-\tau),\tau)\phi_{-1}(x\xi(-\tau))^{-1}y\\&=\Phi_U(x\xi(-\tau),t+\tau)\phi_{-1}(x\xi(-\tau))^{-1}y\\&=\Phi_U(x\xi(t-\tau),\tau)\Phi_U(x\xi(-\tau),t)\phi_{-1}(x\xi(-\tau))^{-1}y\\&=V_1(x\xi t)\phi_{-1}(x\xi(-\tau+t))\Phi_U(x\xi(-\tau),t)\phi_{-1}(x\xi(-\tau))^{-1}y\end{aligned}\tag{6.12}$$

and for $x\in[a_1,2]$, one obtains

$$\begin{aligned}\Phi_U(x,t)V(x)V_1(a_1)^{-1}V_2(a_1)y&=\Phi_U(x,t)V_2(x)y\\&=\Phi_U(x,t)\phi_1(x)^{-1}y\\&=\phi_1(x)^{-1}\phi_1(x)\Phi_U(x,t)\phi_1(x)^{-1}y\\&=V_2(x)V_1(a_1)^{-1}V_2(a_1)\phi_1(x)\Phi_U(x,t)\phi_1(x)^{-1}y.\end{aligned}\tag{6.13}$$

Consider the following system of ordinary differential equations on $]-2,2[\times E^-$

$$\begin{aligned}\dot x&=1-x^2\\ \dot y&=\begin{cases}G(-1)y:=P^-_{-1}(0)(-A+F(-1))y & x\le a_1\\ G(1)y:=V_1(a_1)^{-1}V_2(a_1)P^-_1(0)(-A+F(1))V_2(a_1)^{-1}V_1(a_1)y & a_1<x.\end{cases}\end{aligned}\tag{6.14}$$

Let $(u(t),v(t))$ be a solution of (6.14) which is defined on $[0,T]$. If $u(t)\in\,]-2,a_1]$ for all $t\in[0,T]$, then $u(t)\xi(-\tau)\in\,]-2,b_{-1}[$ and

$$v(t)=\phi_{-1}(u(t)\xi(-\tau))\Phi_U(u(0)\xi(-\tau),t)\phi_{-1}(u(0)\xi(-\tau))^{-1}.$$

In conjunction with (6.12), we obtain

$$\Phi_U(u(0),t)V(u(0))v(0)=V(u(t))v(t),$$

that is, $(u(t), V(u(t))v(t))$ is a solution of $V[\pi]$. Now, suppose that $u(t) \in [a_1, 2[$ for all $t \in [0, T]$. We have

$$V_2(a_1)^{-1}V_1(a_1)v(t) = \phi_1(u(t))\Phi_U(u(0), t)\phi_{-1}(u(0))^{-1}V_2(a_1)^{-1}V_1(a_1)v(t)$$

Using (6.13), we can conclude that

$$\Phi_U(u(0), t)V(u(0))v(0) = V(u(t))v(t),$$

which shows again that $(u(t), V(u(t))v(t))$ is a solution of $V[\pi]$ lying entirely in U. Therefore, $V^{-1}[\pi_U]$ is induced by mild solutions of (6.14).

Proposition 6.24. *$(\xi, \pi_n) \sim V^{-1}[\pi_U]$, where π_n, $n := \dim E^-$, denotes the flow on E^- which is induced by solutions of $\dot{y} = y$.*

Proof. All eigenvalues λ_i, $i \in \{1, \dots, n\}$, of $G(1)$ and $G(-1)$ are positive real numbers, so there are $T_e \in \mathrm{ISO}(\mathbb{R}^n, E^-)$, $e \in \{-1, 1\}$, such that $G(e)$ is a diagonal matrix, namely

$$G(e) = T_e \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} T_e^{-1}.$$

Let G^ν be defined by

$$G^\nu(e) = T_e \begin{pmatrix} \lambda_1^\nu & & 0 \\ & \ddots & \\ 0 & & \lambda_n^\nu \end{pmatrix} T_e^{-1}.$$

$[-1, 1] \times \{0\}$ is an isolated invariant set relative to χ_ν for all $\nu \in [-1, 1]$, where χ_ν is induced by mild solutions of

$$\dot{x} = 1 - x^2$$
$$\dot{y} = \begin{cases} G(-1)^\nu y & x \le a_1 \\ G(1)^\nu y & a_1 \le x. \end{cases}$$

It follows that $V^{-1}[\pi_U] = \chi_1 \sim \chi_0 = (\xi, \pi_n)$. $\square$

5. Calculation of the homotopy index

Proposition 6.25. *Let F be a Banach space, let $\pi = (\xi, \Phi) \in \mathrm{SK}([-2, 2], F)$ such that*

(1) *$([a, b], \{b\})$ is an isolating block for $(\xi, [-1, 1])$;*
(2) *there exists a constant $1 \le M \in \mathbb{R}^+$ such that $\|\Phi(x, t)y\| \le M\|y\|$ whenever $x\xi[0, t]$ is defined with $x\xi[0, t] \subset [a, b]$.*
(3) *there is a strongly π-admissible isolating neighborhood $\tilde{N}$ for $K := [-1, 1] \times \{0\}$ relative to π with $[a, b] \times \{0\} \subset \tilde{N}$.*

Then $h(\pi, K) = \bar{0}$.

Proof. Let

$$N_1 := \{(x, y) \in [a, b] \times F : \|\Phi(x, t)y\| \le 1 \text{ for all } t \ge 0 \text{ with } x\xi t \le b\}$$

$$N_2 := \{(x, y) \in N_1 : x = b\}.$$

Suppose that N_1 is not closed in $]-2,2[\times F$. Then there is a sequence $(x_n, y_n) \to (x_0, y_0)$ in $[a,b] \times F$ such that $(x_n, y_n) \in N_1$ for all $n \in \mathbb{N}$ and $(x_0, y_0) \notin N_1$. We thus have $\|\Phi(x_0, t_0)y_0\| > 1$ for some $t_0 \in \mathbb{R}^+$ with $x_0\xi t_0 < b$. It follows that $x_n\xi t_0 \leq b$ for all $n \in \mathbb{N}$ sufficiently large. Consequently, we have $\|\Phi(x_n, t_0)y_n\| > 1$ for all n large enough, a contradiction to $(x_n, y_n) \in N_1$. N_2 is closed in N_1 and hence also in $]-2,2[\times F$.
Let $(x,y) \in [a,b] \times F$ with $\|y\| \leq \frac{1}{2M}$ and let $t \in \mathbb{R}^+$ with $x\xi[0,t] \subset [a,b]$. It follows that $\|\Phi(x,t)y\| \leq \frac{1}{2}$ and thus $(x,y) \in N_1$. Hence $[-1,1] \times \{0\} \subset \mathrm{Int} N_1 \setminus N_2$.
Let $(x,y) \in N_2$ that is, $(x,y) = (b,y)$. Then $x\xi t \notin [a,b]$ for all $t \in \mathbb{R}^+$ with $x\xi t$ defined, showing that N_2 is N_1-positively invariant.
Let $(x,y) \in N_1$ and $t \in [0,\infty[$ such that $(x,y)\pi t$ is defined and $(x,y)\pi t \notin N_1$. It follows that $x\xi t > b$, so there is an $s \in [0,t]$ with $x\xi s = b$, showing that N_2 is an exit ramp for N_1. Furthermore, there exists an $\varepsilon > 0$ such that

$$N_1 \subset \tilde{N}_\varepsilon := \{(x, \varepsilon^{-1}y) \in]-2,2[\times F :\ (x,y) \in \tilde{N}\}$$

since $\sup_{(x,y)\in N_1} \|y\| \leq 1$. $\tilde{N}$ is a strongly admissible isolating neighborhood and so is $\tilde{N}_\varepsilon$. This implies that the closed subsets N_1 and $\mathrm{cl}(N_1 \setminus N_2)$ are strongly admissible isolating neighborhoods for (π, K). Hence, (N_1, N_2) is a strongly admissible FM-index pair for (π, K).
Define a homotopy $H(x,y,\lambda) : (N_1, N_2) \times [0,1] \to (N_1, N_2)$ by

$$H(x,y,\lambda) := (x, \lambda y).$$

Let $(x,t) \in [a,b] \times \mathbb{R}^+$ such that $x\xi\,[0,t] \subset [a,b]$. It follows from the linearity of $\Phi(x,t)$ that, given $(x,y) \in N_1$ and $\lambda \in [0,1]$, we also have $(x, \lambda y) \in N_1$. Thus, H is well-defined and

$$[(N_1/N_2, \{[N_2]\})] = [([a,b], \{[b]\})] = \bar{0}.$$

□

COROLLARY 6.26. *$h(\pi, [-1,1] \times \{0\}) = \bar{0}$ for all $\pi \in \mathrm{SK}_2$.*

PROOF. We have $\pi \sim (U \oplus S^\alpha)[\pi_U \oplus \pi_S]$ (see Lemma 6.21) and $\pi_U \oplus \pi_S = V[(\xi, \pi_n)] \oplus \pi_S$ (see Lemma 6.24). Recall that π_n is induced by the differential equation $\dot{y} = y$ on E^- where we set $n := \dim E^-$, so $V[\pi_n] \oplus \pi_S$ can be considered as the product of π_S with π_n. Moreover, $h([-1,1] \times \{0\}, \pi_S) = \bar{0}$ has been proved in Proposition 6.25.
It is well-known (see **[5]**) that in the case of product semiflows the homotopy index equals the smash product of the indices of its factors, that is,

$$h([-1,1] \times \{(0)\}, (\pi_n, \pi_S)) = \Sigma^n \wedge h([-1,1] \times \{0\}, \pi_S) = \Sigma^n \wedge \bar{0} = \bar{0}.$$

□

CHAPTER 7

Orientation of trajectories

In the preceding chapters, it has been shown that, under appropriate assumptions, the homotopy Conley index associated with a heteroclinic solution is zero. Let u be such a solution, that is, let there exist equilibria e^- and e^+ with $u(t) \to e^{\pm}$ as $t \to \pm\infty$. Now, both equilibria have the homotopy index of a pointed sphere whose dimension is the Morse index of the respective equilibrium. The reduced singular homology $\tilde{H}_q(S^n)$ with coefficients in $\mathbb{Z}$ is 0 for all q except for $q = n$, where it is isomorphic to $\mathbb{Z}$.
Considering the long exact attractor-repeller sequence associated with u, the connecting homomorphism is thus either -1 or 1 up to isomorphisms (the choice of generators). This number, which can be interpreted as the orientation of u, is not a property of the homotopy index of $\bar{u} := \mathrm{cl}\{u(t) : \ t \in \mathbb{R}\}$ but rather a relation between two heteroclinic solutions connecting the same pair of equilibria.
In what follows, we will, roughly speaking, define isomorphisms between connected simple systems in the homotopy category of pointed spaces which map a pointed sphere to the categorial Conley index of an equilibrium. Among other things, these isomorphisms are natural with respect to the inner structure of the categorial Conley index and with respect to homotopies, that is, $\mathcal{S}$-continuous changes of semiflows.
Subsequently, we prove a formula relating the connecting homomorphism of u to the linearization of the semiflow along u (using the results of the previous chapters).

1. Preliminaries

1.1. Categories of connected simple systems. For the convenience of the reader, we will recall a few concepts from [**4**]. A *connected simple system* is a small category such that, given any two objects, there is exactly one morphism between them.
Now, let $\mathcal{K}$ be an arbitrary category, and define another category $[\mathcal{K}]$. The objects of $[\mathcal{K}]$ are all subcategories of $\mathcal{K}$ which are connected simple systems. Let L be an object of $[\mathcal{K}]$. In this context, a morphism of L will be called an *inner morphism*. A morphism between $\mathcal{K}_1$ and $\mathcal{K}_2$ in $[\mathcal{K}]$ is a family

$$(f_{A,B})_{A \in \mathrm{Obj}(\mathcal{K}_1)\ B \in \mathrm{Obj}(\mathcal{K}_2)}$$

of morphisms in $\mathcal{K}$ such that

$$\begin{tikzcd} A \arrow[r, "f_{A,B}"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f_{A',B'}"] & B' \end{tikzcd}$$

is commutative where the vertical arrows denote the (unique) inner morphisms in $\mathcal{K}_1$ respectively $\mathcal{K}_2$ (here, we do not follow [**4**] exactly).

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be objects of $[\mathcal{K}]$, A (resp. B) be an object of $\mathcal{K}_1$ (resp. $\mathcal{K}_2$) and f be a morphism between A and B. Then there is exactly one morphism F of $\mathcal{K}$ with $f = F(A, B)$; this morphism is denoted by $[f]$.

Let $\mathcal{TOP}$ denote the category of pointed topological spaces and $\mathcal{HT}$ the corresponding homotopy category, that is, morphisms of $\mathcal{HT}$ are equivalence classes of morphisms in $\mathcal{TOP}$, which are continuous, base-point preserving mappings.

As shown in [**4**], there is a singular homology functor on $[\mathcal{HT}]$. The q-th singular homology is denoted by $\hat{H}_q$ or H_q for short.

1.2. Conley indices as a category. Recall that in [**4**] the *categorial Conley-Morse index* is defined as a connected simple system, the objects of which are certain FM-index pairs of an invariant set admitting a strongly admissible isolating neighborhood.

Let (X, d) be a metric space, π a (local) semiflow on X, and S an isolating invariant set admitting a strongly π-admissible isolating neighborhood. Then there is an FM-index pair (N_1, N_2) for (π, S) with the additional property that $\mathrm{cl}(N_1 \setminus N_2)$ is strongly π-admissible. In this case, we say that (N_1, N_2) is a strongly π-admissible isolating neighborhood for (π, K). Note that, in general, we neither need nor make the stronger assumption that N_1 is strongly π-admissible.

Now, the Conley index $\mathcal{C}(\pi, S)$ of (π, S) is an object of $[\mathcal{HT}]$ (see [**4**]). The objects of $\mathcal{C}(\pi, S)$ are all pointed spaces of the form $(N_1/N_2, \{[N_2]\})$ where (N_1, N_2) is a strongly admissible FM-index pair for (π, S). If $(N_1, N_2) \subset (M_1, M_2)$ are strongly admissible FM-index pairs for (π, S), then the inclusion induced (see [**18**]) morphism $(N_1/N_2, \{[N_2]\}) \to (M_1/M_2, \{[M_2]\})$ is a morphism of $\mathcal{C}(\pi, S)$. Indeed, the morphisms of $\mathcal{C}(\pi, S)$ are completely characterized by this property as shown in [**4**, Lemma 4.8].

We will use $H_q\langle \pi, S\rangle := \hat{H}_q(\mathcal{C}(\pi, S))$ to denote the homology Conley index of (π, S) as defined in [**4**, Definition 4.3]. The notation of π is sometimes omitted. Let $(\tilde{X}, \tilde{d})$ be another metric space, $\tilde{\pi}$ a local semiflow on X, and $\tilde{S}$ be an isolating invariant set admitting a strongly $\tilde{\pi}$-admissible isolating neighborhood. Then, given a morphism $[f]:\ \mathcal{C}(\pi, S) \to \mathcal{C}(\tilde{\pi}, \tilde{\mathcal{S}})$, there is a unique induced morphism $H_q\langle f\rangle := H_q\langle [f]\rangle :\ H_q\langle \pi, S\rangle \to H_q\langle \tilde{\pi}, \tilde{S}\rangle$.

1.3. Linearizable semiflows. Let X be a Banach space and let π' be a global semiflow on X generating a C_0-semigroup of linear operators, that is, for every $t \in \mathbb{R}^+$ the map $T(t):\ X \to X$, $T(t) := x\pi' t$, is linear. We will call such a semiflow *linear*.

Suppose there is a direct sum $X = X_1 \oplus X_2$ of invariant subspaces, X_1 is finite-dimensional, $T(t)$ can be uniquely extended to $t \in \mathbb{R}^-$ to form a C_0-group on X_1, and there are constants $M, \delta \in \mathbb{R}^+ \setminus \{0\}$ such that

$$\|T(t)x\| \leq Me^{\delta t}\|x\| \qquad x \in X_1 \quad t \in \mathbb{R}^- \tag{7.1}$$

$$\|T(t)x\| \leq Me^{-\delta t}\|x\| \qquad x \in X_2 \quad t \in \mathbb{R}^+. \tag{7.2}$$

These are the assumptions of [**18**, Theorem I.11.1]. Letting $V^+(x)$ and $V^-(x)$ be defined as in the proof of this theorem, there exists a $\rho \in \mathbb{R}^+$ such that $N_1 := \{x \in X :\ V^+(x) \leq \rho \text{ and } V^-(x) \leq \rho\}$ and $N_2 := \{x \in N_1 :\ V^+(x) = \rho\}$ defines a strongly π'-admissible FM-index pair (N_1, N_2).

Suppose that $\mathcal{U} \subset X$ is an open neighborhood of 0, π a semiflow on $\mathcal{U}$, and $\{0\}$ an isolated invariant set relative to π admitting a strongly π-admissible isolating neighborhood.

DEFINITION 7.1. Let $P := P_\pi : X \to X_1$ denote the projection with $\ker P = X_2$. π is called *strongly linearizable* (at 0) if there exists an $\mathcal{S}$-continuous family $(\pi_\lambda, \{0\})_{\lambda \in [0,1]}$ such that

(1) $\pi_1 = \pi$ and
(2) π_0 is a linear semiflow for which the assumptions above hold;
(3) for every $\lambda \in [0,1]$, there exists a neighborhood $U = U_\lambda$ of 0 such that $\|x_n\|^{-1} P x_n \to 0$ whenever $x_n \in \mathrm{Inv}^+_{\pi_\lambda}(U) \setminus \{0\}$ is a sequence with $x_n \to 0$ as $n \to \infty$.

$\pi' := \pi_0$ is called a linearization of π.

Roughly speaking, the above notion of being strongly linearizable holds for hyperbolic equlibria of our parabolic evolution equations.

PROPOSITION 7.2. *Suppose that the semiflow π on $\mathcal{U} \subset X^\alpha$ is given by mild solutions of a semilinear parabolic equation*

$$\dot{x} + Ax = f(x)$$

such that

(1) *A is sectorial and has compact resolvent;*
(2) *$f : \mathcal{U} \to X$ is locally Lipschitz; $f(0) = 0$, f has a Fréchet derivative $Df(0)$ at 0;*
(3) *$L := A - Df(0)$ is hyperbolic.*

Then π is strongly linearizable.

PROOF. For $\lambda \in [0,1]$, let

$$f_\lambda(x) := (1-\lambda)(f(x) - Df(0)x)$$

and π_λ be the semiflow defined by mild solutions of

$$\dot{x} + Lx = f_\lambda(x),$$

and note that $\pi_1 = \pi$ and $f_0 \equiv 0$.
Then $(\pi_\lambda, \{0\})_{\lambda \in [0,1]}$ is an $\mathcal{S}$-continuous family [**18**, Theorem II.3.5]. As before, let $X = X_1 \oplus X_2$, where X_1 belongs to $\{\mathrm{Re}\,\sigma(l) < 0\}$ and X_2 to $\{\mathrm{Re}\,\sigma(L) > 0\}$. This decomposition of X is the same for all $\lambda \in [0,1]$ since $Df_\lambda(0) = 0$ for all $\lambda \in [0,1]$. Let $P^-(0) : X \to X_1$ and $P^+(0) : X \to X_2$ denote the associated projections.
Let $\lambda \in [0,1]$ be arbitrary but fixed. For $\rho > 0$, set

$$U_{\rho,\lambda} := U_\rho := \{x \in X^\alpha : \left\|P^-(0)(x)\right\|_\alpha + \left\|P^+(0)(x)\right\|_\alpha \le \rho\}.$$

It follows from [**13**, Theorem 5.2.1] that $\mathrm{Inv}^+(U_\rho) \subset S$ provided that ρ is small enough. Here, S denotes the local stable manifold as defined [**13**, Theorem 5.2.1]. It ist tangent to X_2, which means that $\|x_n\|_\alpha^{-1} P(x_n) = \|x_n\|_\alpha^{-1}(x_n - P^+(0)(x_n)) \to 0$ whenever x_n is a sequence in $S \setminus \{0\}$ with $x_n \to 0$ in X^α.
This proves that π_λ is a sequence which satisfies Definition 7.1, so π is indeed strongly linearizable. $\square$

DEFINITION 7.3. Let $f(x) := x - a$ be defined in a neighborhood of a in X^α. Then π is called *strongly linearizable* in a if $f[\pi]$ is strongly linearizable.

2. Orientations and seeds

Throughout this section, let X be a metric space, $e \in X$, and π a local semiflow defined in a neigborhood of e in X such that $\{e\}$ is an isolated invariant set admitting a strongly π-admissible isolating neighborhood.
Until further notice, we set $\mathbb{R}^0 = \{0\} \subset \mathbb{R}$, $D^0 := \{0\}$, and $S^{-1} := \emptyset$. For $n \in \mathbb{N}$, $\mathcal{S}^n$ is an object of $[\mathcal{HT}]$ (a connected simple system), which has itself only one object, namely $(D^n/S^{n-1}, \{[S^{n-1}]\})$, and exactly one morphism: the identity $\mathrm{id}: (D^n/S^{n-1}, \{[S^{n-1}]\}) \to (D^n/S^{n-1}, \{[S^{n-1}]\})$.

Definition 7.4. An (n-)orientation is an ismorphism $o: \mathcal{S}^n \to \mathcal{C}(\pi, S)$ in $[\mathcal{HT}]$.

We will now develop a method which is based on continuous mappings $\mathbb{R}^n \to X$ to obtain orientations or, depending on the point of view, to describe them. These mappings are called *seeds*, and they may or may not induce orientations.
Before defining them, we will introduce a few additional notational shortcuts: A/B denotes the pair $(A/B, [B])$, that is, the explicit notation of the basepoint is omitted in order to keep certain diagrams readable. For every FM-index pair (N_1, N_2) for $(\pi, \{e\})$, define $N_2^{-s} := N_2^{-s}(N_1) := \{x \in N_1 :\ \exists t \in [0, s]\ x\pi t \in N_2\}$ and $N_2^{-\infty} := N_2^{-\infty}(N_1) := \{x \in N_1 :\ \exists t \in \mathbb{R}^+\ x\pi t \in N_2\}$, that is, $N_2^{-\infty} = \bigcup_{s\in\mathbb{R}^+} N_2^{-s}$ (see also [**4**, Proposition 4.6]).

Definition 7.5. Let $n \in \mathbb{N} \cup \{0\}$, $U \subset \mathbb{R}^n$, $f: U \to X$ continuous with $f(0) = e$, and for every strongly π-admissible FM-index pair (N_1, N_2) let there exist a $\lambda \in \mathbb{R}^+$ such that $f^\lambda(x) := f(\lambda x)$ is defined for all $x \in D^n$ and $f^\lambda(x) \in N_2^{-\infty}$ for all $x \in D^n \setminus \{0\}$. Then f is called a *seed* for (π, e).

However, in view of this definition it is not clear whether seeds exist. We will see later on that sufficient conditions for the existence of seeds and natural choices for them exist. Note that U is necessarily a neighborhood of 0 if $f: U \to X$ is a seed.

Lemma 7.6. *Let (N_1, N_2) be a strongly π-admissible FM-index pair for $(\pi, \{e\})$, $\lambda \in \mathbb{R}^+$, and f be a seed such that $f^\lambda(x)$ is defined for all $x \in D^n$ and $f^\lambda(D^n) \subset N_1$.*
Let

$$\Omega := \{g : D^n \to N_1 :\ g \text{ is continuous and } g(0) = e\}$$

be equipped with the maximum metric.
Then there is an $s \in \mathbb{R}^+$ and a neighborhood U_{f^λ} of $f^\lambda_{|D^n}$ in Ω such that $g(D^n, S^{n-1}) \subset (N_1, N_2^{-s})$ for all $g \in U_{f^\lambda}$.

Proof. Let $\tau(x, g) := \sup\{t \in \mathbb{R}^+ :\ g(x)\pi t \in \mathrm{cl}(N_1 \setminus N_2)\}$. We have $\tau(x, f^\lambda) < \infty$ for all $x \in S^{n-1}$ because f is a seed.
Let $x \in S^{n-1}$ and $\varepsilon \in]0, 1]$ with $f^\lambda(x)\pi(\tau(x) + \varepsilon) \in X \setminus \mathrm{cl}(N_1 \setminus N_2)$. Since $X \setminus \mathrm{cl}(N_1 \setminus N_2)$ is an open set, there exist neighborhoods V_x of x in D^n and U_{x,f^λ} of $f^\lambda_{|D^n} \in \Omega$ such that $g(x)\pi(\tau(x) + \varepsilon) \notin \mathrm{cl}(N_1 \setminus N_2)$ for all $(x, g) \in V_x \times U_{x,f^\lambda}$, showing that $\tau(\xi, g) \le \tau(x) + \varepsilon \le \tau(x) + 1$ for all $(\xi, g) \in V_x \times U_{x,f^\lambda}$.
Due to the compactness of S^{n-1}, there are $x_1, \dots, x_n \in S^{n-1}$ such that $S^{n-1} \subset \bigcup_{k=1,\dots,n} V_{x_k}$.
Letting $\tilde{U}_{f^\lambda} := \bigcap_{k=1,\dots,n} U_{x_k,f^\lambda}$, it follows that
$\tau(x, g) \le \max_{k=1,\dots,n} \tau(x_k, f^\lambda) + 1 =: s$ for all $(x, g) \in S^{n-1} \times \tilde{U}_{f^\lambda}$. Hence, for every

$(x, g) \in S^{n-1} \times \tilde{U}_{f^\lambda}$ we have $g(x) \in N_1$ and $g(x)\pi r \notin N_1$ for some $r \in [0, s]$, showing that $g(x) \in N_2^{-s}$. □

LEMMA 7.7. *Let $f : U \to X$, $U \subset \mathbb{R}^n$, be continuous with $f(0) = e$, and suppose that there exist a strongly π-admissible FM-index pair (N_1, N_2) for $(\pi, \{e\})$ and a $\lambda \in \mathbb{R}^+$ such that $f^\lambda(x)$ is defined for all $x \in D^n$ and $f^\lambda(D^n \setminus \{0\}) \subset N_2^{-\infty}$.*
Then f is a seed.

PROOF. Let (M_1, M_2) be a strongly π-admissible FM-index pair for $(\pi, \{e\})$ and let $x \in D^n \setminus \{0\}$. By our assumptions, there exists another (possibly the same) strongly π-admissible FM-index pair (N_1, N_2) for $(\pi, \{e\})$ and a $\lambda \in \mathbb{R}^+$ with $f^\lambda(D^n \setminus \{0\}) \subset N_2^{-\infty}$. The set $N := \mathrm{cl}(N_1 \setminus N_2)$ is an isolating neighborhood for $(\pi, \{e\})$, and $(\tilde{N}_1, \tilde{N}_2) := (N_1 \cap N, N_2 \cap N)$ is again a strongly admissible FM-index pair.
By the continuity of f and because $e \in \operatorname{int} N$, there is a $\tilde{\lambda} \in]0, \lambda]$ such that $f^{\tilde{\lambda}}(D^n) \subset \tilde{N}_1$. We have $N_1 \setminus N_2 = \tilde{N}_1 \setminus \tilde{N}_2$, showing that $f^{\tilde{\lambda}}(D^n \setminus \{0\}) \subset \tilde{N}_2^{-\infty}$.
It follows from [**4**, Lemma 4.8] that there are an $s \in \mathbb{R}^+$ and a strongly π-admissible FM-index pair (L_1, L_2) for $(\pi, \{0\})$ such that L_1 is an isolating neighborhood for $(\pi, \{e\})$ and

$$(M_1, M_2) \subset (M_1, M_2^{-s}) \supset (L_1, L_2) \subset (\tilde{N}_1, \tilde{N}_2^{-s}) \supset (\tilde{N}_1, \tilde{N}_2).$$

We can choose $\tilde{\tilde{\lambda}} \in \left]0, \tilde{\lambda}\right]$ such that $f^{\tilde{\tilde{\lambda}}}(D^n) \subset L_1$.
For every $x \in D^n$, it follows that $f^{\tilde{\tilde{\lambda}}}(x)\pi t \notin \tilde{N}_1 \supset L_1$ for some $t \in \mathbb{R}^+$ because $\tilde{N}_1$ is an isolating neighborhood and $f^{\tilde{\tilde{\lambda}}}(x) \in \tilde{N}_2^{-\infty}$ for all $x \in D^n$. Hence, there exists an $r \in [0, t]$ with $f^{\tilde{\tilde{\lambda}}}(x)\pi r \in L_2$, showing that $f^{\tilde{\tilde{\lambda}}}(x) \in M_2^{-\infty}$. □

DEFINITION 7.8. Let (N_1, N_2) be a strongly π-admissible FM-index pair for $(\pi, \{e\})$, f a seed, and $\lambda \in \mathbb{R}^+$ such that $f^\lambda(x)$ is defined for all $x \in D^n$ and $f^\lambda(D^n \setminus \{0\}) \subset N_2^{-\infty}$.
$\overline{f} := \overline{f}_{N_1, N_2} :\ D^n/S^{n-1} \to N_1/N_2$ denotes the unique morphism in $\mathcal{HT}$ for which

$$\begin{tikzcd} N_1/N_2^{-s} & N_1/N_2 \arrow[l, "\supset"'] \\ D^n/S^{n-1} \arrow[u, "f^\lambda"] \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

commutes whenever $f^\lambda(S^{n-1}) \subset N_2^{-s}$, $s \in \mathbb{R}^+$.

The subscript of $\overline{f}$, although important, is often omitted when the FM-index pair is clear from the context.

DEFINITION 7.9. Let f be a seed for (π, e), and let $\langle f, \pi, e\rangle :\ \mathcal{S}^n \to \mathcal{C}(\pi, \{e\})$ denote the morphism in $[\mathcal{HT}]$ for which

$$\langle f, \pi, e\rangle((D^n/S^{n-1}, \{[S^{n-1}]\}), (N_1/N_2, \{[N_2]\})) = \overline{f}_{N_1, N_2}$$

whenever (N_1, N_2) is a strongly admissible FM-index pair for $(\pi, \{e\})$.
Since $e = f(0)$ by the assumption of f being a seed, we will sometimes write $\langle f, \pi\rangle$.

LEMMA 7.10. *Let (N_1, N_2) be a strongly admissible FM-index pair for $(\pi, \{e\})$. Then $\overline{f}$: $D^n/S^{n-1} \to N_1/N_2$ is well-defined.*

PROOF. There are two parameters involved in Definition 7.8, s and λ. First, we will consider s. Given $r, s \in \mathbb{R}^+$ with $f^\lambda(S^{n-1}) \subset N_2^{-r} \subset N_2^{-s}$, there is a commutative diagram

$$\begin{tikzcd}
N_1/N_2^{-s} & N_1/N_2^{-r} \arrow[l, "\supset"'] & N_1/N_2 \arrow[l, "\supset"'] \\
D^n/S^{n-1} \arrow[u, "f^\lambda"] \arrow[ur, "f^\lambda"] \arrow[urr, "\overline{f}"'] & &
\end{tikzcd}$$

showing that r and s induce the same morphism $\overline{f}$.
Second, it follows from Lemma 7.6 that, for every $\mu \in]0, \lambda]$ there are an $s \in \mathbb{R}^+$ and a neighborhood U of $f^\mu_{|D^n}$ in Ω such that

$$\begin{tikzcd}
N_1/N_2^{-s} & N_1/N_2 \arrow[l, "\supset"'] \\
D^n/S^{n-1} \arrow[u, bend left, "{[f^\mu]}"] \arrow[u, bend right, "g"'] \arrow[ur, "\overline{f}"] &
\end{tikzcd}$$

is defined for all $g \in U$ and commutative whenever g is homotopic to $[f^\mu]$. Since $f^{\tilde{\mu}}(D^n) \subset f^\mu(D^n) \subset N_1$ for all $\tilde{\mu} \leq \mu$, one has $f^{\tilde{\mu}} \in U$ for all $\tilde{\mu} \leq \mu$ large enough. Hence, $\mu \mapsto \overline{f^\mu}_{N_1,N_2}$ is locally constant on $]0, \lambda]$, which is connected. $\square$

Using [**18**, Proposition I.8.2], it is easy to give a direct formula for $\overline{f}$. Let f be a seed for (π, e), (N_1, N_2) be a strongly admissible FM-index pair for $(\pi, \{e\})$, and $\lambda \in \mathbb{R}^+$ be sufficiently small that $f^\lambda(D^n)$ is defined and $f^\lambda(D^n) \subset N_1$. Then, $\overline{f} = [g]_{\mathcal{HT}}$ where $g:\ D^n/S^{n-1} \to N_1/N_2$,

$$g([x]) := \begin{cases} [f^\lambda(x)\pi s] & f^\lambda(x)\pi[0,s] \text{ is defined and } f^\lambda(x)\pi[0,s] \subset N_1 \setminus N_2 \\ [N_2] & \text{otherwise.} \end{cases}$$

LEMMA 7.11. *Let*

$$\Omega := \{g :\ D^n \to X :\ g \text{ is a seed for } (\pi, e)\}$$

be equipped with the maximum metric.
Then $g \mapsto \overline{g}$, is constant on path components of Ω.

PROOF. Let $\lambda \mapsto g_\lambda$, $[0, 1] \to \Omega$ be continuous. It is sufficient to show that $g \mapsto \overline{g}$ is locally constant.
Let (N_1, N_2) be a strongly admissible FM-index pair for $(\pi, \{e\})$ and $\lambda_0 \in [0, 1]$. There exists a $\mu > 0$ such that

$$g^\mu_{\lambda_0}(D^n) \subset \operatorname{int}(N_1 \setminus N_2).$$

Hence, there is a neighborhood U of g_{λ_0} in Ω such that

$$h^\mu(D^n) \subset \operatorname{int}(N_1 \setminus N_2)$$

for all $h \in U$. By Lemma 7.6, there is another neighborhood $\tilde{U} \subset U$ of g_{λ_0} in Ω and an $s \in \mathbb{R}^+$ such that

$$h^\mu(S^{n-1}) \subset N_2^{-s}$$

for all $h \in \tilde{U}$. The continuity of $\lambda \mapsto g_\lambda$ now implies that there is a neighborhood of V of λ_0 in $[0,1]$ such that $g_\lambda \in \tilde{U}$ for all $\lambda \in V$, so

$$\begin{tikzcd} D^n/S^{n-1} \arrow[r, bend left, "g_\lambda"] \arrow[r, bend right, "g_{\lambda_0}"'] & N_1/N_2^{-s} \end{tikzcd}$$

is defined and commutative. This shows that $\overline{g_\lambda}$ is constant on V. □

Lemma 7.12. *Let (N_1, N_2) and (M_1, M_2) be strongly admissible FM-index pairs for $(\pi, \{e\})$ and f a seed.*
Then

$$\begin{tikzcd} M_1/M_2 \arrow[r, "\alpha"] & N_1/N_2 \\ D^n/S^{n-1} \arrow[u, "\overline{f}"] \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

commutes, where α denotes the inner morphism of the categorial Conley index.

Proof. In view of [**4**, Lemma 4.8], it is sufficient to prove our claim in the special case $(M_1, M_2) \subset (N_1, N_2)$. It follows immediately from the definitions of M_2^{-s} and N_2^{-s} that $M_2^{-s} \subset N_2^{-s}$ for all $s \in \mathbb{R}^+$.
By Lemma 7.6, we may choose $s \in \mathbb{R}^+$ and $\lambda \in [0,1]$ such that

$$\begin{tikzcd} M_1/M_2 \arrow[r, "\subset"] \arrow[d, "\subset"] & N_1/N_2 \arrow[d, "\subset"] \\ M_1/M_2^{-s} \arrow[r, "\subset"] & N_1/N_2^{-s} \\ D^n/S^{n-1} \arrow[r, "\mathrm{id}"] \arrow[u, "f^\lambda"] & D^n/S^{n-1} \arrow[u, "f^\lambda"'] \end{tikzcd} \tag{7.3}$$

is defined and commutative. Consequently, composing the vertical arrows,

$$\begin{tikzcd} M_1/M_2 \arrow[r, "\subset"] & N_1/N_2 \\ D^n/S^{n-1} \arrow[r, "\mathrm{id}"] \arrow[u, "\overline{f}"] & D^n/S^{n-1} \arrow[u, "\overline{f}"'] \end{tikzcd}$$

commutes by Definition 7.8. □

Proposition 7.13. *Let $(\pi_k)_{k \in \mathbb{N} \cup \{\infty\}}$ be a family of semiflows such that $\pi_k \to \pi_\infty := \pi$ and let $(N_{1,\infty}, N_{2,\infty})$, $(\tilde{N}_{1,\infty}, \tilde{N}_{2,\infty})$ be strongly π_∞-admissible FM-index pairs for $(\pi_\infty, \{e\})$ such that $N_{1,\infty}$ is a strongly admissible isolating neighborhood for $(\pi_\infty, \{e\})$.*
Further, let $(N_{1,k}, N_{2,k})_{k \in \mathbb{N}}$, $(\tilde{N}_{1,k}, \tilde{N}_{2,k})_{k \in \mathbb{N}}$ be families of strongly π_n-admissible FM-index pairs for $(\pi_k, \{e\})$ such that

$$(\tilde{N}_{1,k}, \tilde{N}_{2,k}) \subset (\tilde{N}_{1,\infty}, \tilde{N}_{2,\infty}) \subset (N_{1,k}, N_{2,k}) \subset (N_{1,\infty}, N_{2,\infty})$$

for all $k \in \mathbb{N}$.
Finally, let $f : D^n \to X$ be a common seed, that is, for every $k \in \mathbb{N} \cup \{\infty\}$ it holds that f is a seed for (π_k, e).

Then there is an $n_0 \in \mathbb{N}$ *such that*

$$\begin{tikzcd} \tilde{N}_{1,k}/\tilde{N}_{2,k} \arrow[rr, "\subset"] & & N_{1,l}/N_{2,l} \\ & D^n/S^{n-1} \arrow[ul, "\overline{f}"] \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

is commutative for all $k, l \in \mathbb{N} \cup \{\infty\}$ *with* $k, l \geq n_0$.

Lemma 7.14. *Let* $\lambda \in]0, 1]$ *such that* $f^\lambda(D^n) \subset N_{1,k}$ *and* $r \in \mathbb{R}^+$. *Then:*

(1) $M := M_{\lambda,r} := \{(x, s) \in S^{n-1} \times [0, r] :\ f^\lambda(x)\pi[0, s] \subset \tilde{N}_{1,\infty}\}$ *is compact.*
(2) $g := g_\lambda :\ [0, r] \times D^n \to \tilde{N}_{1,\infty}/\tilde{N}_{2,\infty}$,

$$g(s, x) := \begin{cases} f^\lambda(x)\pi s & f^\lambda(x)\pi_\infty[0, s] \text{ is defined and } f^\lambda(x)\pi[0, s] \subset \tilde{N}_{1,\infty} \\ [\tilde{N}_{2,\infty}] & \text{otherwise} \end{cases}$$

is continuous.
(3) *There is a* $\tau \in \mathbb{R}^+$ *such that* $g([0, r] \times S^{n-1}) \subset N_{2,k}^{-\tau}/\tilde{N}_{2,\infty}$ *for all* $k \in \mathbb{N} \cup \{\infty\}$ *sufficiently large.*

Proof. (1) $S^{n-1} \times [0, r]$ is compact, so it suffices to prove that M is closed. Let $(x_k, s_k) \to (x_0, s_0)$ in M and $s \in [0, s_0[$. It follows that for all $k \in \mathbb{N}$ large enough $s_k > s$, so $x_k\pi s \in \tilde{N}_{1,\infty}$ and $x_0\pi s \in \tilde{N}_{1,\infty}$. Hence, by the closedness of $\tilde{N}_{1,\infty}$, we have $x_0\pi[0, s_0] \subset \tilde{N}_{1,\infty}$, and thus $(x_0, s_0) \in M$.

(2) This follows from [**18**, Proposition I.8.1].

(3) Let $\tilde{M} := \{(f^\lambda(x), s) :\ (x, s) \in M\}$, $x \in \pi(\tilde{M})$, and note that $\pi(\tilde{M}) \subset \tilde{N}_{1,\infty} \subset N_{1,k}$ for all $k \in \mathbb{N}$. By the assumption that $N_{1,\infty}$ is a (strongly admissible) isolating neighborhood of $\{e\}$ relative to π, there is a $t = t_x \in \mathbb{R}^+$ such that $f^\lambda(x)\pi t \in X \setminus N_{1,\infty}$. Otherwise, there would be a full solution of π lying entirely in $N_{2,\infty}$ (using the strong admissibility), contradicting the assumption of $N_{1,\infty}$ being an isolating neighborhood.

Hence, there are $n_0 = n_0(x)$ and a neighborhood U_x of x in $\pi(\tilde{M})$ such that $U_x\pi_k t \in X \setminus N_{1,\infty} \subset X \setminus N_{1,k}$ for all $k \geq n_0$. Consequently, for every $x \in U_x$, there is an $r \in [0, t_x]$ with $x\pi_k r \in N_{2,k}$. The compactness of $\pi(\tilde{M})$ implies that there are $x_1, \ldots, x_N \in \pi(\tilde{M})$ with $\pi(\tilde{M}) \subset \bigcup_{i=1,\ldots,N} U_{x_i}$. We can choose $\tau := \max_{i=1,\ldots,N} t_{x_i}$ and $n_0 := \max_{i=1,\ldots,N} n_0(x_i)$.

Since for every $(s, x) \in \mathcal{D}(g)$ one has either $g(s, x) \in \pi(\tilde{M})$ or $g(s, x) = [\tilde{N}_{2,\infty}]$, it follows that $g(s, x) \in N_{2,k}^{-\tau}/\tilde{N}_{2,\infty}$ for all $k \geq n_0$.

□

Proof of Proposition 7.13. Let $\tau \in \mathbb{R}^+$ be given by Lemma 7.14, and assume that $f^\lambda(S^{n-1}) \subset N_{2,\infty}^{-s}$ for $\lambda \in [0, 1]$ and $s \in \mathbb{R}^+$. It follows that there is an $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$

$$\begin{tikzcd}
D^n/S^{n-1} \arrow[r, "f^\lambda"] \arrow[d, "\mathrm{id}"] & \tilde{N}_{1,\infty}/\tilde{N}^{-s}_{2,\infty} \\
D^n/S^{n-1} \arrow[r, "{g(s,.)}"] \arrow[d, "\mathrm{id}"] & \tilde{N}_{1,\infty}/\tilde{N}_{2,\infty} \arrow[u, "\subset"'] \arrow[d, "\subset"] \\
D^n/S^{n-1} \arrow[r, "{g(s,.)}"] \arrow[d, "\mathrm{id}"] & N_{1,k}/N^{-\tau}_{2,k} \arrow[d, "\mathrm{id}"] \\
D^n/S^{n-1} \arrow[r, bend left, "{g(0,.)}"] \arrow[r, bend right, "f^\lambda"] & N_{1,k}/N^{-\tau}_{2,k}
\end{tikzcd}$$

commutes in $\mathcal{HT}$.
This shows that

$$\begin{tikzcd}
N_{1,k}/N_{2,k} & \tilde{N}_{1,\infty}/\tilde{N}_{2,\infty} \arrow[l, "\supset"'] \arrow[r, "\subset"] & N_{1,l}/M_{2,l} \\
& D^n/S^{n-1} \arrow[ul, "\overline{f}"] \arrow[u, "\overline{f}"] \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

commutes for all $k, l \geq n_0$.
It follows from Lemma 7.12 that

$$\begin{tikzcd}
\tilde{N}_{1,k}/\tilde{N}_{2,k} \arrow[r, "\subset"] \arrow[rr, bend left, "\subset"] & \tilde{N}_{1,\infty}/\tilde{N}_{2,\infty} \arrow[r, "\subset"] & N_{1,k}/N_{2,k} \\
& D^n/S^{n-1} \arrow[ul, "\overline{f}"] \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

is commuatative and thus also

$$\begin{tikzcd}
\tilde{N}_{1,k}/\tilde{N}_{2,k} \arrow[r, "\subset"] & \tilde{N}_{1,\infty}/\tilde{N}_{2,\infty} \arrow[r, "\subset", "\approx"'] & N_{1,k}/N_{2,k} \\
& D^n/S^{n-1}, \arrow[ul, "\overline{f}"] \arrow[u, "\overline{f}"] \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

where $\approx$ indicates an isomorphism.
Finally, we conclude that

$$\begin{tikzcd}
\tilde{N}_{1,k}/\tilde{N}_{2,k} \arrow[r, "\subset"] & \tilde{N}_{1,\infty}/\tilde{N}_{2,\infty} \arrow[r, "\subset"] & N_{1,l}/N_{2,l} \\
& D^n/S^{n-1} \arrow[ul, "\overline{f}"] \arrow[u, "\overline{f}"] \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

commutes for all $k, l \geq n_0$. $\square$

THEOREM 7.15. *Let Λ be a connected metric space and let $(\pi_\lambda, \{e\})_{\lambda\in\Lambda}$ be an $\mathcal{S}$-continuous family such that there exists a common seed $f : D^n \to X$.*
If there exists a $\lambda_0 \in \Lambda$ such that $\langle f, \pi_{\lambda_0}, e\rangle$ is an isomorphism, then $\langle f, \pi_\lambda, e\rangle$ is an isomorphism for all $\lambda \in \Lambda$.

PROOF. Let $\chi : \Lambda \to \{0,1\}$ be defined by

$$\chi(\lambda) := \begin{cases} 1 & \langle f, \pi_\lambda, e\rangle \text{ is an isomorphism} \\ 0 & \text{otherwise.} \end{cases}$$

It follows from [**18**, Theorem I.12.3] and Proposition 7.13 that χ is locally constant on Λ, which is a connected metric space. □

3. Orientation for fixed points of linearizable semiflows

Throughout this section, let X be a Banach space and π, π' be strongly linearizable semiflows defined in a neighborhood of 0. Moreover, suppose that π' is a linear semiflow, and let $n = \dim X_1$. Recall that the subspaces $X_1 = P_\pi X$ depends on the semiflow π. We will use the notation introduced in the *Preliminaries* section.

LEMMA 7.16. *Let $f : D^n \to X$ be continuous, $f(0) = 0$, and $0 < \theta \in \mathbb{R}^+$ such that*

$$\|P_\pi \circ f(x)\| > \theta \|f(x)\| \tag{7.4}$$

for all $x \neq 0$ in a sufficiently small neighborhood of 0.
Then f is a seed for $(\pi, 0)$.

PROOF. Suppose that f is not a seed.
By Definition 7.1, there exists a neighborhood U of 0 such that

$$\|y_n\|^{-1} P_\pi(y_n) \to 0 \tag{7.5}$$

whenever y_n is a sequence in $\mathrm{Inv}^+(U) \setminus \{0\}$ with $x_n \to 0$ as $n \to \infty$.
Let (N_1, N_2) be a strongly admissible FM-index pair with $N_1 \subset U$. By Lemma 7.7, there is a sequence $0 \neq x_n \to 0$ such that $f(x_n) \subset \mathrm{Inv}^+(N_1) \subset \mathrm{Inv}^+(U)$ for all $n \in \mathbb{N}$.
We have $f(x_n) \neq 0$ for all $n \in \mathbb{N}$ by (7.4). Since π is strongly linearizable, it follows from (7.5) that

$$\|f(x_n)\|^{-1} P_\pi \circ f(x_n) \to 0,$$

a contradiction to (7.4). □

REMARK 3. (7.4) *holds if $f(x) \in X_1$ for all $x \in X$. Moreover,* (7.4) *also holds if f has a Fréchet-derivative $Df(0)$ at 0 with* $\ker P \circ Df(0) = \{0\}$.

COROLLARY 7.17. *Let $f : D^n \to X_1$ be continuous and injective with $f(0) = 0$.*
Further, let Λ be a connected metric space and let $(\pi_\lambda)_{\lambda \in \Lambda}$ be an $\mathcal{S}$-continuous family of strongly linearizable semiflows with $X_1 = X_1(\pi_\lambda)$ being constant.
Then f is a seed for $(\pi_\lambda, 0)$ for all $\lambda \in \Lambda$. Furthermore, if there is a $\lambda_0 \in \Lambda$ such that $\langle f, \pi_{\lambda_0}, 0\rangle$ is an isomorphism, then $\langle f, \pi_\lambda, 0\rangle$ is an isomorphism for all $\lambda \in \Lambda$.

PROOF. f is a seed for every π_λ by Lemma 7.16 and the remark thereafter. Thus, the claim follows from Theorem 7.15. □

PROPOSITION 7.18. *Let (N_1, N_2) be a strongly π'-admissible FM-index pair for $(\pi', \{0\})$, and let $f : D^n \to X_1$ be injective and continuous with $f(0) = 0$.*
Then $\overline{f} : D^n / S^{n-1} \to N_1/N_2$ is an isomorphism in the homotopy category of pointed spaces.

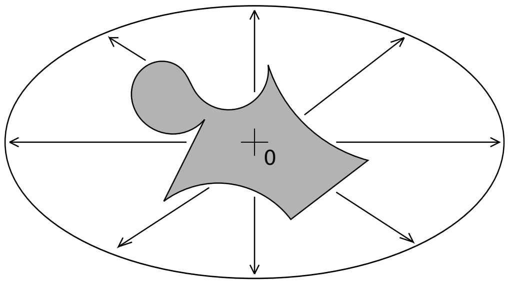

FIGURE 7.1. Homotopy of a seed

PROOF. It is shown in the proof of [**18**, Theorem I.11.1] that there exists an isolating block $B = B_1 \oplus B_2$ with

$$B_1 = \{x \in X_1 : V^+(x) \leq 1\}$$
$$B_2 = \{x \in X_2 : V^-(x) \leq 1\}$$

and $B^- = \partial B_1 \oplus B_2$. $B_1/\partial B_1$ is a strong deformation retraction of B/B^-, that is, the inclusion induced mapping

$$(B_1/\partial B_1, [B_1]) \xrightarrow{\subset} (B/B^-, [B^-])$$

is an isomorphism in the homotopy category of pointed spaces.
There exists a $\lambda \in]0,1]$ such that f^λ is injective and $f^\lambda(D^n) \subset \operatorname{int} B_1$. Moreover, there is a continuous functional $\rho : D^n \setminus \{0\} \to \mathbb{R}^+$ with $f^\lambda(x)\pi'(\rho(x)) \in \partial B_1$ for all $x \in D^n \setminus \{0\}$ (see [**18**, Lemma 3.8]).
Define $g : [0,1] \times D^n \to X_1$ by

$$g(\mu, x) := \begin{cases} f^\lambda(x)\pi'(\mu\,\kappa(x)\,\rho(x)) & x \neq 0 \\ 0 & x = 0, \end{cases}$$

where $\kappa : D^n \to [0,1]$ is continuous, $\kappa(x) = 1$ for all $x \in S^{n-1}$, and there is a neighborhood U of 0 in D^n with $\kappa(x) = 0$ for all $x \in U$. This is illustrated in Figure 7.1: the grey area shows the image of f^λ, the arrows indicate the flow on B_1. Lemma 7.16 and the remark thereafter imply that $g(\mu, .)$ is a seed for every $\mu \in [0,1]$. It follows from Lemma 7.11 that $\overline{f} = \overline{g(0,.)} = \overline{g(1,.)}$.
Both spaces, D^n/S^{n-1} and $B_1/\partial B_1$ are homeomorphic to S^n. Let h be induced by the following commutative diagram in the category of pointed spaces, where the vertical arrows denote isomorphisms:

$$\begin{tikzcd} (D^n/S^{n-1}, 0) \arrow[r, "g(1,.)"] \arrow[d, "\approx"] & (B_1/\partial B_1, 0) \arrow[d, "\approx"] \\ (S^n, o) \arrow[r, "h"] & (S^n, o). \end{tikzcd}$$

$o \in S^n$ can be chosen arbitrary as long as the morphisms are basepoint-preserving.
We now have $h^{-1}(\{o\}) = \{o\}$. Since $\kappa(x) = 0$ in a neighborhood of 0, and by the injectivity of f, there is an open neighborhood of V of o in S^n such that $h_{|V}$ is injective. $h(V)$ is open

by the invariance of domain, so h is a local homeomorphism at o. Therefore, $\deg h = \pm 1$ by [**12**, Proposition 2.2.30]. It follows that $[h]_{\mathcal{HT}}$ is an isomorphism (see [**22**, Theorem VIII.10.1]). Therefore, $\overline{f} = [g(1,.)]_{\mathcal{HT}}$ is also an isomorphism. $\square$

It is now straightforward to formulate the following

PROPOSITION 7.19. *Let $f: D^n \to X_1$, $f(0) = 0$, be injective and continuous. Then f is a seed for $(\pi, 0)$, and $\langle f, \pi, 0\rangle$ is an orientation for $(\pi, 0)$.*

PROOF. Since π is strongly linearizable, there is an $\mathcal{S}$-continuous family $(\pi_\lambda, \{0\})$ with $\pi_1 = \pi$ and $\pi' := \pi_0$ being linear.
It follows from Proposition 7.18 that $\langle f, \pi', 0\rangle$ is an isomorphism. Using Corollary 7.17 and the definition of strong linearizability, one obtains that $\langle f, \pi, 0\rangle$ is also an isomorphism. $\square$

COROLLARY 7.20. *Let $f: D^n \to X$ with $f(0) = 0$. Assume that the Fréchet-derivative $Df(0)$ exists and $P \circ Df(0): \mathbb{R}^n \to X_1$ is an isomorphism.*
Then f and $P \circ Df(0)$ are seeds for $(\pi, 0)$, and $\langle f, \pi, 0\rangle = \langle P \circ Df(0), \pi, 0\rangle$ are orientations.

PROOF. By Lemma 7.16, $g_\lambda: D^n \to X$,

$$g_\lambda(x) := \lambda f(x) + (1 - \lambda) P \circ Df(0)x,$$

is a seed for every $\lambda \in [0, 1]$. We have $g_0 = f$, $g_1 = P \circ Df(0)$, so it follows from Lemma 7.11 that $\overline{g_\lambda}$ is constant.
Finally, $\langle P \circ Df(0), \pi, 0\rangle$ is an orientation by Proposition 7.19. $\square$

One might expect that an orientation is merely a choice of a basis for X_1. The relationship between orientations (induced by the above seeds) and bases is established by the following proposition, which states that compatible bases induce the same orientation and vice versa.

PROPOSITION 7.21. *Let $\Phi_1, \Phi_2 \in \mathcal{L}(\mathbb{R}^n, X_1)$ be ismorphisms. Then $\langle \Phi_1, \pi, e\rangle = \langle \Phi_2, \pi, e\rangle$ if and only if* $\det \Phi_2^{-1}\Phi_1 > 0$.

Let E and F be finite-dimensional normed spaces. For $A, B \in \mathrm{ISO}(E, F)$ let $A \sim B$ (homotopic) iff there exists a family $(C_\lambda)_{\lambda \in [0,1]}$ in $\mathrm{ISO}(E, F)$ such that

(1) $C_0 = A$;
(2) $C_1 = B$;
(3) $\lambda \mapsto C_\lambda$ is continuous.

It is well known [**15**, Proposition 9.36] that $A \sim B$ if and only if $\det A \cdot \det B > 0$.

PROOF. The case $n = 0$ is trivial, so we may assume that $n \geq 1$. Suppose that $\det \Phi_2^{-1}\Phi_1 > 0$. Then, there exists $H \in C([0, 1], \mathrm{ISO}(\mathbb{R}^n, X_1))$ such that $H(0, .) = \Phi_1$ and $H(1, .) = \Phi_2$.
It follows from Lemma 7.16 that $H(\lambda, .)$ is a seed for all $\lambda \in [0, 1]$ and from Lemma 7.11 that $\langle \Phi_1, \pi, 0\rangle = \langle \Phi_2, \pi, 0\rangle$.
In order to prove the only-if part, it is sufficient to show that there are Φ_1, Φ_2 with $\langle \Phi_1, \pi, 0\rangle \neq \langle \Phi_2, \pi, 0\rangle$. Let $\Phi_1 \in \mathrm{ISO}(\mathbb{R}^n, X_1)$ be arbitrary and define $\Phi_2(x_1, \ldots, x_n) := \Phi_1(-x_1, x_2, \ldots, x_n)$ so that $\det \Phi_2^{-1}\Phi_1 = -1$. Further, let (N_1, N_2) be a strongly π-admissible FM-index pair for $(\pi, \{0\})$, $\lambda \in]0, 1]$, and $s \in \mathbb{R}^+$ such that $\Phi_1^\lambda(D^n, S^{n-1}) \subset (N_1, N_2^{-s})$,

Setting $\alpha(x_1,\dots,x_n) := (-x_1,x_2,\dots,x_n)$, it follows that

$$\begin{tikzcd} D^n/S^{n-1} \arrow[r, "\Phi_1^\lambda"] \arrow[d, "\alpha"] & N_1/N_2^{-s} \\ D^n/S^{n-1} \arrow[ur, "\Phi_2^\lambda"] & \end{tikzcd} \tag{7.6}$$

is a commutative diagram in the category of pointed topological spaces. Hence, passing (7.6) to singular homology, we obtain $-1 = H_n(\alpha) = H_n\left(\overline{\Phi_2^\lambda}\right)^{-1} \circ H_n\left(\overline{\Phi_1^\lambda}\right)$ (see [**12**, Section 2.2] for the computation of $H_q(\alpha)$). This shows that $\langle \Phi_1,\pi,0\rangle \neq \langle \Phi_2,\pi,0\rangle$. □

4. The effect of homeomorphisms

Definition 7.22. For every $q \in \mathbb{Z}$, let

$$\mu_q : \mathbb{Z} \to H_q(\mathcal{S}^q)$$

be an isomorphism and $\mu := (\mu_q)_{q\in\mathbb{Z}}$. Then, given an arbitrary morphism $f:\ H_{q+k}[\mathcal{S}^{q+k}] \to H_q(\mathcal{S}^q)$, $k \in \mathbb{Z}$, there is a unique number $\theta(f) := \theta(f,\mu,q,k)$ such that $f\circ\mu_{q+k} = \theta(f,\mu,q,k)\cdot \mu_q$.

Until further notice, we will work with a fixed but arbitrary collection μ of isomorphisms. Let X and Y be Banach spaces and let π be a strongly linearziable local semiflow on X. As in the previous section, let $X_1 = X_1(\pi)$ be defined as in the definition of strong linearizability and choose $n := \dim X_1$. Let $U \subset X$ be a neighborhood of 0 in X, $V \subset Y$ and $f:\ U \to V$ a homeomorphism. Using orientations $o_1:\ \mathcal{S}^n \to \mathcal{C}(\pi,\{0\})$ and $o_2:\ \mathcal{S}^n \to \mathcal{C}(f[\pi],\{f(0)\})$, the action of f can be described by its induced action f^* on $\mathcal{S}^n$, whose singular homology can be expressed by a number $\theta \in \mathbb{Z}$.

Definition 7.23. Let o_1, o_2 be orientations for $(\pi,0)$ resp. $(f[\pi],f(0))$.
$f^* := f^*_{o_1,o_2}$ (we drop the subscript when no confusion can arise) denotes the unique morphism in $\mathcal{HT}$ for which

$$\begin{tikzcd} \mathcal{S}^n \arrow[r, "o_1"] \arrow[d, "f^*"] & (\pi,\{0\}) \arrow[d, "\langle f\rangle"] \\ \mathcal{S}^n \arrow[r, "o_2"] & (f[\pi],\{f(0)\}) \end{tikzcd}$$

is commutative.
Moreover, let $\theta(\langle f\rangle) := \theta(\langle f\rangle,\mu,o_1,o_2) := \theta(H_n(f^*_{o_1,o_2}),\mu,n,0)$.

In general, the morphism $H_q(f^*)$ depends on o_1 and o_2. However, if we assume that $X = Y$, $f(0) = 0$, f is Fréchet-differentiable in 0, and that $Df(0)$ is an isomorphism, then $H_q(f^*)$ depends only on $Df(0)$:

Proposition 7.24. *Suppose that $Df(0) = \mathrm{id}_X$, and let $o:\ D^n \to X_1$ be injective and continuous with $o(0) = 0$.*
Then

(1) *o is a seed for π and $f[\pi]$;*
(2) *$\langle o,\pi\rangle$ and $\langle o,f[\pi]\rangle$ are orientations;*

(3) $\theta(\langle f\rangle, \mu, \langle o, \pi\rangle, \langle o, f[\pi]\rangle) = 1$.

Proof. Letting
$$g_r(x) := r(f^{-1} \circ o(x)) + (1-r)o(x),$$
there is a neighborhood U of 0 in $\mathbb{R}^n$ such that
$$\|Pg_r(x)\| \geq \|P(o(x))\| - \|P(f(o(x)) - o(x))\| \geq \frac{1}{2}\|o(x)\| > 0$$
for all $x \in U \setminus \{0\}$ and all $r \in [0,1]$.
It follows from Proposition 7.19 that g_r is a seed for $(\pi, 0)$ for all $r \in [0,1]$ which induces an orientation $\langle g_r, \pi\rangle$. In view of Lemma 7.11, $\langle g_r, \pi\rangle$ does not depend on r.
Moreover, since g_r is a seed for $(\pi, 0)$, $f \circ g_r$ is a seed for $(f[\pi], 0)$. We thus have $\langle o, f[\pi]\rangle = \langle f \circ g_1, f[\pi]\rangle = \langle f \circ g_0, f[\pi]\rangle = \langle f \circ o, f[\pi]\rangle$.
We need to show that
$$\begin{array}{ccc} \mathcal{S}^n & \xrightarrow{\langle o,\pi\rangle} & \mathcal{C}(\pi, \{0\}) \\ \big\downarrow [\mathrm{id}] & & \big\downarrow [f] \\ \mathcal{S}^n & \xrightarrow{\langle o, f[\pi]\rangle} & \mathcal{C}(f[\pi], \{0\}) \end{array} \tag{7.7}$$
is commutative.
Let (N_1, N_2) be a strongly π-admissible FM-index pair for $(\pi, \{0\})$. Then there are $\lambda \in \,]0,1]$ and $s \in \mathbb{R}^+$ such that
$$\begin{array}{ccc} D^n/S^{n-1} & \xrightarrow{o^\lambda} & N_1/N_2^{-s} \\ \big\downarrow \mathrm{id} & & \big\downarrow f \\ D^n/S^{n-1} & \xrightarrow{(f\circ o)^\lambda} & f(N_1)/f(N_2^{-s}) \end{array}$$
is commutative in $\mathcal{TOP}$. Since o is a seed for $f[\pi]$, we can assume without loss of generality (choosing λ and s large enough) that
$$\begin{array}{ccc} D^n/S^{n-1} & \xrightarrow{o^\lambda} & N_1/N_2^{-s} \\ \big\downarrow \mathrm{id} & & \big\downarrow f \\ D^n/S^{n-1} & \xrightarrow{o^\lambda} & f(N_1)/f(N_2^{-s}) \end{array}$$
is defined. It commutes because $\langle f \circ o, f[\pi]\rangle = \langle o, f[\pi]\rangle$ as we have already seen.
Since f induces an isomorphism $N_1/N_2^{-s} \to f(N_1)/f(N_2^{-s})$ in $\mathcal{HT}$, it follows that $\langle o, f[\pi]\rangle = \langle f \circ o, f[\pi]\rangle$ is an orientation. $\square$

5. Linear skew product semiflows

We will apply the approach to orientations developed in the previous sections to semilinear skew product semiflows as considered in Chapter 6 (using the same notation). In particular, let $n = \dim E^-(-1) = \dim E^-(1)$. Recall that the subspaces $E^\pm(x)$ correspond to the spectral sets $\{\operatorname{Re}\sigma(A - F(x)) > 0\}$ and $\{\operatorname{Re}\sigma(A - F(x)) < 0\}$. Additionally, we will use $E^{+,\alpha}(x) := E^+(x) \cap X^\alpha$.

5.1. Conley index orientations.

LEMMA 7.25. *Let* $\pi = (\xi, \Phi) \in \mathrm{SK}_{-1}([a,b], \alpha, X, A)$ *and let* $\Psi_{-1}, \Psi_1 \in \mathcal{L}(\mathbb{R}^n, X^\alpha)$, *such that* $P^-_{-1}(\pi, 0)\Psi_{-1}$ *and* $P^-_1(\pi, 0)\Psi_1$ *are isomorphisms.*
Then

(1) $o_{-1}(x, y) := (-1, \Psi_{-1}y)$, $(x, y) \in \left]-\frac{1}{2}, \frac{1}{2}\right[\times \mathbb{R}^n$, *is a seed for* $(\pi, (-1, 0))$, *and*
(2) $o_1(y) := (1, \Psi_1 y)$, $y \in \mathbb{R}^n$, *is a seed for* $(\pi, (1, 0))$.

PROOF. Suppose that $\pi = \pi(A, F)$.

(1) Let $U := B_{1/2}[(-1, 0)] \subset [-2, 2] \times X^\alpha$, and let (N_1, N_2) be a strongly π-admissible FM-index pair for $(\pi, (-1, 0))$ with $N_1 \subset U$. Since $\mathrm{Inv}^+(U) \subset \{0\} \times E^{+,\alpha}(-1)$, it follows that $o_{-1}(x, y) \in N_2^{-\infty}$ for all $(x, y) \in \mathcal{D}(o_{-1}) \setminus \{0\}$. Now, Lemma 7.7 implies that o_{-1} is a seed for $(\pi, (-1, 0))$.
(2) Let $X_1 := \{0\} \times X^\alpha$ and (N_1, N_2) be a strongly π-admissible FM-index pair for $(\pi, (1, 0))$ with $N_1 \subset B_1[(1, 0)] \subset \mathbb{R} \times X^\alpha$. Then $(X_1 \cap N_1, X_1 \cap N_2)$ is a strongly π-admissible FM-index pair for $(\tilde{\pi}, 0)$, where $\tilde{\pi}$ is induced by mild solutions of the linear equation

$$\begin{aligned} \dot{x} &= 0 & x &\in \{0\} \\ \dot{y} + Ay &= F(1)y & y &\in X^\alpha. \end{aligned}$$

It follows from Corollary 7.20 that o_1 is a seed for $(\tilde{\pi}, 0)$, that is, there is a $\lambda \in \mathbb{R}^+$ such that $o_1^\lambda(y) = o(\lambda y) \in N_2^{-\infty}$ for all $y \in \mathbb{R}^n \setminus \{0\}$. As before, an application of Lemma 7.7 proves that o_1 is a seed for $(\pi, (-1, 0))$.

□

Until further notice, let μ be given by Definition 7.22, $\pi \in \mathrm{SK}_{-1}$, $\Psi_{-1} \in \mathrm{ISO}(\mathbb{R}^n, E^-(-1))$, $\Psi_1 \in \mathrm{ISO}(\mathbb{R}^n, E^-(1))$, and o_1 and o_{-1} be defined by Lemma 7.25.

DEFINITION 7.26.

$$\bar{\theta}(\pi) := \bar{\theta}(\pi, \mu, \Psi_{-1}, \Psi_1) := \theta(H_{n-1}\langle o_1\rangle^{-1} \circ \partial_n \circ H_n\langle o_{-1}\rangle, \mu, n, 1)$$

where $\partial_q : H_q\mathcal{C}(\pi, \{(-1, 0)\}) \to \mathcal{C}(\pi, \{(1, 0)\})$ denotes the q-th connecting homorphism of the long exact attractor-repeller sequence in singular homology which is associated with $(\pi, [-1, 1] \times \{0\}, \{(1, 0)\}, \{(-1, 0)\})$.

LEMMA 7.27. *Let* $\pi_k \to \pi_\infty$ *be a sequence in* SK_0 *such that the assumptions of [4, Theorem 7.3] hold whenever* $\tilde{N}$ *is a bounded neighborhood of* $[-1, 1] \times \{0\}$. *Suppose that* $\langle o_{-1}, \pi_k, (-1, 0)\rangle$ *(resp.* $\langle o_1, \pi_k, (1, 0)\rangle$*) is an orientation for all* $k \in \mathbb{N} \cup \{\infty\}$ *sufficiently large.*
Then $\bar{\theta}(\pi_k) = \bar{\theta}(\pi_\infty)$ *for all* $k \in \mathbb{N}$ *sufficiently large.*

PROOF. By [**4**, Theorem 7.3], there are strongly admissible FM-index triples $(N_{1,k}, N_{2,k}, N_{3,k})$, $(\tilde{N}_{1,k}, \tilde{N}_{2,k}, \tilde{N}_{3,k})$ for $(\pi_k, [-1, 1], \{-1\}, \{1\})$ and (M_1, M_2, M_3), $(\tilde{M}_1, \tilde{M}_2, \tilde{M}_3)$ for $(\pi_\infty, [-1, 1], \{-1\}, \{1\})$ such that for all $k \in \mathbb{N}$ sufficiently large

$$(\tilde{N}_{1,k}, \tilde{N}_{2,k}, \tilde{N}_{3,k}) \subset (\tilde{M}_1, \tilde{M}_2, \tilde{M}_3) \subset (N_{1,k}, N_{2,k}, N_{3,k}) \subset (M_1, M_2, M_3).$$

We can assume that M_1 is bounded in X so that it is strongly π_∞-admissible by Corollary 6.4.

It follows from Proposition 7.13 that

$$\begin{tikzcd} D^{n+1}/S^n \arrow[r, "\overline{o_{-1}}"] \arrow[d, "\mathrm{id}"] & N_{1,k}/N_{2,k} \arrow[d, "\subset"] \\ D^{n+1}/S^n \arrow[r, "\overline{o_{-1}}"] & M_1/M_2 \end{tikzcd} \tag{7.8}$$

and

$$\begin{tikzcd} D^{n}/S^{n-1} \arrow[r, "\overline{o_{1}}"] \arrow[d, "\mathrm{id}"] & N_{2,k}/N_{3,k} \arrow[d, "\subset"] \\ D^{n}/S^{n-1} \arrow[r, "\overline{o_{1}}"] & M_2/M_3 \end{tikzcd} \tag{7.9}$$

are commutative for all $k \in \mathbb{N}$ sufficiently large.
Moreover, there is a commutative ladder

$$\begin{tikzcd} {} \arrow[r] & H_{q+1}[N_{1,k}/N_{3,k}] \arrow[r] \arrow[d, "\subset"] & H_{q+1}[N_{1,k}/N_{2,k}] \arrow[r, "\tilde{\partial}_{q+1}"] \arrow[d, "\subset"] & H_q[N_{2,k}/N_{3,k}] \arrow[r] \arrow[d, "\subset"] & {} \\ {} \arrow[r] & H_{q+1}[M_1/M_3] \arrow[r] & H_{q+1}[M_1/M_2] \arrow[r, "\partial_{q+1}"] & H_q[M_2/M_3] \arrow[r] & {} \end{tikzcd},$$

where ∂_q and $\tilde{\partial}_q$ denote the respective q-th connecting homomorphism.
It follows that $\bar{\theta}(\pi_k) = \bar{\theta}(\pi_\infty)$ for all k sufficiently large. □

PROPOSITION 7.28. *$\langle o_\nu, \pi, (\nu, 0)\rangle$ is an orientation for every $\pi \in \mathrm{SK}_0$, $\nu \in \{-1, 1\}$. Moreover, for all $\pi_0, \pi_1 \in \mathrm{SK}_0$, it holds that $\bar{\theta}(\pi_0) = \bar{\theta}(\pi_1)$ whenever $\pi_0 \sim \pi_1$ in the sense of Definiton 6.7.*

PROOF. First, assume that $\pi \in \mathrm{SK}_1$, let $\nu \in \{-1, 1\}$, $m = n$ for $\nu = 1$ and $m = n+1$ for $\nu = -1$. Then there is a neighborhood U of $(\nu, 0)$ in $]-2, 2[\times X^\alpha$ such that the restriction of π to U is induced by mild solutions of

$$\dot{x} = 1 - x^2$$
$$\dot{y} + Ay = F(\nu)y.$$

It follows from Corollary 7.20 that $\langle o_\nu, \pi\rangle$ is an orientation.
Now, let $\pi \in \mathrm{SK}_0$. By Lemma 6.11, there is an $\mathcal{S}$-continous family $(\pi_\lambda, \{(\nu, 0)\})$ such that $\pi_1 \in \mathrm{SK}_1$, $\pi_0 = \pi$, and $E^-(\pi_\lambda, \nu)$ are constant. Hence, o_ν is a seed for $(\pi_\lambda, (\nu, 0))$ for every $\lambda \in [0, 1]$. It follows from Theorem 7.15 that $\langle o_\nu, \pi\rangle$ is an orientation for $(\pi, \{\nu\})$, proving the first claim.
In order to show the second claim, let $\pi_0, \pi_1 \in \mathrm{SK}_0$ with $\pi_0 \sim \pi_1$, that is, there exists an $\mathcal{S}$-continuous family $(\pi_\lambda, [-1, 1] \times \{0\})_{\lambda \in [0,1]}$ such that $E^-(\pi_\lambda, -1)$ and $E^-(\pi_\lambda, 1)$ are constant. Therefore, we can choose Ψ_1 and Ψ_{-1} such that o_{-1} (resp. o_1) induces orientations for $(\pi_\lambda, -1)$ (resp. $(\pi_\lambda, 1)$) for all $\lambda \in [0, 1]$.
Suppose that $\bar{\theta}(\pi_\lambda)$ is not constant. Then there is a sequence $\lambda_n \to \lambda_\infty$ in $[0, 1]$ with $\bar{\theta}(\pi_n) \neq \bar{\theta}(\pi_\infty)$, where we set $\pi_n := \pi_{\lambda_n}$, $n \in \mathbb{N} \cup \{\infty\}$. This is a contradiction to Lemma 7.27, showing that $\bar{\theta}(\pi_0) = \bar{\theta}(\pi_1)$. □

5.2. The unstable subbundle. For every $\pi \in \mathrm{SK}_2$, we have defined an invariant subbundle U of $[-2,2]\times X^\alpha$. Let (N_1,N_2,N_3) be an arbitrary FM-index pair for $(\pi,[-1,1]\times\{0\},\{(1,0)\},\{(-1,0)\})$. Due to the invariance of U, $(M_1,M_2,M_3) := (N_1\cap U, N_2\cap U, N_3\cap U)$ is an FM-index pair for $\pi_{|\,\mathrm{im}\,U}$ (recall that we have already defined as semiflow π_U on $E^-(-1)$), which denotes the restriction of π to U.

The inclusion $(M_1,M_2,M_3)\subset(N_1,N_2,N_3)$ induces a commutative ladder in singular homology, namely

$$\begin{array}{ccccccc} \longrightarrow & H_q[M_1,M_3] & \longrightarrow & H_q[M_1,M_2] & \longrightarrow & H_{q-1}[M_2,M_3] & \longrightarrow \\ & {\scriptstyle i}\Big\downarrow\subset & & {\scriptstyle k}\Big\downarrow\subset & & {\scriptstyle l}\Big\downarrow\subset & \\ \longrightarrow & H_q[N_1,N_3] & \longrightarrow & H_q[N_1,N_2] & \longrightarrow & H_{q-1}[N_2,N_3] & \longrightarrow . \end{array} \tag{7.10}$$

LEMMA 7.29. *For every $\pi\in\mathrm{SK}_2$, $\langle o_\nu,\pi_{|\,\mathrm{im}\,U}\rangle$, $\nu\in\{-1,1\}$, induces an orientation for $(\pi_{|\,\mathrm{im}\,U},\{(\nu,0)\})$.*

PROOF. $F(x)$ is constant for all x in a neighborhood $N_{\pm1}$ of ±1. Therefore, $U(\pm1)=E^-(\pm1)$, and so o_{-1} and o_1 can be defined by Lemma 7.25. It follows from Corollary 7.20 that $\langle o_\nu,\pi,(\nu,0)\rangle$ is an orientation for every $\nu\in\{-1,1\}$. □

Lemma 7.29 guarantees that $\bar\theta(\pi_{|\,\mathrm{im}\,U})$ is defined and so we may formulate the following

PROPOSITION 7.30. *For every $\pi\in\mathrm{SK}_2$ it holds that $\bar\theta(\pi)=\bar\theta(\pi_{|\,\mathrm{im}\,U})$.*

PROOF. Let $\nu\in\{-1,1\}$ and (N_1,N_2) be an arbitrary strongly admissible FM-index pair for $(\pi,\{(\nu,0)\})$. Then $(N_1\cap U,N_2\cap U)$ is a strongly admissible FM-index pair for $(\pi_{|\,\mathrm{im}\,U},\{(\nu,0)\})$

By Lemma 7.29, there is an $s\in\mathbb{R}^+$ and a $\lambda\in[0,1]$ such that $o_\nu^\lambda(S^{m-1})\subset N_2^{-s}\cap U\subset N_2^{-s}$, where $m=n$ for $\nu=1$ and $m=n+1$ for $\nu=-1$. Therefore,

$$\begin{tikzcd} D^m/S^{m-1} \arrow[r, "o_\nu^\lambda"] \arrow[dr, "o_\nu^\lambda"'] & (N_1\cap U)/(N_2^{-s}\cap U) \arrow[d, "\subset"] \\ & N_1/N_2^{-s} \end{tikzcd}$$

is commutative in $\mathcal{TOP}$ and thus also

$$\begin{tikzcd} D^m/S^{m-1} \arrow[r, "\overline{o_\nu}"] \arrow[dr, "\overline{o_\nu}"'] & (N_1\cap U)/(N_2\cap U) \arrow[d, "\subset"] \\ & N_1/N_2 \end{tikzcd}$$

in $\mathcal{HT}$.

Now, let (N_1,N_2,N_3) be a strongly admissible FM-index triple for $(\pi,[-1,1]\times\{0\},\{(1,0)\},\{(-1,0)\})$. It follows that $H_q(i)$ and $H_q(l)$ (defined in (7.10)) are isomorphisms since $\overline{o_\nu}$ is an isomorphism by Proposition 7.28 respectively Lemma 7.29. Therefore, the commutativity of (7.10) implies that $\bar\theta(\pi_{|\,\mathrm{im}\,U})=\bar\theta(\pi)$ as claimed. □

The definition

$$\mathrm{sgn}_{\Psi_{-1},\Psi_1}U := \mathrm{sgn}\det\Psi_1^{-1}U(1)U(-1)^{-1}\Psi_{-1}$$

gives U a sign. We define the inverse of an injective and continuous homorphism $A \in \mathcal{L}(E, F)$ on $\operatorname{im} A \subset F$. The definition of $\operatorname{sgn} U$ makes sense because

- $\operatorname{im} \Psi_{-1} = E^-(-1) = \operatorname{im} U(-1)$ and
- $\operatorname{im} \Psi_1 = E^-(1) = \operatorname{im} U(1)$.

Alternatively, one can read the inverses in the above equation as left inverses. In this case, $\operatorname{sgn}_{\Psi_{-1},\Psi_1} U$ is well-defined and agrees with the first definition.
Recall that the definition of θ requires a choice of generators $\mu = (\mu_q)_{q \in \mathbb{Z}}$. Consider the following system

$$\dot{x} = 1 - x^2$$
$$\dot{y} = y$$

of ordinary differential equations on $]-2, 2[\times \mathbb{R}^n$. They define a semiflow χ_n, which is obviously a linear skew product semiflow, that is, $\chi_n \in \mathrm{SK}_2([-2, 2], \mathbb{R}^n)$. Let U_{χ_n} denote the subbundle U which is defined with respect to χ_n (in fact, $U_{\chi_n} = [-2, 2] \times \mathbb{R}^n$).

Definition 7.31. Let $\tilde{\mu}_0 : \mathbb{Z} \to H_0(\mathcal{S}^0)$ be arbitrary, and let $\mu = (\mu_q)_{q \in \mathbb{Z}}$ be such that

$$\mu_0 = \tilde{\mu}_0$$

and for all $n \in \mathbb{N}$

$$\bar{\theta}(\chi_n, \mu, \mathrm{id}_{\mathbb{R}^n}, \mathrm{id}_{\mathbb{R}^n}) = \operatorname{sgn}_{\mathrm{id}_{\mathbb{R}^n}, \mathrm{id}_{\mathbb{R}^n}} U_{\chi_n}.$$

It is clear that μ is well-defined, and the following proposition shows that the definition makes sense.

Proposition 7.32. *For every $\pi \in \mathrm{SK}_2$, for every $\Psi_{-1} \in \mathrm{ISO}(\mathbb{R}^n, E^-(-1))$, and for every $\Psi_1 \in \mathrm{ISO}(\mathbb{R}^n, E^-(1))$, it holds that*

$$\bar{\theta}(\pi, (\mu_q)_{q \in \mathbb{Z}}, \Psi_{-1}, \Psi_1) = \operatorname{sgn}_{\Psi_{-1},\Psi_1} U_\pi \neq 0.$$

Proof. Recall that by Proposition 6.24, there is an isomorphism of bundles $V := V_{\pi_U}$ such that $\tilde{\chi} := V^{-1}[\pi_U] \sim \chi_n$.
As usual, let o_1 and o_{-1} be given by Lemma 7.25. They are seeds for an orientation for $(-1, 0)$ respectively $(1, 0)$. Moreover, $\hat{o}_{-1} := (-1, 0) + \mathrm{id}_{\mathbb{R}^n}$ is a seed for $(\tilde{\chi}, (-1, 0))$ and $\hat{o}_1 := (1, 0) + \mathrm{id}_{\mathbb{R}^n}$ is a seed for $(\tilde{\chi}, (1, 0))$.
Let α be defined by

$$\begin{tikzcd}
\mathcal{S}^{n+1} \arrow[r, "\langle o_{-1} \rangle"] & \mathcal{C}(\pi_{|\operatorname{im} U}, \{(-1,0)\}) \\
\mathcal{S}^{n+1} \arrow[u, "\alpha"] \arrow[r, "\langle \hat{o}_{-1} \rangle"] & \mathcal{C}(\tilde{\chi}, \{(-1,0)\}) \arrow[u, "\langle U \circ V \rangle"']
\end{tikzcd}$$

and β by

$$\begin{tikzcd}
\mathcal{S}^{n} \arrow[r, "\langle o_{1} \rangle"] & \mathcal{C}(\pi_{|\operatorname{im} U}, \{(1,0)\}) \\
\mathcal{S}^{n} \arrow[u, "\beta"] \arrow[r, "\langle \hat{o}_{1} \rangle"] & \mathcal{C}(\tilde{\chi}, \{(1,0)\}). \arrow[u, "\langle U \circ V \rangle"']
\end{tikzcd}$$

It follows from Proposition 7.21 and Corollary 7.20 that

$$\begin{aligned}\alpha &= \operatorname{sgn}\det\begin{pmatrix}1 & 0\\ 0 & \Psi_{-1}^{-1}\circ U(-1)\circ V(-1)\end{pmatrix}\\ &= \operatorname{sgn}\det \Psi_{-1}^{-1}\circ U(-1)\circ V(-1)\end{aligned}$$

and

$$\beta = \operatorname{sgn}\det \Psi_{1}^{-1}\circ U(1)\circ V(1).$$

Since $\operatorname{sgn}\det V(1)\circ V(-1) = \operatorname{sgn}\det V(1)\circ V(1)$ (by homotopy, V is a continuous family of isomorphisms), it follows that

$$\beta\circ\alpha^{-1} = \operatorname{sgn}_{\Psi_{-1},\Psi_1} U,$$

where we denote the mappings by their mapping degree.
In singular homology, the respective attractor-repeller sequences define a commutative diagram:

$$\begin{tikzcd} H_{q+1}\langle \pi_{|\operatorname{im} U}, \{(-1,0)\}\rangle \arrow[r, "\partial_{q+1}"] & H_q\langle \pi_{|\operatorname{im} U}, \{(1,0)\}\rangle \\ H_{q+1}\langle \tilde\chi, \{(-1,0)\}\rangle \arrow[r, "\tilde\partial_{q+1}"] \arrow[u, "H_{q+1}\langle U\circ V\rangle"] & H_q\langle \tilde\chi, \{(1,0)\}\rangle \arrow[u, "H_q\langle U\circ V\rangle"'] \end{tikzcd}$$

It follows from Proposition 7.28 and the choice of μ that that $\bar\theta(\tilde\chi,\mu,\mathrm{id}_{\mathbb{R}^n},\mathrm{id}_{\mathbb{R}^n}) = \bar\theta(\chi_n,\mu,\mathrm{id}_{\mathbb{R}^n},\mathrm{id}_{\mathbb{R}^n}) = 1$.
We obtain a commutative diagram

$$\begin{tikzcd} H_{n+1}(\mathcal{S}^{n+1}) \arrow[r, "\bar\theta(\pi_{|\operatorname{im} U})"] & H_n(\mathcal{S}^n) \\ H_{n+1}(\mathcal{S}^{n+1}) \arrow[r, "1"] \arrow[u, "\cdot\alpha"] & H_n(\mathcal{S}^n), \arrow[u, "\cdot\beta"'] \end{tikzcd}$$

showing that

$$\bar\theta(\pi_{|\operatorname{im} U}) = \alpha\beta = \operatorname{sgn}_{\Psi_{-1},\Psi_1} U.$$

□

5.3. Geometric orientation. Let $\Psi_{-1}\in \mathrm{ISO}(\mathbb{R}^n, E_{-1}^-)$ and $\Psi_1 \in \mathcal{L}\in \mathrm{ISO}(\mathbb{R}^n, E_1^-)$ be arbitrary but fixed as in the previous section. We will define a geometric orientation for every $\pi\in \mathrm{SK}_2$ and then show that this geometric orientation is well-defined for every $\pi \in \mathrm{SK}_0$ and coincides with the (Conley index) orientation of the previous section.

Definition 7.33. For every $\pi = (\xi,\Phi)\in \mathrm{SK}_{-1}$, let $\operatorname{sgn}\pi \in \{-1,1\}$ denote the unique number for which

$$\operatorname{sgn}\pi := \operatorname{sgn}(\pi,\Psi_{-1},\Psi_1) := \lim_{(x,x\xi t)\to(-1+,1-)} \operatorname{sgn}\det \Psi_1^{-1}P\Phi(x,t)\Psi_{-1},$$

where $P = P_1^-(0)$ denotes the unique projection $P\colon\ X\to E^-(1)$ with $\ker P = E^+(1)$.

Note that for every $t\in\mathbb{R}^+$, the spaces $E^-(1)$ and $E^{+,\alpha}(1)$ (resp. $E^-(-1)$ and $E^{+,\alpha}(1)$) are $\Phi(1,t)$-invariant (resp. $\Phi(-1,t)$-invariant) subspaces.

LEMMA 7.34. *Let* $\pi = \pi(A, F) \in \mathrm{SK}_1$ *and* $\delta > 0$ *such that*

$$F(x) = F(-1) \qquad\qquad x \in [-1, -1+\delta]$$
$$F(x) = F(1) \qquad\qquad x \in [1-\delta, 1].$$

Then

$$\operatorname{sgn} \pi = \operatorname{sgn} \det \Psi_1^{-1} P\Phi(-1+\delta, t_0)\Psi_{-1} \neq 0,$$

where $(-1+\delta)\xi t_0 = 1 - \delta$.
In particular, $\operatorname{sgn}\pi$ *is well-defined for every* $\pi \in \mathrm{SK}_1$.

PROOF. Let $x \in]-1, 1[$ and $t \in \mathbb{R}^+$ such that $x \in]-1, -1+\delta]$ and $x\xi t \geq 1 - \delta$. Then there are $t_{-1}, t_1 \in \mathbb{R}^+$ such that $x\xi t_1 = -1 + \delta$, and $(1-\delta)\xi t_1 = x\xi t$. We have

$$\begin{aligned} P\Phi(x,t) &= P\Phi(1-\delta, t_1) \quad \Phi(-1+\delta, t_0) \quad \Phi(x, t_{-1}) \\ &= P\Phi(1-\delta, t_1) \quad P\Phi(-1+\delta, t_0) \quad \Phi(x, t_{-1}). \end{aligned}$$

$P\Phi(1-\delta, t_1)\Phi(-1+\delta, t_0)\Phi((-1+\delta)\xi(-\lambda t_{-1}), \lambda t_{-1})\Psi_{-1}$ is an ismorphism for all $\lambda \in [0,1]$. Otherwise, there would be a $0 \neq \tilde{y} \in E^-(-1)$, an $\tilde{x} \in]-1, -1+\delta]$, and a $\tilde{t} \in \mathbb{R}^+$ with $\tilde{x}\xi\tilde{t} \geq 1 - \delta$ and $\Phi(\tilde{x}, \tilde{t})\tilde{y} \in E^+(1)$. This implies that there exists a full bounded solution through $(\tilde{x}, \tilde{y})$, which contradicts the isolation of $[-1,1] \times \{0\}$ relative to π (see Lemma 6.10).
We have shown that

$$\operatorname{sgn} \det \Psi_1^{-1} P\Phi(x,t)\Psi_{-1} = \operatorname{sgn} \det P\Phi(1-\delta, t_1)P \quad P\Phi(-1+\delta, t_0)\Psi_{-1}.$$

A similar argument applies to $P\Phi(1-\delta, s)P$. It is an isomorphism for all $s \in [0, t_1]$ and homotopic to the identity on $E^-(1)$, showing that

$$\operatorname{sgn} \det \Psi_1^{-1} P\Phi(x,t)\Psi_{-1} = \operatorname{sgn} \det \Psi_1^{-1} P\Phi(-1+\delta, t_0)\Psi_1.$$

□

The following proposition relies on Proposition 7.32.

PROPOSITION 7.35. *Let* $\pi \in \mathrm{SK}_1$. *Then* $\operatorname{sgn}\pi = \bar{\theta}(\pi) \neq 0$.

PROOF. Recall that for every $\pi \in \mathrm{SK}_1$ there is a $\delta = \delta(\pi) > 0$ such that

$$F(x) = F(-1) \qquad\qquad x \in [-1, -1+\delta]$$
$$F(x) = F(1) \qquad\qquad x \in [1-\delta, 1],$$

and we have $U(x) = E^-(-1)$ for all $x \in [-2, -1+\delta]$.
Initially, suppose that $\pi \in \mathrm{SK}_2$. Let $x \in]-1, -1+\delta[$, $t \in \mathbb{R}^+$, and $\pi_U = (\xi, \Phi_U)$. We have

$$\Phi(x,t)y = U(x\xi t)\Phi_U(x,t)U(x)^{-1}y \tag{7.11}$$

for all $y \in E^-(-1)$.

Moreover, we have $\operatorname{sgn}\det \Phi_U(x_0,t_0) = 1$ since $\Phi_U(x,t) \in \mathrm{ISO}(E^-(-1),E^-(-1))$ for all $(x,t) \in \mathcal{D}(\Phi_U)$. It follows from (7.11) that

$$\operatorname{sgn}\det \Psi_1^{-1}P\Phi(x,t)\Psi_{-1} = \operatorname{sgn}\det \Psi_1^{-1}PU(x\xi t)U(x)^{-1}\Psi_{-1}. \tag{7.12}$$

Taking (7.12) to the limit $(x, x\xi t) \to (-1,1)$, we obtain

$$\operatorname{sgn}\pi = \operatorname{sgn} U,$$

proving in conjunction with Proposition 7.32 that

$$\operatorname{sgn}\pi = \operatorname{sgn} U = \bar\theta(\pi).$$

Lemma 6.11 states that for every $\pi_0 \in \mathrm{SK}_1$ there is a $\pi_1 \in \mathrm{SK}_2$ with $\pi_0 \sim \pi_1$. It follows immediately from the differential equation given there that $\operatorname{sgn}\pi_0 = \operatorname{sgn}\pi_1$. Moreover, Proposition 7.28 implies that $\bar\theta(\pi_0) = \bar\theta(\pi_1)$. This proves the claim for every $\pi \in \mathrm{SK}_1$. □

LEMMA 7.36. *Let $\Psi_{\nu,k}$, $\nu \in \{-1,1\}$, be a sequence of homomorphisms in $\mathcal{L}(\mathbb{R}^n, X^1)$ with $\Psi_{\nu,k} \to \Psi_{\nu,\infty} \in \mathrm{ISO}(\mathbb{R}^n, E^-(\nu))$ as $k \to \infty$.*
Then for every $\pi = (\xi,\Phi) \in \mathrm{SK}_0$ one has

$$\lim_{(x,x\xi t,k)\to(-1+,1,\infty)} \operatorname{sgn}\det(P\Psi_{1,k})^{-1}P\Phi(x,t)\Psi_{-1,k} = \bar\theta(\pi,\Psi_{-1,\infty},\Psi_{1,\infty}) \neq 0. \tag{7.13}$$

PROOF. It is clear that $H_k := \Psi_{-1,k}\Psi_{-1,\infty}^{-1}P_{-1}^-(0) + P_{-1}^+(0) \to \mathrm{id}_{X^\alpha}$ in $\mathcal{L}(X^\alpha,X^\alpha)$ as $k\to\infty$, so for large k, H_k is a toplinear isomorphism which takes $E^-(-1) = \operatorname{im}\Psi_{-1,\infty}$ to $\operatorname{im}\Psi_{-1,k}$.
Let $x_k \to -1$ in $[-1,1]$, $t_k \in \mathbb{R}^+$ with $x_k\xi t_k \to 1$, $\pi_0 = \pi(A,F) \in \mathrm{SK}_0$, and $\pi_k := \pi(A,F_k)$ with

$$F_k(x) := \begin{cases} H_kF(-1)H_k^{-1} & -2 + x_k \le x < x_k \\ F(1) & x_k\xi t_k \le x < 2-(x_k\xi t_k) \\ F(x) & \text{otherwise.}\end{cases}$$

We have $F_k \to F$ as $k \to \infty$ in $L^\infty([-2,2],\mathcal{L}(X^\alpha,X))$. Moreover, there is a strongly admissible isolating neighborhood for $[-1,1]\times\{0\}$ relative to π, so we can choose $k_0 \in \mathbb{N}$ such that $[-1,1]\times\{0\}$ is an isolated invariant set relative to π_k for all $k \ge k_0$. Consequently, one has $\pi_k \in \mathrm{SK}_1$ for all $k_0 \le k < \infty$. Moreover, we can assume w.l.o.g. that $P\Psi_{1,k}$ is an isomorphism for $k \ge k_0$.
If $(u(t),v(t))$, $t \in [0,T]$, is a solution of π_0 with $x_k \le u(t) \le x_k\xi t_k$ for all $t \in [0,T]$, then it is also a solution of π_k. Hence, it follows from Lemma 7.34 and Proposition 7.35 that for all $k_0 \le k < \infty$

$$\operatorname{sgn}\det(P\Psi_{1,k})^{-1}P\Phi(x_k,t_k)\Psi_{-1,k} = \bar\theta(\pi_k,\Psi_{-1,k},P\Psi_{1,k}).$$

As shown in the proof of Proposition 7.28, every $\tilde\pi \in \mathrm{SK}_1$ is strongly linearizable in the sense of Definition 7.1 in each of its equilibria. Thus, it follows from Corollary 7.20 and Proposition 7.21 that there is a $k_1 \ge k_0$ such that

$$\bar\theta(\pi_k,\Psi_{-1,k},P\Psi_{1,k}) = \bar\theta(\pi_k,\Psi_{-1,\infty},\Psi_{1,\infty})$$

for all $k_2 \le k < \infty$.
Finally, in view of Lemma 7.27, there is a $k_2 \ge k_1$ such that

$$\bar\theta(\pi_k,\Psi_{-1,\infty},\Psi_{1,\infty}) = \bar\theta(\pi,\Psi_{-1,\infty},\Psi_{1,\infty})$$

for all $k_3 \leq k \leq \infty$. □

An immediate consequence of Lemma 7.36 is

COROLLARY 7.37. $\operatorname{sgn}\pi$ *is well-defined for every* $\pi \in \mathrm{SK}_0$ *and we have* $\bar{\theta}(\pi) = \operatorname{sgn}(\pi)$.

COROLLARY 7.38. *Let* $\pi = (\xi, \Phi) \in \mathrm{SK}_{-1}([a,b], \alpha, X, A)$. *Then* $\operatorname{sgn}\pi$ *is well-defined and we have* $\operatorname{sgn}\pi = \bar{\theta}(\pi)$.
Moreover, Lemma 7.36 holds for every $\pi \in \mathrm{SK}_{-1}$.

PROOF. According to Definition 6.6, there is a semiflow $\tilde{\pi} = (\tilde{\xi}, \Phi) \in \mathrm{SK}_0$ such that $(h(u(t)), v(t))$ is a solution of $\tilde{\pi}$ whenever $(u(t), v(t))$ is a solution of π.
This shows immediately that $\operatorname{sgn}\pi$ is well-defined and $\operatorname{sgn}\pi = \operatorname{sgn}\tilde{\pi}$. It is also clear that

$$\begin{tikzcd} D^{n+1}/S^n \arrow[r, "\overline{o_1}"] \arrow[d, "\mathrm{id}"] & \tilde{N}_1/\tilde{N}_2 \arrow[d, "h\times\mathrm{id}"] \\ D^{n+1}/S^n \arrow[r, "\overline{o_1}"] & \tilde{M}_1/\tilde{M}_2 \end{tikzcd}$$

is commutative whenever $(\tilde{N}_1, \tilde{N}_2)$ is a strongly π-admissible FM-index pair for $(\pi, \{(1,0)\})$ and $(\tilde{M}_1, \tilde{M}_2) = (h \times \mathrm{id})(\tilde{N}_1, \tilde{N}_2)$.
Since h is necessarily strictly monotone increasing,

$$g_\lambda(x) := \lambda(h \times \mathrm{id}) \circ o_{-1}(x) + (1-\lambda)o_{-1}(x)$$

satisfies $g_\lambda(x) \neq (-1, 0)$ for all $x \in D^n \setminus \{0\}$. Given an arbitrary $\lambda \in [0,1]$, it is a straightforward extension of Lemma 7.25 that g_λ is a seed for $(\pi, \{(-1,0)\})$ and $(\tilde{\pi}, \{(-1,0)\})$. Hence, by Lemma 7.11,

$$\begin{tikzcd} D^{n+1}/S^n \arrow[r, "\overline{o_{-1}}"] \arrow[d, "\mathrm{id}"] \arrow[dr, "g_\lambda"] & N_1/N_2 \arrow[d, "h\times\mathrm{id}"] \\ D^{n+1}/S^n \arrow[r, "\overline{o_{-1}}"] & M_1/M_2 \end{tikzcd}$$

commutes in $\mathcal{HT}$ for all $\lambda \in [0,1]$, where (N_1, N_2) is a strongly admissible FM-index pair for $(\pi, \{(-1,0)\})$ and $(M_1, M_2) = (h \times \mathrm{id})(N_1, N_2)$.
Therefore, we have $\bar{\theta}(\pi) = \bar{\theta}(\tilde{\pi})$. The left hand side of (7.13) is unaffected by h, showing that the formula still holds. □

6. Heteroclinic solutions

Recall the assumptions of the beginning of the the previous section. In particular let $u : \mathbb{R} \to X^\alpha$ be a solution with $u(t) \to e^\pm$ as $t \to \pm\infty$. It follows from Theorem 3.2 that $\|u(t) - e^+\|_\alpha^{-1}(u(t) - e^+) \to \eta \in X^1$ as $t \to \infty$. η is an eigenvector of $A - Df(e^+)$ which belongs to an eigenvalue $\lambda > 0$.
Let $E \subset X$ be an A-invariant and $A - Df(e^+)$ invariant subspace[1] with $X = E \oplus \{\eta\}$. By $E = E_1 \oplus E_2$, we mean that E_1 and E_2 are closed linear subspaces of E with $E_1 \cap E_2 = \{0\}$ and $E = E_1 + E_2$. The canonical projection $P : E_1 \oplus E_2 \to E_1$ is given by $P(e_1 \oplus e_2) := e_1$.

[1]This can always be achieved by choosing A appropriately.

Due to the hyperbolicity of $A - Df(e^+)$, there is a decomposition $E = E^-(-1) \oplus E^+(-1)$, where $E^-(-1)$ (resp. $E^+(-1) \cap X^1$) is a $A - Df(e^+)$ invariant subspace and the restriction $\tilde{A}^-$ of $A - Df(e^+)$ to $E^-(-1)$ (resp. $\tilde{A}^+$ of $A - Df(e^+)$ to $E^+(-1)$) satisfies $\operatorname{Re}\sigma(\tilde{A}^-) < 0$ (resp. $\operatorname{Re}\sigma(\tilde{A}^+) > 0$).
It follows from Theorem 4.1 (respectively Propositon 4.4, which is based upon that theorem) that there exists a diffeomorphism $G:\ X^\alpha \to\ ]-2,2[\ \times E^\alpha$, which is defined in a neighborhood of $\overline{u} := \mathrm{cl}\{u(t):\ t \in \mathbb{R}\}$ and satisfies

(1) $G(e^+) = (1,0)$, $G(e^-) = (-1,0)$;
(2) $G(u(t)) \in\ ]-1,1[\ \times \{0\}$ for all $t \in \mathbb{R}$;
(3) $DG(x)y = (0,y)$ for all $y \in E$ and for all x in a neighborhood of e^+.

Note that $E^\alpha = E \cap X^\alpha$ corresponds to X^α in the previous section.
Let the family of semiflows $(\pi_\lambda)_{\lambda\in[0,1]}$ be given by Definition 5.11 such that $G \circ u$ is a solution of π_1. It follows from Theorem 5.12 that $(\pi_\lambda, \overline{G \circ u})_{\lambda\in[0,1]}$ is $\mathcal{S}$-continuous, and, in particular, that $\pi_0 \in \mathrm{SK}_{-1}$.
Finally, it follows from Theorem 6.8 that the homotopy index of $(\pi_0, \overline{G\circ u})$ (a closed subset of the zero section, that is, $\mathbb{R}\times\{0\}$) is $\bar{0}$. However, we will not make direct use of that theorem, which is a partial corollary to Theorem 7.42 in the sense that Proposition 6.25 and Corollary 6.26 can be replaced by the arguments below as far as the singular homology (with coefficients in $\mathbb{Z}$) of the homotopy index is concerned.

Definition 7.39. Let $\{x_1,\dots,x_{n+1}\}$ be a basis for $E^-_X(e^-)$ consisting of eigenvectors of $A - Df(e^-)$, let $\{y_1,\dots,y_n\}$ be a basis for $E^-_X(e^+)$, and let $\Psi_{-1} := (x_1,\dots,x_{n+1})$ and $\Psi_1 := (y_1,\dots,y_n)$ denote corresponding matrices, which we understand as isomorphisms $\mathbb{R}^{n+1} \to E^-_X(e^-)$ (resp. $\mathbb{R}^n \to E^-_X(e^+)$).
Let $P(t)$ denote the canonical projection

$$P(t):\ E^-(-1) \oplus \operatorname{span}\{\dot{u}(t)\} \oplus E^+(-1) \to E^-(-1).$$

$P(t)$ is well defined for large $t\in\mathbb{R}$.
Define

$$\nu(u) := \nu(u,\Psi_{-1}) := (-1)^{i+1}\operatorname{sgn}\tilde{\nu},$$
$$\hat{\Psi} = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1}),$$

and

$$\operatorname{sgn} u := \nu(u)\cdot \lim_{(t,t+\Delta)\to(-\infty,\infty)} \operatorname{sgn}\det \Psi_1^{-1} P(t+\Delta) D\Pi_\Delta(u(t))\hat{\Psi}$$

where $u(-t)\|u(-t)\|_\alpha^{-1} \to \tilde{\nu} x_i \|x_i\|_\alpha^{-1}$ as $t\to\infty$ and $\Pi_t x := x\pi t$.
It is clear that $\operatorname{sgn} u$ depends on the isomorphisms Ψ_{-1} and Ψ_1, that is, $\operatorname{sgn} u = \operatorname{sgn}(u,\Psi_{-1},\Psi_1)$.

$\overline{u} = \mathrm{cl}\{u(t):\ t\in\mathbb{R}\}$ is an isolated invariant set, and $(\overline{u}, \{e^+\}, \{e^-\})$ is an attractor-repeller decomposition ($\{e^+\}$ denotes the attractor). There is a long exact sequence in singular homology associated with the attractor-repeller decomposition. Let $(\partial_q)_{q\in\mathbb{Z}}$ denote the family of connecting homomorphisms of this sequence, that is, $\partial_{q+1}:\ H_{q+1}\langle\pi,\{e_{-1}\}\rangle \to H_q\langle\pi,\{e_1\}\rangle$ for all $q\in\mathbb{Z}$.

Definition 7.40. Let θ be given by Definition 7.22, μ by Definition 7.31, and let

$$\hat{\theta}(\pi,u) := \hat{\theta}(\pi,u,\Psi_{-1},\Psi_1) := \theta(H_n\langle\hat{o}_1\rangle \circ \partial_{n+1} \circ H_{n+1}\langle\hat{o}_{-1}\rangle, \mu, n+1, 1),$$

where we set

$$\hat{o}_{-1}(y) := e^- + \Psi_{-1}(y) \quad y \in \mathbb{R}^{n+1}$$

and

$$\hat{o}_{+1}(y) := e^+ + \Psi_{+1}(y) \quad y \in \mathbb{R}^n.$$

It follows from Proposition 7.2 that π is strongly linearizable at e^+ and e^-, so Proposition 7.19 implies that $\hat{o}_i$, $i \in \{-1,1\}$, induces an orientation. Thus, $\hat{\theta}$ is defined. Let $p_1 : \mathbb{R} \times E \to \mathbb{R}$ (resp. $p_2 : \mathbb{R} \times E \to E$), $p_1(x,y) := x$ (resp. $p_2(x,y) := y$), denote the projection onto the first (resp. second) component.

Proposition 7.41. *$\hat{\theta}(\pi, u, \Psi_{-1}, \Psi_1) = \nu(u, \Psi_{-1}) \cdot \bar{\theta}(\pi_0, \tilde{\Psi}_{-1}, \tilde{\Psi}_1)$, where we set*

$$\tilde{\Psi}_{-1} := p_2 \circ DG(e^-) \circ \hat{\Psi}$$

and

$$\tilde{\Psi}_1 := p_2 \circ DG(e^+) \circ \Psi_1. \tag{7.14}$$

Note that our assumptions at the beginning of this section imply that $(0, \tilde{\Psi}_1 y) = (DG(e^+) \circ \Psi_1)y$ for all $y \in \mathbb{R}^n$.

Proof. Define

$$\begin{aligned} o_{-1}(x,y) &:= (-1 + x, \tilde{\Psi}_1(y)) & (x,y) \in \mathbb{R} \times \mathbb{R}^n \\ o_1(y) &:= (1, \tilde{\Psi}_1(y)) & y \in \mathbb{R}^n \end{aligned}$$

as in Lemma 7.25 and consider the following commutative diagram

$$\begin{tikzcd}
H_{n+1}\langle \pi, \{e_{-1}\}\rangle \arrow[r, "\partial_{n+1}"] \arrow[d, "H_{n+1}\langle B_{-1}\rangle"] \arrow[ddd, bend right=60, "H_{n+1}\langle G\rangle"'] & H_n\langle \pi, \{e_1\}\rangle \arrow[d, "H_n\langle B_1\rangle"'] \arrow[ddd, bend left=60, "H_n\langle G\rangle"] \\
H_{n+1}\langle B_{-1}[\pi], \{(-1,0)\}\rangle \arrow[d, "H_n\langle \mathrm{id}\rangle"] & H_n\langle B_1[\pi], \{(1,0)\}\rangle \arrow[d, "H_n\langle \mathrm{id}\rangle"'] \\
H_{n+1}\langle B_{-1}[\pi], \{(-1,0)\}\rangle \arrow[d, "H_{n+1}\langle GB_{-1}^{-1}\rangle"] & H_n\langle B_1[\pi], \{(1,0)\}\rangle \arrow[d, "H_n\langle GB_1^{-1}\rangle"'] \\
H_{n+1}\langle \pi_1, \{(-1,0)\}\rangle \arrow[r, "\delta_{n+1}"] & H_n\langle \pi_1, \{(1,0)\}\rangle,
\end{tikzcd}$$

where we set

$$B_{-1}(e^- + x) := (-1,0) + DG(e^-)x$$
$$B_1(e^+ + x) := (1,0) + DG(e^+)x.$$

$\delta_q : H_q\langle \pi_1, \{(-1,0)\}\rangle \to H_{q-1}\langle \pi_1, \{(1,0)\}\rangle$ is the connecting homomorphism associated with $(\pi_1, [-1,1] \times \{0\}, \{(1,0)\}, \{(-1,0)\})$.

Applying orientations, we obtain for $i \in \{-1, 1\}$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "H_m\langle \hat{o}_i\rangle \circ \mu_m"] \arrow[d, "\cdot 1"] & H_m\langle \pi, \{e^i\}\rangle \arrow[d, "H_m\langle B_1\rangle"'] \arrow[ddd, bend left=60, "H_m\langle G\rangle"] \\
\mathbb{Z} \arrow[r, "H_m\langle B_i \circ \hat{o}_i\rangle \circ \mu_m"] \arrow[d, "\cdot \alpha_i"] & H_m\langle B_i[\pi], \{(i,0)\}\rangle \arrow[d, "H_m\langle \mathrm{id}\rangle"'] \\
\mathbb{Z} \arrow[r, "H_m\langle o_i\rangle \circ \mu_m"] \arrow[d, "\cdot \beta_i"] & H_m\langle B_i[\pi], \{(i,0)\}\rangle \arrow[d, "H_m\langle GB_1^{-1}\rangle"'] \\
\mathbb{Z} \arrow[r, "H_m\langle o_i\rangle \circ \mu_m"] & H_m\langle \pi_1, \{(i,0)\}\rangle,
\end{tikzcd}$$

where we set

$$m := \begin{cases} n+1 & i = -1 \\ n & i = 1. \end{cases}$$

It follows from Proposition 7.24 that $\beta_{-1} = \beta_1 = 1$. We thus have (relative to these orientations)

$$\hat{\theta}(\pi, u, \hat{o}_{-1}, \hat{o}_1) = \alpha_1 \alpha_{-1} \tilde{\theta}(\delta_{n+1}),$$

where we set $\tilde{\theta}(\delta) := \theta(H_n\langle o_1\rangle^{-1} \circ \delta \circ H_{n+1}\langle o_{-1}\rangle, \nu, n+1, 1)$.
One has $\alpha_1 = 1$ because $B_1 \circ \hat{o}_1 = o_1$. By Proposition 7.21 we further have

$$\alpha_{-1} = \operatorname{sgn}\det\left(\Psi_1^{-1} \circ DG(e_{-1})^{-1} \circ (1, \tilde{\Psi}_{-1})\right),$$

where $(1, \tilde{\Psi}_{-1})(y_1, y_2) = (y_1, \tilde{\Psi}_{-1} y_2)$.
Since $(u(t) - e^+)\|u(t) - e^+\|_\alpha^{-1} \to \tilde{\nu} x_i \|x_i\|_\alpha^{-1}$ in X^α as $t \to -\infty$, one has $DG(e^-)(\tilde{\nu} x_i) = (c, 0)$ for some $0 < c \in \mathbb{R}$, so written as matrices [2]

$$\Psi_{-1}^{-1} DG(e^-)^{-1}((1,0), \tilde{\Psi}_{-1}) \sim (\tilde{\nu}\tilde{x}_i, \tilde{x}_1, \dots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \dots \tilde{x}_{n+1}).$$

Here, $\tilde{x}_k := \Psi_{-1} x_k$ denotes the k-th unity vector in $\mathbb{R}^{n+1}$, and given $C, D \in \mathrm{ISO}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$, we write $C \sim D$ iff $\det C \det D > 0$. This shows that $\alpha_{-1} = (-1)^{i+1}\tilde{\nu} = \nu(u)$.
It follows from Proposition 5.15 that $(\pi_\lambda, [-1,1])_{\lambda \in [0,1]}$ is $\mathcal{S}$-continuous and for every $\lambda \in [0,1]$, $([-1,1], \{1\}, \{-1\})$ is an attractor-repeller decomposition relative to π_λ. Let $\delta_{n+1}^\lambda : H_{n+1}\langle \pi_\lambda, \{(-1,0)\}\rangle \to H_n\langle \pi_\lambda, \{(1,0)\}\rangle$ denote the associated connecting homomorphism in singular homology.
We will show that $\lambda \mapsto \tilde{\theta}(\delta_{n+1}^\lambda) =: \tilde{\theta}_\lambda$ is locally constant. Otherwise, there is a sequence $\lambda_k \to \lambda_0$ in $[0,1]$ such that $\theta_k := \tilde{\theta}(\delta^{\lambda_k}) \neq \tilde{\theta}(\delta^{\lambda_0}) =: \theta_0$. It follows from [**4**, Theorem 7.3] that for all k large enough, there are strongly admissible FM-index triples $(N_{1,k}, N_{2,k}, N_{3,k})$ and $(\tilde{N}_{1,k}, \tilde{N}_{2,k}, \tilde{N}_{3,k})$ for $\pi_k := \pi_{\lambda_k}$, $k \in \mathbb{N} \cup \{0\}$ such that the following diagram (the rows

[2] $(y_1, \dots, y_n)(x_1, \dots, x_n) := x_1 \cdot y_1 + \dots + x_n \cdot y_n$ for $(x_1, \dots, x_n) \in \mathbb{R}^n$

of which are a part of the respective long exact attractor repeller sequence in homology)

$$\begin{array}{ccc} H_{q+1}[N_{1,k}/N_{2,k}] & \xrightarrow{\delta^k_{q+1}} & H_q[N_{2,k}/N_{3,k}] \\ \downarrow \subset & & \downarrow \subset \\ H_{q+1}[\tilde N_{1,0}/\tilde N_{2,0}] & \xrightarrow{\delta^0_{q+1}} & H_q[\tilde N_{2,0}/\tilde N_{3,0}] \end{array}$$

is defined, commutative, and its vertical arrows denote isomorphisms[3].
Now, Proposition 7.13 implies that $\theta_k = \theta_0$ for all $k \in \mathbb{N}$ sufficiently large, a contradiction, and so $\bar\theta(\pi_0, \tilde\Psi_{-1}, \tilde\Psi_1) = \theta_0 = \theta_1 = \tilde\theta(\delta_{n+1})$. □

THEOREM 7.42. $\operatorname{sgn} u := \operatorname{sgn}(u, \Psi_{-1}, \Psi_1)$ *is well-defined and*

$$\partial_{q+1} \circ H_{q+1}\langle \hat o_{-1}\rangle \circ \mu_{q+1} = \begin{cases} \operatorname{sgn} u \cdot H_q\langle \hat o_1\rangle \circ \mu_q & q = n \\ 0 & \textit{otherwise.} \end{cases}$$

Note that the seeds $\hat o_{\pm 1}$ and the sign of u depend on $\Psi_{\pm 1}$.

PROOF. Let $v(t) := p_1 \circ u(t)$. Lemma 5.16 relates the semigroup[4] Π_t to the linear skew product semiflow $\pi_0 = (\xi, \Phi)$, namely

$$p_2 D\tilde\Pi_\Delta(v(t)) = \Phi(v(t), \Delta)p_2,$$

where we set

$$\tilde\Pi_t x := G(x\xi t)\Pi_t G(x)^{-1} = x\pi_1 t.$$

If $w:\ [0,T] \to E^\alpha$ is a solution of $\Phi(v(t), .)$, that is, $w(t) = \Phi(v(t), t)w(0)$, then w is a mild solution of

$$\dot y + Ay = F(t)y$$

with $F(t)y = p_2 D(DG(u(t)) \circ f \circ G^{-1}(v(t)))(0, y)$. For large $t \in \mathbb{R}$, one has $F(t)y = Df(u(t))y$.
Recall that Definition 7.33 relies on a special projection P. In view of the previous remarks, it is clear that P is the canonical projection $P:\ E^-(-1) \oplus E^+(-1) \to E^-(-1)$.
Let $P(t)$ be given by Definition 7.39. Translating to $\mathbb{R} \times E$, we obtain

$$\tilde P(t) := DG(u(t))P(t)DG(u(t))^{-1}.$$

We have

$$P(t)DG(u(t))^{-1}(x, y) = P(t)\tilde x\dot u(t) + Py$$

for some $\tilde x \in \mathbb{R}$, so we can drop the notation of t that is, $\tilde P := \tilde P(t)$, where t is large (so that $P(t)$ is defined) but, apart from that, arbitrary.

[3] The respective inclusion induced morphism in the homotopy category of pointed spaces is a homotopy equivalence and therefore induces an isomorphism in singular homology.
[4] $\Pi_t x := x\pi t$

Defining

$$\begin{aligned}
\tilde{\Psi}_{1,t} &:= DG(u(t))\Psi_1 \\
\tilde{\Psi}_1 &:= DG(e^+)\Psi_1 \\
\tilde{\Psi}_{-1,t} &:= DG(u(t))\hat{\Psi} \\
\tilde{\Psi}_{-1} &:= DG(e^-)\hat{\Psi}
\end{aligned}$$

we further have $\tilde{\Psi}_{1,t} = \tilde{\Psi}_1$ for all $t \in \mathbb{R}$ with $|t|$ sufficiently large, and $\tilde{\Psi}_{-1,t} \to \tilde{\Psi}_{-1}$ as $t \to -\infty$.

It follows from Corollary 7.38 that

$$\operatorname{sgn}\det(p_2\tilde{\Psi}_1)^{-1}P\Phi(v(t),\Delta)p_2\tilde{\Psi}_{-1,t} \to \bar{\theta}(\pi_0, \tilde{\Psi}_{-1}, \tilde{\Psi}_1) \neq 0 \tag{7.15}$$

as $(t, t+\Delta) \to (-\infty, \infty)$.

For fixed parameters t and Δ, one has

$$P\Phi(v(t),\Delta)p_2 = \tilde{P}\,D\tilde{\Pi}_\Delta(v(t)),$$

so it follows from (7.15) that

$$\operatorname{sgn}\det\tilde{\Psi}_1^{-1}\tilde{P}\,D\tilde{\Pi}_\Delta(v(t))\tilde{\Psi}_{-1,t} \to \bar{\theta}(\pi_0, \tilde{\Psi}_{-1}, \tilde{\Psi}_1) \neq 0.$$

We have

$$\tilde{\Psi}_1^{-1}\tilde{P}\,D\tilde{\Pi}_\Delta(v(t))\tilde{\Psi}_{-1,t} = \Psi_1^{-1}P(t)D\Pi_\Delta(v(t))\hat{\Psi},$$

showing that $\operatorname{sgn}(u, \Psi_{-1}, \Psi_1)$ is defined. Using Proposition 7.41,

$$\operatorname{sgn}(u, \Psi_{-1}, \Psi_1) = \hat{\theta}(\pi, u, \Psi_{-1}, \Psi_1).$$

Resolving the definition of $\hat{\theta}$ yields the claim of this theorem. $\square$

CHAPTER 8

Morse homology

This chapter collects several technical theorems which allow for the definition of Morse complexes of isolated invariant sets.

1. Sums of connecting homomorphisms

In the previous chapters, we have computed homotopy and homology of the Conley index along certain heteroclinic solutions of semilinear parabolic equations. If one considers all solutions connecting a given couple of equilibria, their connecting homomorphism is the sum of the homomorphisms associated with each single isolated connecting orbit.

What we have described, is the simplest possible application of Proposition 8.7. It has been shown by McCord [**16**, Theorem 2.5] in the context of flows on locally compact metric spaces.

Throughout this section, let π be a semiflow on a metric space (X, d) and for $i \in \{1, 2\}$ K_i an isolated invariant set admitting a strongly π-admissible isolating neighborhood. Further let (A, A^*) be an attractor-repeller decomposition for both invariant sets: K_1 and K_2, and suppose that there is a neighborhood I of A^* with $K_1 \cap K_2 \cap I = A^*$.

All indices and isolating neighborhoods are to be understood relative to the semiflow π, which is not mentioned explicitly.

We consider singular homology with coefficients in an abelian group G, and denote the homology functor by H_q, $q \in \mathbb{Z}$.

Definition 8.1. Let $U \subset X$ and $f:\ U \to \mathbb{R}$ be continuous. We say that f is locally strictly monotone increasing along π in α if there is a neighborhood V of α in $\mathbb{R}$ such that $f \circ \sigma$ is strictly montone increasing along every solution σ of π with $\operatorname{im} \sigma \subset U$ and $\operatorname{im} f \circ \sigma \subset V$.

Lemma 8.2. *Let B be an isolating block, $\alpha \in \mathbb{R}$, and $f:\ B \to \mathbb{R}$ strictly monotone increasing along π in α.*
Then $\{f \geq \alpha\} := \{x \in B:\ f(x) \geq \alpha\}$ and $\{f \leq \alpha\} := \{x \in B:\ f(x) \leq \alpha\}$ are isolating blocks.

Proof. Letting $x \in \partial\{f \geq \alpha\}$ (resp. $\{f \leq \alpha\}$), it follows that $x \in \partial B$ or $f(x) = \alpha$. Let $\delta_1 \geq 0$, $\delta_2 > 0$, and $\sigma:\ [-\delta_1, \delta_2] \to X$ be a solution of π with $\sigma(0) \in \partial B$. Then π is transversal to $\partial\{f \geq \alpha\}$ (resp. $\partial\{f \leq \alpha\}$), that is, there is a neighborhood V of 0 in $[-\delta_1, \delta_2]$ such that $\sigma(V) \cap \partial\{f \geq \alpha\} = \{\sigma(0)\}$.
In order to show that $\{f \geq \alpha\}$ (resp. $\{f \leq \alpha\}$) is an isolating block, we need to prove that the following situation cannot occur: $\delta_1 > 0$ and $\sigma([-\delta_1, \delta_2]) \subset \{f \geq \alpha\}$ (resp. $\{f \leq \alpha\}$). Suppose that we are given such a solution. Then $\sigma(0) \in \operatorname{int} B$, that is, we can assume w.l.o.g. that $\sigma([-\delta_1, \delta_2]) \subset \operatorname{int} B$. This implies that $f(\sigma(0)) = \alpha$, so for $s > 0$ sufficiently

small, $f(\sigma(s)) > \alpha$ (resp. for $s < 0$ sufficiently small $f(\sigma(s)) < \alpha$), a contradiction. Here, we have used the fact that f is locally strictly monotone increasing along π in α. $\square$

LEMMA 8.3. *Let $N \subset X$ be a strongly π-admissible isolating neighborhood for K.*
Then there are an isolating block B for (π, K), $\alpha < \beta \in \mathbb{R}^+$, and a continuous function $V :\ B \to \mathbb{R}^+$ such that

(1) *π is locally gradient-like with respect to V in every $\xi \in [a, b]$;*
(2) *$V(x) < \alpha$ for all $x \in A^*$; $V(x) > \beta$ for all $x \in A$;*
(3) *$\{\alpha \leq V \leq \beta\} = D_1 \dot{\cup} D_2$, where $K_i \cap D_k = \emptyset$ whenever $\{i, k\} = \{1, 2\}$; D_1 and D_2 are not connected.*

The construction of B and V in Lemma 8.3 is depicted in Figure 8.1.

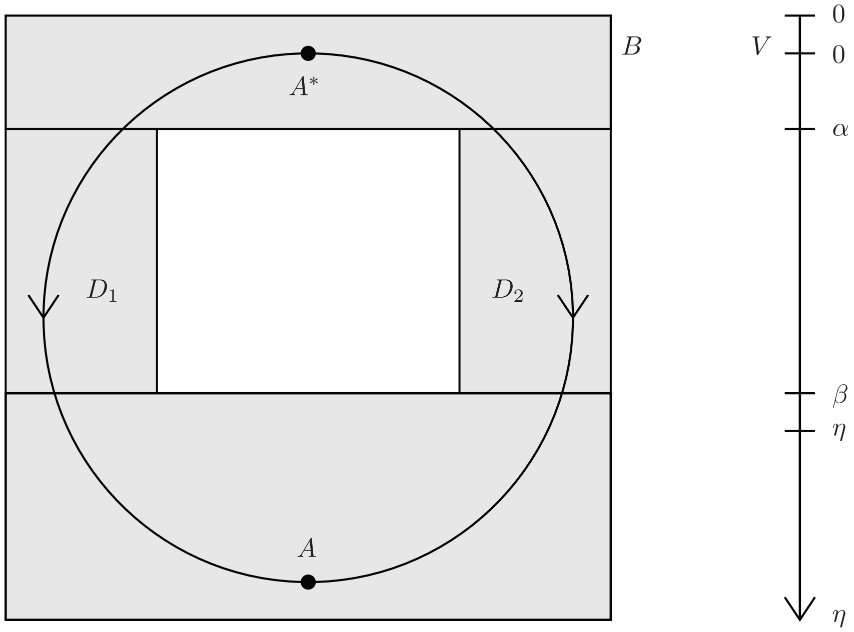

FIGURE 8.1. Idealized construction of B and V

PROOF. Recall that there is a neighborhood I of A^* with $K_1 \cap K_2 \cap I = A^*$. Let $\tilde{N} \subset N$ be an isolating block for A^* with $\tilde{N} \subset I$ and $K_0 := K \cap \operatorname{Inv}^-(\tilde{N})$. There is an open neighborhood L of $K \setminus K_0$ such that $L \cap K_0 = \emptyset$ and $\tilde{\tilde{N}} := \tilde{N} \setminus L$ is still a (strongly admissible) isolating neighborhood for A^*.
Let $g^-, g^+ :\ \tilde{N} \to \mathbb{R}^+$ be defined as in the proof of [**18**, Theorem I.5.1]. These functions are continuous, and $g^-(x) = 0$ for all $x \in K \cap \tilde{\tilde{N}} \subset \operatorname{Inv}^-(\tilde{N})$. For $\eta > 0$ small enough, we have $H_\eta \subset \operatorname{int} \tilde{\tilde{N}}$, where

$$H_\eta := \operatorname{cl}\{x \in \tilde{N} :\ g^+(x) < \eta \text{ and } g^-(x) < \eta\}.$$

There is a family $(B_\delta)_{\delta \in [0,1]}$ of isolating blocks for K with $\sup_{x \in B_\delta} d(x, K) \to 0$ as $\delta \to 0$. We claim that there is a $\delta_0 > 0$ such that

$$g^-(x) < \eta/2 \text{ for all } \delta \in \,]0, \delta_0] \quad x \in B_\delta \cap H_\eta. \tag{8.1}$$

Otherwise, there is a sequence $x_n \in H_\eta \subset \tilde{\tilde{N}}$ with $d(x_n, K) \to 0$ as $n \to \infty$ and $g^-(x_n) \geq \eta/2$ for all $n \in \mathbb{N}$. However, one has $g^-(x) = 0$ for all $x \in K \cap \tilde{\tilde{N}} \subset \operatorname{Inv}^-(\tilde{N})$.

Therefore, choosing δ_0 sufficiently small, we have $g^+(x) = \eta$ for all $x \in (\partial H_\eta) \cap B_{\delta_0}$, which allows us to define continuous extensions $V_\delta : B_\delta \to \mathbb{R}^+$ by

$$V_\delta(x) := \begin{cases} g^+(x) & x \in H_\eta \\ \eta & x \in B_\delta \setminus H_\eta, \quad 0 < \delta \le \delta_0. \end{cases}$$

π is locally gradient-like with respect to V_δ in ξ for all $\xi \in]0, \eta[$ and all $\delta \in]0, \delta_0]$.
Let $0 < \alpha < \beta < \eta$ and $D := D_\delta := \{\alpha \le V_\delta \le \beta\}$. Choosing $\delta \in]0, \delta_0]$ small enough, it follows that $D \cap K_1$ and $D \cap K_2$ cannot lie in the same connected component of D. Otherwise, there would be a sequence $x_n \to x_0 \in K_1 \cap K_2 \cap I \subset A^*$ with $V_\delta(x_n) = V_{\delta_0}(x_n) \in [\alpha, \beta]$ for all $n \in \mathbb{N}$, a contradiction since $V_{\delta_0}(x_0) \in \{0, \eta\}$. □

Fix some $\tilde{\beta} \in]\alpha, \beta[$ and let V be given by Lemma 8.3. In the sequel, we will tacitly use that $\{V \ge \alpha\}$ and $\{V \ge \tilde{\beta}\}$ are strongly admissible FM-index pairs (proved in Lemma 8.2). Moreover, on $\{V \ge \alpha\}$ define

$$\tilde{V}(x) := \begin{cases} V(x) & x \in D_1 \\ \beta & \text{otherwise.} \end{cases}$$

It follows that $\{\tilde{V} \ge \tilde{\beta}\} = D_2 \cup \{V \ge \tilde{\beta}\}$ is an isolating block. Analogously, one obtains that $D_1 \cup \{V \ge \tilde{\beta}\}$ is an isolating block.

LEMMA 8.4. *Let $B_1 \subset B_2$ be isolating blocks such that B_1 is B_2-positively invariant. Then $B_1^- \subset B_2^-$.*

PROOF. For every $x \in B_1^-$, there is an $\varepsilon > 0$ such that $x\pi\,]0, \varepsilon[\cap B_1 = \emptyset$. It follows from the positive invariance that there is an $s \in]0, \varepsilon[$ with $x\pi s \notin B_2$, showing that $x\pi r \in B_2^-$ for some $r \in [0, \varepsilon[$. We have shown that for every $\varepsilon > 0$ there is a $t \in [0, \varepsilon[$ with $x\pi t \in B_2^-$, which is closed, so $x \in B_2^-$. □

LEMMA 8.5. *Let $B_1 \subset B_2$ be isolating blocks such that B_1 is B_2-positively invariant. Then $(B_2, B_1 \cup B_2^-, B_2^-)$ is an FM-index triple with* $\mathrm{Inv}(B_1 \cup B_2^-) = \mathrm{Inv} B_1$.

PROOF. We need to show that $(B_1 \cup B_2^-, B_2^-)$ is an FM-index pair.

(1) Let $x \in B_2^-$ and $t \ge 0$ such that $x\pi[0, t] \subset B_1 \cup B_2^-$. It follows that $t = 0$ since there is an $\varepsilon > 0$ such that $x\pi\,]0, \varepsilon[\cap B_2 = \emptyset$. This shows that B_2^- is $B_1 \cup B_2^-$-positively invariant.
(2) Let $x \in B_1 \cup B_2^-$ and $t \ge 0$ such that $x\pi[0, t]$ is defined and $x\pi t \notin B_1 \cup B_2^-$. Since $B_1 \cup B_2^-$ is B_2-positively invariant, there is an $s \in [0, t]$ with $x\pi s \notin B_2$. B_2^- is an exit ramp for B_2, showing that B_2^- is also an exit ramp for $B_2^- \cup B_1$.
(3) We have $\mathrm{Inv}(B_1 \cup B_2^-) \subset \mathrm{Inv} B_2$, showing that $\mathrm{Inv}(B_1 \cup B_2^-) = \mathrm{Inv}(B_1)$ since $\mathrm{Inv}(B_2) \cap B_2^- = \emptyset$.

□

DEFINITION 8.6. Let K be an isolated invariant set admitting a strongly admissible isolating neighborhood. $H_*\langle \pi, K \rangle$ denotes the graded module $(H_q\langle \pi, K \rangle)_{q \in \mathbb{Z}}$ of the q-th homology of the Conley index, which is defined in [**4**, Definition 4.3] (see also the beginning of chapter 7).

We can understand the connecting morphism associated with (K, A, A^*) as a morphism of degree -1 in the category of graded modules, that is, $\partial : H_*(A^*) \to H_*(A)$ with $\partial = (\partial_q)_{q\in\mathbb{Z}}$, where for each $q \in \mathbb{Z}$, $\partial_q : H_q(A^*) \to H_{q-1}(A)$ is the q-th connecting homomorphism defined by the attractor-repeller sequence.

PROPOSITION 8.7. *Let $\partial : H_*\langle\pi, A^*\rangle \to H_*\langle\pi, A\rangle$ denote the connecting homomorphism for (K, A, A^*) and for $i \in \{1, 2\}$ let $\partial^i : H_*\langle\pi, A^*\rangle \to H_*\langle A\rangle$ denote the connecting morphism for (K_i, A, A^*). Then $\partial = \partial^1 + \partial^2$.*

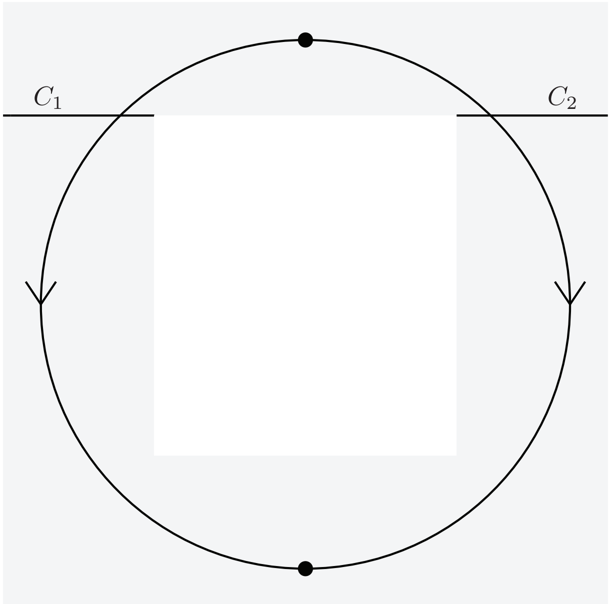

FIGURE 8.2. C_1 and C_2 in the proof of Proposition 8.7

PROOF. Let B, $\alpha, \beta \in \mathbb{R}^+$, D_1, D_2, and $V : B \to \mathbb{R}^+$ be given by Lemma 8.3. Letting $B_1 := \{V \le \alpha\}$, we have $B_1^- \setminus B^- = C_1 \dot\cup C_2$ where we set $C_i := (B_1^- \cap D_i) \setminus B^-$. Clearly, $C_1 \subset D_1$ and $C_2 \subset D_2$ are not connected in B_1^- (this is additionally illustrated in Figure 8.2).
Recall (see [**11**] or [**4**]) that a sequence of chain maps

$$\Gamma_1 \xrightarrow{i} \Gamma_2 \xrightarrow{p} \Gamma_3$$

is called *weakly exact* if $\ker i = 0$, $p \circ i = 0$, and the induced mapping $H_q(\Gamma_2/\operatorname{im} i) \to H_q(\Gamma_3)$ is an isomorphism for all $q \in \mathbb{Z}$.
It is clear that

$$0 \longrightarrow \Delta B_1^-/\Delta\tilde{B}^- \longrightarrow \Delta B_1/\Delta\tilde{B}^- \longrightarrow \Delta B_1/\Delta B_1^- \longrightarrow 0$$

is exact, where Δ denotes the functor which passes a topological space to its singular chain complex and $\tilde{B}^- := B^- \cap B_1$.
The inclusion $B_1^- \subset B_1$ is a cofibration [**18**, Theorem 3.7], so it follows from [**12**, Proposition 2.22] that

$$\Delta B_1^-/\Delta\tilde{B}^- \longrightarrow \Delta B_1/\Delta\tilde{B}^- \longrightarrow \Delta(B_1/B_1^-)/\Delta(B_1^-/B_1^-) \tag{8.2}$$

is weakly exact.

Since C_1 and C_2 are not connected, there are closed neighborhoods $\Omega_{1,1}$ (resp. $\Omega_{2,1}$) of C_1 (resp. C_2) in B_1^- such that $\Omega_{1,1} \cap \Omega_{2,1} = \emptyset$ and

$$H_q(B_1^-, \tilde{B}^-) \approx H_q(\Omega_{1,1}, \Omega_{1,2}) \oplus H_q(\Omega_{2,1}, \Omega_{2,2}) \tag{8.3}$$

where we set $\Omega_{i,2} := \Omega_{i,1} \cap B^-$ (excision).

Combining (8.2) and (8.3), one obtains a long exact sequence

$$\longrightarrow H_{q+1}[B_1/B_1^-] \xrightarrow{\delta_{q+1}} H_q(\Omega_{1,1}, \Omega_{2,1}) \oplus H_q(\Omega_{2,1}, \Omega_{2,2}) \longrightarrow H_q(B_1, \tilde{B}^-) \longrightarrow .$$

Let $\{i,k\} = \{1,2\}$, $B_{2,i} := B_1 \cup D_i \cup \{V \geq \tilde{\beta}\}$, and $B_{3,i} := D_i \cup \{V \geq \tilde{\beta}\}$. Note that $\Omega_{i,2} \subset B^-$ and $\Omega_{i,1} \subset B_{2,1} \cap B_{2,2}$. Moreover, for $x \in \Omega_{i,1} \setminus B^-$ one has $x\pi s \in D_i$ for small s, so $x\pi s \notin B_{2,k}$, showing that $\Omega_{i,1} \subset B_{2,k}^-$, where $\{i,k\} = \{1,2\}$.

Lemma 8.5 implies that $(B_{2,i}, B_{3,i} \cup B_{2,i}^-, B_{2,i}^-)$ is an FM-index triple for (π, K_i, A^*, A). By inclusion (resp. projection), we obtain a commutative diagram

$$\begin{tikzcd}
H_{q+1}[B_{2,i}/(B_{3,i} \cup B_{2,i}^-)] \arrow[r, "\tilde{\partial}^i_{q+1}"] & H_q[(B_{3,i} \cup B_{2,i}^-)/B_{2,i}^-] \arrow[r] & H_q[B_{2,i}/B_{2,i}^-] \\
H_{q+1}[B_1/B_1^-] \arrow[u, "\approx"] \arrow[r, "\delta_{q+1}"] & H_q(\Omega_{1,1}, \Omega_{1,2}) \oplus H_q(\Omega_{2,1}, \Omega_{2,2}) \arrow[u, "j_{i,q} \oplus 0"] \arrow[r] & H_q(B_1, \tilde{B}^-) \arrow[u]
\end{tikzcd}$$

We have $\Phi(\tilde{\partial}^i_{q+1}) = \partial^i_{q+1}$, where Φ denotes the functor which embeds an FM-index pair into its homology Conley index.

Likewise, we obtain a relation between $\partial_{q+1} = \Phi(\tilde{\partial}_{q+1})$ and δ_{q+1}, namely

$$\begin{tikzcd}
H_{q+1}[B/(B_{3,1} \cup B_{3,2} \cup B^-)] \arrow[r, "\tilde{\partial}_{q+1}"] & H_q[(B_{3,1} \cup B_{3,2} \cup B^-)/B^-] \arrow[r] & H_q[B/B^-] \\
H_{q+1}[B_1/B_1^-] \arrow[u, "\approx"] \arrow[r, "\delta_{q+1}"] & H_q(\Omega_{1,1}, \Omega_{1,2}) \oplus H_q(\Omega_{2,1}, \Omega_{2,2}) \arrow[u, "k_{1,q} \oplus k_{2,q}"] \arrow[r] & H_q(B_1, \tilde{B}^-). \arrow[u]
\end{tikzcd}$$

It remains to show that $k_{i,q} = H_q(\alpha_i) \circ j_{i,q}$ for $i \in \{1,2\}$, where $\alpha_i : (B_{3,i} \cup B_{2,i}^-)/B_{2,i}^- \to (B_{3,1} \cup B_{3,2} \cup B^-)/B^-$ denotes the inner morphism. This is clear since $(B_{3,i} \cup B_{2,i}^-, B_{2,i}^-)$, $(B_{3,i}, B_{3,i}^-)$, and $(B_{3,1} \cup B_{3,2} \cup B^-, B^-)$ are FM-index pairs and

$$\begin{tikzcd}
(B_{3,i} \cup B_{2,i}^-)/B_{2,i}^- \arrow[rr, bend left, "\alpha_i"] & B_{3,i}/B_{3,i}^- \arrow[l, "\supset"'] \arrow[r, "\subset"] & (B_{3,1} \cup B_{3,2} \cup B^-)/B^- \\
& (\Omega_{i,1}, \Omega_{i,2}) \arrow[ul, "j_i"] \arrow[u, "\subset"] \arrow[ur, "k_i"'] &
\end{tikzcd}$$

is commutative (the notation of the basepoint for the quotient spaces is, as usual, omitted).

$\square$

2. Morse complex

Let (X, d) be a metric space and π be a semiflow on X. Further let $N \subset X$ be a strongly π-admissible isolating neighborhood for $K := \operatorname{Inv} N$. Define

$$E_n := \{e \in K : \text{ } e \text{ is an isolated equilibrium with } h(\pi, \{e\}) = \Sigma^n\}$$

to be the set of all equilibria of degree n. The corresponding set of connecting orbits

$$C_n \subset K$$

is given by $x \in C_n$ iff there exists a full solution $u : \mathbb{R} \to K$ with $u(0) = x$ such that there are equilibria $e^- \in E_n$ and $e^+ \in E_m$, $0 \leq m < n$, with $u(t) \to e^+$ as $t \to \infty$ and $u(t) \to e^-$ as $t \to -\infty$.

Lemma 8.8. *$E := \bigcup_{n \in \mathbb{N}} E_n$ is finite, and hence $E_n = C_n = \emptyset$ for almost all $n \in \mathbb{N}$.*

Proof. Since K is compact, it follows that E (which is closed) is compact. Each $e \in E$ is isolated, hence, there exists an open set $U_e \subset E$ such that $U_e \cap E = e$. $\{U_e : \ e \in E\}$ is an open covering, so there is a finite subset $E_0 \subset E$ with $E \subset \bigcup_{e \in E_0} U_e$, which implies that E itself is finite. □

Let $N := \max\{n \in \mathbb{N} : \ E_n \neq \emptyset\}$ and assume that $(E_1, \dots, E_N)$ is a Morse decomposition of K. It follows immediately [**18**, Theorem III.1.7] that

$$K = \bigcup_{n \in \mathbb{N}} E_n \cup \bigcup_{n \in \mathbb{N}} C_n.$$

For every $n \in \mathbb{Z}$, define

$$K_n := \bigcup_{k=0}^{n} E_k \cup \bigcup_{k=0}^{n} C_k.$$

for $n \geq 0$ and

$$K_n = \emptyset$$

for $n < 0$.

Let $e \in E_n$ and $f \in E_m$, and set $e < f$ if and only if $n < m$. This defines a partial order on E, and $(\{e\})_{e \in E}$ is a $<$-ordered Morse decomposition. This follows from the assumption of $(E_1, \dots, E_N)$ being a Morse decomposition. Note that this implies in particular that given $e^- \in E_{n+1}$ and $e^+ \in E_n$, the set

$$K_{e^-, e^+} := \{x \in K : \ \exists \text{ full solution } \sigma \text{ through } x \text{ with } \sigma(t) \to e^{\pm} \text{ as } t \to \pm\infty\} \cup \{e^-, e^+\}$$

is an isolated invariant set admitting a strongly admissible isolating neighborhood. Therefore, K_{e^-, e^+} is compact.

Lemma 8.9. *Let K be a compact invariant set, and let $\sigma_n : \ \mathbb{R} \to K$ be a sequence of full solutions.*
Then there is a solution full solution $\sigma : \ \mathbb{R} \to K$ and a subsequence $\sigma_{n(k)}$ with $\sigma_{n(k)}(t) \to \sigma(t)$ as $k \to \infty$ for all $t \in \mathbb{R}$.

Proof. Due to the compactness of K, we can choose a subsequence $a_{1,n}$ of $(\sigma_n)_n$ with $\sup_{k,l \geq n} d(a_{1,k}(0), a_{1,l}(0)) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$.

Suppose that we are given a sequence $(a_{m,n})_n$ with

$$d(a_{m,k}(-m'), a_{m,l}(-m')) \leq \frac{1}{n}$$

for all $k, l \geq n$, all $n \in \mathbb{N}$, and all $m' \in \{0, \dots, m\}$. Let $(a_{m,N(n)})_n$ be an arbitrary subsequence of $(a_{m,n})_n$. Then $N(n) \geq n$ for all $n \in \mathbb{N}$, so

$$d(a_{m,N(k)}(-m'), a_{m,N(l)}(-m')) \leq \frac{1}{n}$$

for all $k, l \geq n$, all $n \in \mathbb{N}$, and all $m' \in \{0, \dots, m\}$.
Inductively, we obtain a family $(a_{m,n})_{(m,n) \in \mathbb{N} \times \mathbb{N}}$ such that for all $m' \in \{0, \dots, m\}$ and all $k, l \geq n$

$$d(a_{m,k}(-m'), a_{m,l}(-m')) \leq \frac{1}{n}. \tag{8.4}$$

Let $b_n := a_{n,n}$, $t \in \mathbb{R}$ be arbitrary, and choose $m \in \mathbb{N}$ with $-m \leq t$. $b_n(-m)$ is a Cauchy sequence by (8.4), so $b_n(-m) \to x_0 \in K$ and $b_n(t) = b_n(-m)\pi(t+m) \to x_0\pi(t+m)$ by continuity. The same argument shows that $\sigma(t) := \lim_{n \to \infty} b_n(t)$ is a solution. □

LEMMA 8.10. *Let e^-, e^+ be as above, and let $\sigma_n : \mathbb{R} \to K_{e^-,e^+}$ be a sequence of solutions with $\sigma_n(t) \to e^\pm$ as $t \to \pm\infty$.*
Then there is a sequence t_k in $\mathbb{R}$, a solution $\sigma : \mathbb{R} \to K_{e^-,e^+}$ with $\sigma(t) \to e^\pm$ as $t \to \pm\infty$, and a subsequence $\sigma_{n(k)}$ with $\sup_{t \in \mathbb{R}} d(\sigma_{n(k)}(t + t_k), \sigma(t)) \to 0$ as $k \to \infty$.

PROOF. Let $\varepsilon > 0$ such that $d(e^-, e^+) > 2\varepsilon$, and set $t_n := \sup\{t \in \mathbb{R} : d(\sigma_n(t), e^-) \leq \varepsilon$. In view of Lemma 8.9, there exist subsequences $\sigma_{n(k)}$ and $t_{n(k)}$, and a full solution $\sigma : \mathbb{R} \to K_{e^-,e^+}$ with $\tilde{\sigma}_k(t) := \sigma_{n(k)}(t_{n(k)} + t) \to \sigma(t)$ as $k \to \infty$ for all $t \in \mathbb{R}$. Since $\sigma(0) \in K_{e^-,e^+}$ with $d(\sigma(0), e^-) \geq \varepsilon$ and $d(\sigma(0), e^+) \geq \varepsilon$, it is clear that $\sigma \not\equiv e^-$ and $\sigma \not\equiv e^+$. Thus, $\sigma(t) \to e^-$ as $t \to -\infty$ and $\sigma(t) \to e^+$ as $t \to \infty$.
Let $\delta \in]0, \varepsilon[$ be arbitrary but fixed. We claim that there is a $t_0 \in \mathbb{R}$ such that $d(\tilde{\sigma}_k(t), e^-) \leq \delta$ for all $t \leq t_0$ and all $k \in \mathbb{N}$. Otherwise there is a sequence $s_k \to -\infty$ and a mapping $n(k) : \mathbb{N} \to \mathbb{N}$ with $d(\tilde{\sigma}_{n(k)}(s_k), e^-) > \delta$ for all $k \in \mathbb{N}$. We can assume w.l.o.g. that $\tilde{\sigma}_{n(k)}(s_k) \to x_0 \in K_{e^-,e^+}$. It follows that $d(x_0\pi t, e^-) = \lim_{k \to \infty} d(\tilde{\sigma}_{n(k)}(s_k + t), e^-) \leq \varepsilon$ for all $t \in \mathbb{R}^+$, so $x_0 = e^-$. However, $d(x_0, e^-) \geq \delta$, a contradiction.
Hence, there exists a $\tau \in \mathbb{R}^-$ such that for all $t \leq \tau$ and all $k \in \mathbb{N}$

$$d(\tilde{\sigma}_k(t), e^-) \leq \delta$$
$$d(\sigma(t), e^-) \leq \delta.$$

Let $\tilde{\pi}$ denote the restriction of π to K_{e^-,e^+}. It is again a semiflow, so it follows from [**18**, Corollary 5.5] that there is an isolating block B for e^+ with $B^- = \emptyset$ and $d(e^+, B) \leq \delta$. Choosing $T \in \mathbb{R}$ large enough, we have $\sigma(t) \in \operatorname{int} B$ for all $t \geq T$. Therefore, $\tilde{\sigma}_k(t) \in B$ for all k large enough. Hence, for all $t \geq T$ and all k large enough, one has

$$d(\tilde{\sigma}_k(t), e^+) \leq \delta$$
$$d(\sigma(t), e^+) \leq \delta.$$

It is easy to prove that $\tilde{\sigma}_k \to \sigma$ uniformly on $[\tau, T]$. We have shown that $\tilde{\sigma}_k \to \sigma$ uniformly in t. □

DEFINITION 8.11. $\Gamma \subset K_{e^-,e^+}$ is called an orbit if there exists a full solution $\sigma : \mathbb{R} \to K_{e^-,e^+}$ such that $\sigma(t) \to e^{\pm}$ as $t \to \pm\infty$ and $\sigma(\mathbb{R}) = \Gamma$.

LEMMA 8.12. *Suppose that for every orbit Γ, $\Gamma \cup \{e^-, e^+\}$ is an isolated invariant set. Then there are finitely many orbits $\Gamma_1, \dots, \Gamma_n$ such that $K_{e^-,e^+} = \{e^-, e^+\} \cup \Gamma_1 \cup \dots \cup \Gamma_n$.*

PROOF. Suppose that the lemma is not true. Then there is a sequence of pairwise distinct orbits Γ_n. Let σ_n be a sequence of solutions with $\Gamma_n = \sigma_n(\mathbb{R})$.
Let σ and t_k be given by Lemma 8.10. By our assumptions, $K_0 := \{e^-, e^+\} \cup \sigma(\mathbb{R})$ is an isolated invariant set. Hence, there is an isolating neighborhood $\tilde{N}$ of K_0. Lemma 8.10 implies that there is a subsequence $\sigma_{n(k)}$ with $\sigma_{n(k)}(t) \in \tilde{N}$ for all $t \in \mathbb{R}$. Thus, we have $\Gamma_{n(k)} \subset K_0$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be arbitrary but fixed and set $\tilde{\sigma} := \sigma_{n(k)}$.
Let $\tau_0 \in \mathbb{R}$ such that $\tilde{\sigma}(\tau_0) \notin \{e^-, e^+\}$. For every $\tau \le \tau_0$, there is a $t = t(\tau)$ such that $\tilde{\sigma}(\tau) = \sigma(\tau - t)$. Let $\tilde{\sigma}(\tau_1) = \sigma(\tau_1 - t_1)$ and $\tilde{\sigma}(\tau_2) = \sigma(\tau_2 - t_2)$ with $\tau_2 < \tau_1 \le \tau_0$. It follows that $\sigma(\tau_1 - t_1) = \sigma(\tau_2 - t_2 + (\tau_1 - \tau_2))$, so $\sigma(\tau_1 - t_1) = \sigma(\tau_1 - t_2)$. Suppose w.l.o.g. that $\delta := t_2 - t_1 > 0$. It follows that for all $k \in \mathbb{N}$ $\sigma(\tau_1 - t_1) = \sigma(\tau_1 - t_1 + k\delta) \to e^+$ as $k \to \infty$, implying that $\sigma(\tau_1 - t_1) = \tilde{\sigma}(\tau_1) = e^+$, a contradiction to $\tau \le \tau_0$. We have proved that $\tilde{\sigma}(\tau) = \sigma(\tau - t)$ for some $t \in \mathbb{R}$ and all $\tau < \tau_0$, and consequently for all $\tau \in \mathbb{R}$.
Therefore, $\Gamma_{n(k)} = \sigma(\mathbb{R})$ for all $k \in \mathbb{N}$, which contradicts the assumption of all Γ_n being pairwise distinct. $\square$

DEFINITION 8.13. Let (A, R) be an attractor-repeller decomposition of an invariant set S, assume that there exists a strongly π-admissible FM-index triple for (π, S, A, R), and define for all $q \in \mathbb{Z}$

$$H_q\langle S, A\rangle := H_q\langle R\rangle,$$

where $H_q\langle R\rangle$ denotes the *homology Conley index* of R (see [**4**, Definition 4.3]).

Let $o \le m \le n$ and let $(K_n, K_m, R_{n,m})$, $(K_n, K_o, R_{n,o})$, and $(K_m, K_o, R_{m,o})$ be attractor-repeller decompositions (denoted in the order: invariant set, attractor, repeller). Then $(R_{n,o}, R_{m,o}, R_{n,m})$ is also an attractor-repeller decomposition. Hence, there is a long exact attractor-repeller sequence in homology

$$\longrightarrow H_q\langle R_{m,o}\rangle \longrightarrow H_q\langle R_{n,o}\rangle \longrightarrow H_q\langle R_{n,m}\rangle \longrightarrow,$$

which (Definition 8.13) can also be written as

$$\longrightarrow H_q\langle K_m, K_o\rangle \xrightarrow{i_q} H_q\langle K_n, K_o\rangle \xrightarrow{p_q} H_q\langle K_n, K_m\rangle \longrightarrow. \tag{8.5}$$

Let $n, n', m, m' \in \mathbb{Z}$ with $m \le m' \le n \le n'$. Then (8.5) gives homomorphisms

$$H_q\langle K_n, K_m\rangle \xrightarrow{i_q} H_q\langle K_{n'}, K_m\rangle$$

and

$$H_q\langle K_n, K_m\rangle \xrightarrow{p_q} H_q\langle K_n, K_{m'}\rangle,$$

which depend on n, m, n', m' and which are natural, that is, they commute with the connecting homomorphisms of long exact attractor-repeller sequences (see [**11**] or [**4**, Section 6]). This is a special case of the commutativity of the homology index braid presented there.

For all $q \in \mathbb{Z}$, let δ_{q+1} be defined by the following section of the long exact sequence

$$\longrightarrow H_{q+1}\langle K_{q+1}, K_{q-1}\rangle \longrightarrow H_{q+1}\langle K_{q+1}, K_q\rangle \xrightarrow{\delta_{q+1}} H_q\langle K_q, K_{q-1}\rangle \longrightarrow$$

It follows from the commutativity of diagram (6.2) in [**4**] that $\delta_q \circ \delta_{q+1} = 0$ for all $q \in \mathbb{Z}$, that is, $\delta = (\delta_q)_{q\in\mathbb{Z}}$ is a boundary operator and

$$WN := (H_q\langle K_q, K_{q-1}\rangle, \delta_q)_{q\in\mathbb{Z}}$$

is a chain complex. Let HW_qN denote its q-th homology.
For every $n \in \mathbb{N}$, (K_{n-1}, E_n) is an attractor-repeller decomposition of K_n. Hence, it follows immediately from Definition 8.13 that

LEMMA 8.14.
$$H_q\langle K_n, K_{n-1}\rangle = H_q\langle E_n\rangle \approx \begin{cases} G^{\#E_n} & n = q \\ 0 & n \neq q. \end{cases}$$

The following proposition and its subsequent lemmas are an adaption of chapter V.1 in [**8**] to Morse decompositions.

PROPOSITION 8.15. *There is an isomorphism*

$$\Theta :\ HWN \simeq H\langle K\rangle.$$

LEMMA 8.16. $H_q\langle K_n, K_m\rangle = 0$ *for* $n \geq m \geq q$ *or* $q > n \geq m$.

PROOF. The proof is given by induction on $n - m$. For $n - m = 0$, $(K^n, K^n, \emptyset)$ is an attractor-repeller decomposition, so we have

$$H_q\langle K_n, K_n\rangle \cong H_q\langle\emptyset\rangle = 0 \quad \forall q \in \mathbb{Z}.$$

Now let $n - m > 0$ and $(K_n, K_m, R_{n,m})$ and $(K_{n-1}, K_m, R_{n-1,m})$ be attractor-repeller decompositions which define $R_{n,m}$ and $R_{n-1,m}$. For $(R_{n,m}, R_{n-1,m}, E_n)$ we obtain an attractor-repeller sequence in homology

$$\longrightarrow H_q\langle R_{n-1,m}\rangle \longrightarrow H_q\langle R_{n,m}\rangle \longrightarrow H_q\langle E_n\rangle \longrightarrow,$$

which can also be written as

$$\longrightarrow H_q\langle K_{n-1}, K_m\rangle \xrightarrow{i_q} H_q\langle K_n, K_m\rangle \longrightarrow H_q\langle K_n, K_{n-1}\rangle \longrightarrow. \tag{8.6}$$

The last sequence is called the long exact sequence of the triple (K_n, K_{n-1}, K_m).
Suppose that $n > m \geq q$ or $q > n > m$. In both cases, it follows from Lemma 8.14 that $H_q\langle K_n, K_{n-1}\rangle = 0$, and so (8.6) implies that $H_q\langle K_n, K_m\rangle = 0$ since i_q is an epimorphism and $H_q\langle K_{n-1}, K_m\rangle = 0$ by induction. □

LEMMA 8.17. $H_q\langle K_n, K_m\rangle \cong H_q\langle K, K_m\rangle$ *provided* $n \geq m \geq q$.

PROOF. Consider the long exact sequence of the triple (K, K_n, K_m) as defined in the proof of Lemma 8.16

$$\longrightarrow H_q\langle K_n, K_m\rangle \longrightarrow H_q\langle K, K_m\rangle \longrightarrow H_q\langle K, K_n\rangle \longrightarrow.$$

It follows from Lemma 8.8 that there is an $r \geq n$ with $K = K_r$. We may thus consider

$$\longrightarrow H_q\langle K_n, K_m\rangle \longrightarrow H_q\langle K_r, K_m\rangle \longrightarrow H_q\langle K_r, K_n\rangle \longrightarrow.$$

We now have $H_q\langle K_r, K_n\rangle = 0$ by Lemma 8.16, showing that

$$H_q\langle K, K_m\rangle = H_q\langle K_r, K_m\rangle \cong H_q\langle K_n, K_m\rangle.$$

□

PROOF OF PROPOSITION 8.15. Let $q \in \mathbb{Z}$, $m < q-1$, and consider the long exact sequence of the triple (K_{q+1}, K_q, K_m)

$$\longrightarrow H_{q+1}\langle K_{q+1}, K_q\rangle \xrightarrow{\partial^1_{q+1}} H_q\langle K_q, K_m\rangle \longrightarrow H_q\langle K_{q+1}, K_m\rangle \longrightarrow 0,$$

where the 0 is justified by Lemma 8.16. The exactness implies that $H_q\langle K_{q+1}, K_m\rangle \cong H_q\langle K_q, K_m\rangle / \operatorname{im} \partial^1_{q+1}$.
Now, consider the long exact sequence of the triple (K_q, K_{q-1}, K_m)

$$\longrightarrow 0 \longrightarrow H_q\langle K_q, K_m\rangle \xrightarrow{p_q} H_q\langle K_q, K_{q-1}\rangle \xrightarrow{\partial^2_q} H_{q-1}\langle K_{q-1}, K_m\rangle \longrightarrow,$$

where the zero is again justified by Lemma 8.16. It follows that $H_q\langle K_q, K_m\rangle / \operatorname{im} \partial^1_{q+1} \cong \operatorname{im} p_q / \operatorname{im}(p_q \circ \partial^1_{q+1})$ and $\operatorname{im} p_q / \operatorname{im}(p_q \circ \partial^1_{q+1}) = \ker \partial^2_q / \operatorname{im}(p_q \circ \partial^1_{q+1})$ by the exactness of the row.
Finally, consider the long exact sequence of the triple (K_{q-1}, K_{q-2}, K_m)

$$\longrightarrow 0 \longrightarrow H_{q-1}\langle K_{q-1}, K_m\rangle \xrightarrow{\tilde{p}_{q-1}} H_{q-1}\langle K_{q-1}, K_{q-2}\rangle \longrightarrow,$$

where the zero is again justified by Lemma 8.16. Since, $\tilde{p}_{q-1}$ is a monomorphism, we obtain $\ker \partial^2_q / \operatorname{im}(p_q \circ \partial^1_{q+1}) = \ker(\tilde{p}_{q-1} \circ \partial^2_q) / \operatorname{im}(p_q \circ \operatorname{im} \partial^1_{q+1})$.
By the commutativity of diagram (6.2) in **[4]** (the braid), we have

$$HW_q N = \ker \delta_{q+1} / \operatorname{im} \delta_q = \ker(\tilde{p}_{q-1} \circ \partial^2_q) / \operatorname{im}(p_q \circ \partial^1_{q+1}) \cong H_q\langle K_{q+1}, K_m\rangle,$$

which shows in conjunction with Lemma 8.17 that

$$HW_q N \cong H_q\langle K_{q+1}, K_{-2}\rangle = H_q\langle K_{q+1}\rangle \cong H_q\langle K\rangle \text{ for all } q \geq 0.$$

Clearly, for $q < 0$ we have $HW_q N = 0$ and $H_q(K) = 0$. □

PROPOSITION 8.18. *For all $n \in \mathbb{N} \cup \{0\}$ and all $q \in \mathbb{Z}$,*

(1) *there is an isomorphism $\iota_{n,q} :\ H_q\langle E_n\rangle \cong \bigoplus_{e \in E_n} H_q\langle\{e\}\rangle$ such that*
(2) *$\iota_{n,q} \circ \delta_n \circ (\iota_{n,q})^{-1} = \left(\partial_{(e,f),q}\right)_{(e,f) \in E_n \times E_{n-1}}$, where $\partial_{(e,f),q}$ denotes the q-th connecting homomorphism of the attractor-repeller sequence for $(\operatorname{Inv}^-_K\{e\} \cap \operatorname{Inv}^+_K(\{f\}), \{f\}, \{e\})$ in singular homology.*

PROOF. Let $E_n = \{e_1, \ldots, e_m\}$ consist of m equilibria. Then for every $\delta > 0$, there is an isolating block $B \subset N$ for E_n with $B \subset \bigcup_{e \in E_n} B_\delta(e) < \delta$. Let $0 < \delta_0 < \min\{d(e,f) : e, f \in E_n\}$, and let $\{C_i :\ i \in I\}$ denote the family of connected components of B_{δ_0}. Each C_i is an isolating block, and $\tilde{B} := \bigcup_{E_n \cap C_i \neq \emptyset} C_i$ is an isolating block.
Let $C(e)$ denote the connected component of $\tilde{B}$, for which $e \in C(e)$, and let $i_{e,n} :\ C(e)/C(e)^- \to \tilde{B}/\tilde{B}^-$ be inclusion induced. Let $\tilde{p}_i :\ C(e) \to C(e)/C(e)^-$ and $\tilde{p} :\ B \to B/B^-$ be the canonical projections, let $j_i :\ C(e) \to B$ be inclusions, and let $\tilde{j}_i :\ C(e_i)/C(e_i)^- \to B/B^-$ be

inclusion induced. Then

$$
\begin{tikzcd}
\bigoplus_{i=1}^m H_q(C(e_i), C(e_i)^-) \arrow[r, "H_q(j_1)+\cdots+H_q(j_m)"] \arrow[d, "H_q(\tilde{p}_1)\oplus\cdots\oplus H_q(\tilde{p}_m)"] & H_q(\tilde{B}, \tilde{B}^-) \arrow[d, "\tilde{p}"] \\
\bigoplus_{i=1}^m H_q[C(e_i)/C(e_i)^-] \arrow[r, "H_q(\tilde{j}_1)+\cdots+H_q(\tilde{j}_m)"] & H_q[\tilde{B}/\tilde{B}^-]
\end{tikzcd}
$$

is commutative, and $H_q(\tilde{p}_1 \oplus \cdots \oplus \tilde{p}_m)$ and $H_q(\tilde{p})$ are isomorphisms due to the choice of isolating blocks. Using a Mayer-Vietoris sequence, one can conclude that $H_q(j_1) + \cdots + H_q(j_m)$ is an isomorphism and thus also $H_q(\tilde{j}_1) + \cdots + H_q(\tilde{j}_m)$. This proves (1).
It follows from Lemma 8.5 that

$$(N_1, N_2, N_3) := (\tilde{B}, C(e_i) \cup \tilde{B}^-, \tilde{B}^-)$$

is a strongly admissible FM-index triple for $(E_n, \{e_i\}, E_n \setminus \{e_i\})$. The following diagram commutes because its morphisms are inclusion induced:

$$
\begin{tikzcd}
& H_q[C(e_i)/C(e_i)^-] \arrow[d, "H_q(\alpha)"] \arrow[dr, "H_q(j_i)"] & & & \\
\arrow[r] & H_q[N_2/N_3] \arrow[r] & H_q[N_1/N_3] \arrow[r] & H_q[N_1/N_2] \arrow[r] & \cdot
\end{tikzcd}
$$

Since α is an inner morphism of $\mathcal{C}(\{e\})$, we have shown that $H_q\langle j_i\rangle : H_q\langle\{e_i\}\rangle \to H_q\langle E\rangle$ equals $i_{e_i,n}$ which is given by the following attractor-repeller sequence:

$$\longrightarrow H_q\langle\{e_i\}\rangle \xrightarrow{i_{e_i,n}} H_q\langle E_n\rangle \longrightarrow H_q\langle E_n \setminus \{e_i\}\rangle \longrightarrow .$$

Let $(e, f) \in E_n \times E_{n-1}$. By the commutativity of the homology index braid [**4**],

$$
\begin{tikzcd}
H_n\langle\{e\}\rangle \arrow[d, "i_{e,n}"] \arrow[dr, "\tilde{\partial}"] & \\
H_n\langle E_n\rangle \arrow[r, "\delta_n"] & H_{n-1}\langle E_{n-1}\rangle,
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
H_n\langle\{e\}\rangle \arrow[r, "\partial_{(e,f),n}"] \arrow[dr, "\tilde{\partial}"] & H_{n-1}\langle\{f\}\rangle \arrow[d, "i_{f,n-1}"] \\
& H_{n-1}\langle E_{n-1}\rangle,
\end{tikzcd}
$$

are commutative. Thus, composing the previous two diagrams, we obtain another commutative diagram

$$
\begin{tikzcd}
H_n\langle\{e\}\rangle \arrow[r, "\partial_{(e,f),n}"] \arrow[d, "i_{e,n}"] & H_{n-1}\langle\{f\}\rangle \arrow[d, "i_{f,n-1}"] \\
H_n\langle E_n\rangle \arrow[r, "\delta_n"] & H_{n-1}\langle E_{n-1}\rangle,
\end{tikzcd}
$$

which shows (2). $\square$

APPENDIX A

Trivial vector bundles

Although one could certainly use the notion of a vector bundle as defined in [**14**], this would create a large overhead due to formalism since the structure of the vector bundles used here is relatively simple. Therefore, definitions restricted to the use case will be given.
Let $[a,b] \subset \mathbb{R}$ be fixed and let E, F denote arbitrary Banach spaces. We will write $E = E_1 \oplus E_2$ iff E_1 and E_2 are closed linear subspaces of E with $E = E_1 + E_2$ and $E_1 \cap E_2 = \{0\}$. Given a linear subspace $E_1 \subset E$, another linear subspace E_2 is called a topological complement iff $E = E_1 \oplus E_2$. In particular, such a complement exists if either $\dim E_1 < \infty$ or $\operatorname{codim} E_1 < \infty$.

Definition A.1. A (trivial) bundle is the cartesian product $[a,b] \times E$ equipped with the product metric.

Taking (trivial) bundles as objects of a category $\mathcal{B} = \mathcal{B}([a,b])$, one needs to define morphisms:

Definition A.2. A morphism in $\mathcal{B}$ is a continuous mapping $G : [a,b] \to \mathcal{L}(E,F)$. G is called a *splitting* if for every $x \in [a,b]$, $G(x)E$ has a topological complement in F.

Given bundles $[a,b] \times E$ and $[a,b] \times \tilde{E}$ and a morphism F between them, F can be applied to $[a,b] \times E$ in the following way: $\hat{F}(x,\eta) := (x, F(x)\eta)$.
If F_1, F_2 are morphisms, then $(F_1 \circ F_2)(x) := F_1(x) \circ F_2(x)$ is again a morphism. In particular, a morphism F is an isomorphism iff for every $x \in [a,b]$ $F(x) \in \mathcal{L}(E,F)$ is an isomorphism and iff the induced mapping $\hat{F}$ is a homeomorphism.

Lemma A.3. *Let $G \in C([a,b], \mathcal{L}(E,F))$ and suppose that $G(x_0)$ is an isomorphism in $\mathcal{L}(E,F)$. Then there is a neighborhood U of x_0 in $[a,b]$ such that $G(x)$ is an isomorphism for all $x \in U$. Moreover, $G(x)^{-1}$ is continuous in x for all $x \in U$.*

Proof. If we set $H_x := (G(x_0) - G(x))G(x_0)^{-1}$, then it follows using the Neumann series that $G(x_0)G(x)^{-1} = \sum_{n=0}^{\infty} H_x^n$ whenever $\|H_x\| < 1$. The set $U := \{x \in [a,b] : \|H_x\| < 1\}$ is open and $\sum_{n=0}^{\infty} H_x^n$ depends continuously on x, showing that $G(x)^{-1}$ is continuous on U. □

Corollary A.4. *$G \in C([a,b], \mathcal{L}(E,F))$ is an isomorphism if and only if for every $x \in [a,b]$ $G(x)$ is an isomorphism in $\mathcal{L}(E,F)$.*

Definition A.5. A subset $U \subset [a,b] \times F$ is called a subbundle if there exists another bundle $[a,b] \times E$ and a splitting monomorphism $G : [a,b] \times E \to [a,b] \times F$ such that $U = \hat{G}([a,b] \times E)$.

Lemma A.6. *$\hat{G} : [a,b] \times E \to U$ is a homeomorphism, and the norms on the fibers are equivalent, that is, there are constants $m, M \in \mathbb{R}^+$ such that $0 \neq m$ and $m\|\eta\|_E \leq \|G(x)\eta\|_F \leq M\|\eta\|_E$ for all $(x,\eta) \in [a,b] \times E$.*

PROOF. It follows from the open mapping theorem that $G(x)$ is an isomorphism for all $x \in [a,b]$. Let $x_0 \in [a,b]$ and let $P: F \to G(x_0)E$ denote a projection. By Lemma A.3, there exists a neighborhood V of x_0 such that for all $x \in V$, $(PG(x))^{-1} \in \mathcal{L}(PG(x)E, F)$ and $G(x)^{-1} = (PG(x))^{-1}P$, that is, $G(x)^{-1}$ is the restriction of the "pseudo" inverse $(PG(x))^{-1}P \in \mathcal{L}(F,E)$ to $G(x)E$. We have $(PG(x))^{-1}P \to G(x_0)^{-1}$ in $\mathcal{L}(F,E)$ as $x \to x_0$. This shows that $G(x_n)^{-1}y_n \to G(x_0)y_0$ whenever $(x_n, y_n) \to (x_0, y_0)$ in U. Hence, $\hat{G}(x)$ is a homeomorphism.
We have $M := \sup_{x\in[a,b]} \|G(x)\| < \infty$ and $m^{-1} := \sup_{x\in[a,b]} \left\|(PG(x))^{-1}P\right\| < \infty$ by continuity, which shows the equivalence of norms. $\square$

Given a splitting monomorphism $U : [a,b] \times E \to [a,b] \times F$, one can speak of a subbundle, identifying U with its image $\hat{U}([a,b] \times E)$. Then the fibers are given by $U(x) := U(x)E$ for $x \in [a,b]$. If $V \subset [a,b]$, then we write $U(V) := \bigcup_{x\in V}\{x\} \times U(x)$.

LEMMA A.7. *Let $U: [a,b] \times E \to [a,b] \times F$ be a subbundle, let $x_0 \in [a,b]$ and let $P: F \to U(x_0)$ be a continuous projection onto $U(x_0)$. Then there exists a neighborhood V of x_0 in $[a,b]$ such that $p: U(V) \to V \times U(x_0)$, $p(x,y) = (x, Py)$, is a homeomorphism and the norms on the fibers are equivalent, that is, there are constants $m, M \in \mathbb{R}^+$ such that $0 \neq m$ and $m\|\eta\| \leq \|P\eta\| \leq M\|\eta\|$ for all $(x,\eta) \in U(V)$.*

PROOF. Define $H_x \in \mathcal{L}(E, U(x_0))$ by $H_x := PU(x)$. It follows as in the proof of Lemma A.6 that there is a closed neighborhood V of x_0 such that H_x is an isomorphism for all $x \in V$. Lemma A.3 implies that H_x^{-1} depends continuously on x. Therefore and by Lemma A.6, $P(x) := H_x U(x)^{-1}$ and $P(x)^{-1} = U(x)H_x^{-1}y$ are isomorphisms and the norms of $P(x)$ and $U(x)H_x^{-1}$ are bounded on V. $\square$

As before, let $U: [a,b] \times E \to B$ be a subbundle, where $B := [a,b] \times F$. Define the quotient bundle B/U to be the disjoint union of the quotients on the fibers, that is,

$$B/U := \bigcup_{x\in[a,b]} \{x\} \times (F/U(x)).$$

It is natural to endow B/U with the quotient topology and to assign to each fiber the norm

$$\|[y]\|_{F/U(x)} := \inf\{\|y - z\|_F : z \in U(x)\} \quad y \in F.$$

LEMMA A.8. *Let $E = E_1 \oplus E_2$ and let $U: [a,b] \times E_1 \to B = [a,b] \times E$ be a subbundle. Then there exists another subbundle $S: [a,b] \times E_2 \to [a,b] \times E$ such that $U \oplus S: [a,b] \times (E_1 \oplus E_2) \to E$, which is defined by $(U \oplus S)(x)(y_1 \oplus y_2) = U(x)y_1 + S(x)y_2$, is an isomorphism.*
Furthermore, if $E = U(\xi) \oplus E_2$ for some $\xi \in [a,b]$, then we can assume that $S(x) = \mathrm{id}_{E_2}$ for all x in a sufficiently small neighborhood of ξ in $[a,b]$.

A consequence of the previous lemma is that B/U is again a metric (metrizable) space, which allows for example to consider the Conley index on B/U.

LEMMA A.9. *Let the assumptions of Lemma A.8 hold, and let $x_0 \in [a,b]$. For every $x_0 \in [a,b]$ there is a neigborhood V of x_0 and an extension $\bar{U}_{x_0} \in C(V, \mathcal{L}(E_1 \oplus E_2, E)$ such that for every $x \in V$, $\bar{U}_{x_0}(x)$ is an isomorphism and $\bar{U}_{x_0}(x)_{|E_1} = U(x)$ for all $x \in V$.*

PROOF. Let $\tilde{E}$ be a topological complement for $U(x_0)$, that is, $E = U(x_0) \oplus \tilde{E}$. There is a $\Phi \in \mathrm{ISO}(\tilde{E}, E_2)$ since $E_2 \cong E/E_1 \cong E/U(x_0) \cong \tilde{E}$. Now, let $y_1 \oplus y_2 \in E_1 \oplus E_2$ and define $\bar{U}(x)(y_1 \oplus y_2) := U(x)y_1 + \Phi(y_2)$. $\bar{U}(x_0)$ is an isomorphism, so by Lemma A.3 there is a neighborhood V of x_0 such that $\bar{U}(x)$ is an isomorphism for all $x \in V$. □

LEMMA A.10. *Under the assumptions of Lemma A.8, there is an isomorphism* $\bar{U} \in C([a,b], \mathcal{L}(E,E)$ *such that* $\bar{U}(x)_{|E_1} = U(x)$ *for all* $x \in [a,b]$.
Furthermore, if $E = U(\xi) \oplus E_2$ *for some* $\xi \in [a,b]$, *then we can assume that* $\bar{U}(x)y = y$ *for all* $y \in E_2$ *and all* x *in a sufficiently small neighborhood of* ξ *in* $[a,b]$.

PROOF. By Lemma A.9 and the compactness of $[a,b]$, there is a decomposition $a = a_0 < a_1 < \cdots < a_n = b$ of $[a,b]$ with $\xi \notin \{a_1, \ldots, a_n\}$ and for every $k \in \{1, \ldots, n\}$ a local extension $\bar{U}_{a_k} \in C([a_{k-1}, a_k], \mathcal{L}(E,E))$.
We prove the claim by induction on n, so we may assume that there is an extension $\bar{U}_{-1} \in C([a, a_{n-1}], \mathcal{L}(E,E))$. Define

$$\bar{U}(x) := \begin{cases} \bar{U}_{-1}(x) & x \le a_{n-1} \\ \bar{U}_{a_n}(x) \circ \bar{U}_{a_n}(a_{n-1})^{-1} \circ \bar{U}_{-1}(a_{n-1}) & x \ge a_{n-1}. \end{cases}$$

We have $\bar{U}_{a_n}(a_{n-1}) \circ \bar{U}_{a_n}(a_{n-1})^{-1} \circ \bar{U}_{-1}(a_{n-1}) = \bar{U}_{-1}(a_{n-1})$, so $\bar{U}$ is well-defined and continuous. Moreover, for all $y \in E_1$ and $x \ge a_{n-1}$, we have

$$\begin{aligned} \bar{U}(x)y &= (\bar{U}_{a_n}(x) \circ \bar{U}_{a_n}(a_{n-1})^{-1} \circ U(a_{n-1}))y \\ &= (\bar{U}(x) \circ U(a_{n-1})^{-1} \circ U(a_{n-1}))y \\ &= \bar{U}(x)y = U(x)y, \end{aligned}$$

so $\bar{U}$ is indeed an extension of U. Corollary A.4 implies that $\bar{U}$ is an isomorphism.
By Lemma A.3, there is a neighborhood $[\tilde{a}, \tilde{b}]$ of ξ such that $U(x) \oplus \mathrm{id}_{E_2}$ is an isomorphism for all $x \in [\tilde{a}, \tilde{b}]$. Choose a continuous mapping $\rho : [a,b] \to [\tilde{a}, \tilde{b}]$ with $\rho(x) = x$ for all $x \in [\tilde{a}, \tilde{b}]$. Therefore,

$$\tilde{U}(x)(y_1 \oplus y_2) := \bar{U}(x) \circ \bar{U}(\rho(x))^{-1}(U(\rho(x))y_1 \oplus y_2)$$

satisfies $\tilde{U}(x)y_2 = y_2$ for all $y_2 \in E_2$ and all $x \in [\tilde{a}, \tilde{b}]$. □

PROOF OF LEMMA A.8. Let the morphism $S: [a,b] \times E_2 \to [a,b] \times E$ be defined by restriction of $\bar{U}$, that is, $S(x)y := \bar{U}(x)y$, $y \in E_2$. □

COROLLARY A.11. *Let the assumptions of Lemma A.8 hold, let* U *and* S *be given by that Lemma, and let the canonical projection* $p: S \to B/U$ *be defined by* $p(x,y) := (x, [y])$.
Then $p \circ S : [a,b] \times E_2 \to B/U$ *is a homeomorphism, and the norms on the fibers are equivalent, that is, there are constants* $m, M \in \mathbb{R}^+$ *such that* $0 \neq m$ *and* $m\|\eta\|_E \le \|[S(x)\eta]\|_{E/U(x)} \le M\|\eta\|_E$ *for all* $(x,\eta) \in [a,b] \times E_2$.

PROOF. For every $x \in [a,b]$, one has $E/U(x) \cong S(x)$, so $p \circ S$ is a bijection, which is continuous as composition of continuous mappings.

Let $q(x, y_1 \oplus y_2) := (x, y_2)$ denote the canonical projection. The following diagram

$$\begin{tikzcd}
{[a,b] \times E} \arrow[r, "(U \oplus S)^{-1}"] \arrow[d, "p"] & {[a,b] \times (E_1 \oplus E_2)} \arrow[d, "q"] \\
{([a,b] \times E)/U} \arrow[r, "(p \circ S)^{-1}"] & {[a,b] \times E_2}
\end{tikzcd}$$

is commutative and $q \circ (U \oplus S)^{-1}$ is continuous. Thus, $p \circ S^{-1}$ is continuous since the quotient topology is final with respect to the projection.

We have $\|[S(x)y]\| = d(S(x)y, U(x)) \leq \|S(x)\|\|y\|$ for all $(x, y) \in [a, b] \times E_2$, so one can choose $M := \sup_{x \in [a,b]} \|S(x)\| < \infty$.

Suppose there are sequences $x_n \to x_0$ in $[a, b]$ and $y_n \in E_2$ with $\|[S(x_n)y_n]\|_{E/U(x)} \to 0$ as $n \to \infty$ and $\|y_n\| = 1$ for all $n \in \mathbb{N}$. Then, there is a sequence $(w_n)_n$ in E_1 with $\|S(x_n)y_n - U(x_n)w_n\| \to 0$ as $n \to \infty$. By Lemma A.6, we have $\sup_{n \in \mathbb{N}} \|w_n\| < \infty$, implying that $\|S(x_0)y_n - U(x_0)w_n\| \to 0$, that is, $\|[S(x_0)y_n]\|_{E/U(x_0)} \to 0$ as $n \to \infty$.

It follows that $(p \circ S)(x_0, y_n) \to (p \circ S)(x_0, 0)$ and so $y_n \to 0$ in E_2 because $p \circ S$ is a homeomorphism. This contradicts the assumption that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

Hence, there exists a constant $m \in \mathbb{R}^+$ with $0 \neq m$ and $m\|y\| \leq \|[S(x)y]\|_{E/U(x)}$. □

LEMMA A.12. *Let* $U : [a, b] \times E_1 \oplus E_2 \to [a, b] \times E$ *be a subbundle, let* $x_0 \in [a, b]$ *and let*

$$E = U(x_0) \oplus E_s.$$

Then there exists a neighborhood V *of* x_0 *in* $[a, b]$ *such that* $p : V \times E_s \to p(V \times E_s) \subset B/U$, $p(x, y) = (x, [y])$ *is a homeomorphism, and the norms on the fibers are equivalent, that is, there are constants* $m, M \in \mathbb{R}^+$ *such that* $0 \neq m$ *and* $m\|\eta\| \leq \|[\eta]\|_{E/U(x)} \leq M\|\eta\|$ *for all* $(x, \eta) \in V \times E_s$.

PROOF. Let $S(x)$ be given by Lemma A.8. In view of Corollary A.11, it is sufficient to consider the morphism $G = G(x)$, which is defined by the composition

$$[a, b] \times E_s \xrightarrow{p} p([a, b] \times E) \xrightarrow{(p \circ S)^{-1}} [a, b] \times E_2.$$

It is clear that $G(x_0)$ is an isomorphism in $\mathcal{L}(E_s, E_2)$. It is thus a consequence of Lemma A.3 that $G(x)$ is an isomorphism in a closed neighborhood V of x_0, showing that there are constants $m, M \in \mathbb{R}^+$ such that $0 \neq m$ and $m\|\eta\| \leq \|G(x)\eta\| \leq M\|\eta\|$ for all $(x, \eta) \in V \times E_s$. □

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