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ON THE GEOMETRY OF FINITE INDEX SUBGROUPS OF GROUPS ACTING PROPERLY ON LOCALLY FINITE TREES AND POLYHEDRAL COMPLEXES

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Introduction

The modular group $\mathrm{Sl}_2(\mathbb{Z})$ can be written as a free product $C_2 * C_3$. Making indirect use of this fact, Millington has classified in [5] the finite index subgroups up to isomorphy. Moreover the author has introduced the notion of the "type" of a subgroup. Roughly speaking, two subgroups have the same type if and only if they are isomorphic as groups and the fundamental domains of their subgroups with respect to the action on the upper halfplane via Moebius transforms are essentially equal. In particular, this concept does not only take into account the isomorphy types, but also the gluing of the fundamental domain which is encoded by the genus and the gluing of the cusps.

It is entirely possible that two isomorphic groups are not of the same type. Jan-Christoph Schlage-Puchta and Matthias Krieger have researched an algorithm which enables the classification of all subgroups of given index up to isomorphy based on the knowledge about the subgroup lattice of each factor of the free product, see [6], [4]. The authors have made tremendous use of the fact that the isomorphy type of a free product does only depend on the frequency of occurence of each factor and not on the order of the factors.

In the first chapter of this work we consider the actions of virtually free groups on trees. This is in particular generalizes the action of $\mathrm{Sl}_2(\mathbb{Z})$ on the upper plane. In this setting, in analogy to the action of $\mathrm{Sl}_2(\mathbb{Z})$ on the upper plane, we are also able to define cusps and to analyze the behaviour of those cusps under transition to subgroups of finite index appropriately. For this purpose we introduce the concepts "cusp order" and "cusp multiplicity" of a finite index subgroup. In the example of the action of $\mathrm{Sl}_2(\mathbb{Z})$ on the upper plane, the fundamental domain of a finite index subgroup Δ can be decomposed into $|\mathrm{Sl}_2(\mathbb{Z})/\Delta|$-many copies of fundamental domains belonging to the $\mathrm{Sl}_2(\mathbb{Z})$-action on the upper plane. The cusp order of a cusp measures the number of copies of the fundamental domains belonging to the $\mathrm{Sl}_2(\mathbb{Z})$-action and being adjacent to that cusp.
On the other hand, the cusp multiplicity measures the total number of cusps, into which a cusp decomposes under transition to the considered subgroup.

We present a geometric and a group theoretic definition for these quantities and show that they are equivalent under certain conditions.

The second chapter of the thesis is concerned with the behaviour of elliptic singularities of groups acting with finite stabilizers on simply connected polyhedral complexes under transition to finite index subgroups. This consideration is strictly more general than the analysis of group actions of virtually free groups on trees. For example $\mathrm{Sl}_3(\mathbb{Z})$ is not virtually free, see [7] p.67 theorem 16 but it acts with finite stabilizers on the quadratic forms over $\mathbb{R}^3$, see [8]. Soulé reduces $\mathrm{Sl}_3(\mathbb{Z})$-set of the quadratic forms to a polyhedral complex, such that $\mathrm{Sl}_3(\mathbb{Z})$ acts cocompactly and with finite stabilizers on it, by retracting the fundamental domain appropriately. We consider polyhedral complexes as certain directed graphs. In turn we will work out some concepts and results for group actions on such graphs. We will apply these methods to the group action of $\mathrm{Sl}_3(\mathbb{Z})$ on the polyhedral complex constructed from the quadratic forms as mentioned above and obtain explicit results for certain finite index subgroups.

CHAPTER 1

Cusps of finite graphs of finite groups

In the present chapter we introduce the notion of "cusps" of a graph of groups and prove some fundamental properties of them and their stabilizers which we will call "parabolic" subgroups.

In the following, let G be a group acting inversion free on a tree $\mathfrak{X}$ in the sense of [7] (the definition can be also found below).

The notion of a cusp for a group action on a manifold is well known. In order to give an idea what a cusp on a graph might be, we look at the following example.

Let $G := \mathrm{Sl}_2(\mathbb{Z})$ be the action on the Hyperbolic plane by Moebius transforms. In this setting, cusps are exactly the points on the boundary of the fundamental domain which compactify it. The standard fundamental domain of the modular group has exactly one cusp which is the point at infinity. However, if we look at translates of the fundamental domain, we see that the translates of that cusp exhaust the whole boundary $\partial\mathbb{H}$ of $\mathbb{H}$. In other words, the cusps lie in the same orbit under the extension of the action $G \curvearrowright \mathbb{H}$ to $G \curvearrowright \mathbb{H} \cup \partial\mathbb{H}$.

We may think of the action of $G = \mathrm{Sl}_2(\mathbb{Z})$ on $\mathbb{H}$ as an action on a geodesic tree $\mathfrak{X}$. $\mathfrak{X}$ can be made up of the translates of the only boundary component of the fundamental domain of $G \curvearrowright \mathbb{H}$, which is not heading to a cusp. The set of ends of $\mathfrak{X}$ will play the role of the boundary and therefore we will define a cusp to be the orbit of an end and a parabolic subgroup to be a stabilizer of one.

Furthermore we will give some conditions to determine when a parabolic group is cyclic and we will prove under certain assumptions that a cyclic parabolic group is self normalizing. This leads to some consequences for the behaviour of cusps of finite index subgroups of G from this fact.

1.1 Terminology, notation and basics

In the whole chapter $\mathfrak{X}$ will denote a tree and G a group acting inversion free on $\mathfrak{X}$ if nothing else is mentioned. There is a natural combinatorial metric on $\mathfrak{X}$ which we denote by D. We recall that an action of G on a tree $\mathfrak{X}$ is called *inversion free* if an element $\gamma \in G$ fixes an edge if and only if it fixes its vertices. If $\mathfrak{Y}$ is a subgraph of $\mathfrak{X}$ we write $\mathfrak{Y} \subseteq \mathfrak{X}$ and by abuse of notation we write $y \in \mathfrak{Y}$ instead of $y \in V(\mathfrak{Y})$. A *ray* is an infinite path without backtracking which has exactly one vertex with only one neighbour. The unique vertex of degree one is called the *origin* of the ray. If we think of rays as rooted trees with root in the origin we obtain a natural ordering. Each ray endowed with this ordering is isomorphic to the positive integers as ordered space and the isomorphism is uniquely determined. Take respect to this isomorphism if we talk of a $r \in \mathfrak{r}$ being large enough. The existence of the isomorphism implies that with this ordering the ray is a totally ordered space.

We define an equivalence relation on the set of rays of $\mathfrak{X}$ by saying two rays are equivalent if and only if they intersect in a ray themselves. This clearly defines an equivalence relation. The set of equivalence classes of rays is called the set of *ends* of $\mathfrak{X}$ which we denote by $\Omega\mathfrak{X}$.
$\mathrm{Aut}(\mathfrak{X})$ induces an action on $\Omega\mathfrak{X}$ via $\alpha^*([\mathfrak{r}]) := [\alpha(\mathfrak{r})]$. Therefore an action $G \curvearrowright \mathfrak{X}$ also induces an action $G \curvearrowright \Omega\mathfrak{X}$. We refer on this action on $\Omega\mathfrak{X}$ if nothing else is mentioned.
We denote by $\mathrm{Aut}^\circ(\mathfrak{X})$ the set of automorphisms which act inversion free on $\mathfrak{X}$. Let $v \in \mathfrak{X}$ a vertex. Then, there exists a bijection between the rays with origin in v and $\Omega\mathfrak{X}$ which we obtain by mapping the rays with origin in v on their equivalence classes. For the reason $\mathfrak{X}$ is a tree and therefore contains no circle, this map is necessarily injective. Surjectivity of the map is trivial. We have to remark that this bijection cannot be interpreted as isomorphism of $\mathrm{Aut}(\mathfrak{X})$-modules in a canonical way because any group action mapping a ray with origin in v to a ray with origin in v has to fix v. This is the reason why we work with me more abstract space $\Omega\mathfrak{X}$.

Definition 1.1.1 (cusps)**.** The set of left G-orbits on the ends $G\backslash\Omega\mathfrak{X}$ is called the set of *cusps* of G on $\mathfrak{X}$.

Definition 1.1.2 (parabolic groups)**.** Let $P \leq G$ such that there exists a $c \in \Omega\mathfrak{X}$ with the property $P = G_c$. Then P is called *parabolic.*

We observe by definition that the parabolic groups associated to one cusp are conjugate by elements of G.

1.2 The structure of the automorphisms preserving ends

In the sequel fix a $c \in \Omega\mathfrak{X}$. We set $\mathrm{Aut}(c) := \{\alpha \in \mathrm{Aut}(\mathfrak{X}) : \ \alpha c = c\}$. Our aim is to construct a group homomorphism between $\mathrm{Aut}(c)$ and $\mathbb{Z}$.
Let $\mathfrak{r}, \mathfrak{s} \in c$, $\mathfrak{r} = (r_i)_{i\in\mathbb{N}_0}$, $\mathfrak{s} = (s_i)_{i\in\mathbb{N}_0}$. Then there exists a smallest integer $l_{\mathfrak{r},\mathfrak{s}}$ with the property $r_l \in \mathfrak{s}$ for all $l \geq l_{\mathfrak{r},\mathfrak{s}}$ as well as a smallest integer $l_{\mathfrak{s},\mathfrak{r}}$ such that $s_l \in \mathfrak{r}$ for all $l \geq l_{\mathfrak{s},\mathfrak{r}}$. We observe

$$\mathfrak{r} \cap \mathfrak{s} = \left(r_{l_{\mathfrak{r},\mathfrak{s}}+i}\right)_{i\in\mathbb{N}_0} \quad \text{and} \quad \mathfrak{r} \cap \mathfrak{s} = \left(s_{l_{\mathfrak{s},\mathfrak{r}}+i}\right)_{i\in\mathbb{N}_0}. \tag{1.2.1}$$

We put

$$\tau(\mathfrak{r}, \mathfrak{s}) := l_{\mathfrak{r},\mathfrak{s}} - l_{\mathfrak{s},\mathfrak{r}}$$

and call this number the *displacement of the transition from* $\mathfrak{r}$ *to* $\mathfrak{s}$. For an arbitrary ray $\mathfrak{r} \in c$ set

$$d_{\mathfrak{r}}(\alpha) := \tau(\mathfrak{r}, \alpha\mathfrak{r})$$

and call it the *displacement of* α.

Definition 1.2.1. Let $\mathfrak{r} \in c \in \Omega\mathfrak{X}$ and $x, y \in \mathfrak{r} = (r_i)_{i\in\mathbb{N}_0}$. Then $x = r_i$ for an $i \in \mathbb{N}_0$ and $y = r_j$ for an $j \in \mathbb{N}_0$. We put $d_{\mathfrak{r}}(x, y) := j - i$. This function may have values in the whole set of integers. We call the value $d_{\mathfrak{r}}(x, y)$ the *directed distance from* x *to* y. The directed distance does not depend on the choice of $\mathfrak{r}$ with $x, y \in \mathfrak{r}$. We may therefore write $d_c(x, y) := d_{\mathfrak{r}}(x, y)$.

Before we prove the independence from the choice of the ray, we recapitulate the following facts: We recall that in a tree $\mathfrak{X}$ for each two distinct vertices there exists a unique geodesic with these vertices being terminal. Moreover, in any tree the paths without backtracking are exactly the geodesics. Hence, for each pair of distinct vertices there exists a unique path without backtracking with these terminal vertices.

Proof. Let $\mathfrak{r} \in c \in \Omega\mathfrak{X}$ and $x, y \in \mathfrak{r} = (r_i)_{i\in\mathbb{N}_0}$. Then $x = r_i$ for an $i \in \mathbb{N}_0$ and $y = r_j$ for an $j \in \mathbb{N}_0$. We first show that the directed distance does not depend on the choice of $\mathfrak{r}$. Without loss of generality let $i \leq j$, otherwise transit to

$d_{\mathfrak{r}}(y,x) = -d_{\mathfrak{r}}(x,y)$. Let $\mathfrak{s} = (s_k)_{k\in\mathbb{N}_0}$ be another ray with $x, y \in \mathfrak{s}$. Then $x = s_k$ and $y = s_l$, for suitable $k, l \in \mathbb{N}_0$. We observe $k \leq l$. On the contary, suppose $k > l$. Then $\mathfrak{r} \cap \mathfrak{s}$ would be a path without backtracking between the origins $0_{\mathfrak{r}}$ of $\mathfrak{r}$ and $0_{\mathfrak{s}}$ of $\mathfrak{s}$ and therefore a geodesic of finite length. Thus, $\mathfrak{r} \cap \mathfrak{s}$ cannot be a ray and we conclude $\mathfrak{s} \notin c$, a contradiction. This forces $d_{\mathfrak{r}}(x,y) \geq 0$ and $d_{\mathfrak{s}}(x,y) \geq 0$. They therefore coincide with the (ordinary) distances measured in $\mathfrak{r}$ and $\mathfrak{s}$ respectively.
Let π be a path without backtracking with terminal vertices x and y. Considering the definition of rays, the uniqueness of such paths forces $\pi \subseteq \mathfrak{r}, \mathfrak{s}$. Denoting the length of π by $l(\pi)$, we obtain

$$d_{\mathfrak{r}}(x,y) = l(\pi) = d_{\mathfrak{s}}(x,y)$$

and the independence of the directed distance is verfied. □

Remark 1. A short analysis of the definition, yields that the directed distance is translation invariant, in equal

$$d_{\mathfrak{r}}(v_{i_0+k}, v_{j_0+k}) = d_{\mathfrak{r}}(v_{i_0}, v_{j_0})$$

for all $\mathfrak{r} = (v_i)_{i\in\mathbb{N}_0},\ i_0, j_0 \in \mathbb{N}_0,\ k \geq \max\{-i_0, -j_0\}$.

Lemma 1.2.2. *The displacement of α does not depend on the choice of the ray $\mathfrak{r}$. We may therefore define $d_c(\alpha) := d_{\mathfrak{r}}(\alpha)$.*

Proof. Let $0_{\mathfrak{r}}$ be the origin of $\mathfrak{r}$. For $x \in \mathfrak{X}$ we set $|x|_{\mathfrak{r}} := d_c(0_{\mathfrak{r}}, x)$. Let $\mathfrak{r}, \mathfrak{s} \in c$. We observe for $\mathfrak{r} = (r_i)_{i\in\mathbb{N}_0}$ and $\mathfrak{s} = (s_i)_{i\in\mathbb{N}_0}$

$$l_{\mathfrak{r},\mathfrak{s}} = |r_{l_{\mathfrak{r},\mathfrak{s}}}|_{\mathfrak{r}} = |s_{l_{\mathfrak{s},\mathfrak{r}}}|_{\mathfrak{r}},$$
$$l_{\mathfrak{s},\mathfrak{r}} = |s_{l_{\mathfrak{s},\mathfrak{r}}}|_{\mathfrak{s}} = |r_{l_{\mathfrak{r},\mathfrak{s}}}|_{\mathfrak{s}}.$$

With this notation and $\mathfrak{s} := \alpha\mathfrak{r}$ and hence $s_i = \alpha r_i$, we get for $r \in \alpha\mathfrak{r} \cap \mathfrak{r}$ and therefore for $\alpha^{-1}\mathfrak{r} \in \mathfrak{r} \cap \alpha^{-1}\mathfrak{r}$:

$$d_c(\alpha^{-1}r, r) = d_c(\alpha^{-1}r_{l_{\mathfrak{r},\alpha\mathfrak{r}}}, r_{l_{\mathfrak{r},\alpha\mathfrak{r}}}) \overset{(1.2.1)}{=} d_c(\alpha^{-1}\alpha r_{l_{\alpha\mathfrak{r},\mathfrak{r}}}, r_{l_{\mathfrak{r},\alpha\mathfrak{r}}}) = l_{\mathfrak{r},\alpha\mathfrak{r}} - l_{\alpha\mathfrak{r},\mathfrak{r}} = d_{\mathfrak{r}}(\alpha),$$

where the first equality is deduced from the translation invariance of the directed distance.
We conclude that the displacement $d_{\mathfrak{r}}(\alpha)$ does indeed not depend on $\mathfrak{r}$.

□

Lemma 1.2.3 (additivity of the directed distance)**.** *Let $x, y, z \in \mathfrak{r}$. Then*

$$d_{\mathfrak{r}}(x,z) = d_{\mathfrak{r}}(x,y) + d_{\mathfrak{r}}(y,z).$$

Proof. Let $\mathfrak{r}$ be a ray and $0_{\mathfrak{r}}$ its origin. In section 1.1 we already have stated that $(\mathfrak{r}, 0_{\mathfrak{r}})$ is a totally ordered space isomorphic to $(\mathbb{N}_0, \leq)$. For a finite chain $x_0 \leq \cdots \leq x_n$ we call the vertices x_0 and x_n its *terminal* vertices. With these thoughts in mind, we observe that one of the following two cases occurs:

(a) y is not terminal: We may assume without loss of generality $x \leq y \leq z$, swapping x, z if necessary and applying the identity $d_{\mathfrak{r}}(v,w) = -d_{\mathfrak{r}}(w,v)$ for $v, w \in \mathfrak{r}$. Now, $\mathfrak{r}$ induces a geodesic with terminal vertices x and z which contains by assumption the vertex y. Thus, the geodesic connecting x and z is the concatenation of geodesics connecting x and y and y and z. We conclude $d_{\mathfrak{r}}(x,z) = d_{\mathfrak{r}}(x,y) + d_{\mathfrak{r}}(y,z)$.

(b) y is terminal: We reduce this case to the first one. For example let $x \leq z \leq y$. Then, we conclude from the first case $d_{\mathfrak{r}}(x,y) = d_{\mathfrak{r}}(x,z) + d_{\mathfrak{r}}(z,y)$. Applying the identity $d_{\mathfrak{r}}(z,y) = -d_{\mathfrak{r}}(y,z)$ and adding $d_{\mathfrak{r}}(y,z)$ on both sides, yields the claim.

□

The following theorem is inspired by [7] p.63 proposition 25:

Theorem 1.2.4. *Let $c \in \Omega\mathfrak{X}$. Then $d_c\colon \operatorname{Aut}(c) \to \mathbb{Z}\colon \alpha \mapsto d_c(\alpha)$ is a well-defined group homomorphism with $\ker d_c = \bigcup_{\mathfrak{r}\in c} \bigcap_{v\in R} \operatorname{Aut}(c)_v$.*

Proof. Lemma 1.2.2 says that d_c is well-defined. We want to establish the homomorphy. For this purpose, let $\alpha, \beta \in \operatorname{Aut}(c)$ and $\mathfrak{r} \in c$ arbitrary. Take $r \in \mathfrak{r}$ sufficiently large such that $(\alpha \circ \beta)^{-1} r, \alpha^{-1} r, r \in \mathfrak{r}$. We then get by the additivity of the directed distance

$$\begin{aligned} d_c(\alpha \circ \beta) &= d_{\mathfrak{r}}\big((\alpha \circ \beta)^{-1} r, r\big) \\ &= d_{\mathfrak{r}}(\beta^{-1}\alpha^{-1} r, \alpha^{-1} r) + d_{\mathfrak{r}}(\alpha^{-1} r, r) \\ &= d_c(\beta) + d_c(\alpha). \end{aligned}$$

It remains to show the equation describing the kernel. Let $\alpha \in \ker d_c$ and $\mathfrak{r} \in c$. For $v \in \mathfrak{r}$ large enough we get $d_{\mathfrak{r}}(\alpha^{-1} v, v) = d_c(\alpha) = 0$. Let $(v)_{\mathfrak{r}}$ be the uniquely determined subray with origin in v. We observe $d(\alpha^{-1} r, r) = 0$ for each $r \in (v)_{\mathfrak{r}}$. We thus infer $\alpha^{-1} r = r$ for all $r \in (v)_{\mathfrak{r}}$. This implies $\alpha \in \operatorname{Aut}(c)_r$ for all $r \in (v)_{\mathfrak{r}}$. We hence infer

$$\alpha \in \bigcap_{r \in (v)_{\mathfrak{r}}} \operatorname{Aut}(c)_r \subseteq \bigcup_{\mathfrak{r}\in c} \bigcap_{r\in\mathfrak{r}} \operatorname{Aut}(c)_r.$$

On the other hand, let $\alpha \in \bigcup_{\mathfrak{r}\in c} \bigcap_{v\in\mathfrak{r}} \operatorname{Aut}(c)_v$. Then, there exists a ray $\mathfrak{r} \in c$ such that for each $v \in \mathfrak{r}$ holds $\alpha \in \operatorname{Aut}(c)_v$. This in particular implies $\alpha v' = v'$ for every $v' \geq v$. We conclude

$$d_c(\alpha) = d_{\mathfrak{r}}(\alpha^{-1} v', v') = 0$$

for $v' \in \mathfrak{r}$ large enough and hence $\alpha \in \ker d_c$. □

Corollary 1.2.5. *Let $G \overset{\theta}{\curvearrowright} \mathfrak{X}$ not necessarily inversion free. Let $c \in \Omega\mathfrak{X}$ and G_c a parabolic group. We denote the restriction of the action θ to G_c by θ_c. We then get*

$$\ker d_c \circ \theta_c \leq \bigcup_{\mathfrak{r}\in c} \bigcap_{v\in\mathfrak{r}} G_v.$$

In future we will write d_c instead of $d_c \circ \theta_c$ if we talk about an action $G \curvearrowright \mathfrak{X}$.

The proof of the corollary is trivial. We observe that $\ker d_c = 1$ means G_c is cyclic.

Let $G(\mathfrak{Y})$ be a finite graph of finite groups with vertex groups $(G_v)_{v\in V(\mathfrak{Y})}$. By [7] there exists an uniquely determined tree $\mathfrak{X}$ which is known as universal cover of $G(\mathfrak{Y})$ and an uniquely determined fundamental group G such that $G \curvearrowright \mathfrak{X}$ inversion free and $\mathfrak{Y} = G\backslash\mathfrak{X}$. Now we can formulate the following:

Corollary 1.2.6. *Let $\mathfrak{Y}$ be a finite path and $G(\mathfrak{Y})$ be a graph of finite groups over $\mathfrak{Y}$. Under the assumption $\gcd((|G_v|)_{v\in\mathfrak{Y}}) = 1$, every parabolic subgroup is cyclic.*

Proof. Let P be a parabolic subgroup. Then, there exists a $c \in \Omega\mathfrak{X}$ such that $P = G_c$. By construction of $\mathfrak{X}$ (see [7]), every ray $c \ni \mathfrak{r} \subseteq \mathfrak{X}$ contains a tuple of vertices $(v)_{j\in J}$ such that there is a bijection φ between J and $V(\mathfrak{Y})$ and $G_{v_j} \cong G_{\varphi(j)}$. Applying Lagrange's theorem, we obtain that the cardinality $\left|\bigcap_{v\in\mathfrak{r}} G_v\right|$ divides $\gcd\big((|G_{v_j}|)_{j\in J}\big) = 1$. We conclude $\ker d_c = 1$ and observe that G_c is indeed cyclic. □

1.3 Properties of cyclic parabolic subgroups

Lemma 1.3.1. *Let $\alpha \in \mathrm{Aut}(\mathfrak{X})$ fixing an end $c \in \Omega\mathfrak{X}$. Then, there exists an $\mathfrak{r} \in c$ such that $\alpha\mathfrak{r} \subseteq \mathfrak{r}$ or $\alpha^{-1}\mathfrak{r} \subseteq \mathfrak{r}$. Furthermore there is an $\mathfrak{r} \in c$ such that $\alpha\mathfrak{r} \subseteq \mathfrak{r}$ if and only if $d_c(\alpha) \geq 0$.*

Proof. Let $\mathfrak{s} \in c$ be an arbitrary ray. Then, there exists a $v \in \mathfrak{s}$ large enough such that $\alpha^{-1}v, v \in \mathfrak{s}$. For $x \in \mathfrak{s}$, let $(x)_\mathfrak{s}$ be the uniquely determined subray of $\mathfrak{s}$ with origin in x. We start with the proof of the first assertion. Without loss of generality, we may assume $d_c(\alpha) \geq 0$, otherwise we pass to α^{-1} and make use of $d_c(\alpha^{-1}) = -d_c(\alpha)$. So, we obtain $0 \leq d_c(\alpha) = d_c(\alpha^{-1}v, v)$ and consequently $|v|_\mathfrak{s} \geq |\alpha^{-1}v|_\mathfrak{s}$. Because $\mathfrak{s}$ is a ray and therefore totally ordered, we obtain $(v)_\mathfrak{s} \subseteq (\alpha^{-1}v)_\mathfrak{s}$. We set $\mathfrak{r} := (\alpha^{-1}v)_\mathfrak{s}$ and observe that

$$\alpha\mathfrak{r} = (\alpha\alpha^{-1}v)_\mathfrak{s} = (v)_\mathfrak{s} \subseteq (\alpha^{-1}v)_\mathfrak{s} = \mathfrak{r}.$$

This proves the first assertion and the sufficiency of the second assertion.

For the converse direction, we assume that there exists a $\mathfrak{r} \in c$ such that $\alpha\mathfrak{r} \subseteq \mathfrak{r}$. Let $r \in \mathfrak{r}$ with $0_\mathfrak{r} = \alpha^{-1}r$. Then $\mathfrak{r} = (0_\mathfrak{r})_\mathfrak{r} = (\alpha^{-1}r)_\mathfrak{r}$. By

$$\mathfrak{r} \supseteq \alpha\mathfrak{r} = \alpha(\alpha^{-1}r)_\mathfrak{r} = (r)_\mathfrak{r},$$

we get

$$0 \leq d_\mathfrak{r}(0_\mathfrak{r}, r) = d_\mathfrak{r}(\alpha^{-1}r, r) = d_\mathfrak{r}(\alpha^{-1}v, v)$$

for each $v \in \mathfrak{r}$, where the last equation is obtained by applying the translation invariance of the directed distance on $\mathfrak{r}$. For $v \in \mathfrak{r}$ large enough, we get $d_c(\alpha) = d_c(\alpha^{-1}v, v) \geq 0$. This completes the proof. □

Put $G_{\mathrm{tor}} := \{\gamma \in G : \mathrm{ord}(\gamma) < \infty\}$. We emphasize that in general G_{tor} has not be a subgroup of G.

Lemma 1.3.2. *Let $G \curvearrowright \mathfrak{X}$ inversion free such that $|G_v| < \infty$ for all $v \in \mathfrak{X}$. Then, for every $c \in \Omega\mathfrak{X}$ holds*

$$\ker d_c = G_c \cap G_{\mathrm{tor}}.$$

Proof. Let $c \in \Omega\mathfrak{X}$. By Corollary 1.2.5 and the hypothesis it is obvious that $\ker d_c \subseteq G_{\mathrm{tor}}$.

To prove the converse direction, we observe that the only torsion element in $\mathbb{Z}$ is 0. Therefore $d_c(\gamma) = 0$ for all $\gamma \in G_c \cap G_{\mathrm{tor}}$ and hence $G_c \cap G_{\mathrm{tor}} \subseteq \ker d_c$. □

We are now able to prove the following characterization for infinite cyclic parabolic groups.

Theorem 1.3.3 (characterization for infinite cyclic parabolic groups)**.** *Let $G \curvearrowright \mathfrak{X}$ inversion free. Assume $G_c \neq \{1\}$. Then G_c is infinite cyclic if and only if $\ker d_c = \{1\}$.*

Proof. Suppose G_c is infinite cyclic. Because an infinite cyclic group has no element of finite order we obtain $G_c \cap G_{\mathrm{tor}} = 1$. Then Lemma 1.3.2 yields

$$\ker d_c = G_c \cap G_{\mathrm{tor}} = \{1\},$$

as desired.

Conversely, assume $\ker d_c = \{1\}$. Then $\{1\} \neq G_c \hookrightarrow \mathbb{Z}$. Therefore G_c is infinite cyclic, as required. □

Definition 1.3.4. Let G be a group and $\{1\} \neq H \leq G$ a subgroup. We define $\mathrm{Com}_G(H) := \{g \in G :\ \exists 1 \neq x \in H : g^{-1}xg \in H\}$. We call this set the *Compensator* of H in G.

We have to remark that in general $\mathrm{Com}_G(H)$ is not necessarily a group. But nevertheless $\mathrm{Com}_G(H)$ has some obvious useful properties:

- $(\mathrm{Com}_G(H))^{-1} = \mathrm{Com}_G(H)$,
- $N_G(H) \subseteq \mathrm{Com}_G(H)$.

Before we proceed, let us introduce some notation. It is widely leant on that in [3]. Let $\alpha \in \mathrm{Aut}(\mathfrak{X})$ acting inversion free on $\mathfrak{X}$. We call α a type 2 automorphism if and only if for any finite $F \subseteq V(\mathfrak{X})$ holds $\alpha F \nsubseteq F$. An infinite backtracking free path is called 2-path if every vertex has degree 2. Fix any vertex in a 2-path. The rooted tree with root in this vertex is isomorphic as ordered space to $\mathbb{Z}$. This property characterizes 2-paths. Let α be a type 2 automorphism. The *direction* $\mathfrak{D}(\alpha)$ of α, as introduced as in [3], is the end associated to the uniquely determined ray generated by the sequence $(\alpha^k(v))_{k\in\mathbb{N}_0}$. Let ℓ be a 2-path. Each automorphism α fixing ℓ induces an automorphism on ℓ, namely the restriction $\alpha_{|\ell}$. Hence, the restriction map

$$\rho_\ell\colon \begin{array}{ccc} \mathrm{Aut}(\mathfrak{X})_\ell & \to & \mathrm{Aut}(\ell) \\ \alpha & \mapsto & \alpha_{|\ell} \end{array}$$

is a homomorphism. Finally, we briefly write $(\Omega\mathfrak{X})^\alpha$ for the set of ends fixed by an automorphism α.

Next we state a customized version of Halin's theorems [3] pp.267-268 theorems 7 and 8.

Theorem 1.3.5 (Halin's theorems; [3] pp.267-268 theorems 7 and 8)**.** *Let $\alpha \in \mathrm{Aut}^\circ(\mathfrak{X})$ be an automorphism of type 2. Then, there is exactly one 2-path $\ell(\alpha)$ which is left invariant by it. In particular,*

$$(\Omega\mathfrak{X})^\alpha = \{\mathfrak{D}(\alpha), \mathfrak{D}(\alpha^{-1})\}.$$

Remark 2.

(1) $\ell(\alpha)$ is sometimes also called the *line* or, due to Serre, the *straight path* for α.
(2) Each 2-path can be identified with a pair of distinct ends.

Lemma 1.3.6. *Let $G \curvearrowright \mathfrak{X}$ inversion free and $|G_v| < \infty$ for all $v \in \mathfrak{X}$. Let Gc be a cusp and G_c be an associated parabolic group with $G_c \neq \{1\}$ and $G_c \cap G_{\mathrm{tor}} = \{1\}$. Then, the following assertions hold:*

$$G_c \subseteq \mathrm{Com}_G(G_c) \subseteq G_c \cup \Big\{\gamma \in \bigcup_{v\in\mathfrak{X}} G_v : 2 \mid \mathrm{ord}(\gamma)\Big\} \subseteq G_c \cup \bigcup_{\substack{v\in\mathfrak{X}:\\ |G_v|\in 2\mathbb{Z}}} G_v, \tag{1.3.1}$$

$$\mathrm{Com}_G(G_c)^{\{2\}} \subseteq G_c, \tag{1.3.2}$$

where $G^{\{2\}} := \{g^2 | g \in G\}$.

Proof. Let $\gamma \in \mathrm{Com}_G(G_c)$. Then there exists an element $p \in G_c$ such that $(\gamma^{-1}p\gamma) \in G_c$. We therefore obtain $p.\gamma c = \gamma c$ and hence $\gamma c \in (\Omega\mathfrak{X})^p$. The hypothesis yields that p acts as automorphism of type 2. An application of Theorem 1.3.5 yields $(\Omega\mathfrak{X})^p = \{\mathfrak{D}(p), \mathfrak{D}(p^{-1})\} \ni c$.

Hence, we have to investigate the following cases:

(a) $c = \mathfrak{D}(p) = \gamma c$,
(b) $c = \mathfrak{D}(p^{-1}) = \gamma c$,

(c) $\mathfrak{D}(p) = c, \mathfrak{D}(p^{-1}) = \gamma c$,
(d) $\mathfrak{D}(p^{-1}) = c, \mathfrak{D}(p) = \gamma c$.

In the cases (a) and (b), we obtain $\gamma \in G_c$ immediately and there is nothing to prove. Without loss of generality, we consider the case (c). Let us assume that $\mathfrak{D}(p) \neq \mathfrak{D}(p^{-1})$ otherwise this case reduces to (a) or (b). So, we have $\mathfrak{D}(p) = c$ and $\mathfrak{D}(p^{-1}) = \gamma c = \gamma \mathfrak{D}(p)$. The fact that γ preserves $\{\mathfrak{D}(p), \mathfrak{D}(p^{-1})\}$ forces $\gamma \mathfrak{D}(p^{-1}) = \mathfrak{D}(p)$. This in particular leads to the observation

$$\gamma^2 c = \gamma^2 \mathfrak{D}(p) = \gamma(\gamma \mathfrak{D}(p)) = \gamma \mathfrak{D}(p^{-1}) = \mathfrak{D}(p) = c$$

and hence $\gamma^2 \in G_c$. This proves (1.3.2).

Let ℓ be the uniquely determined infinite 2-path belonging to the ends $\mathfrak{D}(p)$ and $\mathfrak{D}(p^{-1})$. Because of $\mathfrak{D}(p^{-1}) = \gamma.\mathfrak{D}(p)$ and Lemma 1.3.1 we get $\gamma\ell = \ell$. By assumption the ends generated by ℓ get swapped. Hence, there is a vertex or edge in ℓ such that γ fixes this vertex or edge respectively. Because G acts on $\mathfrak{X}$ inversion free, γ fixes a vertex $v \in \ell \subseteq \mathfrak{X}$ in any case. This leads to $\gamma \in G_v$ and $\operatorname{ord} \rho_\ell(\gamma) = 2$ and therefore $2 \mid \operatorname{ord}(\gamma) < \infty$. This yields the claim. □

Lemma 1.3.7 (sharped version of Lemma 1.3.6)**.** *Under the same hypotheses like in Lemma* 1.3.6 *the following assertion is true:*

$$\operatorname{Com}_G(G_c) \subseteq G_c \sqcup \{\gamma \in G \mid \operatorname{ord}(\gamma) = 2\}. \tag{1.3.3}$$

Proof. By Theorem 1.3.3, G_c is an infinite cyclic group. Hence, there exists an $a \in G_c$ such that $G_c = \langle a \rangle$ and $\operatorname{ord}(a) = \infty$.
Assume $\operatorname{Com}_G(G_c) \neq G_c$. Then, there is a $\gamma \in \operatorname{Com}_G(G_c) - G_c$. (1.3.1) forces $n := \operatorname{ord}(\gamma) < \infty$ and (1.3.2) implies $\gamma^2 \in G_c = \langle a \rangle$. Thus, there exists a $k \in \mathbb{Z}$ with the property $\gamma^2 = a^k$. From that we deduce $1 = \gamma^{2n} = a^{kn}$. Finally, $\operatorname{ord}(a) = \infty$ forces $k = 0$ and therefore $\gamma^2 = 1$. Because of $1 \neq \gamma \in G_c$ we conclude $\operatorname{ord}(\gamma) = 2$, as claimed. □

Lemma 1.3.8. *Let $\mathfrak{X}$ be a tree and $c \in \Omega\mathfrak{X}$. Furthermore let $\alpha \in \operatorname{Aut}^\circ(\mathfrak{X})$ such that $\alpha^k \in \operatorname{Aut}^\circ(\mathfrak{X})_c$ and such that for each finite $F \subseteq V(\mathfrak{X})$ holds $\alpha^k F \not\subseteq F$. Then, also*

$$\alpha \in \operatorname{Aut}^\circ(\mathfrak{X})_c$$

is true.

Proof. By hypothesis α^k is an automorphism of type 2. This is also true for α, for otherwise there would be a finite $F \subseteq V(\mathfrak{X})$ such that $\alpha^k F = \alpha^{k-1}\alpha F \subseteq \alpha^{k-1} F \subseteq \cdots \subseteq F$, a contradiction. Applying Theorem 1.3.5 we obtain $c \in \{\mathfrak{D}(\alpha^k), \mathfrak{D}(\alpha^{-k})\}$. On the other hand, we have $\mathfrak{D}(\alpha) = \mathfrak{D}(\alpha^k)$ and $\mathfrak{D}(\alpha^{-1}) = \mathfrak{D}(\alpha^{-k})$. We conclude $c \in \{\mathfrak{D}(\alpha^k), \mathfrak{D}(\alpha^{-k})\} = \{\mathfrak{D}(\alpha), \mathfrak{D}(\alpha^{-1})\}$ and therefore $\alpha c = c$ and get the claim. □

Theorem 1.3.9. *Let $G \curvearrowright \mathfrak{X}$ inversion free and $|G_v| < \infty$ for each $v \in \mathfrak{X}$. Furthermore let $c \in \Omega\mathfrak{X}$ with $G_c \neq \{1\}$ and $G_c \cap G_{\text{tor}} = 1$. Then it holds:*

$$\operatorname{Com}_G(G_c) = N_G(G_c)$$

and therefore $\operatorname{Com}_G(G_c)$ is a subgroup of G.

Proof. The direction $\supseteq$ is immediate. It remains to prove the opposite direction. Theorem 1.3.3 yields that there exists an $a \in G_c$ such that $\operatorname{ord}(a) = \infty$ and $G_c = \langle a \rangle$. Let $\gamma \in \operatorname{Com}_G(G_c)$. Then there exists a $k \in \mathbb{Z}$ such that $\gamma^{-1} a^k \gamma \in G_c$. Let $F \subseteq V(\mathfrak{X})$ be an arbitrary finite set. $G_c \cap G_{\text{tor}} = \{1\}$, meaning $\operatorname{ord}(x) = \infty$ for all $1 \neq x \in G_c$, forces

$$(\gamma^{-1} a \gamma)^k F = \gamma^{-1} a^k \gamma F \not\subseteq F,$$

for otherwise the set F would be left invariant under the action of $\langle \gamma^{-1} a^k \gamma \rangle$. The uniquely determined geodesic tree F^* generated by F would hence be $\langle \gamma^{-1} a^k \gamma \rangle$-invariant and finite as well. Therefore $\gamma^{-1} a^k \gamma$ would necessarily have a fixed vertex or edge in F^* but the action of G is inversion free. Thus, $\gamma^{-1} a^k \gamma$ would have to fix at least one vertex $v \in F^* \subseteq \mathfrak{X}$ and hence $\gamma^{-1} a^k \gamma \in G_v$ for a certain v. But this chain of thoughts leads to

$$\operatorname{ord}(a^k) = \operatorname{ord}(\gamma^{-1} a^k \gamma) \mid |G_v| < \infty,$$

a contradiction.

Now we are able to apply Lemma 1.3.8. This leads to

$$\gamma^{-1} a \gamma \in G_c$$

and we conclude $\gamma \in N_G(G_c)$.

□

Let G be a group and $\Omega \subseteq G$. For each $k \in \mathbb{N}$ we denote by

$$\Omega^k := \{\omega_1 \cdots \omega_k : \ \omega_i \in \Omega \ \forall 1 \leq i \leq k\}$$

Lemma 1.3.10. *Same hypothesis as in Theorem* 1.3.9. *Then*

$$(N_G(G_c) - G_c)^2 \subseteq G_c \quad \textit{and hence} \quad (\operatorname{Com}_G(G_c) - G_c)^2 \subseteq G_c$$

and therefore $[\operatorname{Com}_G(G_c) : G_c] = [N_G(G_c) : G_c] \in \{1, 2\}$.

Proof. The case $N_G(G_c) = G_c$ is trivial, so we may assume $N_G(G_c) - G_c \neq \emptyset$. Let $\gamma, \gamma' \in N_G(G_c) - G_c$. We then get by the same argument as in the proof of Lemma 1.3.6 that $c = \mathfrak{D}(a)$ or $c = \mathfrak{D}(a^{-1})$ and both γ and γ' leave $\{\mathfrak{D}(a), \mathfrak{D}(a^{-1})\}$ invariant. Without loss of generality, we may assume $c = \mathfrak{D}(a)$. Because $\gamma, \gamma' \notin G_c$ and therefore also $\gamma^{-1} \notin G_c$, we obtain

$$\gamma' c = \mathfrak{D}(a^{-1}) = \gamma^{-1} c.$$

This implies $\gamma\gamma' c = c$ and hence $\gamma\gamma' \in G_c$, as required. □

Theorem 1.3.11. *Let G be a group such that each two elements of order 2 are conjugate and such that $G \curvearrowright \mathfrak{X}$ inversion free with finite vertex stabilizers, meaning $|G_v| < \infty$ for all $v \in \mathfrak{X}$. Let Gc be a cusp and $G_c \neq \{1\}$ be the associated parabolic group with $G_c \cap G_{\mathrm{tor}} = \{1\}$. Furthermore, assume that $G_c \not\subseteq \{[\gamma, g] : \gamma, g \in G, \ \operatorname{ord}(\gamma) = 2\}$. Then*

$$\operatorname{Com}_G(G_c) = G_c.$$

Proof. To obtain a contradiction, suppose $\operatorname{Com}_G(G_c) - G_c \neq \emptyset$. Then, there exists a $\gamma \in \operatorname{Com}_G(G_c)$. Lemma 1.3.7 ensures $\operatorname{ord}(\gamma) = 2$. Lemma 1.3.10 yields

$$\operatorname{Com}_G(G_c) - G_c = \gamma G_c.$$

Let $x \in G_c$ be an arbitrary element. Again by Lemma 1.3.7, we obtain $\operatorname{ord}(\gamma x) = 2$. The hypothesis on G forces the existence of a $g \in G$ such that $\gamma x = g^{-1} \gamma g$ and thus $x = \gamma^{-1} g^{-1} \gamma g = [\gamma, g]$. We conclude

$$G_c \subseteq \{[\gamma, g] : \gamma, g \in G, \ \operatorname{ord}(\gamma) = 2\},$$

the desired contradiction. □

Theorem 1.3.12. *Let $G \curvearrowright \mathfrak{X}$ inversion free such that $|G_v| < \infty$ for all $v \in \mathfrak{X}$ and each parabolic subgroup of G is infinite cyclic. Then, the maximal infinite cyclic subgroups are exactly the parabolic subgroups.*

PROOF. Let P be a maximal infinite cyclic subgroup and a its generating element. Then $\operatorname{ord}(a) = \infty$ and because $|G_v| < \infty$ for all $v \in \mathfrak{X}$ we get that a acts as a type 2 automorphism on $\mathfrak{X}$. Theorem 1.3.5 forces $\emptyset \neq (\Omega\mathfrak{X})^a = \{\mathfrak{D}(a), \mathfrak{D}(a^{-1})\}$. We conclude $a \in G_{\mathfrak{D}(a)}$ and hence $P \leq G_{\mathfrak{D}(a)}$. Because $G_{\mathfrak{D}(a)}$ is cyclic, we infer $P = G_{\mathfrak{D}(a)}$.

For the converse direction let $c \in \Omega\mathfrak{X}$ and G_c fixing c. By hypothesis, G_c is infinite cyclic and therefore generated by an element $a \in G_c$, $\operatorname{ord}(a) = \infty$. Let P be an infinite cyclic group, generated by an element $b \in P$ with $\operatorname{ord}(b) = \infty$, such that $G_c \leq P$. We observe

$$\langle a \rangle = G_c \leq P = \langle b \rangle.$$

This yields $a = b^k$ for a $k \in \mathbb{Z}$ and hence $b^k \in G_c$. Lemma 1.3.8 forces $b \in G_c$ and therefore $P = \langle b \rangle \leq G_c$, and the proof is complete. □

We remark that this theorem also guarantees the existence of maximal infinite cyclic subgroups. We extract the case that G does not contain any element of order 2:

Corollary 1.3.13. *Let $G \curvearrowright \mathfrak{X}$ inversion free such that $|G_v| < \infty$ for all $v \in \mathfrak{X}$ and each parabolic subgroup of G is infinite cyclic. Furthermore assume $2 \nmid |G_v|$ for any $v \in \mathfrak{X}$. Then $\operatorname{Com}_G(G_c) = G_c$.*

Example 1.3.14. Let $G := C_2 * C_2$ where the first factor is generated by c_2 and the second by c_2'. G can be interpreted as fundamental group of the associated graph of groups acting on the universal cover which is a tree. Corollary 1.2.5 guarantees that every parabolic group has to be cyclic. We consider $P := \langle c_2 c_2' \rangle$. We observe that P is a maximal infinite cyclic subgroup and hence by Theorem 1.3.12 it is parabolic. But we see that

$$\langle c_2 \rangle \subseteq \operatorname{Com}_G(P)$$

as well as

$$\langle c_2' \rangle \subseteq \operatorname{Com}_G(P).$$

From that we conclude $\operatorname{Com}_G(P) \neq P$. We remark that in G does not hold that each 2 elements of order 2 are in the same conjugacy class.

Proposition 1.3.15 (parabolic groups intersect trivial)**.** *Let $G \curvearrowright \mathfrak{X}$ such that each parabolic subgroup is either trivial or infinite cyclic. Then, for each two $c, c' \in \Omega\mathfrak{X}$ either*

$$G_{c'} = G_c \quad \textit{or} \quad G_{c'} \cap G_c = \{1\}$$

is true.

PROOF. On the contrary to our claim, suppose there exist $c, c' \in \Omega\mathfrak{X}$ such that $G_{c'} \cap G_c \neq \{1\}$. Then it follows $\{1\} < G_{c'} \cap G_c < G_c$. This forces $G_c \cong \mathbb{Z}$. Hence, G_c contains only the subgroups $\{1\}$ and those of finite index. Because $H := G_{c'} \cap G_c$ can be understood as ideal in $\mathbb{Z}$, we deduce that H is of finite index in G_c. We therefore obtain that G_c/H is a finite cyclic group. Fix an arbitrary $1 \neq \gamma \in G_c$. Because G_c/H is finite, there is a $k \in N_{\geq 1}$ such that

$$H = (\gamma H)^k = \gamma^k H.$$

This yields $\gamma^k \in H \leq G_{c'}$. By choice we have $\operatorname{ord}(\gamma) = \infty$. The same argument as in the proof of Theorem 1.3.9 guarantees that $\gamma^k F \nsubseteq F$ for each finite set F. Thus, Lemma 1.3.8 finally forces $\gamma \in G_{c'}$ and therefore $G_c \subseteq G_{c'}$. Exchanging the roles of c and c' yields the claim.

□

1.4 Cusps of finite index subgroups

In [4] has been made an attempt to compute the “cusp multiplicities” of Fuchsian groups using the “cusp definition” for the hyperbolic plane. Fuchsian groups can be understood as finite index subgroups of free products of groups. We want to generalize the results on the setting of finite index subgroups of virtually free groups. Virtually free groups can be considered as fundamental groups of finite graphs of finite groups. Think of $G \curvearrowright \mathfrak{X}$ as the action of such a fundamental group of a finite graph of groups on its universal cover $\mathfrak{X}$, which has to be a tree.

Let G be a group and $H, K \leq G$. We denote by $[H]_K$ the orbit of H under K where K acts on the set of subgroups of G via conjugation. K acts also on the elements of G by conjugation. For $x \in G$ we will denote an orbit by $[x]_K$. If x and y are conjugate by an element of K, we will write $x \sim_K y$. We use an analogue notation for subgroups; just replace x and y by subgroups.

In the sequel let $\Delta \leq G$ a finite index subgroup, $\mu := (G : \Delta)$. Let $\theta \colon G \to \mathrm{Sym}(\Delta\backslash G) = S_\mu$ be the canonical representation on the Δ-cosets.

Definition 1.4.1 (cusp multiplicity)**.** Let $Gc \in G\backslash\Omega\mathfrak{X}$ be a cusp of G in $\mathfrak{X}$. Then we call $\mathrm{cm}_\Delta(Gc) := |\Delta\backslash Gc|$ the *geometric cusp multiplicity* of Gc in Δ. This number counts the cusps of Δ in $\mathfrak{X}$ which fuse to Gc.

We observe that

$$\mathrm{cm}_\Delta(Gc) = |\Delta\backslash G/G_c| = \big|\{1,\dots,\mu\}\overset{\theta}{/}G_c\big|.$$

Definition 1.4.2. Let P be a parabolic subgroup of G. Then the *grouptheoretical cusp multiplicity* is the number

$$\mathrm{gcm}_\Delta(P) := \big|\{[\Delta \cap \gamma P \gamma^{-1}]_\Delta : \gamma \in G\}\big|.$$

That is the number of Δ-conjugacy classes of parabolic subgroups in Δ into which the G-conjugacy class of the parabolic subgroup P decomposes.

Let $c \in \Omega\mathfrak{X}$. We call the number $\mathrm{co}_\Delta(c) := |\Delta_c\backslash G_c|$ the *cusp order* of c. Furthermore, we set for an infinite parabolic subgroup $P = \langle a\rangle$ in G

$$\mathrm{gco}^P_\Delta(\gamma) := \inf\{|k| : k \in \mathbb{Z} - \{0\},\ \gamma a^k \gamma^{-1} \subseteq \Delta\}.$$

the *grouptheoretical cusp order of* γ *with respect to* P for the subgroup Δ.

We proceed by giving a characterization for the cusp order.

Lemma 1.4.3. *Let $G \curvearrowright \mathfrak{X}$ inversion free such that $|G_v| < \infty$ for all $v \in \mathfrak{X}$ and $c \in \Omega\mathfrak{X}$ such that $G_c \neq \{1\}$ and $G_c \cap G_{\mathrm{tor}} = \{1\}$. Then, $G_c = \langle a\rangle$ is infinite cyclic and*

$$\mathrm{co}_\Delta(c) = \inf\{|k| :\ k \in \mathbb{Z} \smallsetminus \{0\},\ a^k \in \Delta\}.$$

Moreover, it is true that $\mathrm{co}_\Delta(c) < \infty$.

Proof. Because G_c is cyclic the cosets $\Delta_c\backslash G_c$ form a cyclic group. We hence obtain $\Delta_c\backslash G_c = \langle \Delta_c.a\rangle$.

$$|\Delta_c\backslash G_c| \leq |\Delta\backslash G| = \mu < \infty$$

yields $\mathrm{co}_\Delta(c) < \infty$. We conclude

$$|\Delta_c\backslash G_c| = \mathrm{ord}_{\Delta_c\backslash G_c}(\Delta_c.a) = \inf\{|k| :\ k \in \mathbb{Z} \smallsetminus \{0\},\ a^k \in \Delta\},$$

as required. ∎

Lemma 1.4.4. *Let θ be the canonical coset-representation $G \to \mathrm{Sym}(\Delta\backslash G)$ and $\langle a\rangle = P \leq G$ be an infinite cyclic parabolic subgroup. Then*

$$\mathrm{gco}_\Delta^P(\gamma) = |\Delta\gamma.P|$$

where $\Delta\gamma.P$ denotes the orbit of $\Delta\gamma \in \Delta\backslash G \cong \{1,\dots,\mu\}$ under $\theta(P)$. In particular, we obtain $\mathrm{gco}_\Delta^P(\gamma) < \infty$ for all $\gamma \in G$.

Proof. It holds $\mu = |\Delta\backslash G| = \sum_{\Delta\gamma.P\in(\Delta\backslash G)/P} |\Delta\gamma.P|$ therefore $|\Delta\gamma.P| < \infty$ for all $\gamma \in G$. So let $k \in \mathbb{N}_{>0}$ such that $|\Delta\gamma.P| = k$. Then

$$\Delta\gamma.a^k = \Delta\gamma$$

and thus $a^k \in G_{\Delta\gamma} = \gamma^{-1}\Delta\gamma$. The last assertion implies $\mathrm{gco}_\Delta^P(\gamma) \leq k$.

On the other hand, we obtain by the definition the grouptheoretical cusp order $\gamma P^{\mathrm{gco}_\Delta^P(\gamma)} \subseteq \Delta\gamma$. This leads to the equation $\Delta\gamma P^{\mathrm{gco}_\Delta^P(\gamma)} = \Delta\gamma$ and therefore $a^{\mathrm{gco}_\Delta^P(\gamma)} \in \langle a\rangle_{\Delta\gamma}$. We therefore conclude

$$k = |\Delta\gamma.\langle a\rangle| = |\langle a\rangle/\langle a\rangle_{\Delta\gamma}| \;\Big|\; |\langle a\rangle/\langle a^{\mathrm{gco}_\Delta^P(\gamma)}\rangle| = \mathrm{gco}_\Delta^P(\gamma),$$

and the proof is complete. ☐

Lemma 1.4.5. *Let $G \curvearrowright \mathfrak{X}$ inversion free such that $|G_v| < \infty$ for all $v \in \mathfrak{X}$ and $c \in \Omega\mathfrak{X}$ with $G_c = \langle a\rangle$ infinite cyclic. Let $c' \in G.c$. Then there is a $\gamma \in G$ with $c' = \gamma.c$ and*

$$\mathrm{co}_\Delta(c') = \mathrm{gco}_\Delta^{G_c}(\gamma).$$

Proof. We first observe

$$G_{c'} = G_{\gamma c} = \gamma\langle a\rangle\gamma^{-1} = \langle\gamma a\gamma^{-1}\rangle.$$

Lemma 1.4.3 therefore implies

$$\mathrm{co}_\Delta(c') = \inf\{|k| : \; k \in \mathbb{Z}\setminus\{0\},\; \gamma a^k\gamma^{-1} \subseteq \Delta\} = \mathrm{gco}_\Delta^{G_c}(\gamma),$$

as desired. ☐

Theorem 1.4.6. *Let P be an infinite cyclic parabolic group generated by an element a. Then it holds*

$$\mathrm{gcm}_\Delta(P) = \big|\big\{\{[x]_\Delta, [x^{-1}]_\Delta\} : x \in \Delta\colon \exists\gamma \in G\colon\; x \sim_G a^{|\Delta\gamma.P|}\big\}\big|.$$

Proof. It is easy to see that the map

$$\big\{\{[x]_\Delta, [x^{-1}]_\Delta\} : x \in \Delta\colon \exists\gamma \in G\colon\; \mathrm{gco}_\Delta^P(\gamma) < \infty \;\wedge\; x \sim_G a^{\mathrm{gco}_\Delta^P(\gamma)}\big\}$$
$$\longrightarrow \{[\Delta \cap \gamma P\gamma^{-1}]_\Delta : \gamma \in G\colon\; \mathrm{gco}_\Delta^P(\gamma) < \infty\}$$
$$\{[x]_\Delta, [x^{-1}]_\Delta\} \longmapsto [\langle x\rangle]_\Delta$$

is well-defined and a bijection. We apply Lemma 1.4.4 and get the claim. ☐

Theorem 1.4.7. *Let $P \leq G$ be a parabolic subgroup. Then*

$$\begin{array}{rccc} \Lambda\colon & \Delta\backslash G/P & \longrightarrow & \{[\Delta\cap\gamma P\gamma^{-1}]_\Delta : \gamma \in G\} \\ & \Delta\gamma P & \longmapsto & [\Delta\cap\gamma P\gamma^{-1}]_\Delta \end{array}$$

is a well-defined surjective map.

Let $c \in \Omega\mathfrak{X}$ such that $P = G_c$. If in addition $\mathrm{Com}_G(P) = P$, then Λ is also injective, hence bijective, and it is even true that

$$\mathrm{gcm}_\Delta(P) = \mathrm{cm}_\Delta(Gc).$$

PROOF. First, we prove that Λ is a well-defined map. For this purpose, let $\gamma, \gamma' \in G$ such that $\Delta\gamma P = \Delta\gamma' P$. Hence, there exist $\delta \in \Delta$ and $p \in P$ with $\gamma' = \delta\gamma p$. We obtain

$$\begin{aligned}\Lambda(\Delta\gamma' P) &= [\Delta \cap \delta\gamma(pPp^{-1})\gamma^{-1}\delta^{-1}] \\ &= [\delta(\Delta \cap \gamma P\gamma^{-1})]_\Delta \\ &= \Lambda(\Delta\gamma P).\end{aligned}$$

The surjectivity of the map is obvious. Hence, we proceed with the second assertion. To this end, let $c \in \Omega\mathfrak{X}$ and $P = G_c$ and assume the equation $\mathrm{Com}_G(P) = P$ holds. To show the injectivity of Λ, take $\Delta\gamma P, \Delta\gamma' P$ such that

$$[\Delta \cap \gamma P\gamma^{-1}]_\Delta = [\Delta \cap \gamma' P\gamma'^{-1}]_\Delta.$$

Then, there is a $\delta \in \Delta$ with the property

$$\Delta \cap \gamma' P\gamma'^{-1} = \Delta \cap \delta\gamma P\gamma^{-1}\delta^{-1}.$$

Let $x \in \Delta \cap \gamma' P\gamma'^{-1} = \Delta \cap \delta\gamma P\gamma^{-1}\delta^{-1}$. Hence, there exist $p \in P$ and $q \in P$ such that

$$\delta\gamma p\gamma^{-1}\delta^{-1} = x = \gamma' q\gamma'^{-1}.$$

This yields $\gamma'^{-1}\delta\gamma p\gamma^{-1}\delta^{-1}\gamma' = q$. Applying the hypothesis we infer

$$\gamma'^{-1}\delta\gamma \in \mathrm{Com}_G(P) = P.$$

This leads to the conclusion

$$\Delta\gamma' P = \Delta\gamma'(\gamma'^{-1}\delta\gamma)P = \Delta\delta\gamma P = \Delta\gamma P,$$

as claimed. □

Recall that we denote the natural combinatorial metric on $\mathfrak{X}$ by D.

Lemma 1.4.8. *Assume G acts inversion free on a tree $\mathfrak{X}$. Let $c \in \Omega\mathfrak{X}$ such that $G_c = \langle a \rangle$ is infinite cyclic. Let $\ell(a)$ be the unique α-invariant 2-path assigned to a; its existence is assured by Theorem 1.3.5. Then, the action of G on $\mathfrak{X}$ induces an action $G_c \curvearrowright \ell(a)$ and*

$$D(v, \Delta_c v) = \mathrm{co}_\Delta(c) \cdot D(v, G_c v) \quad \forall v \in \ell(a).$$

PROOF. Put $k := \mathrm{co}_\Delta(c)$. It follows immediately from the definition of the cusp order that $\Delta_c = \langle a^k \rangle$. Recall that we denote by d_c the displacement function with respect to c. We then observe

$$D(v, \Delta_c v) = D(v, a^k v) = d_c(a^k) = k \cdot d_c(a) = k \cdot D(v, av) = k \cdot D(v, G_c v),$$

as required. □

Theorem 1.4.9 (geometric interpretation of the cusp order)**.** *Assume G acts inversion free on a tree $\mathfrak{X}$. Let $c \in \Omega\mathfrak{X}$ such that $G_c = \langle a \rangle$ is infinite cyclic. Then,*

$$D(v, \Delta_c v) \leq \mathrm{co}(c) \cdot D(v, G_c v)$$

where the equality occurs if and only if $v \in \ell(a)$ or $\Delta_c = G_c$. In other words, we have

$$\mathrm{co}_\Delta(c) = \sup_{v \in \mathfrak{X}} \frac{D(v, \Delta_c v)}{D(v, G_c v)},$$

where the supremum is attained if and only if $v \in \ell(a)$ or $\Delta_c = G_c$.

PROOF. Take an arbitrary $v \in \mathfrak{X}$. Set $k := \mathrm{co}(c)$. Let $v_0 \in \ell(a)$ such that

$$D(v, v_0) = \min\{D(v, w) : w \in \ell(a)\}.$$

Consider an arbitrary ray in $\mathfrak{X}$ containing v and $a^j v$, $j \in \mathbb{N}_{\geq 1}$. Because a acts as an automorphism and therefore as an isometry, this ray shares the vertices $v_0, a^j v_0$ and the vertices between them with $\ell(a)$. Combining this with Lemma 1.4.8, we compute

$$\begin{aligned}
D(v, \Delta_c v) &= D(v, a^k v) \\
&= D(v, v_0) + D(v_0, a^k v_0) + D(a^k v_0, a^k v) \\
&= 2D(v, v_0) + k \cdot D(v_0, G_c v_0) \\
&= k \cdot \underbrace{\big(2D(v, v_0) + D(v_0, G_c v_0)\big)}_{D(v, G_c v)} - (2k-2) \cdot D(v, v_0) \\
&= k \cdot D(v, G_c v) - \underbrace{(2k-2)}_{\geq 0} \cdot \underbrace{D(v, v_0)}_{\geq 0}.
\end{aligned}$$

For the reason the second term vanishes if and only if $D(v, v_0) = 0$ or $k = 1$ and hence if and only if $v = v_0$ or $\Delta_c = G_c$, this yields the claim. ☐

Corollary 1.4.10. *Under the same hypothesis as in Theorem* 1.4.9 *the following formula is true:*

$$\mathrm{co}(c) = \lim_{n \to \infty} \frac{D(v, \Delta_c^{(n)} v)}{D(v, G_c^{(n)} v)} \quad \forall v \in \mathfrak{X},$$

where $G_c^{(n)} = \{\gamma^n : \ \gamma \in G_c\}$ *and* $\Delta_c^{(n)} := \{\delta^n : \ \delta \in \Delta_c\}$.

PROOF. Let $k := \mathrm{co}(c)$, $v \in \mathfrak{X}$ an arbitrary vertex and $v_0 \in \ell(a)$ the vertex of smallest distance to v. The same arguments as in the proof of Theorem 1.4.9 yield for each $1 \leq m \in \mathbb{N}$

$$D(v, a^m v) = 2D(v, v_0) + m \cdot D(v_0, a v_0).$$

We thus conclude

$$\frac{D(v, \Delta_c^{(n)} v)}{D(v, G_c^{(n)} v)} = \frac{D(v, a^{kn} v)}{D(v, a^n v)} = \frac{2D(v, v_0) + kn \cdot D(v_0, a v_0)}{2D(v, v_0) + n \cdot D(v_0, a v_0)} \xrightarrow{n \to \infty} k,$$

as desired. ☐

CHAPTER 2

Singularities of finite index subgroups of $\mathrm{Sl}_3(\mathbb{Z})$

It is well known that a subgroup Δ_0 of finite index μ of the Modular Group $\Gamma_0 := \mathrm{Sl}_2(\mathbb{Z}) = C_2 * C_3$ can be widely characterized by counting the multiplicities of conjugacy classes of maximal finite subgroups in Δ_0. This has been done by using the fact that Γ can be seen as free product of finite groups and more general by a finite graph of finite groups. The Bass-Serre theory ensures that also Δ_0 can be expressed as finite graph of finite groups. This and the notion of the Euler characteristic of a graph of groups give us pretty much a characterization of finite index subgroups of $\mathrm{Sl}_2(\mathbb{Z})$ or more general of free products of finite groups.

There is no similar result for the subgroups of $\mathrm{Sl}_3(\mathbb{Z})$. It is even true, see [7] p. 67 theorem 16, that $\mathrm{Sl}_3(\mathbb{Z})$ cannot be written as finite graph of finite groups. Hence, let us look for another approach. Remember that $\mathrm{Sl}_2(\mathbb{Z})$ acts on the hyperbolic plane $\mathbb{H}$ via Möbius transforms. If we think of $\mathbb{H}$ as the orbit $\mathrm{Sl}_2(\mathbb{R}).i \cong \mathrm{Sl}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R})$, we get a description which can be generalized in a way resulting in a group action of $\mathrm{Sl}_3(\mathbb{Z})$ as isometries on the symmetric space of the quadratic forms.

The fundamental domain of the action of $\mathrm{Sl}_3(\mathbb{Z})$ on this space is not compact as it is not for the action of $\mathrm{Sl}_2(\mathbb{Z})$ on $\mathbb{H}$. But again as for the action of $\mathrm{Sl}_2(\mathbb{Z})$ on $\mathbb{H}$, there can be constructed a $\mathrm{Sl}_3(\mathbb{Z})$-invariant retract with compact fundamental domain. Following this approach, Soulé has indeed contrived such a cocompact retract. Moreover, he has computed the fundamental domain for this action and even the groups fixing the boundary components of it. In this way he has obtained a description for $\mathrm{Sl}_3(\mathbb{Z})$ in terms of a finite complex of finite groups.

Let Δ_μ be an arbitrary finite index subgroup of $\mathrm{Sl}_3(\mathbb{Z})$ with $(\mathrm{Sl}_3(\mathbb{Z}) : \Delta_\mu) = \mu$. The purpose of this work is to have a deeper insight in the geometric structure of some finite index subgroups of $\mathrm{Sl}_3(\mathbb{Z})$ using Soulé's complex of groups for $\mathrm{Sl}_3(\mathbb{Z})$. Moreover, we are interested in the evolution of the numbers of the maximal finite subgroups of certain types up to Δ_μ-conjugacy in μ. We will be able to compute these sizes by solving certain systems of polynomial equations over finite fields. To realize this approach, we have to find a system of representatives for $\Delta_\mu \backslash \Gamma$ with an adequate explicit description. Hence, it seems necessary to focus on specific subgroups Δ_μ.

For each finite index subgroup, we will also provide a geometric interpretation for these numbers. To this end, we introduce the reduction of a scwol associated to the action of a group on that scwol. This reduction is constructed in a way such that the maximal finite subgroups are the maximal stabilizers of the 0-dimensional vertices of that scwol and vice versa.

$\mathrm{Sl}_3(\mathbb{Z})$ satisfies the congruence subgroup property. Thus, each finite index subgroup of $\mathrm{Sl}_3(\mathbb{Z})$ can be obtained as a preimage of a subgroup of $\mathrm{Sl}_3(\mathbb{Z}/n\mathbb{Z})$, $n \in \mathbb{N}$. Hence, it is advisable to choose prototypes for finite index subgroups Δ_μ which are as "large" as possible. For those reasons, we consider in our work only finite index subgroups which occur as preimages (under the congruence map) of Borel subgroups of $\mathrm{Sl}_3(\mathbb{F}_p)$. Finally, we will lift the results of the computations to preimages of subgroups of $\mathrm{Sl}_3(\mathbb{Z}/d\mathbb{Z})$, $d \in \mathbb{Z}$ square-free. Knowing the vertex group, we can determine the distribution of the isotropy groups of the higher dimensional faces containing the considered vertex.

2.1 Preliminaries

In the whole article, let $\Gamma := \mathrm{Sl}_3(\mathbb{Z})$, $2 \leq d \in \mathbb{Z}$ be an arbitrary integer and $\Delta^{(d)} := \{(a_{i,j})_{(i,j)\in 3\times 3} \in \mathrm{Sl}_3(\mathbb{Z}) : a_{i,j} \equiv 0 \mod d \ \forall i > j\}$.

2.2 A complex of groups for $\mathrm{Sl}_3(\mathbb{Z})$

Let us consider the following right action of Γ on the space

$$X := \{A \in \mathbb{R}^{3\times 3} |\ \det(A) = 1, A = {}^tA, \langle Av, v\rangle > 0 \quad \forall\, 0 \neq v \in \mathbb{R}^3\}$$

of scalar products on $\mathbb{R}^3$:

$$x * \gamma := {}^t\gamma x \gamma, \quad x \in X,\ \gamma \in \Gamma.$$

Soulé constructed in [8] a fundamental domain D for that action by watching out for the in some sense minimal elements of the orbits of that action. He could pass to a 3-dimensional connected compact subset $D' \subseteq D$ also being a deformation retract such that $D' * \Gamma \subseteq X$ is connected itself. Note that Soulé considered in [8] the action on the space

$$\hat{X} := \{A \in \mathbb{R}^{3\times 3} |\ A = {}^tA, \langle Av, v\rangle > 0 \quad \forall\, 0 \neq v \in \mathbb{R}^3\}/\mathbb{R}^\times$$

instead of X. This is obviously equivalent via the Γ-isomorphism:

$$\hat{X} \to X : [A]_{\mathbb{R}^\times} \mapsto \left(\frac{1}{\det A}\right)^{\frac{1}{3}} A$$

Defining $X' := D' * \Gamma$ he obtained the 3-dimensional deformation retract $X' \subseteq X$ such that $\Gamma \curvearrowright X'$ cocompactly. Considering a "nice" triangulation of D' and hence of X' he computed in the same article the associated finite complex of finite groups given by this action.

Before we state his result, let us set for the sake of clarity

$$h(u,v,w) := \begin{pmatrix} 2 & w & v \\ w & 2 & u \\ v & u & 2 \end{pmatrix} \text{ and } h^*(\cdot) := \left(\frac{1}{\det h(\cdot)}\right)^{\frac{1}{3}} h(\cdot).$$

Theorem 2.2.1 ([8], pp. 4-5, theorem 2)**.** *There is a triangulation of the fundamental domain D' with vertices $O := h^*(0,0,0)$, $M := h^*(1,1,1)$, $M' := h^*(1,1,0)$, $N := h^*(1,1,\frac{1}{2})$, $N' := h^*(1,\frac{1}{2},-\frac{1}{2})$, $P := h^*(\frac{2}{3},\frac{2}{3},-\frac{2}{3})$, $Q := h^*(1,0,0)$ which can be continued on the whole space X' such that the action of Γ on X' can be described by the fact that $q_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ maps M', N', Q to M, N, Q and $q_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ maps N, N', M', Q to N', N, M', Q and by the stabilizers of the simplices given by their generators. The triangulation of D' and the stabilizers of the simplices belonging to it can be found in the table below. The stabilizer in the second column is always associated with the underlined simplex.*

simplices	*stabilizer of the underlined simplex*	*isomorphy class*
$\underline{O}$	$\left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$	S_4
$\underline{M}$ M'	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right\rangle$	S_4
$\underline{P}$	$\left\langle \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$	S_4
$\underline{Q}$	$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	D_{12}
$\underline{N}$ N'	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	D_8
$\underline{MN}$ $M'N$ $M'N'$	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right\rangle$	V_4
$\underline{M'P}$	$\left\langle \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	D_8
$\underline{N'P}$	$\left\langle \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$	D_8
$\underline{OM}$	$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle$	S_3
$\underline{OQ}$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$	V_4
$\underline{OP}$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	S_3
$\underline{M'PN'}$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \right\rangle$	V_4

$\underline{MQ}$ $M'Q$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$	C_2
$\underline{N'Q}$ NQ	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$	C_2
$\underline{ON}$	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	C_2
$\underline{ON'}$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$	C_2
$\underline{OM'}$	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	C_2
$\underline{OMN}$	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	C_2
$\underline{OM'N}$	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	C_2
$\underline{OMQ}$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$	C_2
$\underline{OPN'}$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$	C_2
$\underline{OM'P}$	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	C_2
$\underline{ON'Q}$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$	C_2

The remaining simplices have trivial stabilizers: ONQ, $OM'Q$, $OM'N'$, MNQ, $M'NQ$, $M'N'Q$, $OMNQ$, $OM'NQ$, $OM'N'Q$, and $OM'PN'$.

Remark 3. The left action $\Gamma \curvearrowright X'$ defined via $\gamma.x := {}^t\gamma^{-1}x\gamma^{-1}$, $(\gamma, x) \in \Gamma \times X'$, has the same orbits and stabilizers as the right action used above. Even the fundamental domains for these actions coincide for the reason $X' = (X')^{-1}$. For the ease of notation we will always consider the left action on X' instead of the right action.

In the corollary below, we compute some representations for maximal finite subgroups of Γ. For this purpose, we take the restriction of the action $\Gamma \curvearrowright X'$ to suitable simplices.

Corollary 2.2.2. *Set*

$$O_2 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},\ O_3 := \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix},\ O_4 := \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix};$$

$$O_2' := \begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},\ O_3' := \begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ O_4' := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

$$M_2 := h^*(-1,-1,1),\ M_3 := h^*(-1,1,-1),\ M_4 := h^*(1,-1,-1);$$

$$P_2 := h^*\left(\frac{2}{3},-\frac{2}{3},\frac{2}{3}\right),\ P_3 := h^*\left(-\frac{2}{3},\frac{2}{3},\frac{2}{3}\right),\ P_4 := h^*\left(-\frac{2}{3},-\frac{2}{3},-\frac{2}{3}\right);$$

$$M_2' := \begin{pmatrix} \sqrt[3]{2} & 0 & \frac{1}{2^{2/3}} \\ 0 & \sqrt[3]{2} & \frac{1}{2^{2/3}} \\ \frac{1}{2^{2/3}} & \frac{1}{2^{2/3}} & \sqrt[3]{2} \end{pmatrix}, M_3' := \begin{pmatrix} \sqrt[3]{2} & -\frac{1}{2^{2/3}} & 0 \\ -\frac{1}{2^{2/3}} & \sqrt[3]{2} & \frac{1}{2^{2/3}} \\ 0 & \frac{1}{2^{2/3}} & \sqrt[3]{2} \end{pmatrix},$$

$$M_4' := h^*(1,-1,-1),$$

$$M_5' := \begin{pmatrix} \sqrt[3]{2} & 0 & -\frac{1}{2^{2/3}} \\ 0 & \sqrt[3]{2} & \frac{1}{2^{2/3}} \\ -\frac{1}{2^{2/3}} & \frac{1}{2^{2/3}} & \sqrt[3]{2} \end{pmatrix}, M_6' := \begin{pmatrix} \sqrt[3]{2} & \frac{1}{2^{2/3}} & 0 \\ \frac{1}{2^{2/3}} & \sqrt[3]{2} & \frac{1}{2^{2/3}} \\ 0 & \frac{1}{2^{2/3}} & \sqrt[3]{2} \end{pmatrix}$$

Let us denote by ψ the action of Γ on X' (from the left) as described in Remark 3. Then the restrictions $\psi_M^O := \psi_{|\Gamma_M \to \mathrm{Sym}(\Gamma_M.O)}$ and $\psi_P^O := \psi_{|\Gamma_P \to \mathrm{Sym}(\Gamma_P.O)}$, $\psi_O^M := \psi_{|\Gamma_O \to \mathrm{Sym}(\Gamma_O.M)}$ and $\psi_O^P := \psi_{|\Gamma_O \to \mathrm{Sym}(\Gamma_O.P)}$ are isomorphisms and $\psi_Q^M := \psi_{|\Gamma_Q \to \mathrm{Sym}(\Gamma_Q.M)}$ is injective.

Furthermore we have following explicit expressions for the orbits given by

$$\begin{aligned} \Gamma_M.O &= \{O, O_2, O_3, O_4\}, \\ \Gamma_P.O &= \{O, O_2', O_3', O_4'\}, \\ \Gamma_O.M &= \{M, M_2, M_3, M_4\}, \\ \Gamma_O.P &= \{P, P_2, P_3, P_4\}, \\ \Gamma_Q.M &= \{M, M_2', M_3', M_4', M_5', M_6'\}. \end{aligned}$$

ψ_M^O is uniquely determined by

$$\psi_M^O\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (O_2\ O_4),\ \psi_M^O\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} = (O\ O_2\ O_3\ O_4);$$

ψ_P^O is uniquely determined by

$$\psi_P^O\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = (O_3'\ O_4'),\ \psi_P^O\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = (O\ O_2'\ O_3'\ O_4');$$

ψ_O^M is uniquely determined by

$$\psi_O^M\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = (M_2\ M_3),\ \psi_O^M\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = (M\ M_2\ M_3\ M_4);$$

ψ_O^P is uniquely determinded by

$$\psi_O^P\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = (P\ P_2),\ \psi_O^P\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = (P\ P_2\ P_3\ P_4);$$

ψ_Q^M is uniquely determined by

$$\psi_Q^M\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = (M\ M_3')(M_4'\ M_6'),$$

$$\psi_Q^M \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = (M\ M_2'\ M_3'\ M_4'\ M_5'\ M_6').$$

PROOF. Let us proof the corollary for the action of Γ_M on $\Gamma.O$: From Theorem 2.2.1 we know that $|\Gamma_M.O| = |\Gamma_M/\Gamma_{OM}| = 4$. For the reasons $\Gamma_M \cong S_4$ and $\Gamma_M \curvearrowright \Gamma_M/\Gamma_{OM} \cong \Gamma_M.O$ transitively we infer that $\psi_M^O = \psi_{|\Gamma_M \to \mathrm{Sym}(\Gamma_M.O)}$ has to be an isomorphism. Straightforward matrix computations lead to $\Gamma_M.O = \{O, O_2, O_3, O_4\}$ as well as

$$\psi_M^O \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (O_2\ O_4),\ \psi_M^O \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} = (O\ O_2\ O_3\ O_4).$$

This and $\Gamma_M = \left\langle \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right\rangle$ yield the claimed properties for that action. The remaining assertions can be attained in the same manner. □

2.3 The lattice of conjugacy classes of finite subgroups of $\mathrm{Sl}_3(\mathbb{Z})$

Using Theorem 2.2.1 and Theorem 2.3.2 we are able to extract the complete lattice structure of the set of Γ-conjugacy classes of finite subgroups of Γ:

Definition 2.3.1. Let G be an arbitrary group. We denote by $\mathrm{Sub}_{\mathrm{fin}}(\Gamma)$ the set of all finite subgroups of G. Let $\mathcal{A} \leq \mathrm{Aut}(G)$ be an arbitrary subgroup. $\mathcal{A}$ acts on $\mathrm{Sub}(G)$ via $\mu\colon \mathcal{A} \to \mathrm{Sym}(\mathrm{Sub}(G))\colon \alpha \mapsto \mu_\alpha$ and $\mu_\alpha(H) := \alpha(H) \in \mathrm{Sub}(G)$. For the orbit of an element of $H \in \mathrm{Sub}(G)$ we write $[H]_{\mathcal{A}}$. If two elements $H, K \in \mathrm{Sub}(G)$ lie in the same orbit set, we write $H \underset{\mathcal{A}}{\sim} K$.

Obviously, this group action induces a group action on $\mathrm{Sub}_{\mathrm{fin}}(G)$ because every automorphism maps finite groups on finite groups.

On the set of orbits $\mathcal{A}\backslash \mathrm{Sub}(G)$ we define a partial order as follows: For any $[H]_{\mathcal{A}}, [K]_{\mathcal{A}} \in \mathcal{A}\backslash \mathrm{Sub}(G)$ we write

$$[H]_{\mathcal{A}} \leq [K]_{\mathcal{A}} \quad \text{if and only if} \quad \exists \alpha \in \mathcal{A} : H \leq \alpha(K).$$

In particular, we can apply these definitions on the action of G on $\mathrm{Sub}(G)$ via conjugation from the left, by making the choice $\mathcal{A}_G := \{\alpha_g : \alpha_g(H) := gHg^{-1}\}$. We write $G\backslash \mathrm{Sub}(G)$ for the set of orbits of $\mathcal{A}_G$. Adequately, for $H, K \in \mathrm{Sub}(G)$ we set $[H]_G := [H]_{\mathcal{A}_G}$ and we write $H \underset{G}{\sim} K$ if and only if $H \underset{\mathcal{A}_G}{\sim} K$. $[H]_G$ is called the *G-conjugacy class* of H or just the *conjugacy class* of H. If $H \underset{G}{\sim} K$, we say H, K are *G-conjugate*. For an arbitrary $g \in G$ we make the convention ${}^g h := ghg^{-1}$ for every $h \in G$.

Ken-Ichi Tahara classified in [9] all the Γ-conjugacy classes of finite subgroups in Γ. We will only need a reduced version of his main-theorem which we state below:

Theorem 2.3.2 ([9], pp. 170-203)**.** *The following table shows the complete list of the isomorphy classes and conjugacy classes of finite subgroups of* Γ*:*

cardinality of the finite subgroups of Γ	*isomorphy classes appearing*	*number of* Γ*-conjugacy classes appearing*
2	C_2	2
3	C_3	2
4	C_4 V_4	2 4
6	C_6 S_3	1 3
8	D_8	2
12	D_{12} A_4	1 3
24	S_4	3

In particular $\Gamma\backslash\operatorname{Sub}_{\text{fin}}(\Gamma)$ *is finite.*

Let us state some trivial facts because we will often make use of them:

Lemma 2.3.3. *Let* G *be an arbitrary group and* $\Phi\colon G \to G$ *an endomorphism on it. Furthermore let* $H, K \leq G$ *such that* $[H]_G \leq [K]_G$*. Then the assertion* $[\Phi(H)]_G \leq [\Phi(K)]_G$ *is valid.*

Proof. By hypothesis there has to exist a $g \in G$ such that ${}^gH \leq K$. Applying Φ we calculate

$$ {}^g\Phi(H) = \Phi({}^gH) \leq \Phi(K). $$

Thus we conclude $[\Phi(H)]_G \leq [\Phi(K)]_G$. □

Lemma 2.3.4. *Let* G *be an arbitrary group and* $H \leq G$ *an arbitrary subgroup. Then the equalities*

$$ \alpha(C_G(H)) = C_G(\alpha(H)) \text{ and } \alpha(N_G(H)) = N_G(\alpha(H)) $$

are true for every $\alpha \in \operatorname{Aut}(G)$.

Proof. We will only show $\alpha(N_G(H)) = N_G(\alpha(H))$ as the proof of the other assertion can be obtained in the same fashion. To this end, it is sufficient to prove

$$ \alpha(N_G(H)) \leq N_G(\alpha(H)) \quad \forall \alpha \in \operatorname{Aut}(G). $$

So let us fix an $\alpha \in \operatorname{Aut}(G)$ and take an arbitrary $g \in \alpha(N_G(H))$. Then by definition there exists a $g' \in N_\Gamma(H)$ such that $g = \alpha(g')$. This implies $g^{-1} = \alpha(g'^{-1})$ and $g'Hg'^{-1} = H$. Therefore we obtain

$$ g\alpha(H)g^{-1} = \alpha(g'Hg'^{-1}) = \alpha(H), $$

which establishes $g \in N_G(\alpha(H))$ and hence the claim. □

The following lemma is an immediate corollary if we replace $\alpha \in \operatorname{Aut}(G)$ with an inner automorphism.

Lemma 2.3.5. *Let* G *be an arbitrary group and* $H, K \leq G$ *be arbitrary subgroups. Then, if there exists a* $g \in G$ *such that* $gHg^{-1} = K$ *then it is also true that*

$$ gC_G(H)g^{-1} = C_G(K) \text{ and } gN_G(H)g^{-1} = N_G(K). $$

Lemma 2.3.6. *Let* G *be an arbitrary group and* H *be a finite subgroup. Then* $N_G(H)$ *is finite if and only if* $C_G(H)$ *is finite.*

PROOF. "⇒" is trivial because of $C_G(H) \subseteq N_G(H)$.

"⇐" For this direction let us assume $C_G(H)$ is finite and let us look on the following homorphism:

$$\varphi\colon N_G(H) \to \mathrm{Aut}(H)\colon g \mapsto \varphi_g$$

where $\varphi_g(h) := ghg^{-1}$. We observe that $\ker \varphi = C_G(H)$. Because H is finite $\mathrm{Aut}(H)$ has to be finite too and hence $N_G(H)/C_G(H)$ is a finite group. Using $C_G(H)$ is finite we obtain $N_G(H)$ is finite and therefore the claim. □

Lemma 2.3.7. *Let G be an arbitrary group and $H, K \leq G$ such that $H \leq K$ then $C_G(H) \supseteq C_G(K)$.*

PROOF. Let $H \leq K \leq G$. Take an arbitrary $g \in C_G(K)$. Then we have for all $k \in K$

$$gk = kg.$$

Because of $H \subseteq K$ this equation remains in particular true for all $k \in H$. We conclude $g \in C_G(H)$. □

Combining Lemma 2.3.6 and Lemma 2.3.7 we obtain the following corollary:

Corollary 2.3.8. *Let G be an arbitrary group and $H \leq K \in \mathrm{Sub}_{\mathrm{fin}}(G)$. Then, if $C_G(H)$ or $N_G(H)$ is finite, $N_G(K)$ has to be finite.*

We want to define the sign homomorphism on an arbitrary Coxeter group. Therefore we start with the following lemma.

Lemma 2.3.9. *Let G be an arbitrary finitely generated group with*

$$G/[G,G] \cong C_{2^k} \oplus A,$$

where $k \in \mathbb{N}$ and A an abelian group, such that A contains no element of order 2. Then,

$$|\mathrm{Hom}(G, C_2)| \in \{1, 2\}.$$

PROOF. Because C_2 is abelian we clearly have that

$$\begin{aligned}\mathrm{Hom}(G, C_2) &\hookrightarrow \mathrm{Hom}(G/[G,G], C_2)\\ &\cong \mathrm{Hom}(C_{2^k} \oplus A, C_2) = \mathrm{Hom}(C_{2^k}, C_2) \oplus \mathrm{Hom}(A, C_2).\end{aligned}$$

The homomorphy theorem tells us that $\mathrm{Hom}(A, C_2)$ is trivial. On the other hand each $\Phi \in \mathrm{Hom}(C_{2^k}, C_2)$ is determined by the value on the generator of C_{2^k}. Therefore we conclude $|\mathrm{Hom}(G, C_2)| \leq |\mathrm{Hom}(C_{2^k}, C_2)| = 2$. □

Definition 2.3.10. Let W be a Coxeter group with Coxeter system S. For $w \in W$ let $l_S(w)$ be the length of w with respect to S. We define $\mathrm{sgn}_S\colon W \to W$ via $\mathrm{sgn}_S(w) := (-1)^{(l_S(w) \bmod 2)}$. Then sgn_S is a homomorphism. Furthermore if there exists a $k \in \mathbb{N}$ such that

$$W/[W,W] \cong C_{2^k} \oplus A,$$

where $k \in \mathbb{N}$ and A is an abelian group, such that A contains no element of order 2, then sgn_S is independent of the choice of the Coxeter system S. In this case we write for sgn_S just sgn_W or sgn and call this homomorphism the *sign* homomorphism of W. Then, sgn_W is in particular $\mathrm{Aut}(W)$-invariant.

PROOF. We start with the first assertion. Let $w \in W$ and $s \in S$. From [2] p.47 we can immediately derive the following formulas:

$$l_S(sw) = l_S(w) + 1 \text{ or } l_S(sw) = l_S(w) - 1$$

and

$$l_S(ws) = l_S(w) + 1 \text{ or } l_S(ws) = l_S(w) - 1.$$

In particular, we get for $w_1, w_2 \in W$

$$l_S(w_1 w_2) \overset{(2)}{\equiv} l_S(w_1) + l_S(w_2)$$

and thus

$$l_S^*\colon W \to \mathbb{F}_2\colon w \mapsto l_S(w) \bmod 2$$

is a homomorphism. Hence sgn_S needs to be a homomorphism as well. The second assertion is an immediate consequence from Lemma 2.3.9. □

Remark 4. Definition 2.3.10 generalizes the notion of the sign of finite symmetric groups to all Coxeter groups. In particular for a group $G \cong S_n$ the sign homomorphism is independent of the choice of the isomorphism.

Lemma 2.3.11. *Let G be a group such that $G \cong S_4$. If $G = \langle \sigma, \tau \rangle$ with $\mathrm{ord}(\sigma) = 4$ and $\mathrm{ord}(\tau) = 2$. Then, we necessarily have $\mathrm{sgn}_G(\sigma) = -1 = \mathrm{sgn}_G(\tau)$.*

PROOF. Let us begin to show $\mathrm{sgn}_G(\sigma) = -1$. $[G, G] \cong A_4 \leq S_4$ implies $\langle \sigma, [G, G] \rangle = G$. Assume $\mathrm{sgn}_G(\sigma) = 1$. Then,

$$\mathrm{sgn}_G(G) = \langle \mathrm{sgn}_G(\sigma), \mathrm{sgn}_G([G, G]) \rangle = \{1\},$$

a contradiction to the definition of the sign!

Assume $\mathrm{sgn}_G(\tau) = 1$. Then, there exists an $H \cong V_4$ such that $\tau \in H \trianglelefteq G$. On the one hand, Sylows' theorems imply that H lies in all subgroups of G of type D_8. On the other hand Sylow's theorems tell us that there exists a group K of type D_8 such that $\langle \sigma \rangle \leq K$. In particular this enforces $G = \langle \sigma, \tau \rangle \leq K \cong D_8$, a contradiction! □

Lemma 2.3.12. *Let G be an arbitrary group and H be a subgroup of G. Further assume that there exists $S, S' \subseteq H$ with $\langle S \rangle = H = \langle S' \rangle$ such that there is an $\alpha \in \mathrm{Aut}(G)$ with $\alpha(S) = S'$. Then even $\alpha(H) = H$ is true.*

PROOF. We just compute $\alpha(H) = \langle \alpha(S) \rangle = \langle S' \rangle = H$ which has been claimed. □

Corollary 2.3.13. *Let $G \cong D_{12}$ and H be an arbitrary 2-Sylow subgroup of G. Furthermore let $\tau_1 \neq \tau_2 \in H \setminus \mathcal{Z}(G)$. Then $[\tau_1]_G \neq [\tau_2]_G$.*

PROOF. Because $|\mathcal{Z}(G)| = |\mathcal{Z}(D_{12})| = 2$, there exists a 2-Sylow subgroup which contains $\mathcal{Z}(G)$. For the reasons that $\mathcal{Z}(G) \trianglelefteq G$ and all 2-Sylow subgroups are conjugate, $\mathcal{Z}(G)$ lies in every 2-Sylow subgroup and hence $\mathcal{Z}(G) \leq H$. Let $\sigma^\bullet$ be the only non trivial element of $\mathcal{Z}(G)$. Set $S := \{\sigma^\bullet, \tau_1\}$ and $S' := \{\sigma^\bullet, \tau_2\}$ and observe that $\langle S \rangle = H = \langle S' \rangle$. Assume in order to obtain a contradiction that there exists a $g \in G$ such that ${}^g\tau_1 = \tau_2$. Then, we infer

$${}^gS = \{{}^g\sigma^\bullet, {}^g\tau_1\} = \{\sigma^\bullet, \tau_2\} = S'.$$

Now, Lemma 2.3.12 tells us that ${}^gH = H$ and therefore $g \in N_G(H) = H$. Because $H \cong V_4$ is abelian, this implies $\tau_1 = \tau_2$, a contradiction! □

Lemma 2.3.14. *Let G be an arbitrary group acting on a set X. Let $x_0 \in X$ an arbitrary element and set $H := G_{x_0}$. Then, $N_G(H)$ is the largest subgroup of G which leaves the set $\{x \in X : G_x = H\}$ invariant. In particular, if $|\{x \in X : G_x = H\}| = 1$, then $N_G(H) = H$.*

PROOF. Set $M := \{x \in X : G_x = H\}$. Let $x \in M$ and $g \in N_G(H)$. Then

$$G_{g.x} = {}^gH = H$$

and hence $g.x \in M$. Thus, $N_G(H)$ leaves M invariant. Let G_M be the largest subgroup of G leaving M invariant. Take $g \in G_M$ and $x \in M$. We clearly obtain

$g.x \in M$ and therefore ${}^gH = G_{g.x} = H$. This finally implies $g \in N_G(H)$ and thus $N_G(H) = G_M$, as required. □

Definition 2.3.15. Let "$\preceq$" be a partial order on a set M. Let $a, b \in M$ such that $a \preceq b$. We say an inclusion $a \preceq b$ is *simple* if and only if for each $c \in M$ with $a \preceq c \preceq b$ either $c = a$ or $c = b$ is true. We call a pair $(a, b) \in M \times M$ *simple* if and only if $a \preceq b$ and $a \preceq b$ is simple or $b \preceq a$ and $b \preceq a$ is simple.

Definition 2.3.16. For an arbitrary group G we set

$$\mathrm{Sub}^{\circ}_{\mathrm{fin}}(G) := \{H \in \mathrm{Sub}(G) : N_G(H) \in \mathrm{Sub}_{\mathrm{fin}}(G)\}.$$

The action of G on $\mathrm{Sub}(G)$ via conjugation induces an action of G on $\mathrm{Sub}^{\circ}_{\mathrm{fin}}(G)$ because Lemma 2.3.5 tells us that normalizers get mapped on normalizers. $\mathrm{Sub}^{\circ}_{\mathrm{fin}}(G)$ has the following property: If $H \in \mathrm{Sub}^{\circ}_{\mathrm{fin}}(G)$ then for each $\mathrm{Sub}_{\mathrm{fin}}(G) \ni K \geq H$ holds $K \in \mathrm{Sub}^{\circ}_{\mathrm{fin}}(G)$.

PROOF. The property is a reformulation of Corollary 2.3.8. □

Notation 2.3.17. Let G and $\tilde{G}$ arbitrary groups and $\varphi \colon \tilde{G} \to G$ an homomorphism. Then for each $g \in G$ and $x \in \tilde{G}$ we write $({}^g\varphi)(x) := {}^g(\varphi(x))$. This yields a map ${}^g\varphi \colon \tilde{G} \to G$ which is a homomorphism as well. If φ is an mono-/epi-/isomorphism then also ${}^g\varphi$ has to be an mono-/epi-/isomorphism.

We have to introduce some further notation now.

Notation 2.3.18. Let K be a field, $A \in K^{n \times n}$ and λ be an Eigenvalue for A. Let L/K be an arbitrary field extension of K containg λ. Then we denote by $E^L_\lambda(A) := \{v \in L^n : Av = \lambda v\}$ the Eigenspace of A over L according to the Eigenvalue λ.

Notation 2.3.19. Let us make the following convention: Let H be an arbitrary group (not necessary contained in Γ) and $G \leq \Gamma$. Then we set

$${}^GH := \{[K]_G \in G\backslash \mathrm{Sub}(G) : K \cong H\}.$$

If this set consists only of one conjugacy class c we write by abuse of notation ${}^GH$ for an arbitrary representative of c.

Using the notation of Theorem 2.2.1 we make the following settings

$$S_4^1 := \Gamma_O, \quad S_4^2 := \Gamma_M, \quad S_4^3 := \Gamma_P, \quad D_{12} := \Gamma_Q.$$

We think it is worth to remark that S_4^1 is by definition the group of orthogonal matrices with determinant 1 and integer entries.

With the convention above let ${}^{S_4^i}A_4$, $i \in \{1, \ldots, 3\}$ be the A_4-type subgroups of S_4^i, and ${}^{S_4^i}D_8$ be arbitrary 2-Sylow subgroups of S_4^i. We write ${}^{D_{12}}C_6$ for the unique cyclic subgroup of order 6 in D_{12} and ${}^{D_{12}}S_3^1$ and ${}^{D_{12}}S_3^2$ for the only two type S_3 subgroups of D_{12}, for which we know, that they are not conjugate in D_{12}. We set ${}^{S_4^i}S_3$ for an arbitrary type S_3 subgroup of S_4^i, taking into account that all type S_3 subgroups are conjugate in groups of type S_4. Furthermore, we write ${}^{S_4^i}V_4^{\bullet}$ for the type V_4 subgroups being normal in S_4^i and ${}^{S_4^i}V_4^{\circ}$ for that being not normal in S_4^i. Moreover we denote by ${}^{D_{12}}V_4$ an arbitrary 2-Sylow subgroup of D_{12}. Let us write ${}^{S_4^i}C_4$ for an arbitrary type C_4-subgroup of S_4^i regarding the fact that all subgroups of S_4 of type C_4 are conjugate. For the only 3-Sylow subgroup of D_{12} we write ${}^{D_{12}}C_3$ and for an arbitrary 3-Sylow subgroup of S_4^i we write ${}^{S_4^i}C_3$. Finally, let us write ${}^{S_4^i}C_2$ for the set of type C_2 conjugacy classes in S_4^i which contains exactly 2 elements; an arbitrary representative of the S_4^i-conjugacy class of subgroups of order 2 with positive sign with respect to S_4^i is denoted by ${}^{S_4^i}C_2^\bullet$ and an arbitrary representative of the S_4^i-conjugacy class of subgroups of order 2 with negative sign with respect to S_4^i is denoted by ${}^{S_4^i}C_2^\circ$. Finally let us write ${}^{D_{12}}C_2$ for the set of type C_2 conjugacy classes in D_{12} which consists of exactly 3 elements.

Lemma 2.3.20. *There exists an automorphism $\Phi\colon \Gamma \to \Gamma$ such that*

$$\Phi(S_4^1) = S_4^1 \text{ and } \Phi(S_4^2) = S_4^3.$$

PROOF. Let $\varphi\colon \Gamma \to \Gamma\colon \gamma \mapsto ({}^t\gamma)^{-1}$ be the transposition-inversion automorphism. Put

$$\sigma_1 := \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix},\ \sigma_2 := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\tau_1 := \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix},\ \tau_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Theorem 2.2.1 tells us that $S_4^2 = \langle \sigma_1, \sigma_2 \rangle$ and $S_4^3 = \langle \tau_1, \tau_2 \rangle$. Let us put

$$\eta := \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in S_4^1$$

and $\Phi := {}^\eta\varphi$. For the reason S_4^1 consists only of orthogonal matrices φ restricts to the identity on S_4^1. Thus $\eta \in S_4^1$ enforces $\Phi(S_4^1) = S_4^1$. We can easily verify that

$$\Phi(\sigma_1) = \tau_1 \in S_4^3 \text{ and } \Phi(\sigma_2) = \tau_2 \in S_4^3.$$

Therefore we get $\Phi(S_4^2) = S_4^3$ which completes the proof. □

Lemma 2.3.21. *Let $\Gamma \geq H \cong V_4$. Then $C_\Gamma(H) = H$. In particular $N_\Gamma(H)$ is finite.*

PROOF. By virtue of Corollary 2.3.8 it is sufficient to show that $C_\Gamma(H) = H$. To do this let $h_1, h_2, h_3 \in H$ be its elements of order 2. By $\mathrm{ord}(h_i) = 2$ we conclude

$$\mathrm{p}_{\min}(h_i)(X) = X^2 - 1 = (X-1)(X+1)$$

and each h_i has to be diagonalizable over $\mathbb{Q}$.
$\det(h_i) = 1$ implies $\dim E_1^{\mathbb{Q}}(h_i) = 1$ and $\dim E_{-1}^{\mathbb{Q}}(h_i) = 2$. For this reason and by virtue of H is abelian, there exists a simultaneous eigenbasis $(b_j)_{j=1}^3$ for H over $\mathbb{Q}$. Now, $\dim E_1^{\mathbb{Q}}(h_i) = 1$ implies that there exists a j_i such that $h_i b_{j_i} = b_{j_i}$ and even further $E_1^{\mathbb{Q}}(h_i) = \mathbb{Q}b_{j_i}$. On the other hand, the fact that $(b_j)_{j=1}^3$ is an eigenbasis for h_i enforces $h_1 b_j = -b_j$ for every $j \neq j_i$ for otherwise we would have $\dim E_1^{\mathbb{Q}}(h_i) > 1$, contrary to $\dim E_1^{\mathbb{Q}}(h_i) = 1$. We conclude $E_{-1}^{\mathbb{Q}}(h_i) = \bigoplus_{j \neq j_i} \mathbb{Q}b_j$. From this and the fact that the h_i share the same eigenvalues, we may deduce that $h_i \neq h_k$ already forces $E_1^{\mathbb{Q}}(h_i) \neq E_1^{\mathbb{Q}}(h_k)$. Hence the map

$$\{1, \ldots, 3\} \to \{\mathbb{Q}b_1, \mathbb{Q}b_2, \mathbb{Q}b_3\}\colon i \mapsto E_1^{\mathbb{Q}}(h_i)$$

is necessarily a bijection. It follows that

$$\mathbb{Q}^3 = E_1^{\mathbb{Q}}(h_1) \oplus E_1^{\mathbb{Q}}(h_2) \oplus E_1^{\mathbb{Q}}(h_3).$$

Setting $V_i := E_1^{\mathbb{Q}}(h_i)$ this becomes

$$\mathbb{Q}^3 = V_1 \oplus V_2 \oplus V_3$$

where $\dim(V_1) = \dim(V_2) = \dim(V_3) = 1$. Now $\dim(V_i) = 1$ forces the existence of a $v_i \in \mathbb{Z}^3$ such that $V_i \cap \mathbb{Z}^3 = \mathbb{Z}v_i$.

Let us take an arbitrary $\gamma \in C_\Gamma(H)$. Then γ commutes with all the elements in H. Thus γ must leave every h_i-eigenspace and therefore the spaces V_i for $i \in \{1, \dots, 3\}$ invariant. On the other hand $\gamma \in \Gamma = \mathrm{Sl}_3(\mathbb{Z})$ implies that γ leaves also $\mathbb{Z}^3$ invariant. Hence γ leaves the $\mathbb{Z}$-modules $V_i \cap \mathbb{Z}^3$ invariant. We conclude that there exists a $\lambda_i \in \mathbb{Z}$ such that

$$\mathbb{Z}^3 \ni \gamma v_i = \lambda_i v_i \tag{2.3.1}$$

for suitable λ_i. For the reason γ is invertible we conclude that there exists a $\mu_i \in \mathbb{Z}$ with the property

$$\mathbb{Z}^3 \ni \gamma^{-1} v_i = \mu_i v_i.$$

Observing $v_i = \gamma^{-1}\gamma v_i = \mu_i \lambda_i v_i$ we obtain $\mu_i = \lambda_i^{-1}$ and therefore in particular $\lambda_i \in \mathbb{Z}^\times$. By virtue of (2.3.1) we get $1 = \det(\gamma) = \lambda_1\lambda_2\lambda_3$ which has at most four possible solutions in $(\mathbb{Z}^\times)^3$. This establishes the claim of the lemma.

□

Lemma 2.3.22. *Let* $\alpha \in \Gamma$ *such that* $\mathrm{ord}(\alpha) = 3$. *Then* $C_\Gamma(\alpha)$ *is finite. In particular* $N_\Gamma(\langle\alpha\rangle)$ *is finite.*

Proof. Let $(\nu_i)_{i=1}^3$ be the eigenvalues for α. We begin by proving

$$\mathrm{p}_{\min}(\alpha)(X) = X^3 - 1 :$$

To this end, we observe that $\deg \mathrm{p}_{\min}(\alpha) \geq 2$ otherwise α would be a multiple of the identity. From $\mathrm{ord}(\alpha) = 3$ we derive

$$\mathrm{p}_{\min}(\alpha) \mid X^3 - 1 = (X-1)(X-\varepsilon_3)(X-\varepsilon_3^2),$$

where $\varepsilon_3 := \exp(2\pi i/3)$. Combining this with $\deg \mathrm{p}_{\min}(\alpha) \geq 2$ we infer ε_3 or $\varepsilon_3^2 = \overline{\varepsilon_3}$ has to be a zero of $\mathrm{p}_{\min}(\alpha)$. By virtue of $1 = \det(\alpha) = \nu_1\nu_2\nu_3$ this leads in each case to the observation that both, ε_3 and ε_3^2, have to be zeroes of $\mathrm{p}_{\min}(\alpha)$. So we may assume without loss of generality $\nu_2 = \varepsilon_3$ and $\nu_3 = \varepsilon_3^2$. Applying again $1 = \nu_1\nu_2\nu_3$ leads to $\nu_1 = 1$. This establishes $\mathrm{p}_{\min}(\alpha)(X) = X^3 - 1$, which has been our introductory statement. Clearly $\mathrm{p}_{\min}(\alpha)$ decomposes into linear factors over the cyclotomic field $\mathbb{Q}(\varepsilon_3)$. Setting $\nu_i := \varepsilon_3^i,\ i \in \{0, \dots, 2\}$ we thus obtain the following decomposition of $\mathbb{Q}(\varepsilon_3)^3$ into eigenspaces for α:

$$\mathbb{Q}(\varepsilon_3)^3 = \bigoplus_{i=0}^{2} E_{\nu_i}^{\mathbb{Q}(\varepsilon_3)}(\alpha). \tag{2.3.2}$$

In particular we obtain

$$\dim_{\mathbb{Q}(\varepsilon_3)}\left(E_{\nu_i}^{\mathbb{Q}(\varepsilon_3)}(\alpha)\right) = 1 \ \forall i \in \{0, \dots, 2\}.$$

Algebraic number theory tells us that the integer ring for $\mathbb{Q}(\varepsilon_3)$ is $\mathbb{Z}(\varepsilon_3)$. Moreover it is well known that $\mathbb{Z}[\varepsilon_3] = \mathbb{Z}\left[\frac{1+\sqrt{3}i}{2}\right]$ is an Euclidean ring. For the reason that $\mathbb{Z}[\varepsilon_3]$ is an Euclidean ring and because

$$V_i := E_{\nu_i}^{\mathbb{Q}(\varepsilon_3)}(\alpha) \cap \mathbb{Z}[\varepsilon_3]^3$$

is a submodule of the free module $\mathbb{Z}[\varepsilon_3]^3$, we conclude that each $\mathbb{Z}[\varepsilon_3]$-module V_i has to be a free $\mathbb{Z}[\varepsilon_3]$-module of rank 1. In this way for each $i \in \{0, \dots, 2\}$ there exists a $0 \neq v_i \in V_i$ such that

$$V_i = \mathbb{Z}[\varepsilon_3] v_i. \tag{2.3.3}$$

Now, take an arbitrary $\gamma \in C_\Gamma(\alpha)$. Because γ commutes with α, γ leaves all eigenspaces $E_{\nu_i}^{\mathbb{Q}(\varepsilon_3)}(\alpha)$ invariant. Additionally, $\gamma \in \mathrm{Sl}_3(\mathbb{Z})$ forces that γ must leave

$\mathbb{Z}[\varepsilon_3]^3$ invariant. We combine these facts to the observation that γ leaves V_i invariant. (2.3.3) now implies the existence of $\lambda_i, \mu_i \in \mathbb{Z}[\varepsilon]$ for each $i \in \{0, \ldots, 2\}$ such that

$$\gamma v_i = \lambda_i v_i. \tag{2.3.4}$$

and

$$\gamma^{-1} v_i = \mu_i v_i.$$

By virtue of $v_i = \gamma^{-1}\gamma v_i = \mu_i \lambda_i v_i$, we finally obtain $\lambda_i \in \mathbb{Z}[\varepsilon_3]^\times$. For the reasons that all embeddings of $\mathbb{Q}(\varepsilon_3)$ into $\mathbb{C}$ are complex and $[\mathbb{Q}(\varepsilon_3) : \mathbb{Q}] = 2$, Dirichlet's Unit theorem forces that $\mathbb{Z}[\varepsilon_3]^\times$ is finite. In particular there are only finitely many choices for the λ_i and hence for γ. This proves the assertion of the lemma.

□

Lemma 2.3.23. *Let $\alpha \in \Gamma$ such that $\mathrm{ord}(\alpha) = 4$. Then $C_\Gamma(\alpha)$ is finite. In particular $N_\Gamma(\langle\alpha\rangle)$ is finite.*

Proof. This follows almost by the same argument we have used in the proof of Lemma 2.3.22. First, we observe $\mathrm{p}_{\min}(\alpha) \mid X^4 - 1$. As in Lemma 2.3.22 we conclude by $\det(\alpha) = 1$ and $\mathrm{tr}(\alpha) \in \mathbb{Q}$ that

$$\mathrm{p}_{\min}(\alpha) = (X-1)(X-i)(X+i) = (X-1)(X^2+1).$$

Therefore the splitting field of $\mathrm{p}_{\min}(\alpha)$ is necessarily $\mathbb{Q}(i)$. Note that its associated ring of integers $\mathbb{Z}[i]$ is Euclidean. Thus, the same method as in the proof Lemma 2.3.22 yields that $C_\Gamma(\alpha)$ has to be finite. □

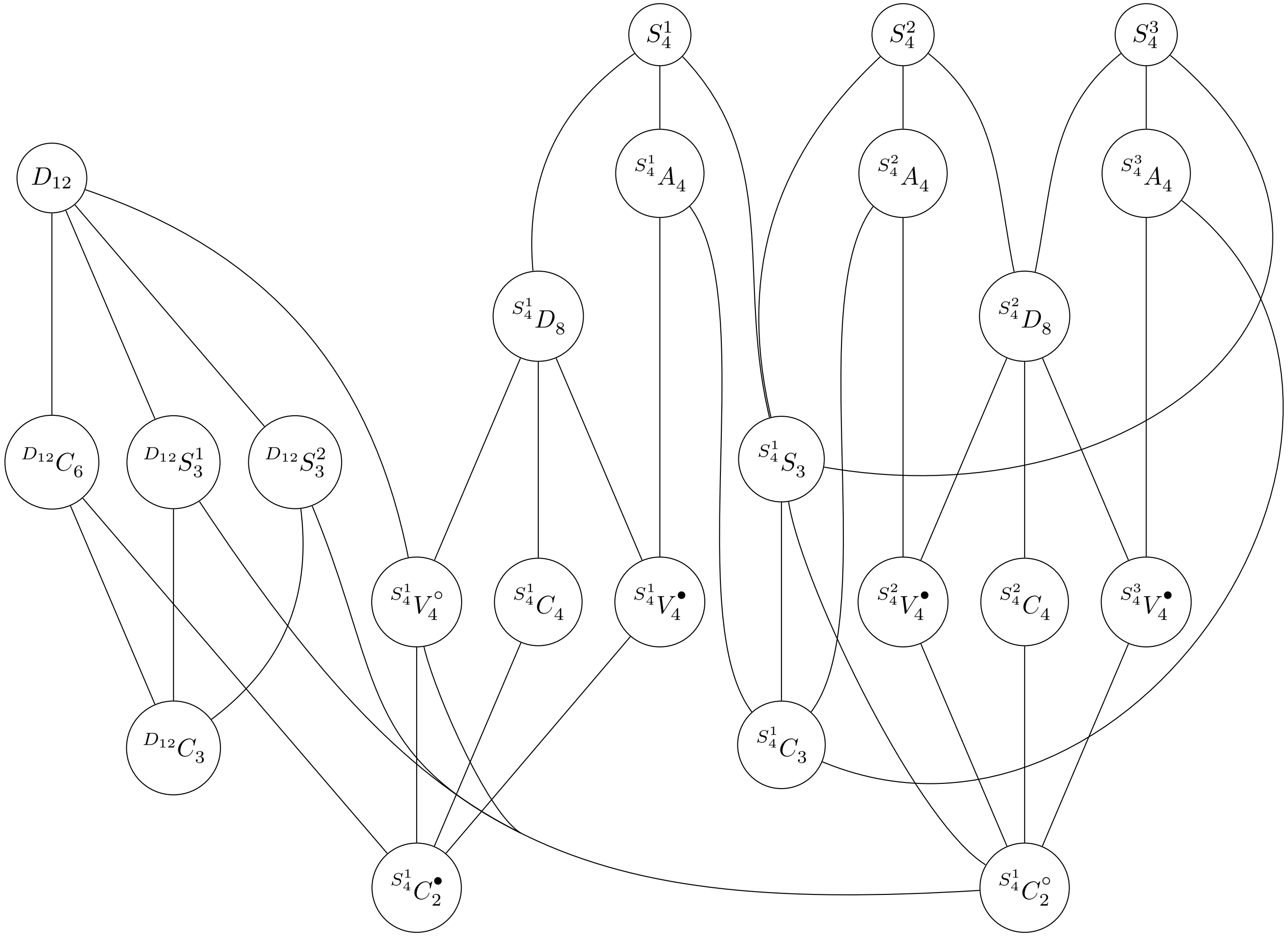
Definition 2.3.24. We call a representative H of a Γ-conjugacy class $c \in \Gamma \backslash \mathrm{Sub}_{\mathrm{fin}}(\Gamma)$ *admissible* if it satisfies $H \leq S_4^i$ for some $i \in \{1, \ldots, 3\}$ or $H \leq D_{12}$.

Theorem 2.3.25. *The lattice of Γ-orbit sets on $\mathrm{Sub}_{\mathrm{fin}}(\Gamma)$ is completely described by the following diagram: The nodes are the representatives of $\Gamma \backslash \mathrm{Sub}_{\mathrm{fin}}(\Gamma)$. Two nodes are on the same level if their labels have the same cardinality. We put a connection line between two nodes if they lie on different levels and if $([A]_\Gamma, [B]_\Gamma)$ are simple with respect to "$\leq$" for their labels $A, B \in \mathrm{Sub}_{\mathrm{fin}}(\Gamma)$. Furthermore we have*

$$\Gamma \backslash \mathrm{Sub}^\circ_{\mathrm{fin}}(\Gamma) = \{[H]_\Gamma \in \Gamma \backslash \mathrm{Sub}_{\mathrm{fin}}(\Gamma) : |H| \geq 3\}.$$

We give a list of the normalizers of the representatives of $\Gamma \backslash \mathrm{Sub}^\circ_{\mathrm{fin}}(\Gamma)$ which are not obviously obtained by this fact:

- $N_\Gamma\left({}^{S_4^1}S_3\right) = {}^{S_4^1}S_3$;
- $N_\Gamma\left({}^{S_4^i}C_4\right) = {}^{S_4^i}D_8, \ i \in \{1,2\}$;
- $N_\Gamma\left({}^{S_4^1}V_4^\circ\right) = {}^{S_4^1}D_8$;
- $N_\Gamma\left({}^{S_4^1}C_3\right) = {}^{S_4^1}S_3$.


S_4^1
S_4^2
S_4^3
D_{12}
${}^{S_4^1}A_4$
${}^{S_4^2}A_4$
${}^{S_4^3}A_4$
${}^{S_4^1}D_8$
${}^{S_4^2}D_8$
${}^{D_{12}}C_6$
${}^{D_{12}}S_3^1$
${}^{D_{12}}S_3^2$
${}^{S_4^1}S_3$
${}^{S_4^1}V_4^\circ$
${}^{S_4^1}C_4$
${}^{S_4^1}V_4^\bullet$
${}^{S_4^2}V_4^\bullet$
${}^{S_4^2}C_4$
${}^{S_4^3}V_4^\bullet$
${}^{D_{12}}C_3$
${}^{S_4^1}C_3$
${}^{S_4^1}C_2^\bullet$
${}^{S_4^1}C_2^\circ$

PROOF. For his convenience, we advise the reader to print out the diagram above. We start our proof by determining admissible representatives for Γ-conjugacy classes of subgroups of order 2:

Let G be arbitrary with $G \cong S_4$. The elements of order 2 in S_4 and therefore of G fall into two conjugacy classes. Because sgn_G is invariant under G-conjugation, the G-conjugacy classes of elements of order 2 are exactly the preimages of the possible values of sgn_G restricted to the set of elements in G having order 2.

Let S_4^1 play the role of G.
Set $\sigma^\bullet := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which can be expressed as square of $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma_O = S_4^1$ and which has therefore positive sign with respect to S_4^1.
Furthermore we define $\sigma^\circ := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. From Theorem 2.2.1 we know that σ° fixes the vertices which have been called O, M and P in that context and hence $\sigma^\circ \in S_4^1 \cap S_4^2 \cap S_4^3$. From Theorem 2.3.2 we also know that there are only two Γ-conjugacy classes of subgroups of type C_2. Assume there exists a $\begin{pmatrix} q & r & s \\ t & u & v \\ x & y & z \end{pmatrix} = \gamma \in \Gamma$ such that ${}^\gamma(\sigma^\bullet) = \sigma^\circ$. Then γ has to solve

$$\begin{pmatrix} q & r & s \\ t & u & v \\ x & y & z \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} q & r & s \\ t & u & v \\ x & y & z \end{pmatrix}.$$

and therefore $q = t$, $r = u$, $s = -v$, $z = 0$. Thus $\gamma = \begin{pmatrix} q & r & s \\ q & r & -s \\ x & y & 0 \end{pmatrix}$. Taking the determinant leads to

$$1 = \det(\gamma) = 2(qsy - rsx),$$

and therefore $2 \in \mathbb{Z}^\times$, a contradiction! We conclude

$$\{[H]_\Gamma : H \cong C_2\} = \{[\langle \sigma^\bullet \rangle]_\Gamma, [\langle \sigma^\circ \rangle]_\Gamma\}. \tag{2.3.5}$$

Due to $\sigma^\bullet, \sigma^\circ \in S_4^1$ this implies $\mathrm{sgn}_{S_4^1}(\sigma^\circ) = -1$.
We proceed to figure out the substructure $\Gamma \backslash \mathrm{Sub}_{\mathrm{fin}}(\Gamma)$ of Γ-conjugacy classes of 2-groups.

From Theorem 2.2.1 we know that

$$S_4^2 = \left\langle \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \sigma^\circ \right\rangle$$

and

$$S_4^3 = \left\langle \alpha_3 := \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \beta_3 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle.$$

Observing $\operatorname{ord}\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 4$ and applying Lemma 2.3.11, we infer

$$\operatorname{sgn}_{S_4^2}(\sigma^\circ) = -1. \tag{2.3.6}$$

Let $\Phi\colon \Gamma \to \Gamma$ be an automorphism such that $\Phi(S_4^1) = S_4^1$ and $\Phi(S_4^2) = S_4^3$. The existence of this automorphism is guaranteed by Lemma 2.3.20. Therefore and by Definition 2.3.10 we obtain

$$\operatorname{sgn}_{S_4^3}(\sigma^\circ) = \operatorname{sgn}_{\Phi(S_4^2)}(\Phi(\sigma^\circ)) = \operatorname{sgn}_{S_4^2}(\sigma^\circ) = -1. \tag{2.3.7}$$

Let us observe that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \sigma^\circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} = \alpha_3^2 \in S_4^3.$$

By virtue of $\operatorname{sgn}_{S_4^3} \alpha_3^2 = 1$ and (2.3.7) we conclude for all $i \in \{1, \ldots, 3\}$:

$$\left[{}^{S_4^i}C_2^\circ\right]_\Gamma = \left[{}^{S_4^3}C_2^\bullet\right]_\Gamma. \tag{2.3.8}$$

Because $\Phi_{|S_4^2 \to S_4^3}\colon S_4^2 \to S_4^3$ is an isomorphism the invariance of the sign under isomorphisms implies:

$$\left[{}^{S_4^2}C_2^\bullet\right]_\Gamma = \left[\Phi^{-1}\left({}^{S_4^3}C_2^\bullet\right)\right]_\Gamma.$$

Putting this together with $\Phi(S_4^1) = S_4^1$ and (2.3.8) for $i = 1$ we obtain

$$\left[{}^{S_4^2}C_2^\bullet\right]_\Gamma = \left[\Phi^{-1}\left({}^{S_4^3}C_2^\bullet\right)\right]_\Gamma = \left[\Phi^{-1}\left({}^{S_4^1}C_2^\circ\right)\right]_\Gamma = \left[{}^{S_4^1}C_2^\circ\right]_\Gamma.$$

Thus (2.3.8) becomes

$$\left[{}^{S_4^2}C_2^\bullet\right]_\Gamma = \left[{}^{S_4^i}C_2^\circ\right]_\Gamma = \left[{}^{S_4^3}C_2^\bullet\right]_\Gamma, \; i \in \{1, \ldots, 3\}. \tag{2.3.9}$$

Let us find admissible representatives and their normalizers for the Γ-conjugacy classes of the type V_4 subgroups. By Theorem 2.3.2 there are exactly four Γ-conjugacy classes of groups of type V_4. Set

$$H_i^\bullet := \{\sigma \in S_4^i : \operatorname{ord}(\sigma) \le 2 \;\wedge\; \operatorname{sgn}_{S_4^i}(\sigma) = 1\}.$$

By definition we have $H_i^\bullet \cong V_4$ and $H_i^\bullet \trianglelefteq S_4^i$. In particular we obtain ${}^{S_4^i}V_4^\bullet = H_i^\bullet$. Combining Lemma 2.3.21 and Theorem 2.3.2 we infer

$$N_\Gamma({}^{S_4^i}V_4^\bullet) = N_\Gamma(H_i^\bullet) = S_4^i. \tag{2.3.10}$$

Let ${}^{S_4^1}V_4^\circ$ be an arbitrary representitive for a type V_4 subgroup of S_4^1 being not normal in S_4^1, choose for example ${}^{S_4^1}V_4^\circ = \langle \sigma^\circ, \sigma^\bullet \rangle$. We now claim that

$$N_\Gamma\left({}^{S_4^1}V_4^\circ\right) = {}^{S_4^1}D_8. \tag{2.3.11}$$

Lemma 2.3.21 ensures that $N_\Gamma\left({}^{S_4^1}V_4^\circ\right)$ is finite. Suppose $N_\Gamma\left({}^{S_4^1}V_4^\circ\right) \cong S_4$. Then there exists an $i_0 \in \{1, \ldots, 3\}$ such that $N_\Gamma\left({}^{S_4^1}V_4^\circ\right) \underset{\Gamma}{\sim} S_4^i$. For the reason that for each group of type S_4 there exists only one subgroup of type V_4 lying normal in it,

${}^{S_4^1}V_4^{\circ}$ would have to be Γ-conjugate to ${}^{S_4^{i_0}}V_4^{\bullet}$. We can exclude the case $i_0 = 1$, for otherwise we would have $\sigma^{\circ} \underset{\Gamma}{\sim} \sigma^{\bullet}$, which is impossible. By virtue of $i_0 \in \{2,3\}$ and

$$[\langle\sigma^{\bullet}\rangle]_\Gamma \leq \left[{}^{S_4^1}V_4^{\circ}\right]_\Gamma = \left[{}^{S_4^{i_0}}V_4^{\bullet}\right]_\Gamma$$

we conclude, taking account of (2.3.9), once more $\sigma^{\bullet} \underset{\Gamma}{\sim} \sigma^{\circ}$, a contradiction. This proves our claim.

Applying Lemma 2.3.5 on (2.3.10) and (2.3.11) we obtain the complete list of Γ-conjugacy classes belonging to the subgroups of type V_4 namely

$$\{[H]_\Gamma : H \cong V_4\} = \left\{\left[{}^{S_4^1}V_4^{\bullet}\right]_\Gamma, \left[{}^{S_4^2}V_4^{\bullet}\right]_\Gamma, \left[{}^{S_4^3}V_4^{\bullet}\right]_\Gamma, \left[{}^{S_4^1}V_4^{\circ}\right]_\Gamma\right\}. \tag{2.3.12}$$

Let us identify admissible representatives for the Γ-Conjugacy Classes of the type C_4 subgroups and compute their normalizers as well:

We begin by recalling that $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in S_4^1$ and $\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \in S_4^2$ are elements of order 4. Therefore their squares lie in ${}^{S_4^1}V_4^{\bullet}$ and ${}^{S_4^2}V_4^{\bullet}$ respectively. Considering (2.3.9), we conclude

$$\left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}^2\right]_\Gamma = [\sigma^{\bullet}]_\Gamma \tag{2.3.13}$$

and

$$\left[\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}^2\right]_\Gamma = [\sigma^{\circ}]_\Gamma. \tag{2.3.14}$$

We consistently choose ${}^{S_4^1}C_4 = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle$ as representative for the S_4^1-conjugacy class of type C_4 subgroups in S_4^1 and ${}^{S_4^2}C_4 = \left\langle \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right\rangle$ as representative for the S_4^2-conjugacy class of type C_4 subgroups in S_4^2. For the reason that Γ contains only two Γ-conjugacy classes of type C_4 and by (2.3.13) and (2.3.14), we obtain already

$$\{[H]_\Gamma : H \cong C_4\} = \left\{\left[{}^{S_4^1}C_4\right]_\Gamma, \left[{}^{S_4^2}C_4\right]_\Gamma\right\}. \tag{2.3.15}$$

We proceed by computing the normalizers of the admissible representatives. To this end, apply Lemma 2.3.23 and observe $N_\Gamma\left({}^{S_4^i}C_4\right)$ is finite for each $i \in \{1, \ldots, 3\}$. Hence by an application of Theorem 2.3.2 combined with the fact that the ${}^{S_4^i}C_4$ are (by definition) admissible, we obtain

$$N_\Gamma\left({}^{S_4^i}C_4\right) = {}^{S_4^i}D_8. \tag{2.3.16}$$

Let us now look for admissible representatives for Γ-conjugacy classes of the type D_8-subgroups.

Due to (2.3.16) we take ${}^{S_4^1}D_8 = N_\Gamma\left({}^{S_4^1}C_4\right)$ and ${}^{S_4^2}D_8 = N_\Gamma\left({}^{S_4^2}C_4\right)$ as candidates for admissible representatives. Because all subgroups of type C_4 are conjugate in groups of type D_8, we infer that

$$\left[{}^{S_4^1}D_8\right]_\Gamma \neq \left[{}^{S_4^2}D_8\right]_\Gamma,$$

for otherwise we would have ${}^{S_4^1}C_4 \underset{\Gamma}{\sim} {}^{S_4^2}C_4$, contrary to (2.3.15). By virtue of Theorem 2.3.2 we are able to deduce

$$\{[H]_\Gamma :\ H \cong D_8\} = \left\{\left[{}^{S_4^1}D_8\right]_\Gamma, \left[{}^{S_4^2}D_8\right]_\Gamma\right\}. \tag{2.3.17}$$

Our next goal is to find all the admissible representatives of type C_3. For this purpose, consider for instance

$${}^{D_{12}}C_3 = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \text{ and } {}^{S_4^1}C_3 = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle.$$

We claim that ${}^{D_{12}}C_3 \underset{\Gamma}{\not\sim} {}^{S_4^1}C_3$. Conversely, suppose that there exists a $\Gamma \ni \gamma = \begin{pmatrix} q & r & s \\ t & u & v \\ x & y & z \end{pmatrix}$ such that

$$\gamma\, {}^{D_{12}}C_3 = {}^{S_4^1}C_3\gamma.$$

Because in a symmetric group each element is conjugate to its inverse, we may assume that γ even satisfies

$$\gamma \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \gamma.$$

A straightforward computation yields that γ is of the form

$$\gamma = \begin{pmatrix} q & r & s \\ q & s-r & -r \\ q & -s & r-s \end{pmatrix}, \quad q, r, s \in \mathbb{Z}.$$

Taking the determinant we obtain

$$1 = \det(\gamma) = 3(-qr^2 + qrs - qs^2),$$

contrary to $3 \notin \mathbb{Z}^\times$.
Finally, an application of Theorem 2.3.2 results in

$$\{[H]_\Gamma : H \cong C_3\} = \left\{\left[{}^{D_{12}}C_3\right]_\Gamma, \left[{}^{S_4^1}C_3\right]_\Gamma\right\}. \tag{2.3.18}$$

By virtue of Lemma 2.3.22 and Theorem 2.3.2 we get immediately

$$N_\Gamma\left({}^{D_{12}}C_3\right) = D_{12}. \tag{2.3.19}$$

This and Theorem 2.3.2 enforce $N_\Gamma\left({}^{S_4^1}C_3\right) \not\cong D_{12}$, for otherwise we would have ${}^{D_{12}}C_3 \underset{\Gamma}{\sim} {}^{S_4^1}C_3$, which contradicts (2.3.18). In this way, Lagrange's theorem and the admissibility of ${}^{S_4^1}C_3$ ensure that

$$N_\Gamma\left({}^{S_4^1}C_3\right) = {}^{S_4^1}S_3. \tag{2.3.20}$$

Next, we determine admissible representatives for the Γ-conjugacy classes being associated with subgroups of type S_3.

To this end, consider the candidates ${}^{D_{12}}S_3^1$, ${}^{D_{12}}S_3^2$ and ${}^{S_4^1}S_3^1$. First we prove

$$\left[{}^{D_{12}}S_3^1\right]_\Gamma \neq \left[{}^{D_{12}}S_3^2\right]_\Gamma.$$

On the contrary, suppose there exists a $\gamma \in \Gamma$ such that ${}^\gamma\left({}^{D_{12}}S_3^1\right) = {}^{D_{12}}S_3^2$. Then we obtain by Lemma 2.3.5

$${}^\gamma(D_{12}) = {}^\gamma\left(N_\Gamma\left({}^{D_{12}}S_3^1\right)\right) = N_\Gamma\left({}^{D_{12}}S_3^2\right) = D_{12}.$$

Hence $\gamma \in N_\Gamma(D_{12})$. On the other hand we deduce, using Corollary 2.3.8, that $\gamma \in D_{12} = N_{D_{12}}(S_3^1)$. We conclude

$${}^{D_{12}}S_3^1 = {}^\gamma\left({}^{D_{12}}S_3^1\right) = {}^{D_{12}}S_3^2,$$

contrary to $|\{[H]_{D_{12}} : D_{12} \geq H \cong S_3\}| = 2$.

Corollary 2.3.8 and the finiteness of $C_\Gamma\left({}^{S_4^1}C_3\right)$ imply that $N_\Gamma\left({}^{S_4^1}S_3\right)$ has to be finite as well. This combined with (2.3.18) and (2.3.20) leads immediately to

$$N_\Gamma\left({}^{S_4^1}S_3\right) = {}^{S_4^1}S_3$$

and additionally in consideration of Lemma 2.3.5 also to

$$\left[{}^{S_4^1}S_3\right]_\Gamma \neq \left[{}^{D_{12}}S_3^j\right]_\Gamma, \ j \in \{1, 2\}.$$

Finally, an application of Theorem 2.3.2 yields

$$\{[H]_\Gamma : H \cong S_3\} = \left\{\left[{}^{D_{12}}S_3^1\right]_\Gamma, \left[{}^{D_{12}}S_3^2\right]_\Gamma, \left[{}^{S_4^1}S_3\right]_\Gamma\right\}. \tag{2.3.21}$$

Theorem 2.3.2 and Lemma 2.3.5 yield admissible representatives for the conjugacy classes of the finite subgroups associated with the remaining isomorphy types. In this manner we obtain

$$\{[H]_\Gamma : H \cong C_6\} = \left\{\left[{}^{D_{12}}C_6\right]_\Gamma\right\}, \tag{2.3.22}$$

$$\{[H]_\Gamma : H \cong D_{12}\} = \{[D_{12}]_\Gamma\}, \tag{2.3.23}$$

$$\{[H]_\Gamma : H \cong A_4\} = \left\{\left[{}^{S_4^i}A_4\right]_\Gamma : \ i \in \{1, \dots, 3\}\right\}, \tag{2.3.24}$$

and

$$\{[H]_\Gamma : H \cong S_4\} = \left\{\left[S_4^i\right]_\Gamma : \ i \in \{1, \dots, 3\}\right\}. \tag{2.3.25}$$

We now turn to the discussion about the positions of the elements in $\Gamma\backslash \mathrm{Sub}_{\mathrm{fin}}(\Gamma)$. We start by analyzing the connection lines belonging to the Γ-conjugacy classes of type C_2. Because each group of order 4 contains an element of order 2 with trivial sign, we get on the one hand

$$[\langle\sigma^\bullet\rangle]_\Gamma \leq \left[{}^{S_4^1}V_4^\bullet\right]_\Gamma, \left[{}^{S_4^1}V_4^\circ\right]_\Gamma, \left[{}^{S_4^1}C_4\right]_\Gamma. \tag{2.3.26}$$

On the other hand we infer by virtue of (2.3.9) that

$$[\langle\sigma^\circ\rangle]_\Gamma \leq \left[{}^{S_4^1}V_4^\circ\right]_\Gamma, \left[{}^{S_4^2}V_4^\bullet\right]_\Gamma, \left[{}^{S_4^3}V_4^\bullet\right]_\Gamma, \left[{}^{S_4^2}C_4\right]_\Gamma, \left[{}^{S_4^1}S_3\right]_\Gamma. \tag{2.3.27}$$

A straightforward computation yields

$$C_\Gamma\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \left\{\begin{pmatrix} \det(A)^{-1} & \\ & \mathrm{A} \end{pmatrix} : \ A \in \mathrm{Gl}_2(\mathbb{Z})\right\} =: G.$$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in {}^{S_4^1}V_4^\bullet$ now implies that $[C_\Gamma(\sigma^\bullet)]_\Gamma = [G]_\Gamma$. On the other hand an analysis of the structure of the generators of D_{12} results in $D_{12} \leq G$. In this way we obtain in particular

$$[\sigma^\bullet]_\Gamma = [\mathcal{Z}(D_{12})]_\Gamma.$$

This leads immediately to

$$[\langle\sigma^\bullet\rangle]_\Gamma \leq \left[{}^{D_{12}}C_6\right]_\Gamma \text{ and } [\langle\sigma^\bullet\rangle]_\Gamma \leq \left[{}^{D_{12}}V_4\right]_\Gamma. \tag{2.3.28}$$

Moreover, Theorem 2.2.1 ensures that $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \in D_{12} \cap S_4^1$. This element is of order 2 and can be represented as transposition. The invariance of the sign under automorphisms enforces that this element has negative sign with respect to S_4^1. We conclude

$$[\langle\sigma^\circ\rangle]_\Gamma = \left[\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle\right]_\Gamma \leq \left[{}^{D_{12}}V_4\right]_\Gamma,$$

which leads by virtue of (2.3.27) and (2.3.26) to

$$\left[{}^{D_{12}}V_4\right]_\Gamma = \left[{}^{S_4^1}V_4^\circ\right]_\Gamma. \tag{2.3.29}$$

From this, (2.3.28) and Corollary 2.3.13 we may derive

$$[\langle\sigma^\circ\rangle] \leq \left[{}^{D_{12}}S_3^j\right]_\Gamma,\ j \in \{1,2\}. \tag{2.3.30}$$

In order to close the chapter about the simple inclusions of Γ-conjugacy classes of subgroups of type C_2, we have to show that there is not any other element in $\Gamma \backslash \mathrm{Sub}_{\mathrm{fin}}(\Gamma)$ with the property that any Γ-conjugacy class of subgroups of type C_2 lies simple in it. We begin by proving

$$[\langle\sigma^\bullet\rangle]_\Gamma \not\leq [S_4^i]_\Gamma,\ i \in \{2,3\}. \tag{2.3.31}$$

Conversely, suppose that there is an $i \in \{2,3\}$ such that

$$[\langle\sigma^\bullet\rangle]_\Gamma \leq [S_4^i]_\Gamma.$$

Then (2.3.8) already implies $[\sigma^\bullet]_\Gamma = [\sigma^\circ]_\Gamma$, a contradiction. Moreover

$$[\langle\sigma^\bullet\rangle]_\Gamma \not\leq \left[{}^{D_{12}}S_3^j\right]_\Gamma,\ j \in \{1,2\},$$

for otherwise, taking account of $[\langle\sigma^\circ\rangle]_\Gamma \leq \left[{}^{D_{12}}S_3^j\right]_\Gamma$, we would necessarily obtain once more $[\sigma^\bullet]_\Gamma = [\sigma^\circ]_\Gamma$.

By virtue of (2.3.28) we derive

$$[\langle\sigma^\circ\rangle]_\Gamma \not\leq \left[{}^{D_{12}}C_6\right]_\Gamma$$

and for the reason that a group of type A_4 contains only one subgroup of type V_4, we must have

$$[\langle\sigma^\circ\rangle]_\Gamma \not\leq \left[{}^{S_4^1}A_4\right]_\Gamma.$$

Because of $[\sigma^\circ]_\Gamma \neq [\sigma^\bullet]_\Gamma$ we see

$$[\langle\sigma^\circ\rangle]_\Gamma \not\leq \left[{}^{S_4^1}C_4\right]_\Gamma.$$

Thus, we infer that we have indeed determined in (2.3.27), (2.3.26), (2.3.28) and (2.3.30) all the connection lines emanating from Γ-conjugacy classes of subgroups of type C_2.

Next we watch out for the simple inclusions of Γ-conjugacy classes of subgroups lying over $[\sigma^\bullet]_\Gamma$. Considering (2.3.31) we immediately observe that

$$\left[{}^{S_4^1}V_4^\bullet\right]_\Gamma, \left[{}^{S_4^1}V_4^\circ\right]_\Gamma, \left[{}^{S_4^1}C_4\right]_\Gamma \not\leq [S_4^i]_\Gamma,\ i \in \{2,3\}.$$

Taking account of (2.3.29) we get

$$\left[{}^{S_4^1}V_4^\circ\right]_\Gamma \leq [D_{12}]_\Gamma$$

and in addition, according to the fact that all subgroups of type V_4 in D_{12} have to be conjugate by Sylow's theorems, we infer

$$\left[{}^{S_4^1}V_4^\bullet\right]_\Gamma \not\leq [D_{12}]_\Gamma.$$

The determination of the remaining simple inclusions of the elements in $\mathrm{Sub}_{\mathrm{fin}}(\Gamma)$ lying properly over $[\sigma^\bullet]_\Gamma$ is straightforward as well as the proof that we have found all of them.

Now, let us find the simple inclusions for the Γ-conjugacy classes c of subgroups of order 4 and 8, for which the following conditions are valid:

- $[\langle\sigma^\circ\rangle]_\Gamma \leq c$ and
- $[\langle\sigma^\bullet\rangle]_\Gamma \not\leq c$.

Such Γ-conjugacy classes cannot be contained in $[S_4^1]_\Gamma$, for otherwise they would contain $[\langle\sigma^\bullet\rangle]_\Gamma$, a contradicition.
This and (2.3.17) forces $\left[{}^{S_4^2}D_8\right]_\Gamma = \left[{}^{S_4^3}D_8\right]_\Gamma$. Moreover, we see $\left[{}^{S_4^i}V_4^\bullet\right]_\Gamma \not\leq [D_{12}]_\Gamma$, $i \in \{2,3\}$, for otherwise we would have $\left[{}^{D_{12}}V_4\right]_\Gamma = \left[{}^{S_4^i}V_4^\bullet\right]_\Gamma$, contrary to (2.3.29). By virtue of this the following list of simple inclusions for the Γ-conjugacy classes, which satisfy the conditions above, is true and complete:

- $\left[{}^{S_4^i}V_4^\bullet\right]_\Gamma \leq \left[{}^{S_4^2}D_8\right]_\Gamma, \left[{}^{S_4^i}A_4\right]_\Gamma,\ i \in \{2,3\};$
- $\left[{}^{S_4^2}C_4\right]_\Gamma \leq \left[{}^{S_4^2}D_8\right]_\Gamma;$
- $\left[{}^{S_4^2}D_8\right]_\Gamma \leq [S_4^i]_\Gamma,\ i \in \{2,3\}.$

It is trivial to give complete lists of simple inclusions for $\left[{}^{S_4^i}A_4\right]_\Gamma,\ i \in \{1,\dots,3\}$ and $\left[{}^{D_{12}}C_6\right]_\Gamma$.

To finish the proof of the proposition, it suffices to find the collection of simple inclusions for Γ-conjugacy classes of subgroups of type C_3 and S_3 and to prove that this collection is exhaustive.

We start by showing

$$\left[{}^{S_4^1}C_3\right]_\Gamma \not\leq [D_{12}]_\Gamma. \tag{2.3.32}$$

If it were true, that $\left[{}^{S_4^1}C_3\right]_\Gamma \leq [D_{12}]_\Gamma$ there would exist a $\gamma \in \Gamma$ satisfying

$${}^\gamma\left({}^{S_4^1}C_3\right) \leq D_{12}.$$

But if this were the case, Sylow's theorems would already force

$$^{\gamma}\left({}^{S_4^1}C_3\right) = {}^{D_{12}}C_3,$$

contradicting (2.3.18).

From this we obtain by the way

$$\left[{}^{S_4^1}S_3\right]_\Gamma \leq [D_{12}]_\Gamma. \tag{2.3.33}$$

On the other hand, Theorem 2.2.1 tells us that S_4^1 shares a common subgroup of type C_3 with S_4^2 and a another common subgroup of type C_3 with S_4^3. Applying Sylow's theorems we conclude

$$\left[{}^{S_4^i}C_3\right]_\Gamma = \left[{}^{S_4^1}C_3\right]_\Gamma \quad \forall i \in \{1, \dots, 3\}. \tag{2.3.34}$$

Hence, we immediately obtain

$$\left[{}^{S_4^1}C_3\right]_\Gamma \leq \left[{}^{S_4^i}A_4\right]_\Gamma \quad \forall i \in \{1, \dots, 3\}. \tag{2.3.35}$$

Taking account of (2.3.18) the relation given in (2.3.34) suggests the following claim.

$$\left[{}^{D_{12}}C_3\right]_\Gamma \not\leq [S_4^i]_\Gamma,\ i \in \{1, \dots, 3\}. \tag{2.3.36}$$

On the contrary, suppose that

$$\left[{}^{D_{12}}C_3\right]_\Gamma \leq [S_4^{i_0}]_\Gamma$$

for some $i_0 \in \{1, \dots, 3\}$. Then there exists a $\gamma \in \Gamma$ such that

$$^{\gamma}\left({}^{D_{12}}C_3\right) \leq S_4^{i_0}.$$

Sylow's theorems allow us to modify our $\gamma \in \Gamma$ in a way such that

$$^{\gamma}\left({}^{D_{12}}C_3\right) = {}^{S_4^{i_0}}C_3.$$

In particular, we obtain

$$\left[{}^{D_{12}}C_3\right]_\Gamma = \left[{}^{S_4^{i_0}}C_3\right]_\Gamma = \left[{}^{S_4^1}C_3\right]_\Gamma,$$

contradicting (2.3.18).

From (2.3.32) and (2.3.36) we derive immediately

$$\left[{}^{D_{12}}S_3^j\right]_\Gamma \not\leq [S_4^i]_\Gamma \tag{2.3.37}$$

Combining (2.3.20) with (2.3.34) yields

$$\left[{}^{S_4^1}S_3\right]_\Gamma \leq [S_4^i]_\Gamma \quad \forall i \in \{1, \dots, 3\}. \tag{2.3.38}$$

So we can give a complete list of simple inclusions for the Γ-conjugacy classes of subgroups having orders 3 or 6.

- $\left[{}^{D_{12}}C_6\right]_\Gamma \leq [D_{12}]_\Gamma$;
- $\left[{}^{S_4^1}C_3\right]_\Gamma \leq \left[{}^{S_4^1}S_3\right]_\Gamma$;
- $\left[{}^{S_4^1}C_3\right]_\Gamma \leq \left[{}^{S_4^i}A_4\right]_\Gamma,\ i \in \{1, \dots, 3\}$;
- $\left[{}^{D_{12}}C_3\right]_\Gamma \leq \left[{}^{D_{12}}C_6\right]_\Gamma, \left[{}^{D_{12}}S_3^j\right]_\Gamma,\ j \in \{1, 2\}$;
- $\left[{}^{S_4^1}S_3\right]_\Gamma \leq [S_4^i]_\Gamma,\ i \in \{1, \dots, 3\}$;

- $\left[{}^{D_{12}}S_3^j\right]_\Gamma \leq [D_{12}]_\Gamma,\ j \in \{1,2\}$.

(2.3.32), (2.3.36) and (2.3.33) enforce that there cannot be another simple inclusion with origin in a group of order 3 or 6.

This completes the proof of the proposition. □

2.4 Scwols and Complexes of groups

Think of a Complex of groups as fundamental domain for a group action, labeled with the stabilizer groups at its singular vertices. We look in this chapter for the construction of, in some sense, minimal complex of groups belonging to a given group. A key feature of such a minimal object should be, that the number conjugacy classes of maximal finite subgroups of the group is equal to the number of vertices labeled by such groups. The main-theorem of this chapter, Theorem 2.4.49, shows that our construction essentially has this property.

The following definition is due to Serre and can be found in [7].

Definition 2.4.1 (graph)**.** A *graph* $\mathfrak{G}$ is a tuple $(V(\mathfrak{G}), \mathfrak{E}(\mathfrak{G}), i, t, ^{-1})$ where

$$i\colon \mathfrak{E}(\mathfrak{G}) \to V(\mathfrak{G}) \text{ and } t\colon \mathfrak{E}(\mathfrak{G}) \to V(\mathfrak{G})$$

are maps and

$$^{-1}\colon \mathfrak{E}(\mathfrak{G}) \to \mathfrak{E}(\mathfrak{G})$$

is an involution without fixed points which satisfies the following condition:

$$i(e^{-1}) = t(e).$$

Remark 5. We sometimes write $V\mathfrak{G}$ or $E\mathfrak{G}$ instead of $V(\mathfrak{G})$ or $E(\mathfrak{G})$ respectively.

Definition 2.4.2. Let $\mathfrak{G}$ be a graph. We say $\mathfrak{H} = (V, \mathfrak{E}, i', t', \kappa)$ is a *subgraph* of $\mathfrak{G}$, if it is a graph having vertices $V \subseteq V(\mathfrak{G})$ and edges $\mathfrak{E} \subseteq \mathfrak{E}(\mathfrak{G})$ such that

$$i' = i_{|\mathfrak{E}\to V} \text{ and } t' = t_{|\mathfrak{E}\to V}$$

and such that the involution κ coincides with the restriction of the involution $^{-1}$ to $\mathfrak{E}$.

Definition 2.4.3 (paths, connected graphs)**.** Let $\mathfrak{G} = (V(\mathfrak{G}), \mathfrak{E}(\mathfrak{G}), i, t, ^{-1})$ be a graph. A *path* is a non-empty tuple of edges

$$(e_k)_{j=1}^n,\ e_k \in \mathfrak{E}(\mathfrak{G})\ \forall 1 \le k \le n,$$

such that $t(e_k) = i(e_{k+1})$ for each $k \in \{1, \dots, n-1\}$ or an 1-tuple (v) for a $v \in V(\mathfrak{G})$. In the last case we say the path is *trivial.* Let π be a path. We set $i(\pi) := i(e_1)$ and $t(\pi) := t(e_n)$ if $\pi = (e_1, \dots, e_n)$ for some edges $e_k \in E(\mathcal{X})$ and $i(\pi) := v$ and $t(\pi) := v$ if $\pi = (v)$ for a $v \in V(\mathfrak{G})$. For vertices $v, w \in V(\mathfrak{G})$ and a path π we say the π *connects* v *to* w if $i(\pi) = v$ and $t(\pi) = w$. For a non-trivial path $\pi = (e_k)_{k=1}^n$ we set

$$\mathfrak{E}(\pi) := \{e_k : 1 \le k \le n\}$$

and

$$V(\pi) := \{v \in V(\mathfrak{G}) : \exists e \in \mathfrak{E}(\pi)\colon v = i(e) \text{ or } v = t(e)\}.$$

If $\pi = (v)$ for some $v \in V(\mathcal{X})$, we put $\mathfrak{E}(\pi) := \emptyset$ and $V(\pi) := \{v\}$. We call $\mathfrak{E}(\pi)$ the set of edges and $V(\pi)$ the set of vertices of π. We say the graph $\mathfrak{G}$ is *connected* if for each two vertices $v, w \in V(\mathfrak{G})$ there exists a path π connecting v to w.

Definition 2.4.4 (concatenation of paths)**.** Let $\mathfrak{G}$ be a graph and $\pi = (e_k)_{k=1}^m$ and $\eta = (f_l)_{l=1}^n$ be paths in $\mathfrak{G}$ such that $t(\pi) = i(\eta)$. If π and η are non-trivial, we define

$$\pi * \eta := (e_1, \dots, e_m, f_1, \dots f_n).$$

If π is trivial, we set $\pi * \eta := \eta$. If η is trivial, we put $\pi * \eta := \pi$. In each case $\pi * \eta$ is a path.

Definition 2.4.5 (backtracking, circle, tree)**.** Let $\mathfrak{G}$ be a graph and π be a path in $\mathfrak{G}$. We say $\pi = (e_k)_{k=1}^n$ has *backtracking* if there exists a $1 \leq k_0 \leq n$ such that $e_{k_0+1} = (e_{k_0})^{-1}$. We say π is a path *without backtracking* if there exists no such k_0. We call π a *circle* if and only if π is a non-trivial path without backtracking satisfying $t(\pi) = i(\pi)$. A graph $\mathfrak{G}$ is said to be a *tree* if it is connected and does not contain any circle.

Remark 6. Consider an arbitrary path π in a graph $\mathfrak{G}$. If there is a backtracking, remove it. If there is not, we have obtained a path without backtracking. Iterating this procedure leads to a path without backtracking after finitely many steps. This path is unique in the sense that it does not depend on the order we removed the backtracking. We call the unique path obtained by this procedure the *path without backtracking associated to* π*.*

Definition 2.4.6 (spanning tree)**.** Let $\mathfrak{G} = (V(\mathfrak{G}), \mathfrak{E}(\mathfrak{G}), i, t, ^{-1})$ be a graph. A subgraph T is called a *spanning tree* if $V(T) = V(\mathfrak{G})$ and if it is a tree.

It is well known that each connected graph contains a spanning tree.

The following definitions can be taken from [1].

Definition 2.4.7 (scwol)**.** A *small category without loops* (briefly a *scwol*) $\mathcal{X}$ is a tuple $(V(\mathcal{X}), E(\mathcal{X}), i, t, \circ)$ consisting of sets $V(\mathcal{X})$, $E(\mathcal{X})$ and maps

$$i\colon E(\mathcal{X}) \to V(\mathcal{X}), \quad t\colon E(\mathcal{X}) \to V(\mathcal{X})$$

and a *composition* map

$$\circ\colon E^{(2)}(\mathcal{X}) \to E(\mathcal{X}),$$

where we have set for $k \in \mathbb{N}$

$$E^{(k)}(\mathcal{Y}) := \left\{ (a_1, \ldots, a_k) \in \prod_{j=1}^{k} E(\mathcal{X}) : i(a_j) = t(a_{j+1}),\ j \in \{1, \ldots, k-1\} \right\},$$

and where the tuple is required to satisfy the following axioms:

(1) $\forall (a, b) \in E^{(2)}(\mathcal{X})\colon i(a \circ b) = i(b)$ and $t(a \circ b) = t(a)$;
(2) $\forall (a, b, c) \in E^{(3)}(\mathcal{X})\colon (a \circ b) \circ c = a \circ (b \circ c)$;
(3) $\forall a \in E(\mathcal{X})\colon i(a) \neq t(a)$.

The elements of $V(\mathcal{X})$ are called *vertices*, those of $E(\mathcal{X})$ are called *arrows*, and those of $E^{(2)}(\mathcal{X})$ *composable* arrows. For composable arrows a, b we will often write ab instead of $a \circ b$.

Definition 2.4.8 (subscwol)**.** A *subscwol* $\mathcal{X}' = (V\mathcal{X}', E\mathcal{X}', i_{|E\mathcal{X}'}, t_{|E\mathcal{X}'}, \circ_{|(E\mathcal{X}')^2})$ of a scwol $\mathcal{X}$ is given by subsets $V(\mathcal{X}') \subseteq V(\mathcal{X})$ and $E(\mathcal{X}') \subseteq E(\mathcal{X})$ such that if $a \in E(\mathcal{X}')$, then $i(a), t(a) \in V(\mathcal{X}')$ and if $a, b \in E(\mathcal{X}')$ are such that $i(a) = t(b)$, then $a \circ b \in E(\mathcal{X}')$.

Definition 2.4.9 (canonical partial order on the set of vertices of the scwol)**.** Let $\mathcal{X}$ be a scwol. Then, the relation given by

$$v \leq w \quad :\Longleftrightarrow \left\{ \begin{array}{c} v = w \\ \text{or } \exists a \in E(\mathcal{Y}) : i(a) = w \text{ and } t(a) = v \end{array} \right\}.$$

is a partial order on $V(\mathcal{X})$. It is called the *canonical parital order* on $V(\mathcal{X})$.

Proof. The axioms satisfied by a scwol enforce that "$\leq$" defines indeed a partial order. □

Remark 7. We will sometimes refer to it as the canonical order on $\mathcal{X}$ or as the order on $\mathcal{X}$. We agree on the following convention: If we write $v \leq w$ for some $v, w \in V(\mathcal{X})$, we explicitly refer to the canonical partial order unless nothing else is specified.

Definition 2.4.10 (dimension of a vertex in a scwol, dimension of a scwol)**.** Let $\mathcal{X}$ be a scwol. For each $v \in V(\mathcal{X})$ we set $E^{(k)}(\mathcal{X}, v) := \{(a_1, \ldots, a_k) \in E^{(k)}(\mathcal{X}) : i(a_k) = v\}$. The *dimension* of a vertex $v \in V(\mathcal{X})$ is the number

$$\dim(v) := \max\{k \in \mathbb{N} : \exists (a_1, \ldots, a_k) \in E^{(k)}(\mathcal{X}, v) \colon i(a_k) = v\}.$$

The number $\dim(\mathcal{X}) := \sup_{v \in V(\mathcal{X})} \dim(v)$ is called the *dimension* of the scwol.

Definition 2.4.11. For an arbitrary scwol $\mathcal{X}$ and any number $k \in \mathbb{N}$ we set

$$V_k(\mathcal{X}) := \{v \in V(\mathcal{X}) : \dim v = k\}.$$

Remark 8. It is clearly true that $V(\mathcal{X}) = \biguplus_{k \in \mathbb{N}} V_k(\mathcal{X})$.

Definition 2.4.12. For an arbitrary scwol $\mathcal{X}$ we define

$$V_{\max}(\mathcal{X}) := \{v \in V(\mathcal{X}) : v \text{ maximal}\}.$$

Lemma 2.4.13 (dimension is a strictly increasing function on $V\mathcal{X}$)**.** *For each two vertices $v, w \in V(\mathcal{X})$ with $v < w$ holds*

$$\dim(v) < \dim(w).$$

Proof. Let $v < w$, $n := \dim v$. By definition there exists a sequence $(a_1, \ldots, a_n)$ such that $a_1 \circ \cdots \circ a_n \in E(\mathcal{Y})$ and $i(a_n) = v$. Furthermore the axioms for scwols imply there exists an arrow $a \in E(\mathcal{Y})$ with $i(a) = w$ and $t(a) = v$. We conclude that

$$a_1 \circ \cdots \circ a_n \circ a \in E(\mathcal{Y})$$

and therefore

$$\dim w \geq n + 1 > n = \dim v.$$

□

Lemma 2.4.14 (each vertice is covered by a maximal one)**.** *Let $\mathcal{X}$ be a scwol such that* $\dim(\mathcal{X}) < \infty$. *For each $v \in V(\mathcal{X})$ there exists a maximal element w with respect to "$\leq$" such that $v \leq w$.*

Proof. Let $n := \dim(\mathcal{X})$. Suppose, contrary to our claim, that there is an element $v \in V(\mathcal{X})$ such that for each $w \geq v$ there exists a $V(\mathcal{X}) \ni w' > w$. Iterative applications of this argument yield a sequence of elements $(v_i)_{i \in \mathbb{N}}$ in $V(\mathcal{X})$ such that $v_i < v_{i+1}$. Take an arbitrary $k > n + 1$. Due to Lemma 2.4.13 we obtain

$$\dim(v_k) \geq \dim(v_1) + k - 1 > n = \dim(\mathcal{X}),$$

which is impossible. □

Definition 2.4.15 (morphisms of scwols)**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be scwols. Then a pair of maps

$$\varphi\colon \begin{array}{ccc} V(\mathcal{X}) & \to & V(\mathcal{Y}) \\ E(\mathcal{X}) & \to & E(\mathcal{Y}) \end{array}$$

is called a *morphism*, if it satisfies the following conditions

(1) $\forall a \in E(\mathcal{X})\colon i_{\mathcal{Y}}(\varphi(a)) = \varphi(i_{\mathcal{X}}(a))$ and $t_{\mathcal{Y}}(\varphi(a)) = \varphi(t_{\mathcal{X}}(a))$; and
(2) $\forall (a, b) \in E^{(2)}(\mathcal{X})\colon \varphi(a \circ b) = \varphi(a) \circ \varphi(b)$.

φ is called an *isomorphism* if, in addition, both maps in φ are bijections. $\mathcal{X}$ and $\mathcal{Y}$ are called *isomorphic* if there is an isomorphism between them. In this case we write briefly $\mathcal{X} \cong \mathcal{Y}$.

Definition 2.4.16 (complex of groups)**.** Let $\mathcal{Y}$ be a scwol. A complex of groups $\mathcal{G}(\mathcal{Y})$ over $\mathcal{Y}$ is given by the following data:

(1) a family of groups $(G_v)_{v\in V(\mathcal{Y})}$ over the vertices of $\mathcal{Y}$, a group G_v is called the *local group at* v;
(2) a family $(\psi_a)_{a\in E(\mathcal{Y})}$ of injective group homomorphisms $\psi_a\colon G_{i(a)} \to G_{t(a)}$ over the arrows of $\mathcal{Y}$;
(3) a family of twisting elements $(g_{a,b})_{(a,b)\in E^{(2)}(\mathcal{Y})}$, $g_{a,b} \in G_{t(a)}$;

with the following compatibility conditions:

(a) $\forall (a,b) \in E^{(2)}(\mathcal{Y})\colon {}^{g_{a,b}}(\psi_{ab}) = \psi_a\psi_b$;
(b) $\forall (a,b,c) \in E^{(3)}(\mathcal{Y})\colon \psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}$

In short we write $\mathcal{G}(\mathcal{Y}) = (G_v, \psi_a, g_{a,b})$.

Definition 2.4.17 (group action on a scwol)**.** An action of a group G on a scwol $\mathcal{X}$ consists of an action $G \curvearrowright V(\mathcal{X})$ and an action $G \curvearrowright E(\mathcal{X})$ with the compatibility conditions

$$g.i(a) = i(g.a) \text{ and } g.t(a) = t(g.a)$$

and

$$g.(a \circ b) = g.a \circ g.b$$

satisfying the following additional axioms:

(1) $\forall a \in E(\mathcal{X}),\ g \in G\colon g.i(a) \neq t(a),$ ("dimension preserving")
(2) $\forall a \in E(\mathcal{X}),\ g \in G\colon g.i(a) = i(a) \Rightarrow g.a = a.$ ("inversion free")

Remark 9. The first condition can be omitted, if the scwol, on which the group acts, is finite. The definition implies that G preserves the canonical partial order on $V(\mathcal{X})$.

Definition 2.4.18 (quotient scwol)**.** Let $\mathcal{X}$ be a scwol and G be a group acting on it. Then the *quotient scwol* $G\backslash\mathcal{X}$ consists of the quotient sets $G\backslash V(\mathcal{X})$, $G\backslash E(\mathcal{X})$ and the induced maps $\bar{i}$ and $\bar{t}$ where $\bar{i}(G.v) := G.i(v)$ and $\bar{t}(G.v) := G.t(v)$. The composition on $G\backslash\mathcal{X}$ is defined as follows: Let $(\alpha,\beta) \in E^{(2)}(G\backslash\mathcal{X})$ and choose arbitrary $a \in \alpha$ and $b \in \beta$. Due to $\bar{i}(\alpha) = \bar{t}(\beta)$ there exists a $g \in G$ such that $i(a) = g.t(b) = t(g.b)$. We set $\alpha * \beta := G.(a \circ g.b)$. It is not hard to see that all the maps are well-defined and that $G\backslash\mathcal{X} := (G\backslash V(\mathcal{X}), G\backslash E(\mathcal{X}), \bar{i}.\bar{t}, *)$ satisfies all the axioms of a scwol.

Because we have not found it anywhere in the literature, we provide a definition for the notion of being a fundamental domain of a group action on a connected scwol.

Definition 2.4.19 (fundamental domain of an action)**.** Let G be a group acting on a finite dimensional connected scwol $\mathcal{X}$. A subscwol $\mathcal{D}$ of $\mathcal{X}$ is called a *fundamental domain for* $G \curvearrowright \mathcal{X}$ if the following properties are satisfied:

(1) $V_{\max}(\mathcal{D})$ is a system of representatives for $G\backslash V_{\max}(\mathcal{X})$,
(2) $\forall a \in E(\mathcal{X})\colon \quad i(a) \in V(\mathcal{D}) \Rightarrow a \in E(\mathcal{D}).$ ("track incidence structure")

Remark 10. Axiom (1) in Definition 2.4.19 can be interpreted in the following way: "There is an open subset of $\mathcal{D}$ such that $\mathcal{X}$ can be covered by disjoint translates of that subset up to sets of measure zero."
It might happen that there are multiple arrows identified between two vertices in a quotient scwol. However, property (2) in Definition 2.4.19 takes account of that fact.

Lemma 2.4.20 (The notion "fundamental domain" is well-defined)**.** *Let $\mathcal{D}$ be a fundamental domain for a group action of a group G on a connected scwol $\mathcal{X}$ with* $\dim(\mathcal{X}) < \infty$*. Then*

$$V(G\backslash\mathcal{X}) = \{G.v : v \in V(\mathcal{D})\} \text{ and } E(G\backslash\mathcal{X}) = \{G.a : a \in E(\mathcal{D})\}.$$

PROOF. We first claim that for each $v \in V(\mathcal{X})$ there are a $v' \in V(\mathcal{D})$ and a $g \in G$ such that $v = g.v'$. To prove this, take an arbitrary $v \in V(\mathcal{X})$. Lemma 2.4.14 guarantees the existence of a $w \in V_{\max}(\mathcal{X})$ with $v \leq w$. By (1) in Definition 2.4.19 there exists a $w' \in V_{\max}(\mathcal{D})$ such that $g.w' = w$. Because the G action on $\mathcal{X}$ preserves the canonical order on $V(\mathcal{X})$, we obtain

$$v' := g^{-1}.v \leq g^{-1}.w = w'.$$

The definition of the canonical order forces that one of the following cases occurs:

(1) there is a $b \in E(\mathcal{D})$ such that $i(b) = w',\ t(b) = v'$ or
(2) $v' = w'$.

In the second case we conclude $g.v' = v,\ v' = w' \in V(\mathcal{D})$, as claimed. So, let us assume the first case occurs. For the reason $i(b) \in V(\mathcal{D})$, we obtain by (2) in Definition 2.4.19 that $b \in E(\mathcal{D})$. Because $\mathcal{D}$ is a subscwol, we get $v' \in V(\mathcal{D})$. We conclude

$$g.v' = gg^{-1}.v = v,$$

which yields the claim.

We get as an immediate consequence

$$V(G\backslash\mathcal{X}) = \{G.v : v \in V(\mathcal{D})\}.$$

Now, we turn to the proof of the second assertion of the lemma. It suffices to show $E(G\backslash\mathcal{X}) \subseteq \{G.a : a \in E(\mathcal{D})\}$. So, let $G.a \in E(G\backslash\mathcal{X})$. Then, $G.i(a) \subseteq V(G\backslash\mathcal{X})$. By the claim above, there is a $v' \in V(\mathcal{D})$ such that $i(a) = g.v'$. Put $a' := g^{-1}.a$. We infer

$$i(a') = g^{-1}.i(a) = v' \in V(\mathcal{D})$$

and therefore $a' \in E(\mathcal{D})$. Hence

$$G.a = G.a' \subseteq \{G.a : a \in E(\mathcal{D})\},$$

as desired.

□

Proposition 2.4.21. *Each group action of a group G on a finite dimensional connected scwol $\mathcal{X}$ has a fundamental domain given by*

$$V(\mathcal{D}) := \{v \in V(\mathcal{X}) : \exists \sigma \in \Sigma\colon v \leq \sigma\},$$
$$E(\mathcal{D}) := \{a \in E(\mathcal{X}) : \exists \sigma \in \Sigma\colon i(a) \leq \sigma\},$$

where Σ is an arbitrary system of representatives for $G\backslash V_{\max}(\mathcal{X})$.

PROOF. The proof is trivial. □

Proposition 2.4.22. *Let G be a group acting on a finite dimensional connected scwol $\mathcal{X}$. Let $\mathcal{D}$ be a fundamental domain for this action. Put $\Sigma := V_{\max}(\mathcal{D})$. Then, $\mathcal{D}$ can be represented as follows:*

$$V(\mathcal{D}) = \{v \in V(\mathcal{X}) : \exists \sigma \in \Sigma\colon v \leq \sigma\},$$
$$E(\mathcal{D}) = \{a \in E(\mathcal{X}) : \exists \sigma \in \Sigma\colon i(a) \leq \sigma\},$$

In particular, each fundamental domain, is uniquely determined by the choice of the system of representatives Σ of the quotient $G\backslash V_{\max}(\mathcal{X})$.

PROOF. The proof is straightforward. □

Lemma 2.4.23. *Let G be a group acting on a finite dimensional connected scwol $\mathcal{X}$ and $\Delta \leq G$ such that $(G : \Delta) < \infty$. Let S be a system of representatives for $\Delta\backslash G$. Furthermore, assume there is a fundamental domain $\mathcal{D}$ for $G \curvearrowright \mathcal{X}$ such that $G_v = \{1\}$ for each $v \in V_{\max}(\mathcal{D})$. Then, the scwol $S.\mathcal{D}$ given by*

$$V(S.\mathcal{D}) := \{s.v : s \in S,\ v \in V(\mathcal{D})\} \text{ and } E(S.\mathcal{D}) := \{s.a : s \in S,\ a \in E(\mathcal{D})\}$$

is a fundamental domain for $\Delta \curvearrowright \mathcal{X}$.

Proof. We begin by proving that $S.\mathcal{D}$ is indeed a subscwol. Obviously, $a \in E(S.\mathcal{D})$ implies $i(a), t(a) \in V(S.\mathcal{D})$. The only non-trivial fact we have to show is $a, b \in E(S.\mathcal{D})$ with $i(a) = t(b)$ implies $a \circ b \in E(S.\mathcal{D})$. To this end, take $a, b \in E(S.\mathcal{D})$ such that $i(a) = t(b)$. By definition, there are $s \in S$ and $b' \in E(\mathcal{D})$ such that $b = s.b'$. We put $a' := s^{-1}a$. Considering $\mathcal{D}$ is a subscwol, we have $i(b'), t(b') \in V(\mathcal{D})$ and thus

$$i(a') = s^{-1}.i(a) = s^{-1}.t(b) = t(b') \in V(\mathcal{D}).$$

Definition 2.4.19 axiom (2) forces $a' \in E(\mathcal{D})$. Because $\mathcal{D}$ is a subscwol, we obtain $a' \circ b' \in E(\mathcal{D})$. We conclude

$$a \circ b = (s.a' \circ s.b') = s.(a' \circ b') \in E(S.\mathcal{D}).$$

Hence $S.\mathcal{D}$ is indeed a subscwol.

Let us show that $S.\mathcal{D}$ satisfies (1) and (2) of Definition 2.4.19. We start with (1). To this end, take an arbitrary $v \in V_{\max}(\mathcal{X})$. Because $\mathcal{D}$ is a fundamental domain, there are $g \in G$ and $v' \in V_{\max}(\mathcal{D})$ such that $g.v' = v$. Considering that S is a system of represantatives for $\Delta\backslash\Gamma$, we get $\Delta g = \Delta s$ for a suitable $s \in S$. We conclude

$$\Delta.v = \Delta g.v' = \Delta.(s.v').$$

Suppose, there are $w, w' \in V_{\max}(S.\mathcal{D})$ and $\delta, \delta' \in \Delta$ such that $\delta'.w' = \delta.w$. Then, there are also $s, s' \in S$ and $v, v' \in V_{\max}(\mathcal{D})$ such that $w = s.v$ and $w' = s'.v'$. In particular, we obtain

$$\delta' s'.v' = \delta s.v.$$

The fact that $V_{\max}(\mathcal{D})$ is a system of representatives for $G\backslash V_{\max}(\mathcal{X})$ now enforces $v = v'$. This yields

$$s^{-1}\delta^{-1}\delta' s' \in G_v = \{1\}$$

and therefore $\Delta s' = \Delta s$. For the reason that S is a system of representatives for $\Delta\backslash\Gamma$, we obtain $s' = s$. This implies

$$w' = s'.v' = s.v = w.$$

We conclude that $V_{\max}(S.\mathcal{D})$ is a system of representatives for $\Delta\backslash V_{\max}(\mathcal{X})$, as required. Property (2) in Definition 2.4.19 for $S.\mathcal{D}$ is an immediate consequence of that of $\mathcal{D}$. □

Definition 2.4.24 (complex of groups associated to an action on a scwol)**.** Let G be a group acting on a scwol $\mathcal{X}$. Let $\mathcal{Y} := G\backslash\mathcal{X}$ the quotient scwol. For each vertex $\bar{v} \in V(\mathcal{Y})$ choose a vertex $v \in \bar{v}$ such that $G.v = \bar{v}$. For each edge $\bar{a} \in E(\mathcal{Y})$ with $i(\bar{a}) = \bar{v}$ we ensure due to the inversion freeness axiom for $\mathcal{X}$ the existence of an unique edge $a \in E(\mathcal{X})$ such that $a \in \bar{a}$ and $i(a) = v$. Take an arbitrary $w \in t(\bar{a})$. Choose an $h_a \in G$ such that $h_a.t(a) = w$. For $\bar{v} \in V(\mathcal{Y})$ let $G_{\bar{v}}$ be the isotropy subgroup of v, and for each $\bar{a} \in E(\mathcal{Y})$, let $\psi_{\bar{a}} \colon G_{i(\bar{a})} \to G_{t(\bar{a})}$ be the homomorphism defined by

$$\psi_{\bar{a}}(g) := h_a g h_a^{-1}$$

which lies by the inversion freeness axiom in $G_{t(\bar{a})}$. For composable edges $(\bar{a}, \bar{b}) \in E^{(2)}(\mathcal{Y})$ define $g_{\bar{a},\bar{b}} := h_a h_b h_{ab}^{-1} \in G_{t(\bar{a})}$. The *complex of groups over $\mathcal{Y}$ associated to the action of G on $\mathcal{X}$* (and the choices above) is

$$\mathcal{G}(\mathcal{Y}) = (G_{\bar{v}}, \psi_{\bar{a}}, g_{\bar{a},\bar{b}}).$$

It is easy to show that all the axioms of a complex of groups are satisfied.

Remark 11. Different choices will result in different complexes of groups. For the interested reader let us mention that other choices for the h_a lead to complexes of groups deduced from $\mathcal{G}(\mathcal{Y})$ by a "coboundary", in particular they are "isomorphic". For the definitions and more detailed information we refer the reader to [1] Chapter III.$\mathcal{C}$.*Complexes of Groups.*

Definition 2.4.25 (developability)**.** A complex of groups $G(\mathcal{Y})$ is called *developable* if there exists a group G and a scwol $\mathcal{X}$ such that G acts on $\mathcal{X}$, $\mathcal{Y} = G\backslash\mathcal{X}$ and $\mathcal{G}(\mathcal{Y})$ is a complex of groups over $\mathcal{Y}$ associated to this action.

Theorem 2.4.26 ([1] pp. 553-554: 3.13 Theorem, 3.15 Corollary)**.** *Each developable complex of groups $\mathcal{G}(\mathcal{Y})$ belongs to an action of a certain group G acting on a simply connected scwol $\mathcal{X}$. G and $\mathcal{X}$ are unique up to isomorphy.*

Remark 12. This theorem allows us to study group actions on simply connected scwols instead of Complexes of Groups. In our view, the first framework allows much more flexibility than the last one.

Definition 2.4.27. We say a group G satisfies *FCoFG* if and only if there are only finitely many G-conjugacy classes of finite subgroups in G, i.e. $G\backslash \operatorname{Sub}_{\mathrm{fin}}(G)$ is finite.

The following lemma is the reason why we consider FCoFG groups.

Lemma 2.4.28 (In FCoFG groups every finite subgroup is dominated by a maximal one)**.** *Let G be a group satisfying FCoFG. Then for each finite group $H \leq G$ there exists a maximal finite subgroup $K \leq G$ such that $H \leq K$.*

Proof. Clearly the size map

$$|\cdot|\colon \operatorname{Sub}_{\mathrm{fin}}(G) \to \mathbb{N}\colon H \mapsto |H|$$

can be pushed down to a well-defined map

$$|\cdot|_*\colon G\backslash \operatorname{Sub}_{\mathrm{fin}}(G) \to \mathbb{N}\colon [H]_G \mapsto |H|.$$

Because $G\backslash \operatorname{Sub}_{\mathrm{fin}}(G)$ is finite, there exists an $m \in \mathbb{N}$ such that $|[H]_G|_* \leq m$ for each $H \in \operatorname{Sub}_{\mathrm{fin}}(G)$. In particular, $|H| \leq m$ for each $H \in \operatorname{Sub}_{\mathrm{fin}}(G)$. Now, suppose contrary to the claim of the lemma, that there exists an element $H \in \operatorname{Sub}_{\mathrm{fin}}(G)$ such that for each $\operatorname{Sub}_{\mathrm{fin}}(G) \ni K \geq H$ there exists a $\operatorname{Sub}_{\mathrm{fin}}(G) \ni L > K$. Then we can construct recursively a sequence of the form $(H_k)_{k\in\mathbb{N}}$ with the properties $H = H_1$, $H_k < H_{k+1}$ and $H_k \in \operatorname{Sub}_{\mathrm{fin}}(G)$ for each $1 \leq k \in \mathbb{N}$. We then have that $|H_k| < |H_{k+1}|$ and via iteration we estimate $|H_k| \geq k$ for every $k \in \mathbb{N}$. Hence, we obtain $|H_{m+1}| \geq m + 1$, a contradiction to $|H_{m+1}| \leq m$. □

Lemma 2.4.29 (finite index subgroups of FCoFG groups are FCoFG)**.** *Let G be a group satisfying FCoFG. Then each subgroup $\Delta \leq G$ with the property $(G : \Delta) < \infty$ also satisfies FCoFG.*

Proof. By hypothesis there exists an $m \in \mathbb{N}$ such that $(G : \Delta) = m$. Pick an arbitrary element $H \in \operatorname{Sub}_{\mathrm{fin}}(\Gamma)$. Clearly, Δ acts on $[H]_G$ via conjugation. In this way, we obtain a map

$$\varphi\colon \Delta\backslash G \to \Delta\backslash[H]_G\colon \Delta g \mapsto [{}^{g}H]_\Delta,$$

which is obviously well-defined and surjective. In particular, $\Delta\backslash[H]_G$ is finite. On the other hand, we have

$$\Delta\backslash \operatorname{Sub}_{\text{fin}}(\Delta) \subseteq \Delta\backslash \operatorname{Sub}_{\text{fin}}(G) \subseteq \bigcup_{[H]_G \in G\backslash \operatorname{Sub}_{\text{fin}}(G)} \Delta\backslash[H]_G.$$

Combining both facts yields the following estimation:

$$|\Delta\backslash \operatorname{Sub}_{\text{fin}}(\Delta)| \leq \sum_{[H]_G \in G\backslash \operatorname{Sub}_{\text{fin}}(G)} |\Delta\backslash[H]_G| \leq |G\backslash \operatorname{Sub}_{\text{fin}}(G)| \cdot m.$$

By hypothesis $|G\backslash \operatorname{Sub}_{\text{fin}}(G)|$ is finite. This enforces together with the estimate above that $|\Delta\backslash \operatorname{Sub}_{\text{fin}}(\Delta)|$ has to be finite as well, which is the desired conclusion.

□

Notation 2.4.30. Let G be a group satisfying FCoFG. In the sequel, $M(G)$ stands for the set

$$M(G) := \{H \in \operatorname{Sub}_{\text{fin}}(G) \colon H \text{ is maximal in } \operatorname{Sub}_{\text{fin}}(G)\}$$

and $M^\circ(G)$ for the set

$$M^\circ(G) := M(G) \cap \operatorname{Sub}^\circ_{\text{fin}}(G).$$

Remark 13. $G \curvearrowright \operatorname{Sub}_{\text{fin}}(G)$ via conjugation. This action leaves $M(G)$ invariant because the conjugation with an element of G preserves the order "$\leq$" on $\operatorname{Sub}_{\text{fin}}(G)$. Hence $G \curvearrowright M(G)$ via conjugation and for a similar reason $G \curvearrowright M^\circ(G)$ via conjugation as well.

Definition 2.4.31. Let G be a group and $\mathcal{X} = (V(\mathcal{X}), E(\mathcal{X}), i_{\mathcal{X}}, t_{\mathcal{X}})$ be a scwol. We say *G acts on $\mathcal{X}$ with finite stabilizers* if and only if G acts on $\mathcal{X}$ and G_v is finite for each $v \in V(\mathcal{X})$.

Definition 2.4.32 (edges in a scwol; edge-graph; edge-path; connected scwol)**.** Let $\mathcal{X}$ be a scwol. Let $\mathcal{E}$ be a subset of $E(\mathcal{X})$. Set

$$\mathcal{E}^+ := \mathcal{E} \times \{1\},\ \mathcal{E}^- := \mathcal{E} \times \{-1\}$$

and finally

$$\mathcal{E}^\pm := \mathcal{E}^+ \uplus \mathcal{E}^-.$$

In this way we obtain maps

$$\iota^+ \colon E(\mathcal{X}) \to E^\pm(X) \colon a \mapsto (a, 1) =: a^+$$

and

$$\iota^- \colon E(\mathcal{X}) \to E^\pm(\mathcal{X}) \colon a \mapsto (a, -1) =: a^-.$$

Furthermore we define incidence maps $i' \colon E^\pm(\mathcal{X}) \to V(\mathcal{X})$ via

$$i'(e) := \begin{cases} t(a), & \text{if } e = a^+ \text{ for some } a \in E(\mathcal{X}), \\ i(a), & \text{if } e = a^- \text{ for some } a \in E(\mathcal{X}) \end{cases}$$

and

$$t'(e) := \begin{cases} i(a), & \text{if } e = a^+ \text{ for some } a \in E(\mathcal{X}), \\ t(a), & \text{if } e = a^- \text{ for some } a \in E(\mathcal{X}). \end{cases}$$

An element $e \in E^\pm(\mathcal{X})$ is called an *edge* of the scwol $\mathcal{X}$. Moreover call the quadruplet $\mathcal{X}^\pm := (V(\mathcal{X}), E^\pm(\mathcal{X}), i', t')$ the *edge-graph* of $\mathcal{X}$. The map

$$^{-1} \colon E^\pm(\mathcal{X}) \to E^\pm(\mathcal{X}) \colon$$

defined via

$$(a^+)^{-1} := a^- \text{ and } (a^-)^{-1} := a^+$$

is clearly an involution without fixed points satisfying $i'_{\mathcal{X}}(e^{-1}) = t'_{\mathcal{X}}(e)$. Thus $(\mathcal{X}^{\pm}, ^{-1})$ is indeed a graph in the sense Definition 2.4.1. For the reason the definition of $^{-1}$ does only depend on $\mathcal{X}$, we may identify $\mathcal{X}^{\pm}$ with $(\mathcal{X}^{\pm}, ^{-1})$. We say a path π is an *edge-path* in $\mathcal{X}$ if it is just a path in the edge-graph $\mathcal{X}^{\pm}$. We say the scwol $\mathcal{X}$ is *connected* if the associated edge-graph $\mathcal{X}^{\pm}$ is connected.

Lemma 2.4.33. *Let $\mathcal{X}$ be a scwol and $\mathfrak{H}$ be a subgraph of $\mathcal{X}^{\pm}$. Then there exists a set $\mathcal{E} \subseteq E(\mathcal{X})$ such that $\mathfrak{E}(\mathfrak{H}) = \mathcal{E}^{+} \uplus \mathcal{E}^{-}$.*

Proof. The set $\mathcal{E} := \{a \in E(\mathcal{X}) : a^{+} \in \mathfrak{E}(\mathfrak{H})\}$ will do the job. □

Our next goal is to construct a scwol $\tilde{\mathcal{X}}$ from $\mathcal{X}$ in a way such that G acts on $\tilde{X}$ and such that for a maximal finite subgroup H the set $\{v \in V(\tilde{\mathcal{X}}) : \ G_v = H\}$ consists of only one element. Furthermore there should be an epimorphism mapping the fundamental group of $\mathcal{X}$ onto that of $\tilde{\mathcal{X}}$ and the stabilizers of the action G on $\tilde{X}$ should be obtained in a canonical way from that of the action G on $\mathcal{X}$.

To this end, let G be a FCoFG group acting with finite stabilizers on a scwol $\mathcal{X}$. Then we can define the following equivalence relation on $V(\mathcal{X})$:

(2.4.1)
$$v \sim w \ :\Leftrightarrow \begin{cases} v = w & \text{or} \\ v \neq w, & G_v \in M^{\circ}(G) \text{ and there is an edge-path } \pi \text{ connecting} \\ & v \text{ to } w \text{ s.t. } G_{\sigma} = G_v \text{ for each } \sigma \in V(\pi). \end{cases}$$

It is easy to verify that "$\sim$" satisfies indeed the axioms of an equivalence relation on $V(\mathcal{X})$. Put
$$E'(\mathcal{X}) := E(\mathcal{X}) \setminus \{a \in E(\mathcal{X}) : G_{i(a)} \in M^{\circ}(G)\}.$$
For each $a, b \in E'(\mathcal{X})$ we write $a \sim b$ if and only if $i_{\mathcal{X}}(a) = i_{\mathcal{X}}(b)$ and $t_{\mathcal{X}}(a) \sim t_{\mathcal{X}}(b)$. This is obviously an equivalence relation on $E'(\mathcal{X})$. Now, we are ready to define the scwol $\tilde{\mathcal{X}}$. For this purpose we set

$$V(\tilde{\mathcal{X}}) := V(\mathcal{X})/\sim \ \text{ and } E(\tilde{\mathcal{X}}) := E'(\mathcal{X})/\sim$$

and furthermore we put

$$i_{\tilde{\mathcal{X}}}([a]) := [i_{\mathcal{X}}(a)] \text{ and } t_{\tilde{\mathcal{X}}}([a]) := [t_{\mathcal{X}}(a)].$$

In order to obtain an appropriate composition on $\tilde{X}$ we need to compute $E^{(2)}(\tilde{X})$. In fact, we can show

$$\forall (\alpha, \beta) \in E^{(2)}(\tilde{\mathcal{X}}) \colon \alpha \times \beta \subseteq E^{(2)}(\mathcal{X}).$$

To prove this take $(\alpha, \beta) \in E^{(2)}(\tilde{\mathcal{X}})$. We then have $t_{\tilde{X}}(\beta) = i_{\tilde{X}}(\alpha)$ and by definition of $\tilde{X}$ we get the relation $t_{\mathcal{X}}(b) \sim i_{\mathcal{X}}(a)$ for any $a \in \alpha$ and any $b \in \beta$. To deduce the claim, it is sufficient to show
$$t_{\mathcal{X}}(b) = i_{\mathcal{X}}(a)$$
for each choice $(a, b) \in \alpha \times \beta$.

On the contrary, suppose that $t_{\mathcal{X}}(b) \neq i_{\mathcal{X}}(a)$ for some $a_0 \in \alpha$ and $b_0 \in \beta$. The definition of "$\sim$" on $V(\mathcal{X})$ tells us that $G_{i_{\mathcal{X}}(a_0)}$ has to be maximal. But this implies $a_0 \notin E'(\mathcal{X})$, contradicting $\alpha \subseteq E'(\mathcal{X})$. This yields the claim.

So we define the composition on $E(\tilde{\mathcal{X}})$ via

$$\bullet \colon E^{(2)}(\tilde{\mathcal{X}}) \to E(\tilde{X}) \colon ([a], [b]) \mapsto [a \circ b].$$

We have to check that "$\bullet$" is a well-defined map. To this end let us take $a' \sim a$ and $b' \sim b$ and observe

$$i_{\tilde{X}}([a'] \bullet [b']) = [i_{\mathcal{X}}(a' \circ b')] = [i_{\mathcal{X}}(b')] = [i_{\mathcal{X}}(b)] = [i_{\mathcal{X}}(a \circ b)] = i_{\tilde{\mathcal{X}}}([a] \bullet [b])$$

and

$$t_{\tilde{X}}([a'] \bullet [b']) = [t_{\mathcal{X}}(a' \circ b')] = [t_{\mathcal{X}}(a')] = [t_{\mathcal{X}}(a)] = [t_{\mathcal{X}}(a \circ b)] = t_{\tilde{\mathcal{X}}}([a] \bullet [b])$$

which is the claim. We proceed by constructing the action on $\tilde{X}$.

It is not hard to see that G preserves the equivalence relation on $V(\mathcal{X})$ because G takes edge-paths to edge-paths, and the property of being a maximal finite subgroup as well as the property of having finite normalizer are invariant under automorphisms, in particular under conjugation. Therefore the action of G on $V(\tilde{\mathcal{X}})$ given by

$$g.[v] := [g.v], \quad \forall (g, v) \in G \times V(\tilde{\mathcal{X}})$$

is well-defined. For the same reasons it turns to be out that the action of G on $E(\mathcal{X})$ leaves the subset $E'(\mathcal{X})$ invariant and we thus obtain that the action of G on $E(\tilde{\mathcal{X}})$ for $[a]$ in $E(\tilde{\mathcal{X}})$ defined by

$$g.[a] := [g.a], \quad \forall (g, a) \in G \times E(\tilde{\mathcal{X}})$$

is also well-defined. In the same manner we observe that the actions $G \curvearrowright V(\tilde{X})$ and $G \curvearrowright E(\tilde{X})$ extend to an action G on $\tilde{X}$.
We call $\tilde{\mathcal{X}}$ the *reduction of* $\mathcal{X}$ associated to the action G on $\mathcal{X}$.

Remark 14. Considering the definition of $\tilde{\mathcal{X}}$ we immediately observe that for each $e \in E'(\mathcal{X})^{\pm}$ the following is true:

$$[e^{-1}] = [e]^{-1}.$$

Definition 2.4.34 (reduction of an edge-path)**.** Let $\mathcal{X}$ be a scwol and G be a group satisfying FCoFG and acting with finite stabilizers on it. Let $v, w \in V(\mathcal{X})$ and $\pi = (e_i)_{i=1}^n$ be an edge-path in $\mathcal{X}$ connecting v to w. Let $(j_k)_{k=1}^m$ be the unique strictly increasing enumeration of $\{i \in [n] : e_i \in E'^{\pm}(\mathcal{X})\}$. We set

$$\tilde{\pi} := \begin{cases} ([v]), & \text{if } \{i \in [n] : e_i \in E'^{\pm}(\mathcal{X})\} = \emptyset, \\ ([e_{j_k}])_{k=1}^m, & \text{otherwise.} \end{cases}$$

$\tilde{\pi}$ is an edge-path in $\tilde{\mathcal{X}}$ connecting $[v] \in V(\tilde{\mathcal{X}})$ to $[w] \in V(\tilde{\mathcal{X}})$. We say $\tilde{\pi}$ is the edge-path obtained by reduction from π.

Proof. Put $J := \{i \in [n] : e_i \in E'^{\pm}(\mathcal{X})\}$ and $m := |J|$. If $J = \emptyset$, there is nothing to prove.
So, let us assume $J \neq \emptyset$. We start our proof by showing that $\tilde{\pi}$ is indeed an edge-path in $\tilde{X}$ with indices in J. Let $(j_k)_{k=1}^m$ be the uniquely determined strictly increasing enumeration of J. Then, we have by definition $\tilde{\pi} = ([e_{j_k}])_{k=1}^m$. So take an arbitrary $1 \leq k \leq m-1$ and the following two cases may occur. In the case $j_k + 1 = j_{k+1}$ we have by hypothesis $t'_{\mathcal{X}}(e_{j_k}) = i'_{\mathcal{X}}(e_{j_k+1}) = i'_{\mathcal{X}}(e_{j_{k+1}})$ and we obtain finally $t'_{\tilde{\mathcal{X}}}([e_{j_k}]) = i'_{\tilde{\mathcal{X}}}([e_{j_{k+1}}])$. If the case $j_k + 1 \neq j_{k+1}$ occurs, we see $e_l \notin E'^{\pm}(\mathcal{X})$ for each $j_k + 1 \leq l \leq j_{k+1} - 1$. By definition $e_l = a_l^+$ or $e_l = a_l^-$ for some $a_l \in E(\mathcal{X}) \setminus E'(\mathcal{X})$. In both cases we have $G_{i_{\mathcal{X}}(a_l)}$ is maximal in $\mathrm{Sub}_{\mathrm{fin}}(G)$ and hence, applying the axioms for an action of G on $\mathcal{X}$, we get additionally $G_{t_{\mathcal{X}}(a_l)} = G_{i_{\mathcal{X}}(a_l)}$. This leads to $i_{\mathcal{X}}(a_l) \sim t_{\mathcal{X}}(a_l)$ and we therefore obtain in both cases to $i'_{\mathcal{X}}(e_l) \sim t'_{\mathcal{X}}(e_l)$. In particular, $(e_{j_k+l})_{l=1}^{j_{k+1}-j_k}$ is an edge path connecting $t'_{\mathcal{X}}(e_{j_k}) = i'_{\mathcal{X}}(e_{j_k+1})$ to $t'_{\mathcal{X}}(e_{j_{k+1}-1}) = i'_{\mathcal{X}}((e_{j_{k+1}}))$ such that $G_{i_{\mathcal{X}}(a_l)} = G_{i_X(a_{j_k+1})} \in M^{\circ}(G)$. We thus conclude $t'_X(e_{j_k}) \sim i'_X(e_{j_{k+1}})$ and therefore $t'_{\tilde{\mathcal{X}}}([e_{j_k}]) = i'_{\tilde{\mathcal{X}}}([e_{j_{k+1}}])$. This proofs that

$\tilde{\pi}$ is indeed an edge-path in $\tilde{\mathcal{X}}$. It remains to show $i'_{\tilde{\mathcal{X}}}(\tilde{\pi}) = [v]$ and $t'_{\tilde{\mathcal{X}}}(\tilde{\pi}) = [w]$. To compute $i'_{\tilde{\mathcal{X}}}(\tilde{\pi})$ consider the following two cases. If $1 = j_1$, we have obviously $i'_{\tilde{\mathcal{X}}}(\tilde{\pi}) = [i'_{\mathcal{X}}(\pi)] = [v]$ as required. If $1 \neq j_1$, we get $e_l \notin E'^{\pm}(\mathcal{X})$ for all $1 \leq l \leq j_1 - 1$. An analogous argumentation to the above one leads to $i'_{\mathcal{X}}(e_1) \sim i'_{\mathcal{X}}(e_{j_1})$ and thus $i'_{\tilde{\mathcal{X}}}(\tilde{\pi}) = [i'_{\mathcal{X}}(\pi)] = [v]$. In the same manner we may compute $t'_{\tilde{\mathcal{X}}}(\tilde{\pi})$ and obtain finally $t'_{\tilde{\mathcal{X}}}(\tilde{\pi}) = [t'_{\mathcal{X}}(\pi)] = [w]$. This completes the proof. □

Corollary 2.4.35. *Same hypothesis as above. If $\mathcal{X}$ is connected, then $\tilde{\mathcal{X}}$ is connected as well.*

Proof. Take arbitrary vertices $\bar{v}, \bar{w} \in V(\tilde{\mathcal{X}})$. Choose an arbitrary $v \in \bar{v}$ and an arbitrary $w \in \bar{w}$. By hypothesis there exists an edge-path π in $\mathcal{X}$ connecting v to w. Now, Definition 2.4.34 guarantees us the existence of an edge-path $\tilde{\pi}$ connecting $\bar{v}$ to $\bar{w}$. This yields the claim. □

Lemma 2.4.36. *Same hypothesis as above. Let π be an arbitrary edge-path and η be the edge-path without backtracking associated to π. Furthermore let $\tilde{\pi}$ or $\tilde{\eta}$ its reductions respectively. Let ω be the edge-path without backtracking associated to $\tilde{\pi}$. Then, ω is also the edge-path without backtracking associated to $\tilde{\eta}$.*

Proof. By Remark 14 it is possible to obtain $\tilde{\eta}$ by iterating the deletion of backtracking from $\tilde{\pi}$. Now, we delete backtracking from $\tilde{\eta}$ until we get to a path without backtracking. The resulting path is the edge-path without backtracking associated to both, $\tilde{\pi}$ and $\tilde{\eta}$. By the uniqueness of the edge-path associated to $\tilde{\pi}$ it has to coincide with ω. □

Lemma 2.4.37. *Let $\mathcal{X}$ be a scwol and G be a group satisfying FCoFG and acting with finite stabilizers on it. Furthermore, let $\bar{v}, \bar{w} \in V(\tilde{\mathcal{X}})$ arbitrary vertices in $\tilde{\mathcal{X}}$ and τ be an arbitrary edge-path in $\tilde{\mathcal{X}}$ connecting $\bar{v}$ to $\bar{w}$. Then for each $v \in \bar{v}$ and $w \in \bar{w}$ there exists an edge-path π in $\mathcal{X}$ connecting v to w such that $\tilde{\pi} = \tau$.*

Proof. Let $\bar{v}, \bar{w} \in V(\tilde{\mathcal{X}})$ and $\tau = (\bar{e}_1, \dots, \bar{e}_m)$ be an edge-path connecting $\bar{v}$ to $\bar{w}$. Fix $v \in \bar{v}$ and $w \in \bar{w}$. We will construct an edge-path π in $\mathcal{X}$, satisfying the conditions above, in a recursive way. Let us start with an arbitrary edge $e_1 \in \bar{e}_1$. We put $v' := i'_{\mathcal{X}}(e_1) \in \bar{v}$. If $v = v'$, just set $\pi_1 := (e_1)$. If we have $v \neq v'$, there exists by definition of the equivalence relation over $V(\mathcal{X})$ an edge-path η_1 in $\mathcal{X}$ connecting v to v', such that $G_\sigma = G_v$ for all $\sigma \in V(\eta_1)$ and $G_v \in M^\circ(G)$, and we set $\pi_1 := \eta_1 * (e_1)$. Let us assume that for $1 \leq k < m$ the edge-path π_k is already defined. We then obtain π_{k+1} as follows. Take $e_{k+1} \in \bar{e}_{k+1}$. We have at least $t'_{\mathcal{X}}(\pi_k) \sim i'_{\mathcal{X}}(e_{k+1})$. If $t'_{\mathcal{X}}(\pi_k) = i'_{\mathcal{X}}(e_{k+1})$, just set $\pi_{k+1} := \pi_k * (e_{k+1})$. In the case $t'_{\mathcal{X}}(\pi_k) \neq i'_{\mathcal{X}}(e_{k+1})$, there exists an edge-path η_{k+1} connecting $t'_{\mathcal{X}}(\pi_k)$ to $i'_{\mathcal{X}}(e_{k+1})$, such that $G_\sigma = G_{t'_{\mathcal{X}}(\pi_k)} \in M^\circ(G)$ for all $\sigma \in V(\eta_{k+1})$, and we set $\pi_{k+1} := \pi_k * \eta_{k+1} * (e_{k+1})$. In this way we have constructed an edge-path π_m connecting v to $t'_{\mathcal{X}}(\pi_m) = t'_{\mathcal{X}}(e_m) \in \bar{w}$. Let $w \in \bar{w}$. If $t'_{\mathcal{X}}(\pi_m) = w$, just set $\pi := \pi_m$. In the case $t'_{\mathcal{X}}(\pi_m) \neq w$ there exists an edge-path η_{m+1} connecting $t'_{\mathcal{X}}(\pi_m)$ to w, such that $G_\sigma = G_{t'_{\mathcal{X}}(\pi_m)} \in M^\circ(G)$ for all $\sigma \in V(\eta_{m+1})$, and we set $\pi := \pi_m * \eta_{m+1}$. By construction π is an edge-path in $\mathcal{X}$ connecting v to w. For the reason that

$$\mathfrak{E}(\eta_j) \subseteq E^{\pm}(\mathcal{X}) \setminus E'^{\pm}(\mathcal{X}) \quad \forall 1 \leq j \leq m+1$$

we obtain for the reductions $\tilde{\pi}_k$ of π_k

$$\tilde{\pi}_1 = ([e_1]) \text{ and } \tilde{\pi}_{k+1} = \tilde{\pi}_k * ([e_k]).$$

and therefore $\tilde{\pi} \equiv ([e_1], \dots, [e_m]) = (\bar{e}_1, \dots, \bar{e}_m) = \tau$. This proves the claim of the lemma. □

The following definitions are taken from [1].

Definition 2.4.38 (universal group associated to a complex of groups)**.** Let $\mathcal{G}(\mathcal{Y}) = (G_v, \psi_a, g_{a,b})$ be a complex of groups over the scwol $\mathcal{Y}$. The *universal group* $F\mathcal{G}(\mathcal{Y})$ is the group given by the following presentation: It is generated by the set

$$\biguplus_{v \in V(\mathcal{Y})} G_v \uplus E^{\pm}(\mathcal{Y})$$

subject to the relations

$$R := \left\{ \begin{array}{c} \text{the relations in the groups } G_v, \\ a^+a^- = 1 = a^-a^+, \\ a^+b^+ = g_{a,b}(ab)^+, \ \forall (a,b) \in E^{(2)}(\mathcal{Y}) \\ \psi_a(g) = a^+ga^-, \ \forall g \in G_{i(a)} \end{array} \right\}.$$

Remark 15. We may regard a scwol $\mathcal{Y}$ as the complex of groups over $\mathcal{U}(Y)$ whose vertex groups are all trivial. We call it the trivial complex of groups over $\mathcal{Y}$. So the universal group over $\mathcal{Y}$ is just $F\mathcal{Y} := F\mathcal{U}(\mathcal{Y})$.

Definition 2.4.39 ($\mathcal{G}(\mathcal{Y})$-path, concatenation of $\mathcal{G}(\mathcal{Y})$-paths)**.** Let $\mathcal{Y}$ be a scwol and $\mathcal{G}(\mathcal{Y}) = (G_v, \psi_a, g_{a,b})$ a complex of groups over $\mathcal{Y}$. Then, a $\mathcal{G}(\mathcal{Y})$*-path* c connecting v to w is a tuple

$$c = (g_0, e_1, g_1, \ldots, e_m, g_m)$$

where $(e_1, \ldots, e_n)$ is an edge-path connecting v to w, $g_0 \in G_v$ and $g_j \in G_{t(e_j)}$ for all $j \in [m]$. We set $i(c) := i'_{\mathcal{X}^{\pm}}(e_1)$ and $t(c) := t'_{\mathcal{X}^{\pm}}(e_m)$. Let $c = (g_0, e_1, g_1, \ldots, e_m, g_m)$ and $c' = (g'_0, e'_1, g'_1, \ldots, e'_n, g'_n)$ be two $\mathcal{G}(\mathcal{Y})$-paths such that $t(c) = i(c')$. The *concatenation* of c with c' is the $\mathcal{G}(\mathcal{Y})$-path

$$c * c' := (g_0, e_1, g_1, \ldots, e_m, g_m g'_0, e'_1, g'_1, \ldots, e'_n, g'_n).$$

Remark 16. For the ease of notation, this definition only covers the case where the considered edge-paths are non-trivial. For the trivial cases just take the obvious definition.

It is easy to verify that the concatenation satisfies the associative law.

Definition 2.4.40. Let $\mathcal{Y}$ be a scwol. For an edge-path $(e_1, \ldots, e_m)$ put $F(c) := e_1 \cdots e_m \in F\mathcal{Y}$ and for a trivial edge-path c put $F(c) := 1 \in F\mathcal{Y}$. Let $\mathcal{G}(\mathcal{Y})$ be a complex of groups over $\mathcal{Y}$. For a $\mathcal{G}(\mathcal{Y})$-path $c = (g_0, e_1, g_1, \ldots, e_m, g_m)$ set $F(c) := g_0 e_1 g_1 \cdots e_m g_m \in F\mathcal{G}(\mathcal{Y})$. It is easy to verify that the maps

$$F\colon \begin{array}{ccc} \{\text{edge-paths}\} & \to & F\mathcal{Y}\colon & c \mapsto F(c) \\ \{\mathcal{G}(\mathcal{Y})\text{-paths}\} & \to & F\mathcal{G}(\mathcal{Y})\colon & c \mapsto F(c) \end{array}$$

are homomorphisms in the sense, that for each pair of edge-paths or $\mathcal{G}(\mathcal{Y})$-paths c, c' with the property $t(c) = i(c')$ it is true that $F(c * c') = F(c)F(c')$.

Definition 2.4.41 (rooted fundamental group of a scwol)**.** Let $\mathcal{Y}$ be a scwol and $v_0 \in V(\mathcal{Y})$.

$$\pi_1(\mathcal{Y}, v_0) := \{F(\pi) \in F\mathcal{Y} : \pi \text{ is an edge-path such that } i(\pi) = v_0 = t(\pi)\}$$

is a subgroup of $F\mathcal{Y}$ and it is called the *fundamental group of* $\mathcal{Y}$ rooted in v_0.

Definition 2.4.42 (rooted fundamental group of a complex of groups)**.** Let $\mathcal{G}(\mathcal{Y}) = (G_v, \psi_a, g_{a,b})$ be a complex of groups over a connected scwol $\mathcal{Y}$ and $v_0 \in V(\mathcal{Y})$. The set

$$\pi_1(\mathcal{G}(\mathcal{Y}), v_0) := \{F(c) \in F\mathcal{G}(\mathcal{Y}) : c \text{ is } \mathcal{G}(\mathcal{Y})\text{-path such that } i(c) = v_0 = t(c)\}$$

is a subgroup of $F\mathcal{G}(\mathcal{Y})$ and it is called the *fundamental group of* $\mathcal{G}(\mathcal{Y})$ rooted in v_0.

Remark 17. Let $\mathcal{Y}$ be a scwol and $\mathcal{U}(\mathcal{Y})$ be the trivial complex of groups over $\mathcal{Y}$. Fix $v_0 \in V(\mathcal{Y})$. Then it is obviously true that

$$\pi_1(\mathcal{Y}, v_0) = \pi_1(\mathcal{U}(\mathcal{Y}), v_0).$$

Definition 2.4.43 (fundamental group of a complex of groups relative to T)**.** Let $\mathcal{G}(\mathcal{Y}) = (G_v, \psi_a, g_{a,b})$ be a complex of groups over a connected scwol $\mathcal{Y}$ and T be a spanning tree in $\mathcal{Y}^{\pm}$. Moreover, let N be the normal hull of the set $\{F(e) : e \in \mathfrak{E}(T)\}$ in $F\mathcal{G}(\mathcal{Y})$. The *fundamental group of* $\mathcal{G}(\mathcal{Y})$ relative to T is the group

$$\pi_1(\mathcal{G}(\mathcal{Y}), T) := F\mathcal{G}(\mathcal{Y})/N.$$

Remark 18. There is an analogue definition for a fundamental group of a scwol relative to T, by identifying $\mathcal{Y}$ with $\mathcal{U}(\mathcal{Y})$. We can just put

$$\pi_1(\mathcal{Y}, T) := \pi_1(\mathcal{U}(\mathcal{Y}), T).$$

Lemma 2.4.44 (equivalence of the definitions, [1] p.549 theorem 3.7)**.** *Let* $\mathcal{G}(\mathcal{Y}) = (G_v, \psi_a, g_{a,b})$ *be a complex of groups over a connected scwol* $\mathcal{Y}$*,* $v_0 \in V(\mathcal{Y})$ *be an arbitrary vertex and* T *be an arbitrary spanning tree in* $\mathcal{Y}$*. Then, the following is true*

$$\pi_1(\mathcal{G}(\mathcal{Y}), v_0) \cong \pi_1(\mathcal{G}(\mathcal{Y}), T).$$

In particular the isomorphy class of the fundamental group over $\mathcal{G}(\mathcal{Y})$ *does neither depend on the root nor the spanning tree.*

Definition 2.4.45 (simply-connected scwol)**.** A connected scwol $\mathcal{X}$ is called *simply-connected* if there exists a vertex $v_0 \in V(\mathcal{X})$ such that $\pi_1(\mathcal{X}, v_0)$ is trivial.

Theorem 2.4.46. *Let* $\mathcal{X}$ *be a connected scwol and* G *be a group satisfying FCoFG and acting with finite stabilizers on it. Let* $\tilde{\mathcal{X}}$ *be the reduction of* $\mathcal{X}$ *associated to the action of* G *on* $\mathcal{X}$*. Fix an arbitrary vertex* $v_0 \in V(\mathcal{X})$*. Then,*

$$\Theta\colon \pi_1(\mathcal{X}, v_0) \to \pi_1(\tilde{\mathcal{X}}, [v_0])\colon F(\pi) \mapsto F(\tilde{\pi})$$

is a well-defined map. Moreover it is a surjective homomorphism.

Proof. Let us introduce some notation before we start with the proof. For an arbitrary scwol $\mathcal{Y}$ denote by $\mathfrak{F}\mathcal{Y}$ be the free group generated by $E^{\pm}(\mathcal{Y})$. By definition $F\mathcal{Y}$ is the quotient $\mathfrak{F}\mathcal{Y}/N_{\mathcal{Y}}$ where $N_{\mathcal{Y}}$ is the normal hull in $\mathfrak{F}\mathcal{Y}$ of the set

$$R_{\mathcal{Y}} := \left\{ \begin{array}{c} a^+a^- = 1 = a^-a^+, \\ a^+b^+ = (ab)^+, \ \forall (a,b) \in E^{(2)}(\mathcal{Y}) \end{array} \right\}.$$

We first observe, applying Corollary 2.4.35, that the reduction $\tilde{\mathcal{X}}$ is connected and thus $\pi_1(\tilde{\mathcal{X}}, [v_0])$ is defined. Next, let us remark, that the reduction induces a surjective homomorphism

$$p\colon \mathfrak{F}\mathcal{X} \to \mathfrak{F}\tilde{\mathcal{X}}\colon e \mapsto \begin{cases} [e], & \text{if } e \in E'(\mathcal{X}); \\ 1, & \text{otherwise.} \end{cases}$$

Further, we infer that p maps $R_{\mathcal{X}}$ to $R_{\tilde{\mathcal{X}}}$ and thus $N_{\mathcal{X}}$ to $N_{\tilde{\mathcal{X}}}$. So we may push the map p down to a surjective map

$$\bar{p}\colon F\mathcal{X} \to F\tilde{\mathcal{X}}.$$

By Definition 2.4.34 the image of the restriction of $\bar{p}$ to $\pi_1(\mathcal{X}, v_0)$ is necessarily a subset of $\pi_1(\tilde{\mathcal{X}}, [v_0])$. Because Θ is just $\bar{p}$ with range restricted to $\pi_1(\tilde{\mathcal{X}}, [v_0])$, we already have that Θ is a well-defined homomorphism. It remains to show that Θ is surjective. For this purpose, consider an arbitrary element $g \in \pi_1(\tilde{\mathcal{X}}, [v])$. Clearly, $g = F(\tau)$ for an appropriate edge-path τ without backtracking in $\tilde{\mathcal{X}}$ such that

$i(\tau) = [v_0] = t(\tau)$. Now, Lemma 2.4.37 guarantees the existence of an edge-path η in $\mathcal{X}$ such that $\tau = \tilde{\eta}$. We therefore obtain finally

$$\Theta(F(\eta)) = F(\tilde{\eta}) = F(\tau) = g,$$

which is the desired conclusion. □

We get the following immediate corollary:

Corollary 2.4.47. *Same hypothesis as above. If $\mathcal{X}$ is simply-connected, then the reduction $\tilde{\mathcal{X}}$ is simply-connected as well.*

Lemma 2.4.48 (reduction preserves the stabilizers)**.** *Let $\mathcal{X}$ be a scwol and G be a group satisfying FCoFG and acting with finite stabilizers on it. Then, G acts on the scwol $\tilde{X}$. If $\mathcal{X}$ is a simply-connected scwol, $\tilde{\mathcal{X}}$ is simply-connected as well. Moreover, the group G even acts with finite stabilizers on $\tilde{\mathcal{X}}$. In fact, the following is true:*

$$\forall v \in V(\mathcal{X})\colon G_{[v]} = G_v.$$

Proof. It is only left to prove the last part of the proposition, everything else has been shown before. For this purpose, fix $v \in V(\mathcal{X})$. If $|[v]| = 1$, g fixes $[v]$ if and only if g fixes v and we are done. Therefore, assume $|[v]| \neq 1$. The definition of the equivalence relation on $V(\mathcal{X})$ ensures that $G_v \in M^\circ(G)$. Therefore, recalling the definition of $M^\circ(G)$, it is sufficient to show $G_{[v]} \leq N_G(G_v)$. To this end, pick an arbitrary $g \in G_{[v]}$. Because g fixes $[v]$, we clearly have $g.v \in [v]$. The definition of the equivalence relation on $V(\mathcal{X})$ guarantees the existence of a path π connecting v to $g.v$ such that $G_\sigma = G_v$ for each $\sigma \in V(\pi)$. In particular, this enforces

$$G_v = G_{g.v} = gG_vg^{-1}$$

and therefore $g \in N_G(G_v)$ as required. □

Now, we are in the position to state and prove the main theorem of this chapter.

Theorem 2.4.49. *Let G be a group satisfying FCoFG and acting with finite stabilizers on a simply-connected scwol $\mathcal{X}$, which has the following additional properties:*

(1) *For each $H \in \mathrm{Sub}_{\mathrm{fin}}(G)$ there exists a $v \in V(\mathcal{X})$ such that $H \leq G_v$;*
(2) *$\forall v, w \in V(\mathcal{X})\colon G_v = G_w \in M^\circ(G)\ \exists$ path π connecting v to $w\colon G_\sigma = G_v\ \forall \sigma \in V(\pi)$.*

By Lemma 2.4.48 the reduction $\tilde{\mathcal{X}}$ is simply-connected and G acts with finite stabilizers on it. Denote by $\mathcal{Y} := G\backslash\tilde{\mathcal{X}}$ and by $\mathcal{G}(\mathcal{Y}) = (G_{\bar{v}}, \psi_{\bar{a}}, g_{\bar{a},\bar{b}})$ an arbitrary complex of groups over $\mathcal{Y}$ associated to that action. Finally set

$$M^\circ(\mathcal{G}(\mathcal{Y})) := \{\bar{v} \in V(\mathcal{Y}) : G_{\bar{v}} \in M^\circ(G)\}.$$

Then, the map

$$\Lambda\colon M^\circ(\mathcal{G}(\mathcal{Y})) \to G\backslash M^\circ(G)\colon \bar{v} \mapsto [G_{\bar{v}}]_G$$

is a bijection.

Proof. It remains to show that Λ is bijective. Everything else follows from Lemma 2.4.48. We begin by proving that Λ is surjective. For this purpose, take an arbitrary conjugacy class $H \in M^\circ(G)$. Condition (1) ensures the existence of a $v \in V(\mathcal{X})$ such that $H \leq G_v$. Because G acts with finite stabilizers on $\mathcal{X}$ we already get by the maximality of H the equality $H = G_v$. Lemma 2.4.48 tells us that $H = G_{[v]}$. In particular, $G_{[v]} \in M^\circ(G)$. Bringing back the definition of the quotient scwol $\mathcal{Y}$ to our mind, we infer, that we necessarily have $\bar{v} := G.[v] \in V(\mathcal{Y})$. Now, the definition of $\mathcal{G}(\mathcal{Y})$ guarantees the existence of an element $[v'] \in G.[v]$ such that $G_{[v']} = G_{\bar{v}}$. We thus obtain

$$[G_{v'}]_G = [G_{\bar{v}}]_G = \Lambda(\bar{v}),$$

as required.

To prove, that Λ is also an injective map, consider $\bar{v}, \bar{w} \in V(\mathcal{Y})$ such that $G_{\bar{v}} = {}^{g}(G_{\bar{w}})$ for some $g \in G$. By definition of $\mathcal{G}(\mathcal{Y})$ there exist $[v], [w] \in V(\tilde{\mathcal{X}})$ such that $\bar{v} = [v],\ \bar{w} = [w]$ and $G_{[v]} = {}^{g}\big(G_{[w]}\big) = G_{[g.w]}$. But then Lemma 2.4.48 ensures that

$$G_v = G_{w'} \quad \forall v \in [v]\ \forall w' \in [g.w].$$

If $v = w'$, we get $[v] = g.[w]$. For $v \neq w'$, condition (2) yields a path π connecting v to w' such that $G_\sigma = G_v \in M^\circ(G)$ for each $\sigma \in V(\pi)$. In particular, $v \sim w'$ which means $[v] = g.[w]$. So we get, no matter the case,

$$\bar{v} = G.[v] = G.[w] = \bar{w},$$

which is the desired conclusion. □

Our next goal, is to show that taking reductions of scwols is compatible with taking actions of subgroups on them. To this end, we need to distinguish the reductions associated to actions of subgroups on a scwol with the action of the original group on it. Hence, we introduce the following notation:

Notation 2.4.50. Let G be a group satisfying FCoFG and $\mathcal{X}$ be a scwol, such that G acts with finite stabilizers on $\mathcal{X}$. We denote the reduction of $\mathcal{X}$ associated to the action of G on $\mathcal{X}$ by $\mathcal{R}_G(\mathcal{X})$. Moreover, we write $\mathcal{R}_G(\pi)$ for the reduction of an edge-path π in $\mathcal{X}$. Finally we set

$$\mathcal{R}_G(E(\mathcal{X})) := E(\mathcal{X}) \smallsetminus \{a \in E(\mathcal{X}) :\ G_{i(a)} \in M^\circ(G)\}$$

and for each subset $\mathcal{E} \subseteq E(\mathcal{X})$

$$\mathcal{R}_G(\mathcal{E}) := \mathcal{R}_G(E(\mathcal{X})) \cap \mathcal{E}.$$

Note that $\mathcal{R}_G(E(\mathcal{X}))$ is exactly the set which we have denoted by $E'(\mathcal{X})$ until now.

Theorem 2.4.51. *Let G be a group satisfying FCoFG and $\mathcal{X}$ be a scwol, such that G acts with finite stabilizers on it. Furthermore, let $\Delta \leq G$, such that $(G : \Delta) < \infty$. Now, put*

$$\mathcal{R}_{\Delta,G}(E(\mathcal{X})) := \mathcal{R}_\Delta(E(\mathcal{X})) \cap \mathcal{R}_G(E(\mathcal{X})).$$

Then the following assertion is true: There exists a pair of surjective maps

$$\varphi\colon \begin{array}{rcl} V(\mathcal{R}_\Delta(\mathcal{X})) & \to & V(\mathcal{R}_\Delta(\mathcal{R}_G(\mathcal{X})))\colon\ \ [v]_{\mathcal{R}_\Delta} \to [[v]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta} \\ \mathcal{R}_{\Delta,G}(E(\mathcal{X}))/\underset{\mathcal{R}_\Delta}{\sim} & \to & E(\mathcal{R}_\Delta(\mathcal{R}_G(\mathcal{X})))\colon\ \ [a]_{\mathcal{R}_\Delta} \to [[a]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta} \end{array},$$

and each map of that pair is Δ-equivariant.

The reason, why we cannot prove $\mathcal{R}_\Delta(\mathcal{R}_G(\mathcal{X})) \cong \mathcal{R}_\Delta(\mathcal{X})$, is that M is maximal finite in G does not imply $M \cap \Delta$ is maximal finite in Δ. If we change the construction such that this problem does not arise, we cannot get exact information how the isotropy groups might change.

Proof. Let $K \leq G$ be an arbitrary subgroup. We write $\underset{\mathcal{R}_K}{\sim}$ for the equivalence relation on $V(\mathcal{X})$ mentioned in (2.4.1). In the same way, let us write $\underset{\mathcal{R}_K}{\sim}$ for the equivalence relation on $\mathcal{R}_K(E(\mathcal{X}))$.

We begin our proof by showing that φ is a pair of well-defined maps. First, consider the "map" between the vertex sets. So, pick $v, w \in V(\mathcal{X})$ such that $v \underset{\mathcal{R}_\Delta}{\sim} w$. If $[v]_{\mathcal{R}_G} = [w]_{\mathcal{R}_G}$ there is nothing to prove. Hence assume $[v]_{\mathcal{R}_G} \neq [w]_{\mathcal{R}_G}$. Now, by definition of the equivalence relation there exists a path π in $\mathcal{X}$ connecting v to w such that $\Delta_v \in M^\circ(\Delta)$ and $\Delta_\sigma = \Delta_v\ \forall \sigma \in V(\pi)$. Reduction of the path associated to the action of G on $\mathcal{X}$ yields a path $\mathcal{R}_G(\pi)$ connecting $[v]_{\mathcal{R}_G}$ and $[w]_{\mathcal{R}_G}$. Applying Lemma 2.4.48 to the action of G on $\mathcal{X}$, we obtain

$$\forall \sigma \in V(\pi)\colon G_{[\sigma]_{\mathcal{R}_G}} = G_\sigma.$$

Hence, we get finally for an arbitrary but fixed $\sigma \in V(\pi)$:

$$\Delta_{[\sigma]_{\mathcal{R}_G}} = G_{[\sigma]_{\mathcal{R}_G}} \cap \Delta = G_\sigma \cap \Delta = \Delta_\sigma.$$

In particular $\mathcal{R}_G(\pi)$ is a path such that $\Delta_{[\sigma]_{\mathcal{R}_G}} = \Delta_{[v]_{\mathcal{R}_G}}$ and $\Delta_{[v]_{\mathcal{R}_G}} = \Delta_v \in M^\circ(\Delta)$. This yields $[v]_{\mathcal{R}_G} \underset{\mathcal{R}_\Delta}{\sim} [w]_{\mathcal{R}_G}$ and thus

$$[[v]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta} = [[w]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta},$$

as required. We proceed by showing that the second entry of φ is well-defined. To this end, take $a, b \in \mathcal{R}_{\Delta,G}(E(\mathcal{X}))$ such that $a \underset{\mathcal{R}_\Delta}{\sim} b$. So, we have $i_{\mathcal{X}}(a) = i_{\mathcal{X}}(b)$ and one of the following cases occurs: If $t_{\mathcal{X}}(a) = t_{\mathcal{X}}(b)$, we have $[a]_{\mathcal{R}_G} = [b]_{\mathcal{R}_G}$. Moreover, $a, b \in \mathcal{R}_{\Delta,G}(E(\mathcal{X}))$ implies that $[a]_{\mathcal{R}_G}, [b]_{\mathcal{R}_G} \in \mathcal{R}_\Delta(E(\mathcal{R}_G(\mathcal{X})))$ and thus

$$[[a]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta} = [[b]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta}.$$

If $t_{\mathcal{X}}(a) \neq t_{\mathcal{X}}(b)$ there exists an edge-path π connecting $t_{\mathcal{X}}(a)$ to $t_{\mathcal{X}}(b)$ such that $\Delta_\sigma = \Delta_{t_{\mathcal{X}}(a)}$ and $\Delta_{t_{\mathcal{X}}(a)} \in M^\circ(\Delta)$. The same argument as for the map between the vertex sets applies and we therefore obtain

$$[[a]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta} = [[b]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta}$$

and the second entry of φ is a well-defined map, as desired.

We now turn to the proof of the surjectivity of those maps. But this is immediate for the first entry of φ. It remains to show the surjectivity of the second one. For this purpose, take $\alpha \in E(\mathcal{R}_\Delta(\mathcal{R}_G(\mathcal{X})))$. Then there exists an $\alpha' \in \mathcal{R}_\Delta(E(\mathcal{R}_G(\mathcal{X})))$ such that $\alpha = [\alpha']_{\mathcal{R}_\Delta}$. By definition of $E(\mathcal{R}_G(\mathcal{X}))$ there exists an $a \in \mathcal{R}_G(E(\mathcal{X}))$ such that $\alpha' = [a]_{\mathcal{R}_G}$. On the other hand, $\alpha' = [a]_{\mathcal{R}_G} \in \mathcal{R}_\Delta(E(\mathcal{R}_G(\mathcal{X})))$ forces $\Delta_{i(\alpha')} \notin M^\circ(\Delta)$ is not maximal in $\mathrm{Sub}_{\mathrm{fin}}(\Delta)$. In this situation an application of Lemma 2.4.48 yields $\Delta_{i(a)} = \Delta_{i(\alpha')}$ and thus $\Delta_{i_{\mathcal{X}}(a)} \notin M^\circ(\Delta)$. This leads to $a \in \mathcal{R}_\Delta(E(\mathcal{X}))$. We infer

$$a \in \mathcal{R}_G(E(\mathcal{X})) \cap \mathcal{R}_\Delta(E(\mathcal{X})) = \mathcal{R}_{\Delta,G}(E(\mathcal{X})).$$

Now, we may insert $[a]_{\mathcal{R}_\Delta}$ in φ and in this way we obtain

$$\alpha = [[a]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta} = \varphi([a]_{\mathcal{R}_\Delta}),$$

which is exactly what we have claimed. The Δ-equivariance of φ is a trivial consequence of the way how we have defined the group action on reductions. Nevertheless, we give a proof of it. So, let us take an arbitrary $v \in V(\mathcal{X})$, $a \in \mathcal{R}_{\Delta,G}(E(\mathcal{X}))$ and $\delta \in \Delta$. We compute

$$\varphi(\delta.[v]_{\mathcal{R}_\Delta}) = \varphi([\delta.v]_{\mathcal{R}_\Delta}) = [[\delta.v]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta} = \delta.[[v]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta} = \delta.\varphi([v]_{\mathcal{R}_\Delta}).$$

and

$$\varphi(\delta.[a]_{\mathcal{R}_\Delta}) = \varphi([\delta.a]_{\mathcal{R}_\Delta}) = [[\delta.a]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta} = \delta.[[a]_{\mathcal{R}_G}]_{\mathcal{R}_\Delta} = \delta.\varphi([a]_{\mathcal{R}_\Delta}),$$

which completes the proof of the proposition. □

2.5 System of Representatives for certain quotients of $\mathrm{Sl}_3(\mathbb{Z})$

We set up the following terminology: Let us denote by Γ the group $\mathrm{Sl}_3(\mathbb{Z})$. For an arbitrary but fixed squarefree non-negative integer d we write

$$\bar{\Gamma}^{(d)} := \mathrm{Sl}_3(\mathbb{Z}/d\mathbb{Z}),$$

$$\bar{\Delta}^{(d)} := \left\{(a_{ij})_{(i,j)\in[3]\times[3]} \in \mathrm{Sl}_3(\mathbb{Z}/d\mathbb{Z}) : \ \forall (i,j) \in [3]\times[3] \colon i > j \colon a_{ij} = 0\right\}$$

and

$$\Delta^{(d)} := \left\{(a_{ij})_{(i,j)\in[3]\times[3]} \in \mathrm{Sl}_3(\mathbb{Z}) : \ \forall (i,j) \in [3]\times[3] \colon i > j \colon a_{ij} \overset{(d)}{\equiv} 0\right\}.$$

If d is a prime, we sometimes call $\bar{\Delta}^{(d)}$ the *Borel subgroup* of $\bar{\Gamma}^{(d)}$. $\Delta^{(d)}$ can be viewed as the preimage of $\bar{\Delta}^{(d)}$ under the *congruence map*

$$\Phi^{(d)} \colon \Gamma \to \bar{\Gamma}^{(d)} \colon (a_{ij})_{i,j} \mapsto (a_{ij} + d\mathbb{Z})_{i,j}.$$

It is a well known fact that the congruence map is an epimorphism of groups. Because $\Phi^{(d)}$ is an epimorphism taking $\Delta^{(d)}$ to $\bar{\Delta}^{(d)}$, we can $\Phi^{(d)}$ push down to an isomorphism between right Γ-sets, namely

$$\tilde{\Phi}^{(d)} \colon \Delta^{(d)}\backslash\Gamma \xrightarrow{\cong} \bar{\Delta}^{(d)}\backslash\bar{\Gamma}^{(d)}. \tag{2.5.1}$$

In particular, $\Delta^{(d)}\backslash\Gamma$ is of finite cardinality.

Let p be a prime. Therefore, p is a squarefree non-negative integer and the notation, we have introduced so far, applies. The main goal of this chapter is to find a system of representatives for the set of cosets $\Delta^{(p)}\backslash\Gamma$.
To this end, we may view $\mathbb{Z}/p\mathbb{Z}$ as field $\mathbb{F}_p$. Based on that and the observations above, we are able to compute the exact number of elements of $\Delta^{(p)}\backslash\Gamma$:

$$\begin{aligned}
\left|\Delta^{(p)}\backslash\Gamma\right| &= \left|\bar{\Delta}^{(p)}\backslash\bar{\Gamma}^{(p)}\right| \\
&= \frac{|\,\mathrm{Gl}_3(\mathbb{F}_p)|}{|\mathbb{F}_p^\times||\bar{\Delta}^{(p)}|} \\
&= \frac{p^3(p^3-1)(p^2-1)(p-1)}{(p-1)(p-1)^2p^3} \\
&= (p^2+p+1)(p+1).
\end{aligned}$$

We will again make use of the isomorphism $\tilde{\Phi}^{(p)}$ during the determination of an appropriate set of representatives for $\Delta^{(p)}\backslash\Gamma$. The idea behind the construction is a slightly modified LR-algorithm for $\mathrm{Gl}_3(\mathbb{F}_p)$: The idea is to apply "permutation" matrices to the set of matrices of the shape $\begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ v & w & 1 \end{pmatrix}$, $u, v, w \in \mathbb{F}_p$ from the right and to look for redundancies under the elimination operations which are induced by the multiplication with elements of $\bar{\Delta}^{(p)}$ from the left. To this end, we have to modify the permutation matrices in a way such that they have determinant one. This can be achieved by modifying one entry by a sign if necessary.

From this background it would maybe be more confident to talk about an "*RL*-construction" instead of an *LR*-construction in this context.

Before we state the result of this consideration, let us introduce some notation.

Notation 2.5.1.

$$\bar{R}_1^{(p)} := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ v & w & 1 \end{pmatrix} : u, v, w \in \mathbb{F}_p \right\}, \quad \bar{R}_2^{(p)} := \left\{ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ u & v & 1 \end{pmatrix} : u, v \in \mathbb{F}_p \right\},$$

$$\bar{R}_3^{(p)} := \left\{ \begin{pmatrix} -1 & 0 & 0 \\ u & 0 & 1 \\ v & 1 & 0 \end{pmatrix} : u, v \in \mathbb{F}_p \right\}, \quad \bar{R}_4^{(p)} := \left\{ \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & u & 0 \end{pmatrix} : u \in \mathbb{F}_p \right\},$$

$$\bar{R}_5^{(p)} := \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & u & 1 \\ 1 & 0 & 0 \end{pmatrix} : u \in \mathbb{F}_p \right\}, \quad \bar{R}_6^{(p)} := \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

Finally let us write $\bar{R}^{(p)} := \biguplus_{i=1}^{6} \bar{R}_i^{(p)}$.

Now we are in the position to state the following theorem.

Theorem 2.5.2. *$\bar{R}^{(p)}$ is a system of representatives for $\bar{\Delta}^{(p)} \backslash \bar{\Gamma}^{(p)}$. In particular, each set $R^{(p)} \subseteq \Gamma$, such that $\bar{R}^{(p)} = \Phi^{(p)}(R^{(p)})$, is a system of representatives for $\Delta^{(p)} \backslash \Gamma$.*

Proof. The second assertion is an immediate consequence of the first one. It remains to prove that $\bar{R}^{(p)}$ is a system of representatives for $\bar{\Delta}^{(p)} \backslash \bar{\Gamma}^{(p)}$. Because we clearly have

$$|\bar{R}^{(p)}| = p^3 + 2p^2 + 2p + 1 = (p^2 + p + 1)(p + 1) = \left| \bar{\Delta}^{(p)} \backslash \bar{\Gamma}^{(p)} \right|,$$

the proof is completed by showing that for each two elements $r, r' \in \bar{R}^{(p)}$, such that

$$\bar{\Delta}^{(p)} r = \bar{\Delta}^{(p)} r',$$

it is already true that $r = r'$. So, take $r, r' \in \bar{R}^{(p)}$ with the property $\bar{\Delta}^{(p)} r = \bar{\Delta}^{(p)} r'$. Then we necessarily have $r r'^{-1} \in \bar{\Delta}^{(p)}$.
Now, we have to consider the cases $r, r' \in R_i$ for a fixed i and the cases $r \in R_i$, $r' \in R_j$, for a fixed pair (i, j) such that $i < j$ and we have to solve the resulting systems of equalities, which is an easy but time-intensive task, if we do it manually. Because this is pretty straightforward and can be quickly done for example with Wolfram Mathematica ([10]), we do not give the details of that calculation. □

Notation 2.5.3. Here and subsequently, we choose the system of representatives $R^{(p)}$ in a way such that the matrix entries have minimal absolute value. Based on this convention, we introduce the following notation:

$$r_1^{(p)}(u, v, w) := \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ v & w & 1 \end{pmatrix}, \quad r_2^{(p)}(u, v) := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ u & v & 1 \end{pmatrix},$$

$$r_3^{(p)}(u, v) := \begin{pmatrix} -1 & 0 & 0 \\ u & 0 & 1 \\ v & 1 & 0 \end{pmatrix}, \quad r_4^{(p)}(u) := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & u & 0 \end{pmatrix},$$

$$r_5^{(p)}(u) := \begin{pmatrix} 0 & 1 & 0 \\ 0 & u & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad r_6^{(p)} := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where

$$-\left\lfloor \frac{p}{2} \right\rfloor \leq u, v, w \leq \left\lfloor \frac{p}{2} \right\rfloor, \text{ if } p \neq 2$$

and $u, v, w \in \{0, 1\}$ in the case $p = 2$.

2.6 The vertex groups for "Borel-subgroups" of $\mathrm{Sl}_3(\mathbb{Z})$

One aim of this chapter is to give a complete list of the vertex groups for reductions of the action of $\Delta^{(p)}$ on the scwol associated to the action of G on the space $\tilde{X}$ described in the introduction of section 2.2. Moreover, we want to carry as much of the results, we will attain, as possible over to the groups $\Delta^{(d)}$, where d is a square-free non-negative integer. Based on Lemma 2.4.48, our attempt to solve this problem at least for primes p, is to determine the set of conjugacy classes of the maximal finite subgroups in $\Delta^{(p)}$. Our first step in this direction will be to determine all the $\Delta^{(p)}$-conjugacy classes of finite subgroups of $\Delta^{(p)}$. We will reduce this problem to the computation of fixpoints of the action of finite subgroups of Γ on $\Delta^{(p)}\backslash\Gamma$. This results in several systems of polynomial equations of degree at most 3, which we will be able to solve.

To obtain from that at least the number vertex groups of a given type, we have to set up some theoretical framework.

In the whole chapter, let G be an arbitrary but fixed group satisfying FCoFG and let $\Delta \leq G$ such that $(G : \Delta) < \infty$. In particular, Lemma 2.4.29 implies that with G also Δ has to satisfy FCoFG. Recall, that the property of a group to satisfy FCoFG, guarantees the existence of maximal finite subgroups of that group, see Lemma 2.4.28. This is our primary reason for considering such groups.

Furthermore, we will make use of the same notation as in Notation 2.3.19 and Theorem 2.3.25.

Before we start to introduce more notation, let us point out that Theorem 2.3.2 ensures that $\Gamma = \mathrm{Sl}_3(\mathbb{Z})$ satisfies FCoFG. Indeed, the class of groups satisfying FCoFG is quite large, as all finitely generated hyperbolic groups satisfy FCoFG, see [1] p.459 theorem 3.2.

Now, let us introduce some notation.

Notation 2.6.1. Let $H_0 \in \mathrm{Sub}_{\mathrm{fin}}(G)$. Here and subsequently, we will write

$$\mathfrak{C}_\Delta(H_0) := \Big\{[H]_\Delta \in \Delta\backslash \mathrm{Sub}_{\mathrm{fin}}(\Delta) : H \underset{G}{\sim} H_0\Big\},$$

$$\mathfrak{M}_\Delta(H_0) := \{[H]_\Delta \in \mathfrak{C}_\Delta(H_0) : H \in M(\Delta)\},$$

and

$$\mathfrak{L}_\Delta(H_0) := \{[H]_\Delta \in \mathfrak{C}_\Delta(H_0) : H \notin M(\Delta)\}.$$

Furthermore, for $H_0, K_0 \in \mathrm{Sub}_{\mathrm{fin}}(G)$ such that $H_0 \leq K_0$, we put

$$\mathfrak{C}_\Delta(H_0, K_0) := \{[H]_\Delta \in \mathfrak{C}_\Delta(H_0) : \exists [K]_\Delta \in \mathfrak{C}_\Delta(K_0) \colon [H]_\Delta < [K]_\Delta\}.$$

For a group G acting on a set Ω from the right, we write $\mathrm{Fix}_\Omega(G)$ for the set of fixed points under that action. If we have to distinguish sets of fixpoints for different actions, we just label that sets with upper indices in an intuitive and appropriate way.

Let us state the following trivial but useful facts:

Remark 19. For subgroups $H, K \leq G$ acting on Ω from the right the following is true:

$$\mathrm{Fix}_\Omega(H) \cap \mathrm{Fix}_\Omega(K) = \mathrm{Fix}_\Omega(\langle H, K\rangle).$$

Remark 20. Let G acting on Ω from the right and $H \leq G$. Then for each $g \in G$ the following relation holds:

$$\mathrm{Fix}_\Omega({}^g H) = \mathrm{Fix}_\Omega(H).g.$$

The following lemmas treat some elementary properties of that notation.

Lemma 2.6.2. *For each $H_0 \in \mathrm{Sub}_{\mathrm{fin}}(G)$ and each $\gamma \in G$ it holds*

$$\mathfrak{C}_\Delta(H_0) = \mathfrak{C}_\Delta({}^\gamma H_0). \tag{2.6.1}$$

Therefore, for arbitrary $[H_0]_G \in G\backslash \mathrm{Sub}_{\mathrm{fin}}(G)$, we may define

$$\mathfrak{C}_\Delta([H_0]_G) := \mathfrak{C}_\Delta(H_0).$$

Moreover, for each $H_0, K_0 \in \mathrm{Sub}_{\mathrm{fin}}(G)$ such that $H_0 \leq K_0$ and each $\gamma \in G,\ \sigma \in G$ it is true, that

$$\mathfrak{C}_\Delta(H_0, K_0) = \mathfrak{C}_\Delta({}^\gamma H_0, {}^\sigma K_0). \tag{2.6.2}$$

So we may introduce the notation

$$\mathfrak{C}_\Delta([H_0]_G, [K_0]_G) := \mathfrak{C}_\Delta(H_0, K_0).$$

Proof. Let us first show (2.6.1). To this end, it is sufficient to show “$\subseteq$”. So, pick an arbitrary $[H]_\Delta \in \mathfrak{C}_\Delta(H_0)$ and an arbitrary $\gamma \in G$. Then, we have by definition $H \subseteq \Delta$ and $H \underset{G}{\sim} H_0$. In particular, we obtain $H \underset{G}{\sim} {}^\gamma H_0$ which yields $H \in \mathfrak{C}_\Delta(H_0)$.

Let us step to the proof of (2.6.2). But this can be immediately seen if we insert $\mathfrak{C}_\Delta(H_0) = \mathfrak{C}_\Delta({}^\gamma H_0)$ and $\mathfrak{C}_\Delta(K_0) = \mathfrak{C}_\Delta({}^\sigma K_0)$ in the expansion of the definition of $\mathfrak{C}_\Delta(H_0, K_0)$. This completes the proof. □

Lemma 2.6.3. *Let $H_0 \in \mathrm{Sub}_{\mathrm{fin}}(G)$. Then the following formulas are true:*

$$\mathfrak{M}_\Delta(H_0) = \mathfrak{C}_\Delta(H_0) \smallsetminus \mathfrak{L}_\Delta(H_0) \tag{2.6.3}$$

and

$$\mathfrak{L}_\Delta(H_0) = \bigcup_{[K_0]_G \in G\backslash \mathrm{Sub}_{\mathrm{fin}}(G)} \mathfrak{C}_\Delta([H_0]_G, [K_0]_G). \tag{2.6.4}$$

Therefore, $\mathfrak{L}_\Delta(H_0)$ and $\mathfrak{M}_\Delta(H_0)$ are constant on the whole G-conjugacy class $[H_0]_G$ and we may define $\mathfrak{L}_\Delta([H_0]_G) := \mathfrak{L}_\Delta(H_0)$ and $\mathfrak{M}_\Delta([H_0]_G) := \mathfrak{M}_\Delta(H_0)$.

Proof. (2.6.3) is obviously true. It remains to prove (2.6.4). To show “$\subseteq$”, pick $[H]_\Delta \in \mathfrak{L}_\Delta(H_0)$. Because $H_0 \notin M(\Delta)$ there exists a $K_0 \in \mathrm{Sub}_{\mathrm{fin}}(\Delta)$ such that $H < K_0$. But then $[H]_\Delta \in \mathfrak{C}_\Delta(H_0, K_0) = \mathfrak{C}_\Delta([H_0]_G, [K_0]_G)$, as required.

For the converse direction, fix an arbitrary $[K_0]_G \in G\backslash \mathrm{Sub}_{\mathrm{fin}}(G)$ and take an arbitrary $[H]_\Delta \in \mathfrak{C}_\Delta([H_0]_G, [K_0]_G) = \mathfrak{C}_\Delta(H_0, K_0)$. By definition, there exists an $K \in \mathrm{Sub}_{\mathrm{fin}}(\Delta)$ such that $K \underset{G}{\sim} K_0$ and $H < K$. In particular, $H \notin M(\Delta)$. We therefore conclude $[H]_\Delta \in \mathfrak{L}_\Delta(H_0)$, which completes the proof. □

Lemma 2.6.4. *The following equation is true:*

$$\Delta\backslash M^\circ(\Delta) \cap \{[H]_\Delta \in \Delta\backslash \mathrm{Sub}_{\mathrm{fin}}(\Delta) : N_G(H) < \infty\} = \biguplus_{[H]_G \in G\backslash \mathrm{Sub}^\circ_{\mathrm{fin}}(G)} \mathfrak{M}_\Delta([H]_G).$$

PROOF. We start with the proof of the direction “$\subseteq$”. To this end, take a Δ-conjugacy class $[H]_\Delta \in \Delta\backslash M^\circ(\Delta) \cap \{[H]_\Delta \in \Delta\backslash \mathrm{Sub}_{\mathrm{fin}}(\Delta) : N_G(H) < \infty\}$. This implies $H \in M(\Delta)$, $[H]_\Delta \in \mathfrak{C}_\Delta([H]_G)$ and $N_G(H) < \infty$. We therefore conclude $H \in \mathfrak{M}_\Delta([H]_G)$ with $[H]_G \in G\backslash \mathrm{Sub}^\circ_{\mathrm{fin}}(G)$ as required.

For the converse direction let $[H]_\Delta \in \mathfrak{M}_\Delta([H_0]_G)$ for one $H_0 \in \mathrm{Sub}^\circ_{\mathrm{fin}}(G)$. By definition, we get $H \in M(\Delta)$. On the other hand, $H_0 \in \mathrm{Sub}^\circ_{\mathrm{fin}}(G)$ and $H \underset{G}{\sim} H_0$ forces $H \in \mathrm{Sub}^\circ_{\mathrm{fin}}(G)$. This leads to $H \in M(\Delta) \cap \mathrm{Sub}^\circ_{\mathrm{fin}}(G) \subseteq M^\circ(\Delta)$, and therefore

$$[H]_\Delta \in \Delta\backslash M^\circ(\Delta)$$

as claimed. □

We have destilled the following lemma from the diploma thesis [4] pp. 40-43.

Lemma 2.6.5. *Let $H_0 \in \mathrm{Sub}_{\mathrm{fin}}(G)$. Then, its normalizer $N_G(H_0)$ acts from the right on $\mathrm{Fix}_{\Delta\backslash G}(H_0)$ and the map*

$$\Xi_{H_0} : \begin{array}{ccc} \mathrm{Fix}_{\Delta\backslash G}(H_0)/N_G(H_0) & \to & \mathfrak{C}_\Delta(H_0) \\ \Delta\gamma.N_G(H_0) & \mapsto & [{}^\gamma H_0]_\Delta \end{array}$$

is a bijection.

Remark 21. If there is not any possiblity of confusion, we write will write Ξ instead of Ξ_{H_0}.

PROOF. We begin by proving the first assertion. To this end, let $\Delta\gamma \in \mathrm{Fix}_{\Delta\backslash G}(H_0)$ and $g \in N_G(H_0)$. For the reason $gH_0 = H_0 g$, we obtain

$$\Delta\gamma g H_0 = \Delta\gamma H_0 g = \Delta\gamma g.$$

From that, we immediately deduce $\Delta\gamma.g \in \mathrm{Fix}_{\Delta\backslash G}(H_0)$, as desired.

Now, we turn to the proof that Ξ is a bijection. For this purpose, we first have to prove that Ξ is a well-defined. So take an arbitrary $\gamma \in G$ such that $\Delta\gamma \in \mathrm{Fix}_{\Delta\backslash G}(H_0)$. We infer

$$\Xi(\Delta\delta\gamma g.N_G(H_0)) = \left[{}^{\delta\gamma}({}^g H_0)\right]_\Delta = \left[{}^{\delta\gamma}H_0\right]_\Delta = [{}^\gamma H_0]_\Delta = \Xi(\Delta\gamma.N_G(H_0))$$

for all $\delta \in \Delta$ and $g \in N_G(H_0)$. On the other hand, $\Delta\gamma \in \mathrm{Fix}_{\Delta\backslash G}(H_0)$ implies $\gamma H_0 \subseteq \Delta\gamma H_0 = \Delta\gamma$ and therefore

$${}^\gamma H_0 \subseteq \Delta.$$

We thus obtain $\Xi(\Delta\gamma.N_G(H_0)) = [{}^\gamma H_0]_\Delta \in \mathfrak{C}_\Delta(H_0)$, and Ξ is indeed well-defined.

Our next step is to show that Ξ is injective. So, take $\gamma, \sigma \in G$ such that there is an $\delta \in \Delta$ with the property ${}^\gamma H_0 = {}^{\delta\sigma}H_0 \subseteq \Delta$. But this means $\gamma^{-1}\delta\sigma \in N_G(H_0)$ and therefore

$$\Delta\gamma.N_G(H_0) = \Delta\gamma(\gamma^{-1}\delta\sigma).N_G(H_0) = \Delta\delta\sigma.N_G(H_0) = \Delta\sigma.N_G(H_0),$$

and Ξ is injective.

It remains to show that Ξ is surjective. To this end, consider a Δ-conjugacy class $[{}^\gamma H_0]$ such that ${}^\gamma H_0 \subseteq \Delta$. We rewrite the last condition as $\gamma H_0 \subseteq \Delta\gamma$. But this forces

$$\Delta\gamma H_0 \subseteq \Delta\gamma$$

and hence even $\Delta\gamma H_0 = \Delta\gamma$. In particular, we see $\Delta\gamma \in \mathrm{Fix}_{\Delta\backslash G}(H_0)$. We thus may insert $\Delta\gamma N_G(H_0)$ in Ξ. This yields

$$\Xi(\Delta\gamma N_G(H_0)) = [{}^{\gamma}H_0]_\Delta,$$

and therefore the claim. □

For the readers convenience we restate the following well-known fact:

Lemma 2.6.6. *For an arbitrary non-negative integer n let us denote by Φ_n the n-th cyclotomic polynomial. Let p be a prime such that* $\gcd(n,p) = 1$. *Then, Φ_n is irreducible over $\mathbb{F}_p$ if and only if $p + n\mathbb{Z}$ generates the group $(\mathbb{Z}/n\mathbb{Z})^\times$.*

There are two D_{12} conjugacy classes of type S_3, and we have not specified which of them we denote by ${}^{D_{12}}S_3^1$ and ${}^{D_{12}}S_3^2$. In fact, there has been no necessity to do this for the proof of Theorem 2.3.25 and thus until now. This becomes reasonable if we consider the diagram there and observe that it is symmetric around the nodes ${}^{D_{12}}S_3^1$ and ${}^{D_{12}}S_3^2$.

For the main theorem of this chapter, we have to make this choice. A simple computation shows, that

$$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \text{ and } \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$$

are subgroups of D_{12} of type S_3 and not equal. For the reason that there are as many D_{12}-conjugacy classes as subgroups of type S_3, they cannot be conjugate in D_{12}. Due to this background, we set

$${}^{D_{12}}S_3^1 := \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

and

$${}^{D_{12}}S_3^2 := \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle.$$

Theorem 2.6.7. *The actions of the following elements of* $\mathrm{Sub}_{\mathrm{fin}}(\Gamma)$ *on* $\Delta^{(p)}\backslash\Gamma$ *from the right have the following fixpoints:*

type	Γ-*class of*	*group (element of that class)*	*representatives for* $\mathrm{Fix}_{\Delta^{(p)}\backslash\Gamma}$(*"group"*)	$\lvert \mathrm{Fix}_{\Delta^{(p)}\backslash\Gamma}(\cdot) \rvert$
C_2	${}^{S_4^1}C_2^\bullet$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$	$R^{(2)}$, $p=2$, $\{r_1^{(p)}(x,0,0), r_3^{(p)}(0,x), r_4^{(p)}(x) : \bar{x} \in \mathbb{F}_p\} \cup \{r_2^{(p)}(0,0), r_5^{(p)}(0), r_6^{(p)}\}$, $p \neq 2$.	21, $p=2$, $3(p+1)$, $p \neq 2$.
	${}^{S_4^1}C_2^\circ$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$	$\{r_1^{(2)}(x,0,1), r_2^{(2)}(x,1) : \bar{x} \in \mathbb{F}_2\} \cup \{r_5^{(2)}(1)\}$, $p=2$, $\{r_1^{(p)}(x,0,-1), r_1^{(p)}(x,2x,1), r_2^{(p)}(x,1)\} \cup \{r_2^{(p)}(0,-1)\}, r_5^{(p)}(\pm 1)\}$, $p \neq 2$.	5, $p=2$, $3(p+1)$, $p \neq 2$.
C_3	${}^{D_{12}}C_3$	$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$	$\{r_1^{(3)}(x,0,-1), r_2^{(3)}(x,-1) : \bar{x} \in \mathbb{F}_3\} \cup \{r_5^{(3)}(-1)\}$, $p=3$, $\{r_1^{(p)}(0,0,t), r_2^{(p)}(0,t), r_5^{(p)}(t) : t^2 - t + 1 \overset{(p)}{\equiv} 0\}$, $p \overset{(3)}{\equiv} 1$, $\emptyset$, $p \overset{(3)}{\equiv} -1$.	7, $p=3$, 6, $p \overset{(3)}{\equiv} 1$, 0, $p \overset{(3)}{\equiv} -1$.
	${}^{S_4^1}C_3$	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$	$\{r_1^{(3)}(1,1,-1)\}$, $p=3$, $\{r_1^{(p)}(t-1,1,-1), r_1^{(p)}(1,t^2,t), r_1^{(p)}(t-1,t^2,t) : t^2 - t + 1 \overset{(p)}{\equiv} 0\}$, $p \overset{(3)}{\equiv} 1$, $\emptyset$, $p \overset{(3)}{\equiv} -1$.	1, $p=3$, 6, $p \overset{(3)}{\equiv} 1$, 0, $p \overset{(3)}{\equiv} -1$.
C_4	${}^{S_4^1}C_4$	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$	$\{r_1^{(2)}(1,x,x), r_3^{(2)}(x,1)\} \cup \{r_4^{(2)}(1) : \bar{x} \in \mathbb{F}_2\}$, $p=2$, $\{r_1^{(p)}(t,0,0), r_3^{(p)}(0,t), r_4^{(p)}(t) : t^2 \overset{(p)}{\equiv} -1\}$, $p \overset{(4)}{\equiv} 1$, $\emptyset$, $p \overset{(4)}{\equiv} -1$.	5, $p=2$, 6, $p \overset{(4)}{\equiv} 1$, 0, $p \overset{(4)}{\equiv} -1$.
	${}^{S_4^2}C_4$	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle$	$\{r_1^{(2)}(1,0,1)\}$, $p=2$, $\{r_1^{(p)}(t,0,1), r_1^{(p)}(1,t+1,t), r_1^{(p)}(-t,t+1,t) : t^2 \overset{(p)}{\equiv} -1\}$, $p \overset{(4)}{\equiv} 1$, $\emptyset$, $p \overset{(4)}{\equiv} -1$.	1, $p=2$, 6, $p \overset{(4)}{\equiv} 1$, 0, $p \overset{(4)}{\equiv} -1$.
V_4	${}^{S_4^1}V_4^\bullet$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	$R^{(2)}$, $p=2$, $\{r_1^{(p)}(0,0,0), r_2^{(p)}(0,0), r_3^{(p)}(0,0), r_4^{(p)}(0), r_5^{(p)}(0), r_6^{(p)}\}$, $p \neq 2$.	21, $p=2$, 6, $p \neq 2$.

	${}^{S_4^1}V_4^{\circ}$	$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$	$\{r_1^{(2)}(x,0,1), r_2^{(2)}(x,1) : \bar{x} \in \mathbb{F}_p\} \cup \{r_5^{(2)}(1)\},\ p=2,$ $\{r_1^{(p)}(0,0,t), r_2^{(p)}(0,t), r_5^{(p)}(t) : t^2 \overset{(p)}{\equiv} 1\},\ p \neq 2.$	$5,\ p=2,$ $6,\ p \neq 2.$
	${}^{S_4^2}V_4^{\bullet}$	$\left\langle \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$	$\{r_1^{(2)}(1,1,0), r_1^{(2)}(1,0,1), r_3^{(2)}(1,1)\},\ p=2,$ $\{r_1^{(p)}(t,1,0), r_1^{(p)}(t,0,1), r_3^{(p)}(t,1) : t^2 \overset{(p)}{\equiv} 1\},\ p \neq 2$	$3,\ p=2,$ $6,\ p \neq 2.$
	${}^{S_4^3}V_4^{\bullet}$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \right\rangle$	$\{r_1^{(2)}(0,1,1), r_1^{(2)}(1,1,1), r_2^{(2)}(1,1)\},\ p=2,$ $\{r_1^{(p)}(0,1,\pm 1), r_1^{(p)}(-1,-1,1), r_1^{(p)}(-1,1,-1), r_2^{(p)}(\pm 1,1)\},\ p \neq 2.$	$3,\ p=2,$ $6,\ p \neq 2.$
C_6	${}^{D_{12}}C_6$	$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \right\rangle$	$\{r_1^{(3)}(0,0,-1), r_2^{(3)}(0,-1), r_5^{(3)}(-1)\},\ p=3,$ $\{r_1^{(p)}(0,0,t), r_2^{(p)}(0,t), r_5^{(p)}(t) : t^2-t+1 \overset{(p)}{\equiv} 0\},\ p \overset{(3)}{\equiv} 1,$ $\emptyset,\ p \overset{(3)}{\equiv} -1.$	$3,\ p=3,$ $6,\ p \overset{(3)}{\equiv} 1,$ $0,\ p \overset{(3)}{\equiv} -1.$
S_3	${}^{D_{12}}S_3^1$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$	$\{r_2^{(3)}(x,-1) : x \in \mathbb{F}_3\} \cup \{r_1^{(3)}(0,0,-1), r_5^{(3)}(-1)\},\ p=3,$ $\emptyset,\ p \neq 3.$	$5,\ p=3,$ $0,\ p \neq 3.$
	${}^{D_{12}}S_3^2$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$	$\{r_1^{(3)}(x,0,-1) : x \in \mathbb{F}_3\} \cup \{r_2^{(3)}(0,-1), r_5^{(3)}(-1)\},\ p=3,$ $\emptyset,\ p \neq 3.$	$5,\ p=3,$ $0,\ p \neq 3.$
	${}^{S_4^1}S_3$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle$	$\{r_1^{(3)}(1,1,-1)\},\ p=3,$ $\emptyset,\ p \neq 3.$	$1,\ p=3,$ $0,\ p \neq 3.$
D_8	${}^{S_4^1}D_8$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	$\{r_3^{(2)}(x,1) : x \in \mathbb{F}_2\} \cup \{r_1^{(2)}(1,0,0), r_1^{(2)}(1,1,1), r_4^{(2)}(1)\},\ p=2,$ $\emptyset,\ p \neq 2.$	$5,\ p=2,$ $0,\ p \neq 2.$
	${}^{S_4^2}D_8$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$	$\{r_1^{(2)}(1,0,1)\},\ p=2,$ $\emptyset,\ p \neq 2.$	$1,\ p=2,$ $0,\ p \neq 2.$
D_{12}	D_{12}	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right\rangle$	$\{r_1^{(3)}(0,0,-1), r_2^{(3)}(0,-1), r_5^{(3)}(-1)\},\ p=3,$ $\emptyset,\ p \neq 3.$	$3,\ p=3,$ $0,\ p \neq 3.$

Any element of $\mathrm{Sub}_{\mathrm{fin}}(\Gamma)$ *having another isomorphy type than the above ones acts fixpoint free on* $\Delta^{(p)}\backslash\Gamma$.

PROOF. We start with the following two observations: First, the set of fixpoints of a group acting on some set is just the intersection of the sets of fixpoints its generators. Second, we observe that for $H_0 \in \mathrm{Sub}_{\mathrm{fin}}(\Gamma)$ the number $|\operatorname{Fix}_{\Delta^{(p)}\backslash\Gamma}(H_0)|$ does only depend on the Γ-conjugacy class of H_0. Therefore, in Theorem 2.6.7, we have chosen the representatives H of those Γ-conjugacy classes in a way such that the fixpoints of the right-action of H on $\Delta^{(p)}\backslash\Gamma$ are as easy to compute as possible. To throw out redundancies, we describe the right-action of H on $\Delta^{(p)}\backslash\Gamma$ by the multiplication of elements from H on $R^{(p)}$ from the right. More precisely, we compute the fixpoints in the following way: Let $H = \langle \alpha_i |\ i \in I\rangle$. Then, $\Delta^{(p)}\gamma H = \Delta^{(p)}\gamma$ if and only if $\Delta^{(p)}\gamma\alpha_i\gamma^{-1} = \Delta$ and thus

$$^{\gamma}\alpha_i \in \Delta^{(p)}\ \forall i \in I.$$

The last condition results in several systems of polynomial equations in $\mathbb{F}_p$ for the matrix entries of γ, if we require that γ is an element of $R^{(p)}$. We will illustrate, how the computation works, for the two hardest examples. The other cases are similiar and easier to deal with. We begin with the conjugacy class $\left[{}^{S_4^1}C_3\right]_\Gamma$ and its representative $H_1 := \left\langle \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$. Clearly, the fixpoints of H_1 are the fixpoints of its generator. Hence, we have to solve the equations given by the condition

$$^{\gamma}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \in \Delta, \tag{2.6.5}$$

where $\gamma \in R^{(p)}$.
γ can be of the shapes $r_1^{(p)}(u,v,w), r_2^{(p)}(u,v), r_3^{(p)}(u,v), r_4^{(p)}(u), r_5^{(p)}(u)$ or $r_6^{(p)}$ respectively. Thus, we have to solve in fact 6 systems of polynomial equations over $\mathbb{F}_p$. We will demonstrate this only for one such system. So, let us assume that γ has the shape $r_1(u,v,w)$. Under that assumption, (2.6.5) becomes

$$\begin{pmatrix} u & -1 & 0 \\ u^2+v-uw & w-u & -1 \\ vw+u\big(v-w^2\big)+1 & w^2-v & -w \end{pmatrix} \in \Delta^{(p)},$$

where

$$-\left\lfloor \frac{p}{2} \right\rfloor \le u,v,w \le \left\lfloor \frac{p}{2} \right\rfloor,\ \text{if } p \neq 2$$

and $u,v,w \in \{0,1\}$ in the case $p = 2$. By the definition of $\Delta^{(p)}$, we obtain the system of equations below:

$$u^2+v-uw \overset{(p)}{\equiv} 0 \tag{2.6.6}$$

$$vw+u\big(v-w^2\big)+1 \overset{(p)}{\equiv} 0 \tag{2.6.7}$$

$$v \overset{(p)}{\equiv} w^2, \tag{2.6.8}$$

where u,v,w are as above.
We insert (2.6.8) in (2.6.7) and (2.6.6) and obtain

$$w^3+1 \overset{(p)}{\equiv} 0 \tag{2.6.9}$$

and

$$u^2-uw+w^2 \overset{(p)}{\equiv} 0. \tag{2.6.10}$$

This means that $\bar{w} := w + p\mathbb{Z}$ has to be a zero of $\Phi_6(X) = X^2 - X + 1 = \Phi_3(-X)$ over $\mathbb{F}_p$ or $w \overset{(p)}{\equiv} -1$.

To derive appropriate consequences from that observations, we state the following

Lemma 2.6.8. *For $f(t, X) := t^2 - tX + X^2 \in \mathbb{F}_p[t, X]$ the following assertions hold:*

(1) $f(t, X) = f(X, t)$,
(2) $f(1, X) = \Phi_6(X)$,
(3) $f(1, X + 1) = \Phi_3(X) = f(-1, X)$,
(4) $f(X - 1, X) = \Phi_6(X)$.

In particular, each two elements $a, b \in \mathbb{F}_p$ with $f(a, b) = 0$ have to satisfy $f(b, a) = 0$ as well.

Proof of the lemma. The proof is trivial. //

We put $f(t, X) := t^2 - tX + X^2$. Hence, we may rewrite (2.6.9) and (2.6.10) as

$$f(\bar{u}, \bar{w}) = 0 \text{ and } (\Phi_6(\bar{w}) = 0 \vee \bar{w} = -1), \tag{2.6.11}$$

where $\bar{w} := w+p\mathbb{Z}$. If $p \neq 3$, Lemma 2.6.6 tells us, that Φ_3 and by $\Phi_6(X) = \Phi_3(-X)$ also Φ_6, have zeroes in $\mathbb{F}_p$ if and only if $|\langle p + 3\mathbb{Z}\rangle_{\mathbb{F}_3^\times}| \neq 2$ and hence if and only if $p \overset{(3)}{\equiv} 1$. We thus distinguish between the following three cases:

1. $p \overset{(3)}{\equiv} -1$: In that case, Φ_3 and Φ_6 have no zeroes in $\mathbb{F}_p$. Suppose to obtain a contradiction that the system of equations above has solutions. Then, by virtue of (2.6.9), $X^3 + 1 = (X + 1)\Phi_6(X)$ has exactly one solution, namely $w \overset{(p)}{\equiv} -1$. Hence, (2.6.8) forces $v \overset{(p)}{\equiv} 1$. Therefore, (2.6.6) becomes $u^2 + u + 1 \overset{(p)}{\equiv} 0$ and u has to be a zero for Φ_3, a contradiction.

2. $p \overset{(3)}{\equiv} 1$: Lemma 2.6.6 implies that Φ_6 has two zeroes $X_1, X_2 \in \mathbb{F}_p$. Because
$$\Phi_6(X) = (X - X_1)(X - X_2) = X^2 - (X_i + X_j)X + X_iX_j,$$
a coefficient comparison yields $X_iX_j = 1$ and $X_i+X_j = 1$. Hence, $X_i \neq X_j$ for otherwise we would have $X_1^2 = 1$ and $2X_1 = 1$. This would imply $X_i = \pm 1$ and $\pm 2 = 2X_i = 1$. This might only happen if $p = 3$, contrary to our assumption. Therefore, X_1 and X_2 are indeed different. By a similar reasoning, we infer $X_i \neq -1$. Therefore, we have to consider the following cases:

   If $\bar{w} = -1$ we have to find $\bar{u} \in \mathbb{F}_p$ such that $f(\bar{u}, -1) = 0$. But
   $$f(t, -1) = \Phi_3(t) = \Phi_6(t + 1)$$
   has exactly the zeroes $X_i - 1$. Hence $\bar{u}$ has to attain one of the values $X_i - 1$.

   In the cases $\bar{w} = X_i$, (2.6.11) becomes $0 = f(\bar{u}, X_i)$. In particular, $\bar{u}$ has to be a zero of the quadratic polynomial $p_i(t) = f(t, X_i)$. Lemma 2.6.8 tells us that $p_i(1) = f(1, X_i) = \Phi_6(X_i) = 0$ and
   $$p_i(X_i - 1) = f(X_i - 1, X_i) = \Phi_6(X_i) = 0$$
   and therefore $\bar{u}$ has to attain the values 1 or $X_i - 1$.

   (2.6.8) guarantees that in each of the cases $\bar{v}$ does only depend on $\bar{w}$.

3. $p = 3$: In this case $\Phi_6(X) = X^2 - X + 1 = X^2 + 2X + 1 = (X+1)^2$. Therefore $\bar{w} = -1$ is a zero of $X^3 + 1$. By $f(\bar{u}, -1) = f(-1, \bar{u}) = \Phi_3(\bar{u}) = \Phi_6(-\bar{u})$ we obtain $-\bar{u} = -1$ and thus $\bar{u} = 1$.

This yields all elements of $\mathrm{Fix}_{\Delta^{(p)}\backslash\Gamma}\left\langle \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$ represented by elements of the form $r_1^{(p)}(u, v, w)$.

Let us proceed with the second example. We consider the Γ-conjugacy class $\left[{}^{S_4^2}C_4\right]_\Gamma$ and its representative $H_2 := \left\langle \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle$. As above, we realize that we have to determine $\gamma \in R^{(p)}$ such that

$$
{}^{\gamma}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \in \Delta^{(p)}. \tag{2.6.12}
$$

Again, we only compute the solutions of the shape $r_1^{(p)}(u, v, w)$. The remaining solutions are obtained by similar but easier calculations. So the condition (2.6.12) becomes

$$
\begin{pmatrix} u & -1 & 0 \\ -v + u(u + w - 1) + 1 & -u - w + 1 & 1 \\ -vw + w + u(v + (w-1)w) - 1 & -w^2 + w - v & w \end{pmatrix} \in \Delta,
$$

where

$$
-\left\lfloor \frac{p}{2} \right\rfloor \leq u, v, w \leq \left\lfloor \frac{p}{2} \right\rfloor, \text{ if } p \neq 2
$$

and $u, v, w \in \{0, 1\}$ in the case $p = 2$. Those matrices are elements of $\Delta^{(p)}$ if and only if they satisfy the following system of equations:

$$
-v + u(u + w - 1) + 1 \overset{(p)}{\equiv} 0 \tag{2.6.13}
$$

$$
-vw + w + u(v + (w-1)w) - 1 \overset{(p)}{\equiv} 0 \tag{2.6.14}
$$

$$
-(w-1)w \overset{(p)}{\equiv} v. \tag{2.6.15}
$$

We insert (2.6.15) in (2.6.14) and (2.6.13) and obtain

$$
w^3 - w^2 + w - 1 \overset{(p)}{\equiv} 0
$$

$$
w^2 - w + 1 - u + u^2 + uw \overset{(p)}{\equiv} 0.
$$

Recalling $\Phi_4(X) = X^2 + 1$, the last system of equations can be rewritten as

$$
(w-1)\Phi_4(w) \overset{(p)}{\equiv} 0 \tag{2.6.16}
$$

$$
(u + w - 1)^2 + u + w - uw \overset{(p)}{\equiv} 0. \tag{2.6.17}
$$

To solve this system of equations we will make use of the lemma below:

Lemma 2.6.9. *The polynomial* $f(t, X) := (t + X - 1)^2 + t + X - tX \in \mathbb{F}_p[t, X]$ *has the following properties:*

(1) $f(t, X) = f(X, t)$
(2) $f(1, X) = \Phi_4(X)$.

(3) $f(X,-X) = \Phi_4(X)$.

Proof of the lemma. The proof is trivial.

//

Now, we put $f(t,X) := (t+X-1)^2 + t + X - tX$. With that notation, the system of equations consisting of (2.6.16) and (2.6.17) becomes

$$f(\bar{u},\bar{w}) = 0 \text{ and } (\Phi_4(\bar{w}) = 0 \vee \bar{w} = 1). \tag{2.6.18}$$

Lemma 2.6.6 tells us for p odd that Φ_4 is irreducible in $\mathbb{F}_p$ if and only if $|\langle p\rangle_{(\mathbb{Z}/4\mathbb{Z})^\times}| = 2$. Thus, Φ_4 has zeroes if and only if $p \overset{(4)}{\equiv} 1$. We therefore consider the following three cases.

1. $p \overset{(4)}{\equiv} -1$: In that case Φ_4 is irreducible over $\mathbb{F}_p$. We claim that there is no solution for (2.6.18). Suppose contrary to our claim, that there is a solution $(\bar{u},\bar{w}) \in (\mathbb{F}_p)^2$ of that system. Then, $\bar{w} = 1$ because Φ_4 is irreducible. Applying Lemma 2.6.9 we necessary have,
$$0 = f(\bar{u},1) = \Phi_4(u),$$
a contradiction to the irreducibility of Φ_4. Hence, (2.6.18) has indeed no solution in that case.
2. $p \overset{(4)}{\equiv} 1$: Lemma 2.6.6 guarantees us that Φ_4 has two zeroes X_1 and X_2. We thus get
$$X^2 + 1 = \Phi_4(X) = (X - X_1)(X - X_2) = X^2 - (X_1 + X_2)X + X_1X_2.$$
Coefficient comparison yields $X_1 + X_2 = 0$ and $X_1X_2 = 1$. In particular $X_1 \neq X_2$ and $X_i \neq 1,\ i \in \{1,2\}$, for otherwise we would have $p = 2$, a contradiction to $p \overset{(4)}{\equiv} 1$. The structure of (2.6.18) suggests that we should consider the following cases: In the case $\bar{w} = 1$, Lemma 2.6.9 yields that (2.6.18) becomes
$$0 = f(\bar{u},1) = \Phi_4(u).$$
Because $f(t,1)$ is a polynomial of degree 2, u has to attain one of the values $X_i,\ i \in \{1,2\}$. If $\Phi(\bar{w}) = 0$, we have $\bar{w} = X_i$ for one $i \in \{1,2\}$. We put $p_i(t) := f(t,X_i)$. Thus, (2.6.18) gets to $p_i(\bar{u}) = 0$. For the reason $p_i(t)$ is a polynomial of degree 2, it has at most two zeroes in $\mathbb{F}_p$. Lemma 2.6.9 yields
$$p_i(1) = f(1,X_i) = \Phi_4(X_i) = 0$$
and
$$p_i(-X_i) = f(-X_i,X_i) = f(X_i,-X_i) = \Phi_4(X_i) = 0.$$
So, $\bar{u} \in \{1, X_i\}$.
3. $p = 2$: If $p = 2$ occurs, (2.6.18) is equivalent to $\bar{u} = 1 = \bar{w}$.

This is the claim for H_2. □

Let d be a square-free non-negative integer. So far, we have computed the fixpoints for the right actions of finite subgroups of Γ on the quotients $\Delta^{(p)}\backslash\Gamma$. With that in our mind, we can derive the fixpoints of those actions on $\Delta^{(d)}\backslash\Gamma$ just by considering direct products. Using the Cauchy-Frobenius formula, we will obtain the numbers $|\mathfrak{C}_{\Delta^{(d)}}(\cdot)|$. On the road to our aim to determine the cardinalities of the sets $\mathfrak{M}_{\Delta^{(d)}}(H)$ for $H \in \mathrm{Sub}_{\mathrm{fin}}(\Gamma)$ we have to describe them in an appropriate way. By virtue of Lemma 2.6.3 it is enough to determine the sizes $|\mathfrak{C}_{\Delta^{(d)}}(H,K)|$ for suitable $H, K \in \mathrm{Sub}_{\mathrm{fin}}(\Gamma)$. In Lemma 2.6.5 we have found a method to compute $\mathfrak{C}_{\Delta^{(d)}}(\cdot)$. But this only works for the reason we can express the term ${}^{\gamma}H \subseteq \Delta$ in an equivalent way using fixpoints. In general, there is no possibility to get such a description for $\mathfrak{C}_{\Delta^{(d)}}(H, K)$ because there is an additional relation $H < K$ which has to be regarded. However, this problem will be solved if we find a way to force that relation to be trivial. For example, this is indeed the case, if $N_{\Delta^{(d)}}(H) = K$ is true.

In the sequel, let $d = \prod_{l=1}^{r} p_l$ the decomposition of d into primes. Because d is square-free, each of the primes occurs exactly once. For a non-negative integer $k \in \mathbb{N}$ we abbreviate an element $a + k\mathbb{Z} \in \mathbb{Z}/k\mathbb{Z}$ with $\bar{a}^{(k)}$. For a matrix $\gamma = (a_{ij})_{(i,j)\in[3]\times[3]}$ we denote by $\bar{\gamma}^{(k)}$ the matrix $(\bar{a}_{ij}^{(k)})_{(i,j)\in[3]\times[3]} \in \mathrm{Mat}_3(\mathbb{Z})$. Using that notation, the Chinese remainder theorem tells us that the map

$$\begin{array}{ccc} \mathbb{Z}/d\mathbb{Z} & \xrightarrow{\cong} & \prod_{l=1}^{r} \mathbb{Z}/p_l\mathbb{Z}\colon \\ \bar{a}^{(d)} & \mapsto & (\bar{a}^{(p_l)})_{l=1}^{r} \end{array} \tag{2.6.19}$$

is an isomorphism. It is a well known fact that this isomorphism gives rise to an isomorphism of groups:

$$\varrho^{(d)}\colon \begin{array}{ccc} \mathrm{Sl}_3(\mathbb{Z}/d\mathbb{Z}) & \xrightarrow{\cong} & \prod_{l=1}^{r} \mathrm{Sl}_3(\mathbb{Z}/p_l\mathbb{Z}) \\ \bar{\gamma}^{(d)} & \mapsto & \left(\bar{\gamma}^{(p_l)}\right)_{l=1}^{r}. \end{array} \tag{2.6.20}$$

The only non trivial part of the proof of this assertion is to ensure the surjectivity of $\varrho^{(d)}$. But the surjectivity is an immediate consequence of the following consideration: Given, there is a matrix $\gamma \in \mathrm{Mat}_3(\mathbb{Z})$ satisfying a system of equations

$$\overline{\det\gamma}^{(p_l)} = \bar{1}^{(p_l)}, \quad l \in \{1, \ldots, r\};$$

we obtain $p_l \mid \det(\gamma) - 1$ for each $l \in \{1, \ldots, r\}$, and hence $d \mid \det(\gamma) - 1$. The last relation can be rewritten as

$$\overline{\det\gamma}^{(d)} = \bar{1}^{(d)},$$

which is the desired conclusion.

Clearly, we have

$$\varrho^{(d)}(\Delta^{(d)}) = \prod_{l=1}^{r} \Delta^{(p_l)}. \tag{2.6.21}$$

Thus (2.6.20) yields

$$\bar{\Delta}^{(d)} \backslash \bar{\Gamma}^{(d)} \cong \prod_{l=1}^{r} \bar{\Delta}^{(p_l)} \backslash \bar{\Gamma}^{(p_l)},$$

where the isomorphy is meant to be between Γ-sets with respect to the canonical Γ-actions from the right.

For this reason and by (2.5.1), the product map

$$\Delta^{(d)} \backslash \Gamma \xrightarrow{\cong} \prod_{l=1}^{r} (\Delta^{(p_l)} \backslash \Gamma) \tag{2.6.22}$$

has be an isomorphism between right Γ-sets, as well.

An element $x = (x_1, \ldots, x_r) \in \prod_{l=1}^{r} \Delta^{(p_l)} \backslash \Gamma$ is fixed by an element $g \in \Gamma$ if and only if

$$(x_1, \ldots, x_r) = x = x.g = (x_1.g, \ldots, x_r.g),$$

and therefore, if and only if g fixes each $x_i \in \Delta^{(p_l)} \backslash \Gamma$. This leads to the following lemma:

Lemma 2.6.10. *The restriction of the product map in (2.6.22) to* $\mathrm{Fix}_{\Delta^{(d)}\backslash\Gamma}(H)$ *induces an isomorphism between the right* $N_\Gamma(H)$*-sets*

$$\mathrm{Fix}_{\Delta^{(d)}\backslash\Gamma}(H) \text{ and } \prod_{l=1}^{r} \mathrm{Fix}_{\Delta^{(p_l)}\backslash\Gamma}(H).$$

By a slight abuse of notation we rewrite this assertion as

$$\mathrm{Fix}_{\Delta^{(d)}\backslash\Gamma}(H) \cong \prod_{l=1}^{r} \mathrm{Fix}_{\Delta^{(p_l)}\backslash\Gamma}(H).$$

Proof. This is a direct consequence of Lemma 2.6.5 combined with (2.6.22) and the statement in front of the lemma. □

For the computation of the cardinality of $\mathfrak{C}_{\Delta^{(d)}}([H]_\Gamma)$ for an element of $H \in \mathrm{Sub}^\circ_{\mathrm{fin}}(\Gamma)$, we will make use of the following lemma:

Lemma 2.6.11. *Let G be a group acting on a set Ω from the right. Furthermore, let $H \in \mathrm{Sub}^\circ_{\mathrm{fin}}(G)$. Then $N_G(H)$ and thus also $N_G(H)/H$ act on $\mathrm{Fix}_\Omega(H)$ from the right and the following formula holds:*

(2.6.23)
$$|\mathrm{Fix}_\Omega(H)/N_G(H)| = \frac{1}{(N_G(H):H)}\Bigg\{|\mathrm{Fix}_\Omega(H)| + \sum_{\substack{gH\in N_G(H)/H,\\ gH\neq H}} |\mathrm{Fix}_\Omega(\langle H, g\rangle)|\Bigg\}.$$

If $|N_G(H)/H| = q$ for a suitable prime q, then this formula can be simplified to

(2.6.24)
$$\begin{aligned}|\mathrm{Fix}_\Omega(H)/N_G(H)| &= \tfrac{1}{q}|\mathrm{Fix}_\Omega(H)| + \Big(1-\tfrac{1}{q}\Big)|\mathrm{Fix}_\Omega(N_G(H))| \\ &= |\mathrm{Fix}_\Omega(N_G(H))| + \tfrac{1}{q}\Big\{|\mathrm{Fix}_\Omega(H)| - |\mathrm{Fix}_\Omega(N_G(H))|\Big\}.\end{aligned}$$

Proof. In the same manner as in the proof of Lemma 2.6.5, it can be shown that $N_G(H)$ and $N_G(H)/H$ act on $\Omega' := \mathrm{Fix}_\Omega(H)$ from the right. By $gH = Hg$ for each $g \in N_G(H)$ we get obviously

$$\mathrm{Fix}_\Omega(H)/N_G(H) = \mathrm{Fix}_\Omega(H)/(N_G(H)/H),$$

in particular the fixpoints on Ω' of both actions coincide, i.e.

$$\mathrm{Fix}^G_{\Omega'}(\langle g\rangle) = \mathrm{Fix}^{N_G(H)/H}_{\Omega'}(\langle gH\rangle).$$

Therefore, it suffices to compute $|\mathrm{Fix}_\Omega(H)/(N_G(H)/H)|$ which can be easily done using the Cauchy-Frobenius formula and Remark 19:

$$\begin{aligned}|\mathrm{Fix}_\Omega(H)/(N_G(H)/H)| &= \frac{1}{(N_G(H):H)}\Bigg\{|\mathrm{Fix}_{\Omega'}(H)| + \sum_{\substack{gH\in N_G(H)/H,\\ gH\neq H}} |\mathrm{Fix}_{\Omega'}(\langle gH\rangle)|\Bigg\} \\ &= \frac{1}{(N_G(H):H)}\Bigg\{|\mathrm{Fix}_\Omega(H)| + \sum_{\substack{gH\in N_G(H)/H,\\ gH\neq H}} |\mathrm{Fix}_\Omega(\langle H, g\rangle)|\Bigg\}.\end{aligned}$$

Under the assumption $N_G(H)/H \cong C_q$ we obtain (2.6.24) by the following consideration: $\langle gH\rangle = N_G(H)/H$ if and only if $gH \neq H$. In this case (2.6.23) becomes

$$\begin{aligned}|\operatorname{Fix}_\Omega(H)/N_G(H)| &= \frac{1}{q}\Big\{|\operatorname{Fix}_\Omega(H)| + (q-1)\,|\operatorname{Fix}_\Omega(N_G(H))|\Big\}\\ &= \frac{1}{q}|\operatorname{Fix}_\Omega(H)| + \Big(1-\frac{1}{q}\Big)|\operatorname{Fix}_\Omega(N_G(H))|\\ &= |\operatorname{Fix}_\Omega(N_G(H))| + \frac{1}{q}\Big\{|\operatorname{Fix}_\Omega(N_G(H))| - |\operatorname{Fix}_\Omega(H)|\Big\},\end{aligned}$$

which is the desired conclusion. □

Lemma 2.6.5 combined with the just proved statement and Lemma 2.6.10 leads immediately to

Lemma 2.6.12. *Let $H \in \operatorname{Sub}^\circ_{\text{fin}}(\Gamma)$, then*

$$|\mathfrak{C}_{\Delta^{(d)}}(H)| = \frac{1}{(N_\Gamma(H):H)}\left\{\prod_{l=1}^{r}|\operatorname{Fix}_{\Delta^{(p_l)}\backslash\Gamma}(H)| + \sum_{\substack{gH\in N_\Gamma(H)/H,\\ gH\neq H}}\ \prod_{l=1}^{r}|\operatorname{Fix}_{\Delta^{(p_l)}\backslash\Gamma}(\langle H,g\rangle)|\right\}.$$

Under the additional assumption that $|N_\Gamma(H)/H| = q$ for a suitable prime q, this formula can be simplified to

$$|\mathfrak{C}_{\Delta^{(d)}}(H)| = \frac{1}{q}\prod_{l=1}^{r}|\operatorname{Fix}_{\Delta^{(p_l)}\backslash\Gamma}(H)| + \frac{q-1}{q}\prod_{l=1}^{r}|\operatorname{Fix}_{\Delta^{(p_l)}\backslash\Gamma}(N_\Gamma(H))|.$$

Remark 22. The lemma above tells us that we obtain the cardinalities $|\mathfrak{C}_{\Delta^{(d)}}([H_0]_\Gamma)|$ from the numbers $|\operatorname{Fix}_{\Delta^{(p)}\backslash\Gamma}(H_0)|$ for all $H_0 \in \operatorname{Sub}^\circ_{\text{fin}}(\Gamma)$, where we already have computed the last ones in Theorem 2.6.7.

In order to derive the numbers $|\mathfrak{M}_{\Delta^{(d)}}([\cdot]_\Gamma)|$, we need a series of technical lemmas which allows us to determine the cardinalities $|\mathfrak{C}_{\Delta^{(d)}}([\cdot]_\Gamma,[\cdot]_\Gamma)|$ or $|\mathfrak{L}_{\Delta^{(d)}}([\cdot]_\Gamma)|$, from those of $\mathfrak{C}_{\Delta^{(d)}}([\cdot]_\Gamma)$. Here, “$\cdot$” stands for arbitrary elements in $\operatorname{Sub}^\circ_{\text{fin}}(\Gamma)$. For the sake of brevity we have introduce the following notions:

Definition 2.6.13. Let $H_0 \le K_0 \in \operatorname{Sub}_{\text{fin}}(G)$. We say H_0 or $[H_0]_G$ satisfies *property* $\big(\mathcal{N}_{[K_0]_G}\big)$ *in* G if and only if the following assertion holds:

$$\forall (H,K) \in [H_0]_G \times [K_0]_G\colon \quad (H \le K \Rightarrow H \trianglelefteq K).$$

Definition 2.6.14. Let $H_0 \in \operatorname{Sub}_{\text{fin}}(G)$. We say H_0 has *property* $(\mathcal{P})$ if and only if for each $G\backslash\operatorname{Sub}_{\text{fin}}(G) \ni [K_0]_G > [H_0]_G$ the following assertion is true: $[K_0]_G$ is a maximal element in the set $\{[K]_G \in G\backslash\operatorname{Sub}_{\text{fin}}(G) : \mathfrak{C}_\Delta([K]_G) \neq \emptyset\}$ with respect to the order induced by that on $G\backslash\operatorname{Sub}(G)$.

We recall the following well known elemental group theoretical fact:

Remark 23. Let K be a finite group and q be the lowest prime dividing it. Then each $H \le K$ such that $(K:H) = q$ satisfies $H \trianglelefteq K$.

PROOF. Consider the kernel of the action $K \to \operatorname{Sym}(K/H)\colon k \mapsto (k'H \mapsto kk'H)$. □

Lemma 2.6.15. *Let $H_0 \le K_0 \in \operatorname{Sub}_{\text{fin}}(G)$. Let q be the smallest prime dividing $|K_0|$. Then, $(K_0 : H_0) = q$ implies that H_0 satisfies $\big(\mathcal{N}_{[K_0]_G}\big)$ in G.*

PROOF. Let $(H,K) \in [H_0]_G \times [K_0]_G$ such that $H \le K$. By Lagrange's theorem we have

$$q = (K_0 : H_0) = \frac{|K_0|}{|H_0|} = \frac{|K|}{|H|} = (K:H).$$

But now, the elemental group theoretical fact recalled in Remark 23 yields $H \trianglelefteq K$ and therefore the claim. □

Lemma 2.6.16. *We denote by Ξ_{H_0} the map defined in Lemma 2.6.5. Let K_0 be a finite group and $H_0 \leq K_0$ such that $N_G(H_0) = K_0$. Under the assumption that H_0 satisfies $\big(\mathcal{N}_{[K_0]_G}\big)$, the following assertions are true:*

$$\mathfrak{C}_\Delta(H_0, K_0) = \{[H]_\Delta \in \mathfrak{C}_\Delta(H_0) \colon N_G(H) \subseteq \Delta\}. \tag{2.6.25}$$

and hence

$$\mathfrak{C}_\Delta(H_0, K_0) = \Xi_{H_0}\big(\mathrm{Fix}_{\Delta\backslash G}(K_0)\big). \tag{2.6.26}$$

Proof. We begin by proving (2.6.25). Therefore, we first show that $\mathfrak{C}_\Delta(H_0, K_0)$ is contained in $\{[H]_\Delta \in \mathfrak{C}_\Delta(H_0) \colon N_G(H) \subseteq \Delta\}$. To this end, take an arbitrary $[H]_\Delta \in \mathfrak{C}_\Delta(H_0, K_0)$. Then, by definition, $H < K \subseteq \Delta$ for an appropriate K such that $[K]_\Delta \in \mathfrak{C}_\Delta(K_0)$. Now, $\big(\mathcal{N}_{[K_0]_G}\big)$ implies that $H \trianglelefteq K$ and hence $K = N_K(H) \leq N_G(H)$. On the other hand, there exists a $\gamma \in G$ such that $H = {}^\gamma H_0$. We combine these assertions to

$$[K_0]_\Delta \ni K \leq N_G(H) = {}^\gamma N_G(H_0) = {}^\gamma K_0.$$

But conjugation with elements in Δ is an automorphism in G and therefore all "$\leq$" are in fact equalities. In particular, $N_G(H) = K \subseteq \Delta$ which is the desired conclusion. For the converse direction pick an arbitrary $[H]_\Delta \in \mathfrak{C}_\Delta(H_0)$ such that $N_G(H) \subseteq \Delta$. Now, $H \underset{G}{\sim} H_0$ implies $N_G(H) \underset{G}{\sim} N_G(H_0) = K_0$. This together with $N_G(H) \subseteq \Delta$ implies $[N_G(H)]_\Delta \in \mathfrak{C}_\Delta(K_0)$ and therefore $[H]_\Delta \in \mathfrak{C}_\Delta(H_0, K_0)$, and the proof of (2.6.25) is complete.

It remains to show (2.6.26). For this purpose, pick an arbitrary $\gamma \in G$ such that

$$[{}^\gamma H_0]_\Delta \in \{[H]_\Delta \in \mathfrak{C}_\Delta(H_0) \colon N_G(H) \subseteq \Delta\}.$$

By definition and Lemma 2.3.5 we have $\gamma N_G(H_0)\gamma^{-1} \subseteq \Delta$. Analysing the proof of Lemma 2.6.5, we infer that this condition can be rewritten as $\Delta\gamma = \Delta\gamma N_G(H)$ or equivalently as $\Delta\gamma \in \mathrm{Fix}_{\Delta\backslash G}(N_G(H_0))$. We thus conclude

$$\Xi^{-1}([{}^\gamma H_0]_\Delta) = \Delta\gamma.N_G(H_0) = \Delta\gamma.$$

This combined with (2.6.25) yields $\Xi^{-1}\big(\mathfrak{C}_\Delta(H_0, K_0)\big) = \mathrm{Fix}(K_0)$, and (2.6.26) is proved. □

Lemma 2.6.17. *Let $H_0 \in \mathrm{Sub}^\circ_{\mathrm{fin}}(G)$ such that H_0 satisfies property $(\mathcal{P})$ and $[H_0]_G$ has property $\big(\mathcal{N}_{[K_0]_G}\big)$ for each $[K_0]_G > [H_0]_G \colon \mathfrak{C}_\Delta(K_0) \neq \emptyset$. Then,*

$$\begin{array}{rccc} \Lambda\colon & \mathfrak{L}_\Delta(H_0) & \to & \biguplus\limits_{\substack{[K_0]_G > [H_0]_G \colon \\ \mathfrak{C}_\Delta(K_0) \neq \emptyset}} \mathfrak{C}_\Delta(K_0) \colon \\ & [H]_\Delta & \mapsto & [N_\Delta(H)]_\Delta \end{array}$$

is a well-defined surjective map.

Proof. First, we have to prove that Λ is indeed well-defined. For this purpose, we take any $[H]_\Delta \in \mathfrak{L}_\Delta(H_0)$. Our goal is to show that there exists a $K_0 > H_0$ such that $[N_\Delta(H)]_\Delta \in \mathfrak{C}_\Delta(K_0)$. The definition of $\mathfrak{L}_\Delta(H_0)$ yields the existence of an $[K_0]_\Delta$ such that $[K_0]_G > [H_0]_G$ and $[H]_\Delta \in \mathfrak{C}_\Delta(H_0, K_0)$. Hence, there is an $[K]_\Delta \in \mathfrak{C}_\Delta(K_0)$ such that $H < K$. In particular, $\mathfrak{C}_\Delta(K_0)$ cannot be empty. Because $[H_0]_G$ satisfies property $\big(\mathcal{N}_{[K_0]_G}\big)$, we infer

$$K_0 \underset{G}{\sim} K = N_K(H) \leq N_\Delta(H) \subseteq \Delta. \tag{2.6.27}$$

But property $(\mathcal{P})$ forces

$$[N_\Delta(H)]_G \leq [K_0]_G.$$

Both assertions combine to

$$|N_\Delta(H)| = |K_0| = |K|.$$

So, (2.6.27) can be strengthened to $[N_\Delta(H)]_\Delta = [K]_\Delta \in \mathfrak{C}_\Delta(K_0)$, as required.

Let us now turn our attention to the proof of the surjectivity of Λ. To this end, pick $[K]_\Delta \in \mathfrak{C}_\Delta(K_0)$ for some $[K_0]_G > [H_0]_G \colon \mathfrak{C}_\Delta(K_0) \neq \emptyset$. Thus, there is a $\gamma \in G$ such that ${}^\gamma K_0 = K \subseteq \Delta$. For the reason ${}^\gamma H_0 \leq {}^\gamma K_0 = K \subseteq \Delta$, we get by property $\left(\mathcal{N}_{[K_0]_G}\right)$

$$K = N_K({}^\gamma H_0) \leq N_\Delta({}^\gamma H_0).$$

Because H_0 satisfies property $(\mathcal{P})$, we may conclude in the same manner as above

$$K = N_\Delta({}^\gamma H_0).$$

From that we derive

$$[K]_\Delta = \Lambda([{}^\gamma H_0]_\Delta),$$

which completes the proof. □

Under further assumptions we can also ensure that the map Λ, mentioned in Lemma 2.6.17, is in addition injective. This is subjective of the following

Lemma 2.6.18. *We use the same notation and hypotheses as in Lemma* 2.6.17. *Under the additional assumption that for each* $[K_0]_G > [H_0]_G$ *with* $\mathfrak{C}_\Delta(K_0) \neq \emptyset$ *there is a representative* $K \in [K_0]_G$ *such that the map*

$$\Theta_K \colon \begin{array}{ccc} \{[H]_K : \ H \cong H_0\} & \to & G\backslash \operatorname{Sub}_{\text{fin}}(G) \\ [H]_K & \mapsto & [N_G(H)]_G \end{array}$$

is an injection, the map Λ *in Lemma* 2.6.17 *is even a bijection.*

Proof. We only need to show that Λ is injective. The remaining statements are consequences of Lemma 2.6.17. For this purpose, take any $H, H' \in \mathfrak{L}_\Delta(H_0)$ such that

$$[N_\Delta(H)]_\Delta = [N_\Delta(H')]_\Delta.$$

By definition, there exists a $\delta \in \Delta$ with the property

$$N_\Delta(H) = {}^\delta N_\Delta(H') = N_\Delta({}^\delta H').$$

We put $H'' := {}^\delta H'$ and rewrite the assertion as

$$N_\Delta(H) = N_\Delta(H'').$$

Lemma 2.6.17 tells us that $\mathfrak{C}_\Delta(N_\Delta(H)) \neq \emptyset$ and $N_\Delta(H) > H$ just because $[N_\Delta(H)]_\Delta$ is an element of the co-domain of Λ. Now, pick a $K \in [N_\Delta(H)]_G$ such that Θ_K is an injective map. Then, there exists a $\gamma \in G$ such that

$$K = {}^\gamma N_\Delta(H) = {}^\gamma N_\Delta(H'').$$

By virtue of $H, H'' \leq N_\Delta(H)$ we obtain ${}^\gamma H, {}^\gamma H'' \leq {}^\gamma N_\Delta(H) = K$.

At this point let us claim that

$$[{}^\gamma H]_K = [{}^\gamma H'']_K. \tag{2.6.28}$$

On the contrary, suppose that $[{}^\gamma H]_K \neq [{}^\gamma H'']_K$. For the reason ${}^\gamma H \cong H_0 \cong {}^\gamma H''$ we may apply Θ_K on both sides. Because Θ_K is an injective map this yields:

$$[N_\Gamma({}^\gamma H)]_G = \Theta_K([{}^\gamma H]_K) \neq \Theta_K([{}^\gamma H'']_K) = [N_\Gamma({}^\gamma H'')]_G.$$

According to Lemma 2.3.5 this can be only true if

$$[H]_G = [{}^\gamma H]_G \neq [{}^\gamma H'']_G = [H'']_G,$$

contradicting $H \underset{G}{\sim} H_0 \underset{G}{\sim} H''$.

We continue to show $[H]_\Delta = [H']_\Delta$. (2.6.28) guarantees the existence of a $k \in K$ such that ${}^{k\gamma}H = {}^{\gamma}H''$. We rewrite this assertion as

$$ {}^{\gamma^{-1}k\gamma}H = H''. $$

Since $k \in K = \gamma N_\Delta(H)\gamma^{-1}$, there exists an $x \in N_\Delta(H) \subseteq \Delta$ such that $k = \gamma x \gamma^{-1}$. We conclude

$$ H'' = {}^{\gamma^{-1}\gamma x \gamma^{-1}\gamma}H = {}^{x}H, $$

and thus $[H']_\Delta = [H'']_\Delta = [H]_\Delta$, as claimed. □

Lemma 2.6.19. *Let $H_0 < K_0 \in \mathrm{Sub}_{\mathrm{fin}}(G)$ such that for each $\Delta\gamma N_G(H_0) \in \mathrm{Fix}_{\Delta\backslash G}(H_0)/N_G(H_0)$ the condition*

$$ \Delta\gamma.N_G(H_0) \cap \mathrm{Fix}_{\Delta\backslash G}(K_0) \neq \emptyset $$

is satisfied. Then, it is already true that

$$ \mathfrak{C}_\Delta(H_0, K_0) = \mathfrak{C}_\Delta(H_0). $$

Proof. Let the maps Ξ_{H_0} and Ξ_{K_0} be defined as in Lemma 2.6.5 and take an arbitrary $[H]_\Delta \in \mathfrak{C}_\Delta(H_0)$. By definition, there is a $\gamma \in G$ such that $H = {}^{\gamma}H_0 \subseteq \Delta$. Taking the preimage under the map Ξ_{H_0} we obtain

$$ \Xi_{H_0}^{-1}([H]_\Delta) = \Delta\gamma N_G(H_0) \in \mathrm{Fix}_{\Delta\backslash G}(H_0)/N_G(H_0). $$

By hypothesis, there exists a $\gamma' \in G$ such that $\Delta\gamma' \in \Delta\gamma.N_G(H_0) \cap \mathrm{Fix}_{\Delta\backslash G}(K_0)$. This in particular yields $\Delta\gamma' \in \mathrm{Fix}_{\Delta\backslash G}(K_0) \subseteq \mathrm{Fix}_{\Delta\backslash G}(H_0)$ and thus

$$ [{}^{\gamma'}H_0]_\Delta = \Xi_{H_0}(\Delta\gamma' N_G(H_0)) \in \mathfrak{C}_\Delta(H_0) $$

as well as

$$ [{}^{\gamma'}K_0]_\Delta = \Xi_{K_0}(\Delta\gamma' N_G(K_0)) \in \mathfrak{C}_\Delta(K_0). $$

On the other hand, $\Delta\gamma.N_G(H_0) = \Delta\gamma'.N_G(H_0)$ forces

$$ [H]_\Delta = \Xi_{H_0}(\Delta\gamma.N_G(H_0)) = \Xi_{H_0}(\Delta\gamma'.N_G(H_0) = [{}^{\gamma'}H_0]_\Delta. $$

$H_0 < K_0$ hence implies

$$ [H]_\Delta = [{}^{\gamma'}H_0]_\Delta < [{}^{\gamma'}K_0]_\Delta \in \mathfrak{C}_\Delta(K_0). $$

This proves $[H]_\Delta \in \mathfrak{C}_\Delta(H_0, K_0)$ and therefore $\mathfrak{C}_\Delta(H_0) \subseteq \mathfrak{C}_\Delta(H_0, K_0)$, as required. □

Lemma 2.6.20. *Let $H_0 \leq K_0 \leq G$ such that $N_G(H_0)$ is finite and $N_G(H_0) = N_G(K_0)$. Furthermore, let Ξ_{H_0} and Ξ_{K_0} be the associated maps from Lemma* 2.6.5*. Then, we have*

$$ \Xi_{K_0}^{-1}(\mathfrak{C}_\Delta(K_0)) \subseteq \Xi_{H_0}^{-1}(\mathfrak{C}_\Delta(H_0)). \tag{2.6.29} $$

Now, let $(K_i)_{i\in I}$ be an arbitrary family with $H_0 \leq K_i \leq G$ and $N_G(K_i) = N_G(H_0)$. Under the assumption

$$ \mathrm{Fix}_{\Delta\backslash G}(H_0) \subseteq \bigcup_{i\in I} \mathrm{Fix}_{\Delta\backslash G}(K_i) $$

it is even true that

$$ \mathfrak{C}_\Delta(H_0) = \bigcup_{i\in I} \mathfrak{C}_\Delta(H_0, K_i). \tag{2.6.30} $$

PROOF. We begin by proving (2.6.29). $H_0 \leq K_0$ clearly implies $\mathrm{Fix}_{\Delta\backslash G}(K_0) \subseteq \mathrm{Fix}_{\Delta\backslash G}(H_0)$. We put $L := N_G(H_0)$ and rewrite the hypothesis as $L = N_G(K_0)$. We thus obtain

$$\Xi_{K_0}^{-1}(\mathfrak{C}_\Delta(K_0)) = \mathrm{Fix}_{\Delta\backslash G}(K_0)/L \subseteq \mathrm{Fix}_{\Delta\backslash G}(H_0)/L = \Xi_{H_0}^{-1}(\mathfrak{C}_\Delta(H_0))$$

which already yields (2.6.29). To prove (2.6.30) take a family $(K_i)_i \in I$ with $H_0 \leq K_i$ and $N_G(H_0) = N_G(K_i)$ for all $i \in I$ such that

$$\mathrm{Fix}_{\Delta\backslash G}(H_0) \subseteq \bigcup_{i\in I} \mathrm{Fix}_{\Delta\backslash G}(K_i).$$

Obviously, it is sufficient to show "$\subseteq$" to obtain the equality in (2.6.30). We set again $L := N_G(H_0)$. So, we may derive from the hypothesis that

$$\Xi_{H_0}^{-1}(\mathfrak{C}_\Delta(H_0)) = \mathrm{Fix}_{\Delta\backslash G}(H_0)/L \subseteq \bigcup_{i\in I} \mathrm{Fix}_{\Delta\backslash G}(K_i)/L \subseteq \bigcup_{i\in I} \Xi_{K_i}^{-1}(\mathfrak{C}_\Delta(K_i)).$$

But (2.6.29) yields $\Xi_{K_i}^{-1}(\mathfrak{C}_\Delta(K_i)) \subseteq \Xi_{H_0}^{-1}(\mathfrak{C}_\Delta(H_0))$ for each $i \in I$. Combining both assertions we get

$$\Xi_{H_0}^{-1}(\mathfrak{C}_\Delta(H_0)) = \bigcup_{i\in I} \Xi_{K_i}^{-1}(\mathfrak{C}_\Delta(K_i)).$$

(2.6.29) tells us that the domains of Ξ_{K_i} are all contained in that of Ξ_{H_0}. Therefore we may apply Ξ_{H_0} on both sides. This leads to

$$\mathfrak{C}_\Delta(H_0) = \bigcup_{i\in I} \Xi_{H_0}\Xi_{K_i}^{-1}(\mathfrak{C}_\Delta(K_i)).$$

Hence, it remains to show $\Xi_{H_0}\Xi_{K_i}^{-1}(\mathfrak{C}_\Delta(K_i)) \subseteq \mathfrak{C}_\Delta(H_0, K_i)$ for each $i \in I$. For this purpose, fix an $i \in I$ and pick an arbitrary $c \in \Xi_{H_0}\Xi_{K_i}^{-1}(\mathfrak{C}_\Delta(K_i))$. Then, there exists a $\gamma \in \Gamma$ such that

$$c = \Xi_{H_0}\Xi_{K_i}^{-1}([{}^\gamma K_i]_\Delta) = \Xi_{H_0}(\Delta\gamma N_G(K_i)) = \Xi_{H_0}(\Delta\gamma N_G(H_0)) = [{}^\gamma H_0]_\Delta,$$

where the definition of $\mathfrak{C}_\Delta(K_i)$ and the choice of γ implies that ${}^\gamma K_i \subseteq \Delta$. We hence infer $c = [{}^\gamma H_0]_\Delta < [{}^\gamma K_i]_\Delta \in \mathfrak{C}_\Delta(K_i)$. This assertion can be rewritten as $c \in \mathfrak{C}_\Delta(H_0, K_i)$, and the lemma follows. □

We are now in the position to state and prove the conclusion from Theorem 2.6.7:

Theorem 2.6.21. *Let $d = \prod_{l=1}^r p_l$ be an arbitrary square-free integer. Then, the numbers $|\mathfrak{C}_{\Delta^{(d)}}(H)|$ and $|\mathfrak{M}_{\Delta^{(d)}}(H)|$ can be computed for each $H \in \mathrm{Sub}^\circ_{\mathrm{fin}}(\Gamma)$. The results of those computations are given by the table below:*

type	Γ-*class*	$\lvert \mathrm{Fix}_{\Delta^{(p)}\backslash\Gamma}(\cdot) \rvert$	$\lvert \mathfrak{C}_{\Delta^{(d)}}([\cdot]_\Gamma) \rvert$	$\lvert \mathfrak{M}_{\Delta^{(d)}}([\cdot]_\Gamma) \rvert$
C_3	${}^{D_{12}}C_3$	$7,\quad p = 3,$ $6,\quad p \overset{(3)}{\equiv} 1,$ $0,\quad p \overset{(3)}{\equiv} -1.$	$5 \quad d = 3,$ $0, \quad \exists p \mid d\colon p \overset{(3)}{\equiv} -1,$ $15 \cdot 6^{r-2}, \quad 3 \mid d,\ d \neq 3 \wedge \forall p \mid d\colon p \overset{(3)}{\not\equiv} -1,$ $3 \cdot 6^{r-1}, \quad \forall p \mid d\colon p \overset{(3)}{\equiv} 1.$	$0, \quad d = 3,$ $0, \quad 3 \nmid d,$ $6^{r-1}, \quad$ *otherwise.*
	${}^{S_4^1}C_3$	$1,\quad p = 3$ $6,\quad p \overset{(3)}{\equiv} 1$ $0,\quad p \overset{(3)}{\equiv} -1$	$1, \quad d = 3,$ $0, \quad \exists p \mid d\colon p \overset{(3)}{\equiv} -1,$ $3 \cdot 6^{r-2}, \quad 3 \mid d,\ d \neq 3 \wedge \forall p \mid d\colon p \overset{(3)}{\not\equiv} -1,$ $3 \cdot 6^{r-1}, \quad \forall p \mid d\colon p \overset{(3)}{\equiv} 1.$	$0, \quad d = 3,$ $0, \quad \exists p \mid d\colon p \overset{(3)}{\equiv} -1,$ $3 \cdot 6^{r-2}, \quad 3 \mid d,\ d \neq 3 \wedge \forall p \mid d\colon p \overset{(3)}{\not\equiv} -1,$ $3 \cdot 6^{r-1}, \quad \forall p \mid d\colon p \overset{(3)}{\equiv} 1.$
C_4	${}^{S_4^1}C_4$	$5,\quad p = 2,$ $6,\quad p \overset{(4)}{\equiv} 1,$ $0,\quad p \overset{(4)}{\equiv} -1.$	$5, \quad d = 2,$ $0, \quad \exists p \mid d\colon p \overset{(4)}{\equiv} -1,$ $15 \cdot 6^{r-2}, \quad 2 \mid d,\ d \neq 2 \wedge \forall p \mid d\colon p \overset{(4)}{\not\equiv} -1,$ $3 \cdot 6^{r-1}, \quad \forall p \mid d\colon p \overset{(4)}{\equiv} 1.$	$0, \quad d = 2,$ $0, \quad \exists p \mid d\colon p \overset{(4)}{\equiv} -1,$ $15 \cdot 6^{r-2}, \quad 2 \mid d,\ d \neq 2 \wedge \forall p \mid d\colon p \overset{(4)}{\not\equiv} -1,$ $3 \cdot 6^{r-1}, \quad \forall p \mid d\colon p \overset{(4)}{\equiv} 1.$
	${}^{S_4^2}C_4$	$1,\quad p = 2,$ $6,\quad p \overset{(4)}{\equiv} 1,$ $0,\quad p \overset{(4)}{\equiv} -1.$	$1, \quad d = 2,$ $0, \quad \exists p \mid d\colon p \overset{(4)}{\equiv} -1,$ $3 \cdot 6^{r-2}, \quad 2 \mid d,\ d \neq 2 \wedge \forall p \mid d\colon p \overset{(4)}{\not\equiv} -1,$ $3 \cdot 6^{r-1}, \quad \forall p \mid d\colon p \overset{(4)}{\equiv} 1.$	$0, \quad d = 2,$ $0, \quad \exists p \mid d\colon p \overset{(4)}{\equiv} -1,$ $3 \cdot 6^{r-2}, \quad 2 \mid d,\ d \neq 2 \wedge \forall p \mid d\colon p \overset{(4)}{\not\equiv} -1,$ $3 \cdot 6^{r-1}, \quad \forall p \mid d\colon p \overset{(4)}{\equiv} 1.$
V_4	${}^{S_4^1}V_4^{\bullet}$	$21,\quad p = 2,$ $6,\quad p \neq 2.$	$6, \quad d = 2,$ $21 \cdot 6^{r-2}, \quad 2 \mid d,\ d \neq 2,$ $6^{r-1}, \quad d$ *odd.*	$1, \quad d = 2,$ $21 \cdot 6^{r-2}, \quad 2 \mid d,\ d \neq 2,$ $6^{r-1}, \quad d$ *odd.*
	${}^{S_4^1}V_4^{\circ}$	$5,\quad p = 2,$ $6,\quad p \neq 2.$	$5, \quad d = 2,$ $15 \cdot 6^{r-2}, \quad 2 \mid d,\ d \neq 2,$ $3 \cdot 6^{r-1}, \quad d$ *odd.*	$0, \quad d \in \{2, 3\},$ $15 \cdot 6^{r-2}, \quad 2 \mid d,\ d \neq 2,$ $3 \cdot 6^{r-1}, \quad d$ *odd*$,\ d \neq 3.$

	${}^{S_4^2}V_4^\bullet$	$3,\ p=2,$ $6,\ p\neq 2.$	$1,\ d=2,$ $3\cdot 6^{r-2},\ 2\mid d,\ d\neq 2,$ $6^{r-1},\ d$ *odd.*	$0,\ d=2,$ $3\cdot 6^{r-2},\ 2\mid d,\ d\neq 2,$ $6^{r-1},\ d$ *odd.*
	${}^{S_4^3}V_4^\bullet$	$3,\ p=2,$ $6,\ p\neq 2.$	$1,\ d=2,$ $3\cdot 6^{r-2},\ 2\mid d,\ d\neq 2,$ $6^{r-1},\ d$ *odd.*	$0,\ d=2,$ $3\cdot 6^{r-2},\ 2\mid d,\ d\neq 2,$ $6^{r-1},\ d$ *odd.*
C_6	${}^{D_{12}}C_6$	$3,\ p=3,$ $6,\ p\overset{(3)}{\equiv}1,$ $0,\ p\overset{(3)}{\equiv}-1.$	$3,\ d=3,$ $0,\ \exists p\mid d\colon p\overset{(3)}{\equiv}-1,$ $9\cdot 6^{r-2},\ 3\mid d,\ d\neq 3\wedge\forall p\mid d\colon p\overset{(3)}{\not\equiv}-1,$ $3\cdot 6^{r-1},\ \forall p\mid d\colon p\overset{(3)}{\equiv}1.$	$0,\ d=3,$ $0,\ \exists p\mid d\colon p\overset{(3)}{\equiv}-1,$ $9\cdot 6^{r-2},\ 3\mid d,\ d\neq 3\wedge\forall p\mid d\colon p\overset{(3)}{\not\equiv}-1,$ $3\cdot 6^{r-1},\ \forall p\mid d\colon p\overset{(3)}{\equiv}1.$
S_3	${}^{D_{12}}S_3^1$	$5,\ p=3,$ $0,$ *otherwise.*	$4,\ d=3$ $0,$ *otherwise.*	$1,\ d=3$ $0,$ *otherwise.*
	${}^{D_{12}}S_3^2$	$5,\ p=3,$ $0,$ *otherwise.*	$4,\ d=3$ $0,$ *otherwise.*	$1,\ d=3$ $0,$ *otherwise.*
	${}^{S_4^1}S_3$	$1,\ p=3,$ $0,$ *otherwise.*	$1,\ d=3,$ $0,$ *otherwise.*	$1,\ d=3,$ $0,$ *otherwise.*
D_8	${}^{S_4^1}D_8$	$5,\ p=2,$ $0,$ *otherwise.*	$5,\ d=2,$ $0,$ *otherwise.*	$5,\ d=2,$ $0,$ *otherwise.*
	${}^{S_4^2}D_8$	$1,\ p=2,$ $0,$ *otherwise.*	$1,\ d=2,$ $0,$ *otherwise.*	$1,\ d=2,$ $0,$ *otherwise.*
D_{12}	D_{12}	$3,\ p=3,$ $0,$ *otherwise.*	$3,\ d=3,$ $0,$ *otherwise.*	$3,\ d=3,$ $0,$ *otherwise.*

Any Γ*-classes of elements of* $\mathrm{Sub}^\circ_{\mathrm{fin}}(\Gamma)$*, different from those, have no representative being contained in* $\Delta^{(d)}$*.*

PROOF. We begin by a verification of 4-th column which contains the sizes of $\mathfrak{C}_{\Delta^{(d)}}([H_0]_\Gamma)$ for all $[H_0]_\Gamma \in \Gamma\backslash \operatorname{Sub}^{\circ}_{\mathrm{fin}}(\Gamma)$. By Lemma 2.6.5, we have $|\mathfrak{C}_{\Delta^{(d)}}(H)| = |\operatorname{Fix}_{\Delta^{(d)}\backslash\Gamma}(H)/N_G(H)|$ for each $H \in \operatorname{Sub}^{\circ}_{\mathrm{fin}}(\Gamma)$. We therefore obtain those numbers in a straight-forward way from Theorem 2.6.7, Theorem 2.3.25 and Lemma 2.6.12. We illustrate the computation of that numbers at the example $\left[{}^{S^1_4}C_3\right]_\Gamma$: Take $H \in \left[{}^{S^1_4}C_3\right]_\Gamma$. Let $a, b \in \mathbb{F}_3$ and put

$$\delta_{a,b} := \begin{cases} 1, & a = b, \\ 0, & a \neq b. \end{cases}$$

Theorem 2.3.25 tells us that $N_\Gamma(H) \cong S_3$. Theorem 2.6.7 yields

$$\left|\operatorname{Fix}_{\Delta^{(p)}\backslash\Gamma}\left({}^{S^1_4}C_3\right)\right| = \delta_{\bar 0,\bar p} + 6\delta_{\bar 1,\bar p}$$

and also

$$\left|\operatorname{Fix}_{\Delta^{(p)}\backslash\Gamma}\left({}^{S^1_4}S_3\right)\right| = \delta_{\bar 0,\bar p}.$$

We hence obtain by Lemma 2.6.12 and Theorem 2.6.7

$$\begin{aligned} |\mathfrak{C}_{\Delta^{(d)}}([H]_\Gamma)| &= \frac{1}{2}\Big\{ \underbrace{\prod_{l=1}^{r}(\delta_{\bar 0,\bar p_l} + 6\delta_{\bar 1,\bar p_l})}_{\substack{=0, \text{ if } p_l \overset{(3)}{\equiv} -1 \\ \text{for some } l}} + \underbrace{\prod_{l=1}^{r}\delta_{\bar 0,\bar p_l}}_{=0, \text{ if } d\neq 3} \Big\} \\ &= \begin{cases} 1, & \text{if } d = 3; \\ 0, & \text{if } \exists l \in \{1,\dots,r\}\colon p_l \overset{(3)}{\equiv} -1; \\ \frac{1}{2}1\cdot 6^{r-1}, & \text{if } 3 \mid d,\ d \neq 3,\ \forall 3 \neq p \mid d\colon p \overset{(3)}{\equiv} 1; \\ \frac{1}{2}6^r, & \text{if } \forall p \mid d\colon p \overset{(3)}{\equiv} 1. \end{cases} \end{aligned}$$

It is left to verify, that the sizes $|\mathfrak{M}_{\Delta^{(d)}}([\cdot]_\Gamma)|$ are given by the numbers in the last column. Of course, we will already make also use of the results for $|\mathfrak{C}_{\Delta^{(d)}}([\cdot]_\Gamma)|$. To compute the numbers $|\mathfrak{M}_{\Delta^{(d)}}([H_0]_\Gamma)|$ for each element in $[H_0]_\Gamma \in \Gamma\backslash \operatorname{Sub}^{\circ}_{\mathrm{fin}}(\Gamma)$, it is therefore sufficient to express $|\mathfrak{M}_{\Delta^{(d)}}(H_0)|$ in terms of $|\mathfrak{C}_{\Delta^{(d)}}(\cdot)|$. For his convenience, we advise the reader to print out the diagram from Theorem 2.3.25 and the table above.

1. $\left[{}^{D_{12}}C_3\right]_\Gamma$: We have to consider the following cases: If $d \neq 3$, we have by the results of the 4-th column $\mathfrak{C}_{\Delta^{(d)}}\left({}^{D_{12}}S_3^j\right) = \emptyset$, $j \in \{1, 2\}$, and $\mathfrak{C}_{\Delta^{(d)}}(D_{12}) = \emptyset$. Theorem 2.3.25 thus implies that ${}^{D_{12}}C_3$ has property $(\mathcal{P})$. Therefore, the only Γ-conjugacy class $[K_0]_\Gamma \in \operatorname{Sub}^{\circ}_{\mathrm{fin}}(\Gamma)$ which lies over $\left[{}^{D_{12}}C_3\right]_\Gamma$ such that $\mathfrak{C}_{\Delta^{(d)}}(K_0) \neq \emptyset$ might occur, is $\left[{}^{D_{12}}C_6\right]_\Gamma$. Hence, we put $K_0 := {}^{D_{12}}C_6$. And the last fact clearly forces that ${}^{D_{12}}C_3$ has property $\left(\mathcal{N}_{[K_0]_\Gamma}\right)$. Because there is only one subgroup of isomorphy type C_3 in K_0, the map Θ_{K_0} in Lemma 2.6.18 is necessarily injective. Therefore, the hypotheses for Lemma 2.6.18 are satisfied. We thus obtain

$$\left|\mathfrak{L}_{\Delta^{(d)}}\left({}^{D_{12}}C_3\right)\right| = |\mathfrak{C}_{\Delta^{(d)}}(K_0)|.$$

Now, an application of Lemma 2.6.3 forces

$$\left|\mathfrak{M}_{\Delta^{(d)}}\left({}^{D_{12}}C_3\right)\right| = \left|\mathfrak{C}_{\Delta^{(d)}}\left({}^{D_{12}}C_3\right)\right| - \left|\mathfrak{C}_{\Delta^{(d)}}\left({}^{D_{12}}C_6\right)\right|.$$

This yields the required cardinality. If $d = 3$, we have $\mathfrak{C}_{\Delta^{(d)}}(D_{12}) \neq \emptyset$. In particular, property $(\mathcal{P})$ is not satisfied. Thus, the criterion Lemma 2.6.18

cannot be applied. Luckily, Lemma 2.6.20 might work here. We hence prepare ourselves to apply Lemma 2.6.20. To this end, consider

$$H_0 := \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \in \left[{}^{S_4^1}C_3\right]_\Gamma,$$

$$K_1 := \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \in \left[{}^{D_{12}}S_3^1\right]_\Gamma,$$

and

$$K_2 := \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle \in \left[{}^{D_{12}}S_3^2\right]_\Gamma.$$

Theorem 2.6.7 tells us that

$$\mathrm{Fix}_{\Delta^{(3)}\backslash\Gamma}(H_0) \subseteq \mathrm{Fix}_{\Delta^{(3)}\backslash\Gamma}(K_1) \cup \mathrm{Fix}_{\Delta^{(3)}\backslash\Gamma}(K_2).$$

So, we may apply Lemma 2.6.20. This yields

$$\mathfrak{C}_{\Delta^{(3)}}(H_0) = \mathfrak{C}_{\Delta^{(3)}}(H_0, K_1) \cup \mathfrak{C}_{\Delta^{(3)}}(H_0, K_2) \subseteq \mathfrak{L}_{\Delta^{(3)}}(H_0).$$

We conclude

$$\left|\mathfrak{M}_{\Delta^{(3)}}\left({}^{D_{12}}C_3\right)\right| = |\mathfrak{M}_{\Delta^{(3)}}(H_0)| = 0.$$

2. $\left[{}^{S_4^1}C_3\right]_\Gamma$: If $d \neq 3$, $\mathfrak{C}_{\Delta^{(d)}}\left({}^{S_4^1}S_3\right) = \emptyset$ immediately ensures that

$$\left|\mathfrak{M}_{\Delta^{(d)}}\left({}^{S_4^1}C_3\right)\right| = \left|\mathfrak{C}_{\Delta^{(d)}}\left({}^{S_4^1}C_3\right)\right|.$$

Now, let us assume $d = 3$. Let $(H_0, K_0) \in \left[{}^{S_4^1}C_3\right]_\Gamma \times \left[{}^{S_4^1}S_3\right]_\Gamma$ such that $H_0 \leq K_0$. Because $(K_0 : H_0) = 2$, H_0 satisfies $\left(\mathcal{N}_{[K_0]_\Gamma}\right)$. On the other hand, Theorem 2.3.25 tells us that $[H_0]_\Gamma$ has property $(\mathcal{P})$. The Sylow theorems guarantee that the map Θ_{K_0} from Lemma 2.6.18 is injective. Thus, we may apply Lemma 2.6.18 and obtain

$$|\mathfrak{L}_{\Delta^{(3)}}(H_0)| = |\mathfrak{C}_{\Delta^{(3)}}(K_0)|.$$

Therefore, we get

$$|\mathfrak{M}_{\Delta^{(3)}}(H_0)| = |\mathfrak{C}_{\Delta^{(3)}}(H_0)| - |\mathfrak{C}_{\Delta^{(3)}}(K_0)|.$$

3. $\left[{}^{S_4^i}C_4\right]_\Gamma$, $i \in \{1,2\}$: First, we treat the case $d \neq 2$. For the reason $\left|\mathfrak{C}_\Delta\left({}^{S_4^i}D_8\right)\right| = 0$, Theorem 2.3.25 implies

$$\mathfrak{M}_{\Delta^{(d)}}\left({}^{S_4^i}C_4\right) = \mathfrak{C}_{\Delta^{(d)}}\left({}^{S_4^i}C_4\right).$$

For $d = 2$, $\left[{}^{S_4^i}D_8\right]_\Gamma$ is the only element in $\Gamma\backslash\mathrm{Sub}^\circ_{\mathrm{fin}}(\Gamma)$ such that $[K_i]_\Gamma >$ $\left[{}^{S_4^i}C_4\right]_\Gamma$ and $\mathfrak{C}_{\Delta^{(2)}}(K_i) \neq \emptyset$. Let H_i be an representative of $\left[{}^{S_4^i}C_4\right]_\Gamma$ and K_i be an representative of $\left[{}^{S_4^i}D_8\right]_\Gamma$ such that $H_i \leq K_i$. Then, $(K_i : H_i) = 2$ implies that H_i has necessarily the property $\left(\mathcal{N}_{[K_i]_\Gamma}\right)$. Moreover, $N_\Gamma(H_i) = K_i$. Thus, Lemma 2.6.16 forces

$$|\mathfrak{L}_{\Delta^{(2)}}(H_i)| = |\mathfrak{C}_{\Delta^{(2)}}(H_i, K_i)| = |\,\mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}(K_i)|.$$

We thus conclude

$$|\mathfrak{M}_{\Delta^{(2)}}(H_i)| = |\mathfrak{C}_{\Delta^{(2)}}(H_i)| - |\,\mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}(K_i)|.$$

$\left[{}^{S_4^1}V_4^\bullet\right]_\Gamma$: If $d \neq 2$, we observe $\mathfrak{C}_{\Delta^{(d)}}([K_0]_\Gamma) = \emptyset$ for each $\Gamma\backslash\operatorname{Sub}_{\mathrm{fin}}(\Gamma) \ni [K_0]_\Gamma > \left[{}^{S_4^1}V_4^\bullet\right]_\Gamma$. This leads to

$$\mathfrak{M}_{\Delta^{(d)}}\left({}^{S_4^1}V_4^\bullet\right) = \mathfrak{C}_{\Delta^{(d)}}\left({}^{S_4^1}V_4^\bullet\right).$$

So, let us assume $d = 2$. We choose representatives $H_0 \in \left[{}^{S_4^1}V_4^\bullet\right]_\Gamma$ and $K_0 \in \left[{}^{S_4^1}D_8\right]_\Gamma$. As for the groups of type C_4 there is only one Γ-class which lies over $\left[{}^{S_4^1}V_4^\bullet\right]_\Gamma$, namely $\left[{}^{S_4^1}D_8\right]_\Gamma$. Therefore, H_0 satisfies the properties $(\mathcal{P})$ and $\left(\mathcal{N}_{[K_0]_G}\right)$. Now, we have to show that the map Θ_{K_0} in Lemma 2.6.18 is injective. To this end, we consider the domain of Θ_{K_0} which is given by the set

$$\{[H]_{K_0} : H \cong D_8\} = \left\{\left[{}^{S_4^1}V_4^\bullet\right]_{K_0}, \left[{}^{S_4^1}V_4^\circ\right]_{K_0}\right\}.$$

Theorem 2.3.25 tells us that

$$\Theta_{K_0}\left(\left[{}^{S_4^1}V_4^\bullet\right]_{K_0}\right) = \left[S_4^1\right]_\Gamma$$

and

$$\Theta_{K_0}\left(\left[{}^{S_4^1}V_4^\circ\right]_{K_0}\right) = \left[{}^{S_4^1}D_8\right]_\Gamma.$$

In particular, Θ_{K_0} is injective. We may thus apply Lemma 2.6.18 on H_0 and get

$$|\mathfrak{L}_{\Delta^{(2)}}(H_0)| = |\mathfrak{C}_{\Delta^{(2)}}(K_0)|.$$

This forces

$$|\mathfrak{M}_{\Delta^{(2)}}(H_0)| = |\mathfrak{C}_{\Delta^{(2)}}(H_0)| - |\mathfrak{C}_{\Delta^{(2)}}(K_0)|.$$

4. $\left[{}^{S_4^i}V_4^\bullet\right]_\Gamma$, $i \in \{2,3\}$: For $d \neq 2$ we obtain by the same reasoning as for $\left[{}^{S_4^1}V_4^\bullet\right]_\Gamma$

$$\mathfrak{M}_{\Delta^{(d)}}\left({}^{S_4^i}V_4^\bullet\right) = \mathfrak{C}_{\Delta^{(d)}}\left({}^{S_4^i}V_4^\bullet\right).$$

So, let us assume $d = 2$. We know

$$\left|\mathfrak{C}_{\Delta^{(2)}}\left({}^{S_4^i}V_4^\bullet\right)\right| = 1.$$

For the reason, $\mathfrak{C}_{\Delta^{(2)}}\left({}^{S_4^2}D_8\right) \neq \emptyset$ and by the diagram in Theorem 2.3.25 it is true that $\mathfrak{C}_{\Delta^{(2)}}\left({}^{S_4^i}V_4^\bullet, {}^{S_4^2}D_8\right) \neq \emptyset$. This leads to

$$\left|\mathfrak{M}_{\Delta^{(2)}}\left({}^{S_4^i}V_4^\bullet\right)\right| = \left|\mathfrak{C}_{\Delta^{(2)}}\left({}^{S_4^i}V_4^\bullet\right)\right| - \left|\mathfrak{C}_{\Delta^{(2)}}\left({}^{S_4^i}V_4^\bullet, {}^{S_4^2}D_8\right)\right| \leq 1 - 1 = 0.$$

We want to point out that $\Theta_{{}^{S_4^2}D_8}$ is not injective here and we therefore cannot apply Lemma 2.6.18.

5. $\left[{}^{S_4^1}V_4^\circ\right]_\Gamma$: If $d \neq 2, 3$, we have $\mathfrak{C}_{\Delta^{(d)}}([K_0]_\Gamma) = \emptyset$ for all $\Gamma\backslash\operatorname{Sub}^\circ_{\mathrm{fin}}(\Gamma) \ni [K_0]_\Gamma > \left[{}^{S_4^1}V_4^\circ\right]_\Gamma$. We therefore get

$$\mathfrak{M}_{\Delta^{(d)}}\left({}^{S_4^1}V_4^\circ\right) = \mathfrak{C}_{\Delta^{(d)}}\left({}^{S_4^1}V_4^\circ\right).$$

We now turn our attention to the case $d = 3$. We then have $\mathfrak{C}_{\Delta^{(3)}}([D_{12}]_\Gamma) \neq \emptyset$. We put

$$H_0 := \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle \in \left[{}^{S_4^1}V_4^\circ\right]_\Gamma,$$

$$K_0 := \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right\rangle \in [D_{12}]_\Gamma$$

and Theorem 2.3.25 tells us that $N_\Gamma(H_0) \in \left[{}^{S_4^1}D_8\right]_\Gamma$. It is obviously true that $H_0 < K_0$. By Theorem 2.6.7, we are able to verify, for instance using [10], that the hypotheses for H_0, K_0 in Lemma 2.6.19 are satisfied. From that we immediately get $C_{\Delta^{(3)}}(H_0, K_0) = C_{\Delta^{(3)}}(H_0)$. This forces

$$|\mathfrak{M}_{\Delta^{(3)}}(H_0)| = |\mathfrak{C}_{\Delta^{(3)}}(H_0)| - |\mathfrak{C}_{\Delta^{(3)}}(H_0, K_0)| = 0.$$

It remains to consider the case $d = 2$. The only Γ-conjugacy class $[K]_\Gamma$ over $\left[{}^{S_4^1}V_4^\circ\right]_\Gamma$ such that $\mathfrak{C}_{\Delta^{(2)}}([K]_\Gamma) \neq \emptyset$ is $\left[{}^{S_4^1}D_8\right]_\Gamma$. For the reason $N_\Gamma\left({}^{S_4^1}V_4^\circ\right) \in \left[{}^{S_4^1}D_8\right]_\Gamma$ the hypotheses for Lemma 2.6.16 are satisfied. This yields

$$\left|\mathfrak{M}_{\Delta^{(2)}}\left({}^{S_4^1}V_4^\circ\right)\right| = \left|\mathfrak{C}_{\Delta^{(2)}}\left({}^{S_4^1}V_4^\circ\right)\right| - \left|\mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}\left({}^{S_4^1}D_8\right)\right| = 5 - 5 = 0.$$

6. $\left[{}^{D_{12}}C_6\right]_\Gamma$: First, consider the case $d \neq 3$. So, for each Γ-class $[K]_\Gamma \in \mathrm{Sub}_{\mathrm{fin}}(\Gamma)$, such that $[K]_\Gamma > \left[{}^{D_{12}}C_6\right]_\Gamma$, holds

$$\mathfrak{C}_{\Delta^{(d)}}([K]_\Gamma) = \emptyset.$$

We conclude

$$\mathfrak{M}_{\Delta^{(d)}}\left(\left[{}^{D_{12}}C_6\right]_\Gamma\right) = \mathfrak{C}_{\Delta^{(d)}}\left(\left[{}^{D_{12}}C_6\right]_\Gamma\right).$$

Now, let us assume $d = 3$. Under that assumption it is clearly true that

$$\mathfrak{L}_{\Delta^{(3)}}\left({}^{D_{12}}C_6\right) = \mathfrak{C}_{\Delta^{(3)}}\left({}^{D_{12}}C_6, D_{12}\right).$$

Furthermore, ${}^{D_{12}}C_6$ has property $(\mathcal{N}_{[D_{12}]_G})$ and $N_\Gamma\left({}^{D_{12}}C_6\right) = D_{12}$. We may thus apply Lemma 2.6.16 and obtain

$$\left|\mathfrak{M}_{\Delta^{(3)}}\left(\left[{}^{D_{12}}C_6\right]_\Gamma\right)\right| = \left|\mathfrak{C}_{\Delta^{(3)}}\left({}^{D_{12}}C_6\right)\right| - |\,\mathrm{Fix}_{\Delta^{(3)}}(D_{12})|.$$

7. $\left[{}^{D_{12}}S_3^j\right]_\Gamma$, $j \in 1, 2$: For $d \neq 3$ is nothing to show. So, let us assume $d = 3$. A similar sequence of arguments, as for $\left[{}^{D_{12}}C_6\right]_\Gamma$ in the case $d = 3$, yields

$$\left|\mathfrak{M}_{\Delta^{(3)}}\left(\left[{}^{D_{12}}S_3^j\right]_\Gamma\right)\right| = \left|\mathfrak{C}_{\Delta^{(3)}}\left({}^{D_{12}}S_3^j\right)\right| - |\,\mathrm{Fix}_{\Delta^{(3)}}(D_{12})|.$$

For the remaining cases we have $\mathfrak{M}_{\Delta^{(d)}}(H) = \mathfrak{C}_{\Delta^{(d)}}(H)$ for each $[H]_\Gamma \in \mathrm{Sub}^\circ_{\mathrm{fin}}(\Gamma)$, and the proof is complete. □

Theorem 2.2.1 and Theorem 2.3.25 together with Theorem 2.4.49 tell us that Theorem 2.6.21 counts indeed the number of maximal vertex groups which are not of type C_2 in a "reduced" complex of groups for $\Delta^{(p)}$. For $\Delta^{(p)}$, p prime, we are even able to compute those groups explicitly. For a general square-free integer

$d = \prod_{l=1}^{r} p_l$ the problem arises to determine the preimages of certain elements under the product maps

$$\mathrm{Fix}_{\Delta^{(d)}\backslash\Gamma}(H) \to \prod_{l=1}^{r} \mathrm{Fix}_{\Delta^{(p_l)}\backslash\Gamma}(H).$$

As long as we are not able to give an explicit system of representatives for $\Delta^{(d)}\backslash\Gamma$, an explicit description for the maximal vertex groups of $\Delta^{(d)}$, which are not of type C_2, can hardly be given. The results of Theorem 2.6.7 and Theorem 2.6.21 allow us to compute maximal $\Delta^{(p)}$-conjugacy classes of finite subgroups with finite Normalizers in Γ.

The subject of the following theorem is the computation of the exact values for the components of the decomposition of $\Delta^{(p)}\backslash M^{\circ}\big(\Delta^{(p)}\big)$ into $\mathfrak{M}_{\Delta^{(p)}}([H]_\Gamma)$, see Lemma 2.6.4, where p is an arbitrary non-negative prime. By Theorem 2.3.25, we know that any non-trivial finite subgroup of any type but C_2 has finite Normalizer in $\Delta^{(p)}$.

Theorem 2.6.22. *Let Ξ_H be defined as in Lemma 2.6.5 for each $H \in \mathrm{Sub}_{\mathrm{fin}}(\Gamma)$. For non-negative primes $p \in \mathbb{N}$ the sets $\mathfrak{M}_{\Delta^{(p)}}([H]_\Gamma)$ can be explicitly given for each $[H]_\Gamma \in \Gamma \backslash \mathrm{Sub}^{\circ}_{\mathrm{fin}}(\Gamma)$:*

type	Γ*-class*	*group (element of that class)*	*system of representatives for* $\Xi^{-1}_{(\cdot)}(\mathfrak{M}_{\Delta^{(p)}}(\cdot))$
C_3	${}^{D_{12}}C_3$	$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$	$\emptyset, \quad p \overset{(3)}{\not\equiv} 1,$ $\emptyset, \quad p \overset{(3)}{\equiv} 1.$
	${}^{S_4^1}C_3$	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$	$\emptyset, \quad p \overset{(3)}{\not\equiv} 1,$ $\{r_1^{(p)}(t-1,1,-1), r_1^{(p)}(1,t^2,t), r_1^{(p)}(t-1,t^2,t)\},\ t^2-t+1 \overset{(p)}{\equiv} 0, \quad p \overset{(3)}{\equiv} 1.$
C_4	${}^{S_4^1}C_4$	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$	$\emptyset, \quad p \overset{(4)}{\not\equiv} 1,$ $\{r_1^{(p)}(t,0,0), r_3^{(p)}(0,t), r_4^{(p)}(t)\},\ t^2 \overset{(p)}{\equiv} -1, \quad p \overset{(4)}{\equiv} 1.$
	${}^{S_4^2}C_4$	$\left\langle \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle$	$\emptyset, \quad p \overset{(4)}{\not\equiv} 1,$ $\{r_1^{(p)}(t,0,1), r_1^{(p)}(1,t+1,t), r_1^{(p)}(-t,t+1,t)\},\ t^2 \overset{(p)}{\equiv} -1, \quad p \overset{(4)}{\equiv} 1.$
V_4	${}^{S_4^1}V_4^{\bullet}$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	$r_6^{(2)}, \quad p = 2,$ $r_6^{(p)}, \quad p \neq 2.$
	${}^{S_4^1}V_4^{\circ}$	$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$	$\emptyset, \quad p \in \{2,3\},$ $\{r_1^{(p)}(0,0,1), r_2^{(p)}(0,1), r_5^{(p)}(1)\} \quad p \neq 2.$
	${}^{S_4^2}V_4^{\bullet}$	$\left\langle \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$	$\emptyset, \quad p = 2,$ $\{r_1^{(p)}(1,1,0)\}, \quad p \neq 2.$
	${}^{S_4^3}V_4^{\bullet}$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \right\rangle$	$\emptyset, \quad p = 2,$ $\{r_1^{(p)}(0,1,1)\}, \quad p \neq 2.$

C_6	${}^{D_{12}}C_6$	$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \right\rangle$	$\begin{array}{rr} \emptyset, & p \overset{(3)}{\not\equiv} 1, \\ \{r_1^{(p)}(0,0,t), r_2^{(p)}(0,t), r_5^{(p)}(t)\},\ t^2-t+1 \overset{(p)}{\equiv} 0, & p \overset{(3)}{\equiv} 1. \end{array}$
S_3	${}^{D_{12}}S_3^1$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$	$\begin{array}{rr} \{r_2^{(3)}(1,-1)\}, & p=3, \\ \emptyset, & p \neq 3. \end{array}$
	${}^{D_{12}}S_3^2$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\rangle$	$\begin{array}{rr} \{r_1^{(3)}(1,0,-1)\}, & p=3, \\ \emptyset, & p \neq 3. \end{array}$
	${}^{S_4^1}S_3$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle$	$\begin{array}{rr} \{r_1^{(3)}(1,1,-1)\}, & p=3, \\ \emptyset, & p \neq 3. \end{array}$
D_8	${}^{S_4^1}D_8$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$	$\begin{array}{rr} \{r_3^{(2)}(x,1) : x \in \mathbb{F}_2\} \cup \{r_1^{(2)}(1,0,0), r_1^{(2)}(1,1,1), r_4^{(2)}(1)\}, & p=2, \\ \emptyset, & p \neq 2. \end{array}$
	${}^{S_4^2}D_8$	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$	$\begin{array}{rr} \{r_1^{(2)}(1,0,1)\}, & p=2, \\ \emptyset, & p \neq 2. \end{array}$
D_{12}	D_{12}	$\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right\rangle$	$\begin{array}{rr} \{r_1^{(3)}(0,0,-1), r_2^{(3)}(0,-1), r_5^{(3)}(-1)\}, & p=3, \\ \emptyset, & p \neq 3. \end{array}$

PROOF. An analysis of Theorem 2.6.21 yields that the only non-trivial cases are $\left[{}^{D_{12}}S_3^j\right]_\Gamma$, $j \in 1, 2$ for $p = 3$ and $\left[{}^{S_4^1}V_4^\bullet\right]_\Gamma$ for $p = 2$. Let us first consider $\mathfrak{M}_{\Delta^{(p)}}(H_j)$ for $p = 3$, where $H_j := {}^{D_{12}}S_3^j$. We observe $\mathrm{Fix}_{\Delta^{(3)}\backslash\Gamma}(D_{12}) \subsetneq \mathrm{Fix}(H_j)$. Theorem 2.6.21 tells us that there is only one element in $\mathfrak{M}_{\Delta^{(3)}}(H_j)$. Because H_j satisfies $N_\Gamma(H_j) = D_{12}$ and property $\left(\mathcal{N}_{[D_{12}]_G}\right)$, we obtain by Lemma 2.6.16 that the only element in $\mathfrak{M}_{\Delta^{(3)}}(H_j)$ has to be parametrized by some element $\Delta\gamma_j \in \mathrm{Fix}_{\Delta^{(3)}\backslash\Gamma}(H_j) \setminus \mathrm{Fix}_{\Delta^{(3)}\backslash\Gamma}(D_{12})$. Theorem 2.6.7 yields $|\mathrm{Fix}_{\Delta^{(3)}\backslash\Gamma}(H_j) \setminus \mathrm{Fix}_{\Delta^{(3)}\backslash\Gamma}(D_{12})| = 2$. For the reason $D_{12} = N_\Gamma(H_j)$ acts from the right on $\mathrm{Fix}_{\Delta^{(3)}\backslash\Gamma}(H_j)$, we therefore obtain that the two elements in $\mathrm{Fix}_{\Delta^{(3)}\backslash\Gamma}(H_j) \setminus \mathrm{Fix}_{\Delta^{(3)}\backslash\Gamma}(D_{12})$ form one orbit. We conclude

$$\Xi_{H_j}^{-1}(\mathfrak{M}_{\Delta^{(3)}}(H_j)) = \{\Delta^{(3)}\gamma_j D_{12}\},$$

as desired.

We now turn to the computation of $\mathfrak{M}_{\Delta^{(p)}}(H)$ for $p = 2$ and $H = {}^{S_4^1}V_4^\bullet$. To this end, let us consider the action of $N_G(H)/H = S_4^1/H$ from the right on $\Omega := \mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}(H)$. This yields

$$|\Omega| = \sum_{k|(N_\Gamma(H):H)} \sum_{\substack{\Delta^{(2)}\gamma.N_\Gamma(H)\in\Omega/N_\Gamma(H):\\ |\Delta^{(2)}\gamma.N_\Gamma(H)|=k}} |\Delta^{(2)}\gamma.N_\Gamma(H)| \tag{2.6.31}$$

$$= \sum_{k|6} k \cdot |\{\Delta^{(2)}\gamma.S_4^1 \in \Omega/S_4^1 : |\Delta^{(2)}\gamma.S_4^1| = k\}|.$$

Now, consider the sets $\Omega_k := \{\Delta^{(2)}\gamma.S_4^1 \in \Omega/S_4^1 : |\Delta^{(2)}\gamma.S_4^1| = k\}$. We clearly have $\Omega_1 = \mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}(S_4^1) = \emptyset$. The orbit formula implies that $|\Delta^{(2)}\gamma.S_4^1| = k$ if and only if $|(S_4^1)_{\Delta^{(2)}\gamma}| = 24/k$. Because the only subgroup of S_4^1 of index 2 is ${}^{S_4^1}A_4$, we obtain for $\Delta^{(2)}\gamma.S_4^1 \in \Omega_2$ that $(S_4^1)_{\Delta\gamma} = {}^{S_4^1}A_4$. We therefore deduce

$$\Omega_2 \subseteq \mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}\left({}^{S_4^1}A_4\right)/S_4^1.$$

Because $\mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}\left({}^{S_4^1}A_4\right) = \emptyset$, we conclude $\Omega_2 = \emptyset$ as well.

On the other hand, every subgroup of index 3 in S_4^1 has to be S_4^1-conjugate to ${}^{S_4^1}D_8$. By Remark 20, we infer

$$\Omega_3 \subseteq \left(\bigcup_{\sigma\in S_4^1} \mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}\left({}^{S_4^1}D_8\right)\sigma\right)/S_4^1 = \mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}\left({}^{S_4^1}D_8\right)/S_4^1.$$

Because $\mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}(S_4^1) = \emptyset$, we also obtain

$$\mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}\left({}^{S_4^1}D_8\right) \subseteq \Omega_3.$$

We therefore get

$$\Omega_3 = \mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}\left({}^{S_4^1}D_8\right)/S_4^1 = \Xi^{-1}_{{}^{S_4^1}D_8}\left(\mathfrak{C}_{\Delta^{(2)}}\left({}^{S_4^1}D_8\right)\right).$$

Inserting these observations into (2.6.31), leads to

$$|\Omega| = 3|\mathfrak{C}_{\Delta^{(2)}}\left({}^{S_4^1}D_8\right)| + 6|\Omega_6|.$$

$|\Omega| = 21$ and $|\mathfrak{C}_{\Delta^{(2)}}\left({}^{S_4^1}D_8\right)| = 5$ enforce $|\Omega_6| = 1$. On the other hand, each element of Ω associated with an element in $\mathfrak{M}_{\Delta^{(2)}}(H)$ cannot lie in $\mathrm{Fix}_{\Delta^{(2)}\backslash\Gamma}\left({}^{S_4^1}D_8\right)$, for otherwise that element can also be associated with an element of $\mathfrak{C}_{\Delta^{(2)}}\left({}^{S_4^1}D_8\right)$. This violates the definition of $\mathfrak{M}_{\Delta^{(2)}}(H)$.

We conclude $\Xi_H^{-1}(\mathfrak{M}_{\Delta^{(2)}}(H)) \subseteq \Omega_6$. Because

$$1 = |\mathfrak{M}_{\Delta^{(2)}}(H)| \leq |\Omega_6| = 1,$$

we can choose an arbitrary element $\Delta^{(2)}\gamma \in \Omega_6$ and get $\mathfrak{M}_{\Delta^{(2)}}(H) = \{\Delta^{(2)}\gamma\}$. The orbit Ω_6 can be easily determined for example with [10], and the proof is complete. □

Recall that Γ acts on $X = \{A \in \mathbb{R}^{3\times 3} |\ \det(A) = 1, A = {}^tA, \langle Av, v\rangle > 0 \quad \forall\, 0 \neq v \in \mathbb{R}^3\}$. The Γ-space X', see Theorem 2.2.1, is a polyhedral complex. Hence there exists a scwol $\mathcal{X}_\Gamma$ such that its geometric realization can be identified with X'. In addition, X is a symmetric space and therefore geodesic complete. This forces that the action of each finite index subgroup of Γ on the scwol $\mathcal{X}_\Gamma$ satisfies the hypotheses for Theorem 2.4.49. Because X' is simply connected the scwol $\mathcal{X}_\Gamma$ has to be simply connected as well.

To consider the consequences of Theorem 2.4.49 in this setting, we introduce and recall the following notation.

Notation 2.6.23. Let Δ be an arbitrary subgroup of Γ such that $(\Gamma : \Delta) < \infty$. We denote by $\mathcal{R}_\Delta(\mathcal{X}_\Gamma)$ the reduction of $\mathcal{X}_\Gamma$ associated to the action $\Delta \curvearrowright \mathcal{X}_\Gamma$, as defined on page 48 and in Notation 2.4.50. Furthermore we agree on $\mathcal{G}(\mathcal{Y}(\Delta))$ to be the complex of groups associated to the action $\Delta \curvearrowright \mathcal{R}_\Delta(\mathcal{X}_\Gamma)$ over the scwol $\mathcal{Y}(\Delta) := \Delta\backslash\mathcal{R}_\Delta(\mathcal{X}_\Gamma)$.

Corollary 2.6.24. *Let Δ be an arbitrary subgroup of Γ such that $(\Gamma : \Delta) < \infty$. By Theorem 2.4.49 the map*

$$\Lambda\colon M^\circ(\mathcal{G}(\mathcal{Y}(\Delta))) \to \Delta\backslash M^\circ(\Delta)$$

induces for each Γ-conjugacy-class $[H_0]_\Gamma \in \Gamma\backslash \mathrm{Sub}^\circ_{\mathrm{fin}}(\Gamma)$ a bijection

$$\Lambda_{[H_0]_\Gamma}\colon \begin{array}{ccc} \{\bar v \in M^\circ(\mathcal{G}(\mathcal{Y}(\Delta))) : \Delta_{\bar v} \underset{\Gamma}{\sim} H_0\} & \to & \mathfrak{M}_\Delta([H_0]_\Gamma) \\ \bar v & \mapsto & [\Delta_{\bar v}]_\Delta \end{array}.$$

PROOF. This is a direct consequence of Theorem 2.4.49 and the introductory remark in front of Corollary 2.6.24. □

Definition 2.6.25 ([1] p.532 1.17)**.** Let $\mathcal{X}$ be a scwol and $v \in V(\mathcal{X})$. The *upper link* $\mathrm{Lk}^v(\mathcal{X})$ of v consists of a set of vertices

$$V(\mathrm{Lk}^v(\mathcal{X})) := \{a \in E(\mathcal{X}) : t(a) = v\}$$

and arrows

$$E(\mathrm{Lk}^v(\mathcal{X})) := \{(a,b) \in E^{(2)}(\mathcal{X}) : t(a) = v\}.$$

Moreover, we define $\mathrm{edge}^v(\mathcal{X})$ to be

$$\mathrm{edge}^v(\mathcal{X}) := \{a \in V(\mathrm{Lk}^v(X)) : \dim_{\mathcal{X}}(i_{\mathcal{X}}(a)) = 1\}.$$

Remark 24. Let $\mathcal{X}$ be a scwol, $v \in V(\mathcal{X})$ and G be a group acting on X. Then, this action induces actions $G_v \curvearrowright \mathrm{Lk}^v(\mathcal{X})$ and $G_v \curvearrowright \mathrm{edge}^v(\mathcal{X})$. For each $\gamma \in G$ the actions $\Gamma_v \curvearrowright \mathrm{Lk}^v(\mathcal{X})$ and $\Gamma_{\gamma.v} \curvearrowright \mathrm{Lk}^{\gamma.v}(\mathcal{X})$ are isomorphic and hence the actions $\Gamma_v \curvearrowright \mathrm{edge}^v(\mathcal{X})$ and $\Gamma_{\gamma.v} \curvearrowright \mathrm{edge}^{\gamma.v}(\mathcal{X})$ are isomorphic as well.

Lemma 2.6.26. *Let G be a group acting on a scwol $\mathcal{X}$. For any $v \in V(\mathcal{X})$ the following statement is true:*

$$\{g \in G : \exists a, a' \in \mathrm{Lk}^v(\mathcal{X})\colon g.a = a'\} \subseteq G_v,$$

and therefore, for each $a \in V(Lk^v(\mathcal{X}))$ the stabilizer G_v acts transistively on

$$G.a \cap V(\mathrm{Lk}^v(\mathcal{X})).$$

In particular, this implies

$$G.a \cap V(\mathrm{Lk}^v(\mathcal{X})) = G_v.a.$$

Lemma 2.6.27. *Let G be a group satisfying FCoFG acting with finite stabilizers on a finite dimensional connected scwol $\mathcal{X}$ such that $G_v \notin M^\circ(G)$ for all $v \in V_{\max}(\mathcal{X})$. Let $\mathcal{D}$ be a fundamental domain $\mathcal{D}$ for that action. Let $\mathcal{R}_G(\mathcal{X})$ be the reduction of the action of G on $\mathcal{X}$. Then, the subscwol $\mathcal{R}_G(\mathcal{D})$, which we define via*

$$V(\mathcal{R}_G(\mathcal{D})) := \{[v]_{\mathcal{R}_G} : v \in V(\mathcal{D})\},$$
$$E(\mathcal{R}_G(\mathcal{D})) := \{[a]_{\mathcal{R}_G} : a \in \mathcal{R}_G(E(\mathcal{X})) \cap E(\mathcal{D})\},$$

is a fundamental domain for the induced action of G on $\mathcal{R}_G(\mathcal{X})$.

PROOF. We begin by proving that $\mathcal{R}_G(\mathcal{D})$ is a subscwol. Clearly, $[a] \in E(\mathcal{R}_G(\mathcal{D}))$ implies $i([a]), t([a]) \in V(\mathcal{R}_G(\mathcal{D}))$. It remains to show that for each pair of arrows $[a], [b] \in E(\mathcal{R}_G(\mathcal{D}))$ satisfying $i([a]) = t([b])$ it is also true that $[a] \circ [b] \in E(\mathcal{R}_G(\mathcal{D}))$. For this purpose, take such $[a]$, $[b]$. By definition of $\mathcal{R}_G(\mathcal{D})$, there are elements

$$a' \in [a] \cap \mathcal{R}_G(E(\mathcal{X})) \cap E(\mathcal{D}) \text{ and } b' \in [b] \cap \mathcal{R}_G(E(\mathcal{X})) \cap E(\mathcal{D}).$$

a' and b' necessarily satisfy $a' \sim b'$. In the case $i(a') = t(b')$, we have

$$[a] \circ [b] = [a' \circ b'] \in E(\mathcal{R}_G(\mathcal{D})),$$

as claimed. Hence, let us assume $i(a') \neq t(b')$ occurs. By definition of the equivalence relation "$\sim$" (p. 47 (2.4.1)), this forces $G_{i(a')} \in M^\circ(G)$. This yields $a' \notin \mathcal{R}_G(E(\mathcal{X}))$, a contradiction. Thus, $\mathcal{R}_G(\mathcal{D})$ is indeed a subscwol.

Now, let us show that $\mathcal{R}_G(\mathcal{D})$ satisfies the axioms of a fundamental domain. We begin with Definition 2.4.19 (1). Because $G_v \notin M^\circ(G)$ for each $v \in V_{\max}(\mathcal{X})$, we clearly have $[v] \in V_{\max}(\mathcal{R}_G(\mathcal{X}))$ if and only if $v \in V_{\max}(\mathcal{X})$ and $[v] \in V_{\max}(\mathcal{R}_G(\mathcal{D}))$ if and only if $v \in \mathcal{R}_G(\mathcal{D})$. If $|[v]| = 1$ for each $[v] \in V_{\max}(\mathcal{R}_G(\mathcal{X}))$, we obtain $V_{\max}(\mathcal{R}_G(\mathcal{X})) = V_{\max}(\mathcal{X})$ and $V_{\max}(\mathcal{R}_G(\mathcal{D})) = V_{\max}(\mathcal{D})$, which yields indeed Definition 2.4.19 (1).
Hence, it is left to show $|[v]| = 1$ for each $[v] \in V_{\max}(\mathcal{R}_G(\mathcal{X}))$. Suppose there is a $v' \neq v$ with $v' \in [v]$. Then $v' \sim v$. By the definition of "$\sim$", we get $G_{v'} = G_v \in M^\circ(G)$, a contradiction to our hypothesis.
Obviously, $\mathcal{R}_G(\mathcal{D})$ inherits property Definition 2.4.19 (2) from $\mathcal{D}$. Thus, $\mathcal{R}_G(D)$ is a fundamental domain for $G \curvearrowright \mathcal{R}_G(\mathcal{X})$, as claimed. □

Let $\Delta \leq G$ be a finite index subgroup and $\mathfrak{M}_\Delta([H_0]_G) \neq \emptyset$. For the link of an arbitrary vertex in $\mathcal{X}$, we want to compute the stabilizers regarding $\Delta \curvearrowright \mathcal{X}$, given the stabilizers for $G \curvearrowright \mathcal{X}$ on a fundamental domain for $G \curvearrowright \mathcal{X}$. To obtain a compact notation, it seems necessary to label each vertex and each edge with its isotropy group with respect to the current action.

Definition 2.6.28. Let G be a group acting on an arbitrary scwol $\mathcal{X}$. Let $\mathcal{V}$ be a subset of $V(\mathcal{X})$ and $\mathcal{E}$ be subset of $E(\mathcal{X})$. We set

$$\mathrm{L}_G\,\mathcal{V} := \{(v, G_v) : v \in \mathcal{V}\} \text{ and } \mathrm{L}_G\,\mathcal{E} := \{(a, G_a) : a \in \mathcal{E}\}.$$

We call an element of $\mathrm{L}_G V(\mathcal{X})$ a *labeled vertex* and an element of $\mathrm{L}_G E(\mathcal{X})$ a *labeled arrow*. Furthermore, we call for each $v \in V(\mathcal{X})$ the set

$$\mathrm{L}_G \mathrm{Lk}^v(\mathcal{X}) := \{(a, G_a) : a \in V(\mathrm{Lk}^v(\mathcal{X}))\}$$

the *labeled link* of v. $\mathrm{L}_G \mathrm{Lk}^v(\mathcal{X})$ is a subset of $E(\mathcal{X}) \times \mathrm{Sub}_{\mathrm{fin}}(G)$.

G acts on $V(\mathcal{X}) \times \mathrm{Sub}_{\mathrm{fin}}(G)$ via

$$g.(v, H) := (g.v, {}^gH), \quad (v, H) \in V(\mathcal{X}) \times \mathrm{Sub}_{\mathrm{fin}}(G)$$

and on $E(\mathcal{X}) \times \mathrm{Sub}_{\mathrm{fin}}(G)$ via

$$g.(a, H) := (g.a, {}^gH), \quad (a, H) \in E(\mathcal{X}) \times \mathrm{Sub}_{\mathrm{fin}}(G).$$

Finally, we call for each $H \in \mathrm{Sub}_{\mathrm{fin}}(G)$ the map χ_H given by

$$\chi_H \colon \begin{array}{ccc} E(\mathcal{X}) \times \mathrm{Sub}_{\mathrm{fin}}(G) & \to & E(\mathcal{X}) \times \mathrm{Sub}_{\mathrm{fin}}(G) \\ (a, K) & \mapsto & (a, H \cap K), \end{array}$$

the *intersection with* H.

Remark 25. Let us use the same notation as above. Then, we have G-isomorphisms

$$\begin{array}{ccc} V(\mathcal{X}) & \xrightarrow{\cong} & \mathrm{L}_G V(\mathcal{X})\colon \\ v & \mapsto & (v, G_v) \end{array} \text{ and } \begin{array}{ccc} E(\mathcal{X}) & \xrightarrow{\cong} & \mathrm{L}_G E(\mathcal{X})\colon \\ a & \mapsto & (a, G_a). \end{array}$$

Lemma 2.6.29. *Let $\mathcal{X}$ be a scwol and G be a group satisfying FCoFG and acting with finite stabilizers on it. Denote by φ the pair of maps*

$$\varphi\colon \begin{cases} \begin{array}{ccc} V(\mathcal{X}) & \to & V(\mathcal{R}_G(\mathcal{X}))\colon \\ v & \mapsto & [v] \end{array} \\ \\ \begin{array}{ccc} \mathcal{R}_G(E(\mathcal{X})) & \to & E(\mathcal{R}_G(\mathcal{X})\colon \\ a & \mapsto & [a]. \end{array} \end{cases}$$

Then, for each $[v] \in \mathcal{V}(\mathcal{R}_G(\mathcal{X}))$ the following formula is true:

$$V\Big(\mathrm{Lk}^{[v]}(\mathcal{R}_G(\mathcal{X}))\Big) = \bigcup_{v' \in [v]} \varphi\Big(\mathcal{R}_G(E(\mathcal{X})) \cap V(\mathrm{Lk}^{v'}(\mathcal{X}))\Big).$$

PROOF. The proof is trivial. □

Remark 26. The reduction of $\mathcal{R}_G$ does not necessarily preserve dimensions. In particular, this means if we compute $\mathrm{L}_G\{a \in V(\mathrm{Lk}^v(\mathcal{X})) : \dim_{\mathcal{X}} i(a) = 1\}$, we do not have the complete information about $\mathrm{L}_G\{[a] \in V(\mathrm{Lk}^{[v]}(\mathcal{X})) : \dim_{\mathcal{R}_G(\mathcal{X})} i([a]) = 1\}$.

Lemma 2.6.30 (Determination of the link, given a fundamental domain)**.** *Let G be a group acting on a finite dimensional connected scwol $\mathcal{X}$. Let $\mathcal{D}$ be a fundamental for that action. Then, for each $v_0 \in V(\mathcal{X})$ the vertices of $\mathrm{Lk}^{v_0}(\mathcal{X})$ can be represented as follows:*

$$V(\mathrm{Lk}^{v_0}(\mathcal{X})) = \bigcup_{\substack{(\gamma,v) \in G \times V(\mathcal{D})\colon \\ \gamma.v = v_0}} \gamma.V(\mathrm{Lk}^v(\mathcal{D})).$$

Moreover, for each family $(\mathcal{L}^v)_{v \in V(\mathcal{D})}$ such that

$$\mathcal{L}^v \subseteq V(\mathrm{Lk}^v(\mathcal{D}))$$

and $\bigcup_{a\in\mathcal{L}^v} G_v.a \supseteq V(\mathrm{Lk}^v(\mathcal{D}))$ *the following statement is true:*

$$V(\mathrm{Lk}^{v_0}(\mathcal{X})) = \bigcup_{\substack{(\gamma,v)\in G\times V(\mathcal{D}):\\ \gamma.v=v_0}} \bigcup_{a'\in\mathcal{L}^v} \gamma G_v.a' = \bigcup_{\substack{(\gamma,v')\in G\times V(\mathcal{D}):\\ \gamma.v=v_0}} \bigcup_{a\in\gamma.\mathcal{L}^v} G_{v_0}.a.$$

PROOF. The proof consists of straightforward applications of Lemma 2.4.20. □

Lemma 2.6.31. *Let G be a group acting on a finite dimensional connected scwol $\mathcal{X}$. Let $\mathcal{D}$ be an arbitrary fundamental domain for this action. And denote by $\mathcal{Y}_G$ the quotient scwol. Furthermore, let ϱ_G be the quotient morphism*

$$\varrho_G\colon \begin{array}{ccccc} V(\mathcal{X}) & \to & V(\mathcal{Y}_G)\colon & v & \mapsto & G.v \\ E(\mathcal{X}) & \to & E(\mathcal{Y}_G)\colon & a & \mapsto & G.a \end{array}.$$

Then, for each $\bar{v}\in V(\mathcal{Y}_G)$ there is a $v_0\in V(\mathcal{D})\cap\bar{v}$ such that

$$V(\mathrm{Lk}^{\bar{v}}(\mathcal{Y}_G)) = \varrho_G(V(\mathrm{Lk}^{v_0}(\mathcal{X}))).$$

PROOF. Let $\bar{v}\in V(\mathcal{Y}_G)$. By Lemma 2.4.20, we clearly have $\bar{v}\cap V(\mathcal{D})\neq\emptyset$. Pick a fixed $v_0\in\bar{v}\cap V(\mathcal{D})$. Then $v'=g.v_0$ for suitable $g\in G$. In particular, we get $V(\mathrm{Lk}^{v'}(\mathcal{X})) = g.V(\mathrm{Lk}^{v_0}(\mathcal{X}))$ and therefore

$$\varrho_G(V(\mathrm{Lk}^{v'}(\mathcal{X}))) = \varrho_G(V(\mathrm{Lk}^{v_0}(\mathcal{X}))).$$

We hence conclude

$$\begin{aligned} V(\mathrm{Lk}^{\bar{v}}(\mathcal{Y}_G)) &= \bigcup_{v'\in\bar{v}} \{G.a\in E(\mathcal{Y}_G) : t(a)=v'\} \\ &= \bigcup_{v'\in\bar{v}} \varrho_G(V(\mathrm{Lk}^{v'}(\mathcal{X}))) \\ &= \varrho_G(V(\mathrm{Lk}^{v_0}(\mathcal{X}))), \end{aligned}$$

as desired. □

Proposition 2.6.32. *Let G be a group satisfying FCoFG, which acts with finite stabilizers on a scwol $\mathcal{X}$. Let $\Delta\leq G$ with $(G:\Delta)<\infty$ and S be a system of representatives for $\Delta\backslash G$. Furthermore, let $v\in V(\mathcal{X})$ and $H_0\in\mathrm{Sub}_{\mathrm{fin}}(G)$ such that $H_0\leq G_v$. Moreover, assume there is a $s\in S$ such that ${}^sH_0\in M(\Delta)$. Finally, denote by χ_{H_0} the intersection with H_0. Then,*

$$\mathrm{L}_\Delta \mathrm{Lk}^{s.v}(\mathcal{X}) = s.\chi_{H_0}(\mathrm{L}_G \mathrm{Lk}^v(\mathcal{X})).$$

PROOF. Take $s\in S$ as above. We first prove $\Delta_{s.v}={}^sH_0$. To this end, we observe $H_0\leq G_v$ forces ${}^sH_0\subseteq G_{s.v}$. But we also have ${}^sH_0\subseteq\Delta$ by the choice of s. We thus obtain

$$H_0\leq G_{s.v}\cap\Delta=\Delta_{s.v}.$$

On the other hand, $\Delta_{s.v}=G_{s.v}\cap\Delta$ is finite by hypothesis. For the reason that ${}^sH_0$ is a maximal finite subgroup in Δ, we hence conclude ${}^sH_0=\Delta_{s.v}$, as desired.

Let us show "$\subseteq$". So, let us take $(a,\Delta_a)\in\mathrm{L}_\Delta\mathrm{Lk}^v(\mathcal{X})$ and put $a':=s^{-1}a$. We clearly have $a'\in V(\mathrm{Lk}^v(\mathcal{X}))$. From

$$\begin{aligned} s.\chi_{H_0}(a',G_{a'}) &= s.(a',H_0\cap G_{a'}) \\ &= (a,{}^sH_0\cap G_{s.a'}) \\ &= (a,\Delta_v\cap G_a) \\ &= (a,\Delta_a), \end{aligned}$$

we deduce $(a, \Delta_a) \in s.\chi_{H_0}(\mathrm{L}_G \operatorname{Lk}^v(H_0))$, as required.

For the converse direction, take $s.(a', H_0 \cap G_{a'})$ with $a' \in \operatorname{Lk}^v(\mathcal{X})$. Putting $a := s.a' \in V(\operatorname{Lk}^{s.v}(\mathcal{X}))$, we obtain by the same computation as above

$$s.(a', H_0 \cap G_{a'}) = (a, \Delta_a) \in \mathrm{L}_\Delta \operatorname{Lk}^{s.v}(\mathcal{X}),$$

as claimed. This completes the proof. □

The 1-dimensional local structure of the complex of groups associated to $\Delta^{(d)} \curvearrowright \mathcal{X}_\Gamma$ over $\mathcal{Y}_{\Delta^{(d)}} := \Delta^{(d)} \backslash \mathcal{X}_\Gamma$ can be partially described by the table given below. Partially has two meanings here: The first one is that the hypotheses for Proposition 2.6.32 have to be satisfied to give the computation below an importance. Proposition 2.6.32 enables us to convert local data of the complex of groups associated to $\Gamma \curvearrowright \mathcal{X}_\Gamma$ over $\Gamma\backslash\mathcal{X}_\Gamma$ into local data for the complex of groups associated to $\Delta^{(d)} \curvearrowright \mathcal{X}_\Gamma$ over $\Delta^{(d)}\backslash\mathcal{X}_\Gamma$. Anything else can be derived from that data by the lemmas Lemma 2.6.27, Lemma 2.6.29, Lemma 2.6.30 and Lemma 2.6.31. This brings up the second meaning of "partially". We will only compute the 1-dimensional local structure, and for the sake of overview, we will restrict ourselves to give derived sizes of that data. The result will be Theorem 2.6.39.

Definition 2.6.33. Let $\mathcal{X}$ be a scwol, G be a FCoFG-group acting with finite stabilizers on it, and $\Delta \leq G$ such that $(G : \Delta) < \infty$. Furthermore, let $v \in V(\mathcal{X})$, $H_0 \leq G_v$ and $[L]_G \in G\backslash \operatorname{Sub}_{\mathrm{fin}}(G)$. Then, for each $a \in V(\operatorname{Lk}^v(\mathcal{X}))$ we set

$$\alpha^{v,H_0}_{[L]_G}(a) := \alpha^{(v,H_0)}_{[L]_G}(a) := |\{(a', H_0 \cap G_{a'}) : a' \in G_v.a,\ [H_0 \cap G_{a'}]_G = [L]_G\}|,$$

$$\beta^v_{\Delta,[L]_G}(a) := |\{(a', \Delta_{a'}) \in \mathrm{L}_\Delta \operatorname{Lk}^v(\mathcal{X}) : a' \in G_v.a,\ [\Delta_{a'}]_G = [L]_G\}|.$$

Remark 27.

(1) From a geometric perspective, $\beta^v_{\Delta,[L]_G}(a)$ counts the number of faces containing v, which lie in the G-orbit of the face a, such that their stabilizers under the Δ-action are G-conjugate to L.

(2) $\alpha^{v,H_0}_{[L]_G}(a) = \alpha^{v,H_0}_{[L]_G}(a')$ and $\beta^v_{\Delta,[L]_G}(a) = \beta^v_{\Delta,[L]_G}(a') \quad \forall a' \in G_v.a$.

(3) $\alpha^{v,H_0}_{[L]_G}(a) = \alpha^{\gamma.(v,H_0)}_{[L]_G}(a') \quad \forall \gamma \in G\ \forall a' \in G.a \cap V(\operatorname{Lk}^{\gamma v}(\mathcal{X}))$.

Proof. The only non-trivial assertion is (3). It is clearly true that

$$\alpha^{(v,H_0)}_{[L]_G}(a) = \alpha^{\gamma.(v,H_0)}(\gamma a).$$

Now, (2) combined with Lemma 2.6.26 implies

$$\alpha^{\gamma.(v,H_0)}(\gamma a) = \alpha^{\gamma.(v,H_0)}(a') \quad \forall a' \in G.a \cap V(\operatorname{Lk}^{\gamma v}(\mathcal{X})),$$

as required. □

Proposition 2.6.34. *Let $\mathcal{X}$ be scwol, G be a FCoFG-group acting with finite stabilizers on it, and $\Delta \leq G$ such that $(G : \Delta) < \infty$. Furthermore, let $v \in V(\mathcal{X})$ and $H_0 \leq G_v$ such that $\mathfrak{M}_\Delta([H_0]_G) \neq \emptyset$. Take an arbitrary $[L]_G \leq [H_0]_G$. Then, for each $s \in \Delta\gamma N_G(H) \in \Xi^{-1}_{H_0}(\mathfrak{M}_\Delta([H_0]_G))$ and $a \in V(\operatorname{Lk}^v(\mathcal{X}))$ the following formula is true:*

$$\alpha^{v,H_0}_{[L]_G}(a) = \beta^{s.v}_{\Delta,[L]_G}(b) \quad \forall b \in G.a \cap V(\operatorname{Lk}^{s.v}(\mathcal{X})).$$

Proof. We set for each $a \in V(\operatorname{Lk}^v(\mathcal{X}))$:

$$A^{v,H_0}_{[L]_G}(a) := \{(a', H_0 \cap G_{a'}) : a' \in G_v.a,\ [H_0 \cap G_{a'}]_G = [L]_G\},$$

$$B^v_{\Delta,[L]_G} := \{(a', \Delta_{a'}) \in \mathrm{L}_\Delta \operatorname{Lk}^v(\mathcal{X}) : a' \in G_v.a,\ [\Delta_{a'}]_G = [L]_G\}.$$

We start by proving the first assertion of the theorem. To this end, take a $s \in \Delta\gamma N_G(H) \in \Xi_{H_0}^{-1}(\mathfrak{M}_\Delta([H_0]_G))$ and consider the map

$$\Phi_s\colon \begin{array}{ccc} \chi_{H_0}(G_v.(a,G_a)) & \to & s.\chi_{H_0}(G_v.(a,G_a)) \\ (a',H_0\cap G_{a'}) & \mapsto & s(a',H_0\cap G_{a'}). \end{array}$$

This map is obviously a bijection. By Proposition 2.6.32, we also have

$$\mathrm{im}(\Phi_s) \subseteq \mathrm{L}_\Delta \,\mathrm{Lk}^{s.v}(\mathcal{X}).$$

This in particular means

$$\mathrm{im}(\Phi_s) = \{(a',\Delta_{a'}) \in \mathrm{L}_\Delta\,\mathrm{Lk}^{s.v}(\mathcal{X}) : a' \in G_{sv}.sa\}$$

and

$$^{s}(H_0 \cap G_{a'}) = \Delta_{sa'}.$$

Hence, the restriction of Φ_s on $A^{v,H_0}_{[L]_G}(a)$ maps onto $B^{s.v}_{\Delta,[L]_G}(s.a)$ and therefore

$$\alpha^{v,H_0}_{[L]_G}(a) = \beta^{s.v}_{\Delta,[L]_G}(s.a),$$

as required. Take an arbitrary $b \in G.a \cap V(\mathrm{Lk}^{s.v}(\mathcal{X}))$. By Lemma 2.6.26, we have $b \in G_{sv}.sa$. For the reason, that $\beta^{s.v}_{\Delta,[L]_G}$ is constant along $G_{s.v}$-orbits, this forces

$$\beta^{s.v}_{\Delta,[L]_G}(s.a) = \beta^{s.v}_{\Delta,[L]_G}(b).$$

This yields the claim. □

Proposition 2.6.35. *Let $\mathcal{X}$ be a finite dimensional connected scwol, G be a FCoFG-group acting with finite stabilizers on it. Let $\mathcal{D}_G$ the fundamental domain of that action, and $\Delta \leq G$ such that $(G:\Delta) < \infty$. Furthermore, let $v \in V(\mathcal{X})$ such that $\Delta_v \in M(\Delta)$. Then, there are $v_0 \in V(\mathcal{D}_G)$ and $H_0 \leq G_{v_0}$ such that*

$$\beta^{v}_{\Delta,[L]_G}(a) = \alpha^{v_0,H_0}_{[L]_G}(a_0) \quad \forall a \in V(\mathrm{Lk}^{v}(\mathcal{X})),\ \forall a_0 \in G.a \cap V(\mathrm{Lk}^{v_0}(\mathcal{X})).$$

Proof. Let $v \in V(\mathcal{X})$ such that Δ_v is maximal finite in Δ. Take an arbitrary $a \in V(\mathrm{Lk}^{v}(\mathcal{X}))$. Lemma 2.4.20 ensures there is a $v_0 \in V(\mathcal{D}_G)$ and a $\gamma \in G$ such that $v = \gamma.v_0$. Put $H_0 := {}^{\gamma^{-1}}\Delta_v$ and $b_0 := \gamma^{-1}a$. Proposition 2.6.34 now implies that

$$\beta^{v}_{\Delta,[L]_G}(a) = \alpha^{v_0,H_0}_{[L]_G}(b_0).$$

Take an arbitrary $a_0 \in G.a \cap V(\mathrm{Lk}^{v_0}(\mathcal{X}))$. By Lemma 2.6.26, we have $a_0 \in G_{v_0}.b_0$. For the reason, that $\alpha^{v_0,H_0}_{[L]_G}$ is constant along G_{v_0}-orbits, this forces

$$\alpha^{v_0,H_0}_{[L]_G}(b_0) = \alpha^{v_0,H_0}_{[L]_G}(a_0).$$

Combining both equations, we conclude

$$\beta^{v}_{\Delta,[L]_G}(a) = \alpha^{v_0,H_0}_{[L]_G}(a_0), \quad \forall a_0 \in G.a \cap V(\mathrm{Lk}^{v_0}(\mathcal{X})),$$

as claimed. □

Lemma 2.6.36. *Let G be a group and $H, K \leq G$ subgroups. Then,*

$$N_G(H)N_G(K) \subseteq \{\gamma \in G : [H \cap {}^{\gamma}K]_G = [H \cap K]_G\}.$$

Remark 28. $N_G(H)N_G(K)$ has not to be a subgroup here.

Proof. Let $h \in N_G(H)$ and $k \in N_G(K)$. We compute

$$[H \cap {}^{hk}K]_G = \left[({}^{h^{-1}}H) \cap K\right]_G = [H \cap K]_G$$

and conclude $N_G(H)N_G(K) \subseteq \{\gamma \in G : [H \cap {}^{\gamma}K]_G = [H \cap K]_G\}$, as required. □

Lemma 2.6.37. *Let $\mathcal{X}$ be a scwol, G be a FCoFG-group acting with finite stabilizers on it, and $\Delta \leq G$ such that $(G:\Delta) < \infty$. Furthermore, let $v \in V(\mathcal{X})$, $H_0 \leq G_v$ and $[L]_G \in G\backslash \mathrm{Sub}_{\mathrm{fin}}(G)$. Then, for any $a \in V(\mathrm{Lk}^v(\mathcal{X}))$ such that*

$$|N_{G_v}(H_0)| \cdot |N_{G_v}(G_a)| \geq |G_v| \cdot |N_{G_v}(H_0) \cap N_{G_v}(G_a)|,$$

we have

$$\alpha^{v,H_0}_{[L]_G}(a) \in \left\{0, \frac{|G_v|}{|G_a|}\right\}.$$

Proof. Take v, H_0, $[L]_G$ and a as above. If $[H_0 \cap G_{a'}]_G \neq [L]_G$ for each $a' \in G_v.a$, we clearly have $\alpha^{v,H_0}_{[L]_G}(a) = 0$, as desired.
Hence, we assume there is at least one $a_0 \in G_v.a$ such that $[H_0 \cap G_{a_0}]_G = [L]_G$. We take a $g_0 \in G_v$ such that $a_0 = g_0 a$. By hypothesis, we have

$$|N_{G_v}(H_0)N_{G_v}(G_a)| = \frac{|N_{G_v}(H_0)| \cdot |N_{G_v}(G_a)|}{|N_{G_v}(H_0) \cap N_{G_v}(G_a)|} \geq |G_v|$$

and therefore $G_v = N_{G_v}(H_0)N_{G_v}(G_a)$. Lemma 2.6.36 forces that

$$[H_0 \cap G_{a_0}]_{G_v} = [H_0 \cap {}^{g_0}G_a]_{G_v} = [H_0 \cap G_a]_{G_v} = [H_0 \cap {}^{g}G_a]_{G_v} \quad \forall g \in G_v.$$

In particular, we obtain

$$[H_0 \cap G_{a'}]_G = [L]_G \quad \forall a' \in G_v.a.$$

This implies

$$\alpha^{v,H_0}_{[L]_G}(a) = |G_v.(a, H_0 \cap G_a)| = |G_v.a| = \frac{|G_v|}{|G_a|},$$

as required. □

Definition 2.6.38. Let $v \in V(\mathcal{X}_\Gamma)$. We call $H_0 \in \mathrm{Sub}(\Gamma)$ *v-admissible* if and only if $H_0 \in \mathrm{Sub}^\circ_{\mathrm{fin}}(\Gamma) \cap \mathrm{Sub}(\Gamma_v)$ and there is a square-free integer $d \in \mathbb{Z}$ such that $\mathfrak{M}_{\Delta^{(d)}}([H_0]_\Gamma) \neq \emptyset$.

Theorem 2.6.39. *Consider the action of $\Gamma \curvearrowright \mathcal{X}_\Gamma$. Let $\mathcal{D}_\Gamma$ be the fundamental domain for that action given by Theorem 2.2.1. Let d be an arbitrary square-free integer. Then, the sizes $|\mathfrak{M}_{\Delta^{(d)}}([H]_\Gamma)|$ can be computed for every $H \in \mathrm{Sub}^\circ_{\mathrm{fin}}(\Gamma)$. For each $v \in V(\mathcal{D}_\Gamma)$, $a \in \mathrm{edge}^v(\mathcal{D}_\Gamma)$, for each v-admissible $H_0 \in \mathrm{Sub}^\circ_{\mathrm{fin}}(\Gamma)$ and for each $1 \neq [L]_\Gamma \leq [H_0]_\Gamma$ we compute the numbers $\alpha^{v,H_0}_{[L]_\Gamma}(a)$. By Remark 27, $\alpha^{v,H_0}_{[L]_\Gamma}$ depends not on the exact choice of H_0 itself, but on $[H_0]_{\Gamma_v}$. The results of that computation can be found in the table below. As a consequence of Proposition 2.6.34 and Proposition 2.6.35, these numbers determine all the functions $\beta^w_{\Delta,[L]_\Gamma}$ with $w \in M^\circ(\Delta)$.*

We use the same the notation as in Theorem 2.2.1 and Notation 2.3.19. In addition, we put

$$ {}^{N}D_8 := \Gamma_N = \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle, $$

$$ {}^{{}^{N}D_8}V_4^1 := \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \right\rangle $$

and

$$ {}^{{}^{N}D_8}V_4^2 := \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle. $$

The rows belonging to a fixed vertex v and a fixed edge a of the table below consist of those entries, for which there exist at least one v-admissible H_0 and one $1 \neq [L]_\Gamma \leq [H_0]_\Gamma$ such that $\alpha^{v,H_0}_{[L]_\Gamma}(a) \neq 0$.

"vertex" $v \in V_0(\mathcal{D}_\Gamma)$	Γ_v	"edge" $a \in \mathrm{edge}^v(\mathcal{D}_\Gamma)$	$[H_0]_{\Gamma_v}$, H_0 v-admissible	$[L]_\Gamma$	$\alpha^{v,H_0}_{[L]_\Gamma}(a)$
O	S_4^1	OM	${}^{S_4^1}C_3$	${}^{S_4^1}C_3$	1
			${}^{S_4^1}V_4^\circ$	${}^{S_4^1}C_2^\circ$	4
			${}^{S_4^1}S_3$	${}^{S_4^1}C_2^\circ$	3
				${}^{S_4^1}S_3$	1
			${}^{S_4^1}D_8$	${}^{S_4^1}C_2^\circ$	4
		OQ	${}^{S_4^1}V_4^\circ$	${}^{S_4^1}V_4^\circ$	2
			${}^{S_4^1}V_4^\bullet$	${}^{S_4^1}C_2^\bullet$	6
			${}^{S_4^1}C_4$	${}^{S_4^1}C_2^\bullet$	2
			${}^{S_4^1}S_3$	${}^{S_4^1}C_2^\circ$	6
			${}^{S_4^1}D_8$	${}^{S_4^1}C_2^\bullet$	4
				${}^{S_4^1}V_4^\circ$	2
		OP	${}^{S_4^1}C_3$	${}^{S_4^1}C_3$	1
			${}^{S_4^1}V_4^\circ$	${}^{S_4^1}C_2^\circ$	4
			${}^{S_4^1}S_3$	${}^{S_4^1}C_2^\circ$	3
				${}^{S_4^1}S_3$	1
			${}^{S_4^1}D_8$	${}^{S_4^1}C_2^\circ$	4
		ON	${}^{S_4^1}V_4^\circ$	${}^{S_4^1}C_2^\circ$	4
			${}^{S_4^1}S_3$	${}^{S_4^1}C_2^\circ$	6
			${}^{S_4^1}D_8$	${}^{S_4^1}C_2^\circ$	4
		ON'	${}^{S_4^1}V_4^\circ$	${}^{S_4^1}C_2^\circ$	4
			${}^{S_4^1}S_3$	${}^{S_4^1}C_2^\circ$	6
			${}^{S_4^1}D_8$	${}^{S_4^1}C_2^\circ$	4
		OM'	${}^{S_4^1}V_4^\circ$	${}^{S_4^1}C_2^\circ$	4
			${}^{S_4^1}S_3$	${}^{S_4^1}C_2^\circ$	6
			${}^{S_4^1}D_8$	${}^{S_4^1}C_2^\circ$	4

M	S_4^2	MN	${}^{S_4^2}V_4^\circ$	${}^{S_4^3}V_4^\bullet$	2
			${}^{S_4^2}V_4^\bullet$	${}^{S_4^1}C_2^\circ$	6
			${}^{S_4^2}C_4$	${}^{S_4^1}C_2^\circ$	2
			${}^{S_4^2}S_3$	${}^{S_4^1}C_2^\circ$	6
			${}^{S_4^2}D_8$	${}^{S_4^1}C_2^\circ$	4
				${}^{S_4^3}V_4^\bullet$	2
		OM	${}^{S_4^2}C_3$	${}^{S_4^1}C_3$	1
			${}^{S_4^2}V_4^\circ$	${}^{S_4^1}C_2^\circ$	4
			${}^{S_4^2}S_3$	${}^{S_4^1}C_2^\circ$	3
				${}^{S_4^1}S_3$	1
			${}^{S_4^2}D_8$	${}^{S_4^1}C_2^\circ$	4
		MQ	${}^{S_4^2}V_4^\circ$	${}^{S_4^1}C_2^\circ$	4
			${}^{S_4^2}S_3$	${}^{S_4^1}C_2^\circ$	6
			${}^{S_4^2}D_8$	${}^{S_4^1}C_2^\circ$	4
M' $= q_1.M$	${}^{q_1}S_4^2$	$M'N$ $M'N'$	${}^{({}^{q_1}S_4^2)}V_4^\circ$	${}^{S_4^3}V_4^\bullet$	2
			${}^{({}^{q_1}S_4^2)}V_4^\bullet$	${}^{S_4^1}C_2^\circ$	6
			${}^{({}^{q_1}S_4^2)}C_4$	${}^{S_4^1}C_2^\circ$	2
			${}^{({}^{q_1}S_4^2)}S_3$	${}^{S_4^1}C_2^\circ$	6
			${}^{({}^{q_1}S_4^2)}D_8$	${}^{S_4^1}C_2^\circ$	4
				${}^{S_4^3}V_4^\bullet$	2
		$M'P$	${}^{({}^{q_1}S_4^2)}V_4^\circ$	${}^{S_4^1}C_2^\circ$	2
				${}^{S_4^3}V_4^\bullet$	1
			${}^{({}^{q_1}S_4^2)}V_4^\bullet$	${}^{S_4^2}V_4^\bullet$	3
			${}^{({}^{q_1}S_4^2)}C_4$	${}^{S_4^1}C_2^\circ$	2
				${}^{S_4^2}C_4$	1
			${}^{({}^{q_1}S_4^2)}S_3$	${}^{S_4^1}C_2^\circ$	3
			${}^{({}^{q_1}S_4^2)}D_8$	${}^{S_4^2}V_4^\bullet$	2
				${}^{S_4^2}D_8$	1
		$M'Q$	${}^{({}^{q_1}S_4^2)}V_4^\circ$	${}^{S_4^1}C_2^\circ$	4
			${}^{({}^{q_1}S_4^2)}S_3$	${}^{S_4^1}C_2^\circ$	6
			${}^{({}^{q_1}S_4^2)}D_8$	${}^{S_4^1}C_2^\circ$	4
		OM'	${}^{({}^{q_1}S_4^2)}V_4^\circ$	${}^{S_4^1}C_2^\circ$	4
			${}^{({}^{q_1}S_4^2)}V_4^\bullet$	${}^{S_4^1}C_2^\circ$	12
			${}^{({}^{q_1}S_4^2)}C_4$	${}^{S_4^1}C_2^\circ$	4
			${}^{({}^{q_1}S_4^2)}D_8$	${}^{S_4^1}C_2^\circ$	12

P	S_4^3	$M'P$	${}^{S_4^3}V_4^\circ$	${}^{S_4^1}C_2^\circ$	2
				${}^{S_4^2}V_4^\bullet$	1
			${}^{S_4^3}V_4^\bullet$	${}^{S_4^3}V_4^\bullet$	3
			${}^{S_4^3}C_4$	${}^{S_4^1}C_2^\circ$	2
				${}^{S_4^2}C_4$	1
			${}^{S_4^3}S_3$	${}^{S_4^1}C_2^\circ$	3
			${}^{S_4^3}D_8$	${}^{S_4^3}V_4^\bullet$	2
				${}^{S_4^2}D_8$	1
		$N'P$	${}^{S_4^3}V_4^\circ$	${}^{S_4^1}C_2^\circ$	2
				${}^{S_4^2}V_4^\bullet$	1
			${}^{S_4^3}V_4^\bullet$	${}^{S_4^3}V_4^\bullet$	3
			${}^{S_4^3}C_4$	${}^{S_4^1}C_2^\circ$	2
				${}^{S_4^2}C_4$	1
			${}^{S_4^3}S_3$	${}^{S_4^1}C_2^\circ$	3
			${}^{S_4^3}D_8$	${}^{S_4^3}V_4^\bullet$	2
				${}^{S_4^2}D_8$	1
		OP	${}^{S_4^3}C_3$	${}^{S_4^1}C_3$	1
			${}^{S_4^3}V_4^\circ$	${}^{S_4^1}C_2^\circ$	4
			${}^{S_4^3}S_3$	${}^{S_4^1}C_2^\circ$	3
				${}^{S_4^1}S_3$	1
			${}^{S_4^3}D_8$	${}^{S_4^1}C_2^\circ$	4
Q	D_{12}	OQ	${}^{D_{12}}V_4$	${}^{S_4^1}C_2^\bullet$	2
				${}^{S_4^1}V_4^\circ$	1
			${}^{D_{12}}S_3^1$, ${}^{D_{12}}S_3^2$	${}^{S_4^1}C_2^\circ$	3
			${}^{D_{12}}C_6$	${}^{S_4^1}C_2^\bullet$	3
			D_{12}	${}^{S_4^1}V_4^\circ$	3
		MQ $M'Q$	${}^{D_{12}}V_4$	${}^{S_4^1}C_2^\circ$	2
			${}^{D_{12}}S_3^2$	${}^{S_4^1}C_2^\circ$	6
			D_{12}	${}^{S_4^1}C_2^\circ$	6
		NQ $N'Q$	${}^{D_{12}}V_4$	${}^{S_4^1}C_2^\circ$	2
			${}^{D_{12}}S_3^1$	${}^{S_4^1}C_2^\circ$	6
			D_{12}	${}^{S_4^1}C_2^\circ$	6
N	${}^{N}D_8$	MN $M'N$	${}^{({}^N D_8)}V_4^1$	${}^{S_4^1}C_2^\circ$	2
			${}^{({}^N D_8)}V_4^2$	${}^{S_4^3}V_4^\bullet$	2
			${}^{({}^N D_8)}C_4$	${}^{S_4^1}C_2^\circ$	2
			${}^{({}^N D_8)}D_8$	${}^{S_4^3}V_4^\bullet$	2
		NQ	${}^{({}^N D_8)}V_4^1$	${}^{S_4^1}C_2^\circ$	4
			${}^{({}^N D_8)}D_8$	${}^{S_4^1}C_2^\circ$	4
		ON	${}^{({}^N D_8)}V_4^2$	${}^{S_4^1}C_2^\circ$	4
			${}^{N}D_8$	${}^{S_4^1}C_2^\circ$	4

$N' = q_1.N$	${}^{q_1}({}^{N}D_8)$	$M'N'$	${}^{q_1}\left({}^{({}^{N}D_8)}V_4^1\right)$	${}^{S_4^1}C_2^\circ$	2
			${}^{q_1}\left({}^{({}^{N}D_8)}V_4^2\right)$	${}^{S_4^3}V_4^\bullet$	2
			${}^{q_1}\left({}^{({}^{N}D_8)}C_4\right)$	${}^{S_4^1}C_2^\circ$	2
			${}^{q_1}({}^{N}D_8)$	${}^{S_4^3}V_4^\bullet$	2
		$N'P$	${}^{q_1}\left({}^{({}^{N}D_8)}V_4^1\right)$	${}^{S_4^2}V_4^\bullet$	1
			${}^{q_1}\left({}^{({}^{N}D_8)}V_4^2\right)$	${}^{S_4^3}V_4^\bullet$	1
			${}^{q_1}\left({}^{({}^{N}D_8)}C_4\right)$	${}^{S_4^2}C_4$	1
			${}^{q_1}({}^{N}D_8)$	${}^{S_4^2}D_8$	1
		$N'Q$	${}^{q_1}\left({}^{({}^{N}D_8)}V_4^1\right)$	${}^{S_4^1}C_2^\circ$	4
			${}^{q_1}({}^{N}D_8)$	${}^{S_4^1}C_2^\circ$	4
		ON'	${}^{q_1}\left({}^{({}^{N}D_8)}V_4^1\right)$	${}^{S_4^1}C_2^\circ$	4
			${}^{q_1}({}^{N}D_8)$	${}^{S_4^1}C_2^\circ$	4

PROOF. We give the proof for a non-trivial example. Let us consider the vertex O and the edge OQ. We know that

$$[\Gamma_{OQ}]_{S_4^1} = \left[{}^{S_4^1}V_4^\circ\right]_{S_4^1}.$$

Applying a suitable representation $\Phi\colon S_4^1 \to S_4$, we may assume that

$$K := \Phi(\Gamma_{OQ}) = \langle (1\ 2), (3\ 4)\rangle.$$

Let us consider the following cases:

- $H_0 = \Gamma_{OQ} \underset{S_4^1}{\sim} {}^{S_4^1}V_4^\circ$: For $H_0 = \Gamma_{OQ}$ we obtain $\Phi(H_0) = K$. For each $\sigma \in S_4$ we have

  $$^{\sigma}K = \langle (\sigma.1\ \sigma.2), (\sigma.3\ \sigma.4)\rangle.$$

  In particular, $K \cap {}^{\sigma}K \neq 1$ if and only if $(\sigma.1\ \sigma.2) \in \{(1\ 2), (3\ 4)\}$. This yields that

  $$K \cap {}^{\sigma}K \neq 1 \quad\Longleftrightarrow\quad {}^{\sigma}\langle (1\ 2), (3\ 4)\rangle = \langle (1\ 2), (3\ 4)\rangle \quad\Longleftrightarrow\quad \sigma \in N_{S_4}(K).$$

  We conclude

  $$\alpha^{O,H_0}_{\left[{}^{S_4^1}V_4^\circ\right]_G}(OQ) = |N_{S_4^1}(\Gamma_{OQ}).(OQ,\Gamma_{OQ})| = \frac{|N_{S_4^1}(\Gamma_{OQ})|}{|\Gamma_{OQ}|} = 2.$$

- $H_0 = {}^{S_4^1}V_4^\bullet$: In this case, we have $N_{S_4^1}(H_0) = S_4^1$ and we obtain by Lemma 2.6.37

  $$\alpha^{O,H_0}_{\left[{}^{S_4^1}C_2^\bullet\right]_G}(OQ) = \frac{|\Gamma_O|}{|\Gamma_{OQ}|} = 6.$$

- $H_0 \underset{S_4^1}{\sim} {}^{S_4^1}C_4$: Because each two groups of type C_4 are conjugate in S_4^1 we can assume that $H_0 \leq N_{S_4^1}(\Gamma_{OQ}) \cong D_8$. For the reason that there is only one group of type C_4 in D_8, we obtain $\Phi(H_0) = \langle (1\ 3\ 2\ 4)\rangle$. Hence, $\pi \in \Phi(H_0) \cap {}^{\sigma}K$ implies $\mathrm{sgn}(\pi) = 1$ and $\mathrm{ord}(\pi) \in \{1, 2\}$. This yields $\pi \in \{1, (1\ 2)(3\ 4)\}$ and thus $\Phi(H_0) \cap {}^{\sigma}K \in \{\{1\}, \langle (1\ 2)(3\ 4)\rangle\}$. Because

$\Phi(H_0) \cap K = \langle(1\ 2)(3\ 4)\rangle$, we have $\Phi(H_0) \cap {}^{\sigma}K = \langle(1\ 2)(3\ 4)\rangle$ if and only if $\sigma \in C_{S_4}\big((1\ 2)(3\ 4)\big) = N_{S_4}(K)$. This implies

$$\alpha^{O,H_0}_{\left[{}^{S_4^1}C_2^{\bullet}\right]_G}(OQ) = \big|N_{S_4^1}(\Gamma_{OQ}).(a, \Phi^{-1}\big((1\ 2)(3\ 4)\big)\big| = \frac{|N_{S_4}(\Gamma_{OQ})|}{|\Gamma_{OQ}|} = 2.$$

- $H_0 \underset{S_4^1}{\sim} {}^{S_4^1}S_3$: We see $\frac{|N_{S_4^1}(H_0)||N_{S_4^1}(\Gamma_{OQ})|}{|\Gamma_O|} = 2$. On the other hand, it holds $|N_{S_4^1}(H_0) \cap N_{S_4^1}(\Gamma_{OQ})| \in \{1, 2\}$ due to Lagrange's theorem. Therefore Lemma 2.6.37 forces
$$\alpha^{O,H_0}_{\left[{}^{S_4^1}C_2^{\circ}\right]_G}(OQ) = \frac{|\Gamma_O|}{|\Gamma_{OQ}|} = 6.$$
- $H_0 = N_{S_4^1}(\Gamma_{OQ}) \underset{S_4^1}{\sim} {}^{S_4^1}D_8$: The choice of Φ and the definition of H_0 force $\Phi(H_0) = \langle(1\ 3\ 2\ 4), (12)\rangle$. Because H_0 contains all order 2 elements with positive sign, we obtain
$$|H_0 \cap {}^{g}(\Gamma_{OQ})| \in \{2, 4\}.$$
Obviously $\Gamma_{OQ} \subseteq H_0$. For the reason both H_0 and Γ_{OQ} contain only two elements with negative sign, we have ${}^{\sigma}K \leq \Phi(H_0)$ if and only if ${}^{\sigma}(1\ 2), {}^{\sigma}(3\ 4) \in \{(1\ 2), (3\ 4)\}$. This is the case if and only if ${}^{\sigma}\big((1\ 2)(3\ 4)\big) = (1\ 2)(3\ 4)$ or $\sigma \in C_{S_4}\big((1\ 2)(3\ 4)\big) = \Phi(H_0)$ respectively. In particular, we obtain
$$\alpha^{O,H_0}_{\left[{}^{S_4^1}V_4^{\circ}\right]_G}(OQ) = |H_0.(OQ, \Gamma_{OQ})| = \frac{|H_0|}{|\Gamma_{OQ}|} = 2.$$

This completes the proof of the example. The other cases can be verified by similar arguments.

□

Nomenclature

$\Delta^{(d)}$	invertible upper triangular matrices mod d, page 17
Γ	the group $\mathrm{Sl}_3(\mathbb{Z})$, page 17
$\mathcal{G}(\mathcal{Y})$	complex of groups over the scwol $\mathcal{Y}$, page 42
$\mathcal{Z}(G)$	the center of the group G, page 24
$M^\circ(\mathcal{G}(\mathcal{Y}))$	the set of vertices v whose vertex groups G_v are maximal finite and satisfy $N_G(G_v) < \infty$, page 52
$\mathrm{Sub}^\circ_{\mathrm{fin}}(G)$	the set of subgroups of G with finite normalizers, page 25
$\tilde{\mathcal{X}}, \mathcal{R}_G(\mathcal{X})$	the scwol obtained by reduction of $\mathcal{X}$ associated to an action $G \curvearrowright \mathcal{X}$, page 47
$\mathcal{X}$	scwol, page 40
$\mathcal{X}_\Gamma$	a scwol with geometric realization X', page 84
Ξ, Ξ_{H_0}	the map considered in Lemma 2.6.5, page 59
$M(G)$	set of maximal finite subgroups in G, page 46
$M^\circ(G)$	set of maximal finite subgroups H in G with $N_G(H)$ finite, page 46
$V_k(\mathcal{X})$	the set of k-dimensional vertices of $\mathcal{X}$, page 41
$V_{\max}(\mathcal{X})$	maximal vertices with respect to the canonical order on $\mathcal{X}$, page 41
X	the set of positive definite quadratic forms over $\mathbb{R}^3$ of determinant 1, page 17
X'	simply connected cocompact simplical complex on which Γ acts on, page 17
$\mathrm{L}_G\, E(\mathcal{X})$	the set of labeled arrows, page 85
$\mathrm{L}_G\, \mathrm{Lk}^v(\mathcal{X})$	the labeled link of a vertex v, page 86
$\mathrm{L}_G\, V(\mathcal{X})$	the set of labeled vertices, page 85

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