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DISSERTATION

# Intrinsic Ultracontractivity of Schrödinger Semigroups in $L^2(\mathbb{R}^n)$

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# Contents

# Chapter 1

# Introduction

To appreciate the mathematical model presented in this document it is necessary to understand the basic theory of Quantum Mechanics first. Therefore we present this short introduction.

## 1.1 Modelling Quantum Mechanics

Summarized under the term *Copenhagen interpretation* the quantum model was developed in the first part of the 20th century by a group of theoretical physists mainly consisting of Niels Bohr, Werner Heisenberg, Max Born, Ernst Pasqual Jordan, Erwin Schrödinger and Paul Dirac. In 1932 John von Neumann contributed significantly by releasing his groundbreaking textbook *Mathematical Foundations of Quantum Mechanics*. The Hilbert space $\mathrm{L}^2(\mathbb{R}^n)$ and differential operators defined within were presented as the perfect mathematical environment to model the newly developed theories in quantum physics. Quantum mechanics served as a catalyst for mathematical fields such as operator theory and functional analysis in the years following.

Generally speaking in quantum mechanics the objects are at the scale of atoms or subatomic particles such as electrons. The term quantum system or short system is often used when refering to a union of several individual particles. The atom is the prime example. A mathematical model of a system's evolution over time reads as following.

Starting at time $t = 0$ a normed function $u_0 \in \mathrm{L}^2(\mathbb{R}^n)$ is interpreted as a probability density of the position of the system's electrons. In particular

$$\int_A |u_0(x)|^2 \mathrm{d}x \in [0, 1]$$

is interpreted as the probabilty of at least one of the system's electrons being contained in $A \subseteq \mathbb{R}^n$. The term *wave function* is often used in quantum mechanics for such a probability density function.
The system's evolution over time is described by *Schrödinger's equation*. An operator $H\colon \mathcal{D}(H) \subseteq \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ is called a Hamilton operator or Hamiltonian if it characterizes the complete energy of the system such that its eigenvalues are interpreted as the system's possible energy levels. In this document we refer to $H$ as a *Schrödinger operator* since most of our references do. The system's evolution over time is given by the unique solution $u \in C^1\big((0,\infty), \mathrm{L}^2(\mathbb{R}^n)\big) \cap C\big([0,\infty), \mathcal{D}(H)\big)$ to the intial value problem

$$\begin{cases} \mathrm{i}u'(t) & = Hu(t) \text{ for } t > 0, \\ u(0) & = u_0 \end{cases}$$

in $\mathrm{L}^2(\mathbb{R}^n)$. Again $|u(t)|^2$ as a function on $\mathbb{R}^n$ serves as a probability density function of the position of the system's electrons at time $t$.

## 1.2 Schrödinger operators in $\mathrm{L}^2(\mathbb{R}^n)$

Let us focus on the definition of an operator $H$ in $\mathrm{L}^2(\mathbb{R}^n)$ to qualify as a Schrödinger operator of a system. We present a short introduction in contrast to a detailed analysis in later sections.

In quantum mechanics a system's energy is not continuous but discrete. Every system owns energy levels with a smallest among them. The latter is called the system's *ground state energy* which is considered to be the most important of all. To qualify as a Schördinger operator $H$ should be defined such that its pure point spectrum is not empty, real-valued and bounded from below. Please mind that modelling free electrons instead of a system requires the spectrum of $H$ to be purely absolutely continuous since free electrons do not have any bound states.

Let us consider the simple case of a hydrogen atom. Our system consists of a single electron and an atomic nucleus. We neglect any magnetic fields at this point although we are dealing with a moving electron. Formally we define $H$ by $-\Delta + q(x)$ in $\mathrm{L}^2(\mathbb{R}^3)$ where the negative Laplacian characterizes the kinetic energy and the multiplication with a measurable function $q$ characterizes the electric potential of the electron resulting from the charge difference to the nucleus. In contrast to the free electron $q$ is responsible for the existence of a pure point spectrum of $H$. For a spectrum contained in $\mathbb{R}$ the operator $H$ has to be self-adjoint which requires $q$ to be real-valued. Also the spectrum shall be bounded from below. So $q$ is usually assumed to be non-negative. Last but not least we demand the spectrum of $H$ to be discrete and ideally consists only of eigenvalues. Typically that is the spectrum of compact operators. But $H$ is not even bounded in $\mathrm{L}^2(\mathbb{R}^3)$. We demand that $q(x) \to \infty$ is satisfied for $|x| \to \infty$ which gives a compact resolvent of $H$.

By focusing on

$$\mathbb{R}^{3m} = \underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{m \textit{ times}}$$

instead of $\mathbb{R}^3$ we approach the more complicated situation of a system with $m$ electrons. Here $x \in \mathbb{R}^{3m}$ is treated as $x = (x_1, \ldots, x_m)$ where $x_k = (x_{k1}, x_{k2}, x_{k3}) \in \mathbb{R}^3$ characterize the spatial coordinates of the $k$-th electron. The Hilbert space is given by $\mathrm{L}^2(\mathbb{R}^{3m})$. Again we neglect any magnetic fields. So the Laplacian in $H$ is defined as expected but for the definition of electric potential one has to take into consideration that not only the nucleus and the electrons interact but also the electrons with each other.

Although all the major arguments in this document unfortunately only hold for the non-magnetic case we give some insights on magnetic Schrödinger operators as well. Magnetic fields are included to the model by adding a magnetic vector potential $\mathbf{a}$ to the formal definition

$$H(\mathbf{a}) = (\nabla - \mathrm{i}\mathbf{a})^2 + q(x)$$

for $\mathbf{a} = (a_1, \ldots, a_n)$ with real-valued functions. Most of the arguments refering to the spectrum being pure point and bounded from below hold in the magnetic case as well. Hence we will present the arguments in the most general way possible and only distinguish between the magnetic and non-magentic case when necessary.

## 1.3 Intrinsic ultracontractivity of $\mathrm{e}^{-tH}$

Let $H\colon \mathcal{D}(H) \subseteq \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ be a non-magnetic Schrödinger operator with a ground state energy $E_0$. As we will see the eigenspace of $E_0$ is one-dimensional spanned by an eigenfunction $\varphi$ that is strictly positive almost everywhere in $\mathbb{R}^n$. This function is called the *ground state* of $H$. At the core of this document we give a condition of $q$ in $H$ such that to every energy level $\lambda$ of the system there is a $C(\lambda) > 0$ satisfying

$$|u(x)| \;\leq\; C(\lambda)\;\varphi(x) \tag{1.1}$$

almost everywhere in $\mathbb{R}^n$ for every normed eigenfunction $u$ of $H$ to $\lambda$. We give a precise definition of the constant $C(\lambda)$. So the probability of the position of the system's electrons in every energy level can be estimated simply by using the system's ground state $\varphi$.

We study strongly continuous Schrödinger semigroups $\{\mathrm{e}^{-tH} \mid t \geq 0\}$ in $\mathrm{L}^2(\mathbb{R}^n)$ to prove (1.1) which might be irritating in the context of quantum mechanics at first glance. Usually such a semigroup solves the abstract evolution equation $u'(t) + Hu(t) = 0$ instead of Schrödinger's equation. However it was Leonard Gross who introduced Logarithmic Sobolev inequalities for $H$ as a tool to prove the intrinsic ultracontractivity of $\{\mathrm{e}^{-tH} \mid t \geq 0\}$ which gives a constant $C_t > 0$ at every time $t > 0$ such that

$$|\mathrm{e}^{-tH}u(x)| \leq C_t\;\|u\|_2\;\varphi(x)$$

holds almost everywhere in $\mathbb{R}^n$ for every $u \in \mathrm{L}^2(\mathbb{R}^n)$. For the special case that $u$ is an eigenfunction of $H$ to $\lambda$ we conclude that

$$\mathrm{e}^{-tH}u = \lim_{k\to\infty}(\mathrm{Id} + \frac{t}{k}H)^{-k}u = \lim_{k\to\infty}(1 + \frac{t\lambda}{k})^{-k}u = \mathrm{e}^{-t\lambda}u$$

is true in $\mathrm{L}^2(\mathbb{R}^n)$ which offers $\varphi(x)^{-1}|u(x)| \;\leq\; C_t\mathrm{e}^{t\lambda}\|u\|_2$. Further arguments on a universal lower boundary regarding $t$ for fixed $x$ finally give (1.1).

# Chapter 2

# Definitions and Spectral Analysis

This section is devoted to the study of Schrödinger operators $H(\mathbf{a})$ in $\mathrm{L}^2(\mathbb{R}^n)$. We define the operators in the most general way possible such that $-H(\mathbf{a})$ generates a strongly continuous semigroup $\left\{\mathrm{e}^{-tH(\mathbf{a})} \mid t \geq 0\right\}$ of contractions in $\mathrm{L}^2(\mathbb{R}^n)$. We follow the procedure of H. Leinfelder and C. G. Simader in [13]. Furthermore we analyze the spectrum of $H(\mathbf{a})$ and present a generalization of the Perron-Frobenius theory to ensure the existence of a ground state $\varphi$ in the non-magnetic case.

## 2.1 The Schrödinger Operator $H(\mathbf{a})$ in $\mathrm{L}^2(\mathbb{R}^n)$

Let us start with a comment on the notation. We use $\mathrm{L}^2(\mathbb{R}^n)$ for the set of complex-valued and measurable functions $u$ on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |u(x)|^2 \, \mathrm{d}x < \infty$$

is satisfied. For real-valued functions in $\mathrm{L}^2(\mathbb{R}^n)$ we explicitly write $\mathrm{L}^2(\mathbb{R}^n, \mathbb{R})$.

Let the magnetic vector potential $\mathbf{a} = (a_1, \ldots, a_n)$ be given such that $a_j \in \mathrm{L}^2_{loc}(\mathbb{R}^n, \mathbb{R})$ is satisfied. Also the electric potential $q \in \mathrm{L}^1_{loc}(\mathbb{R}^n, \mathbb{R})$ is assumed to be non-negative almost everywhere in $\mathbb{R}^n$. By

$$\left\{u \in \mathrm{L}^2(\mathbb{R}^n) \mid \forall j \in \{1, \ldots, n\} : \ \partial_j u - \mathrm{i} a_j u \in \mathrm{L}^2(\mathbb{R}^n) \text{ and } q^{\frac{1}{2}} u \in \mathrm{L}^2(\mathbb{R}^n)\right\}$$

we define a subspace $\mathcal{D}\big(h(\mathbf{a})\big)$. Mind that $u \in \mathcal{D}\big(h(\mathbf{a})\big)$ is weakly differentiable but $\partial_j u$ is not necessarily an element of $\mathrm{L}^2(\mathbb{R}^n)$ if $a_j$ does not map $\mathbb{R}^n$ to 0 almost everywhere. Furthermore $\mathcal{D}\big(h(\mathbf{a})\big)$ is dense in $\mathrm{L}^2(\mathbb{R}^n)$ since $C_c^\infty(\mathbb{R}^n)$ is contained. We define a sesquilinear form $h(\mathbf{a})$ by

$$h(\mathbf{a})(u,v) = \Big\{\sum_{j=1}^{n} \langle \partial_j u - \mathrm{i}a_j u, \partial_j v - \mathrm{i}a_j v\rangle\Big\} + \langle q^{\frac{1}{2}}u, q^{\frac{1}{2}}v\rangle$$

for $u, v \in \mathcal{D}\big(h(\mathbf{a})\big)$ where the inner product of $\mathrm{L}^2(\mathbb{R}^n)$ is denoted by $\langle\cdot,\cdot\rangle$. The form $h(\mathbf{a})$ satisfies almost all of the properties to be an inner product of $\mathcal{D}\big(h(\mathbf{a})\big)$. It lacks positive definiteness. Therefore we define

$$\langle u, v\rangle_{h(\mathbf{a})} = h(\mathbf{a})(u,v) + \langle u, v\rangle$$

for $u, v \in \mathcal{D}\big(h(\mathbf{a})\big)$. In the usual way $\|u\|_{h(\mathbf{a})} = \sqrt{\langle u, u\rangle_{h(\mathbf{a})}}$ for $u \in \mathcal{D}\big(h(\mathbf{a})\big)$ defines a norm on $\mathcal{D}\big(h(\mathbf{a})\big)$. Let us prove completeness.

**Lemma 2.1.1**
*The form domain $\mathcal{D}\big(h(\mathbf{a})\big)$ is complete with respect to $\|\cdot\|_{h(\mathbf{a})}$.*

*Proof.* Let $(u_n) \subset \mathcal{D}\big(h(\mathbf{a})\big)$ be a Cauchy sequence with respect to $\|\cdot\|_{h(\mathbf{a})}$. Then $\big((\partial_j - \mathrm{i}a_j)u_n\big)$, $(q^{\frac{1}{2}}u_n)$ and $(u_n)$ are all Cauchy sequences in $\mathrm{L}^2(\mathbb{R}^n)$. Since $\mathrm{L}^2(\mathbb{R}^n)$ is complete there are functions $u$, $v$ and $w_j$ in $\mathrm{L}^2(\mathbb{R}^n)$ such that $(u_n)$ converges to $u$, $(q^{\frac{1}{2}}u_n)$ converges to $v$ and $\big((\partial_j - \mathrm{i}a_j)u_n\big)$ converges to $w_j$ in $\mathrm{L}^2(\mathbb{R}^n)$.

i.) From the convergence of $(u_n)$ to $u$ in $\mathrm{L}^2(\mathbb{R}^n)$ we conclude that this convergence also holds pointwise almost everywhere in $\mathbb{R}^n$ for a suitable subsequence of $(u_n)$. We keep the notation for esthetic reasons. Then $(q^{\frac{1}{2}}u_n)$ converges pointwise almost everywhere in $\mathbb{R}^n$ to $q^{\frac{1}{2}}u$. Therefore $v$ is equal to $q^{\frac{1}{2}}u$ almost everywhere in $\mathbb{R}^n$.

ii.) For $\varphi \in C_c^\infty(\mathbb{R}^n)$ we infer

$$\langle w_j, \varphi\rangle = \lim_{n\to\infty} \langle (\partial_j - \mathrm{i}a_j)u_n, \varphi\rangle = \langle u, (\mathrm{i}a_j - \partial_j)\varphi\rangle$$

is true. Then $-\langle u, \partial_j\varphi\rangle = \langle w_j + ia_j u, \varphi\rangle$ follows. Since $w_j$ and $a_j u$ are both functions in $\mathrm{L}^1_{loc}(\mathbb{R}^n)$ we see that $u$ is weakly differentiable with $\partial_j u = w_j + ia_j u$. Hence $\partial_j u - ia_j u$ is equal to $w_j \in \mathrm{L}^2(\mathbb{R}^n)$ almost everywhere in $\mathbb{R}^n$.

iii.) Finally we conclude that $u$ is contained in $\mathcal{D}\big(h(\mathbf{a})\big)$ such that $(u_n)$ converges to $u$ with respect to $\|\cdot\|_{h(\mathbf{a})}$ which proves the claim.

□

We define an operator $H(\mathbf{a})$ in $\mathrm{L}^2(\mathbb{R}^n)$ associated to the form $h(\mathbf{a})$. First a set $\mathcal{D}\big(H(\mathbf{a})\big)$ is defined by

$$\Big\{u \in \mathcal{D}\big(h(\mathbf{a})\big) \mid \exists v \in \mathrm{L}^2(\mathbb{R}^n):\; h(\mathbf{a})(u,w) = \langle v, w\rangle \text{ for every } w \in \mathcal{D}\big(h(\mathbf{a})\big)\Big\}$$

which is not empty since the null function is contained. Further $\mathcal{D}\big(H(\mathbf{a})\big)$ is a subspace in $\mathrm{L}^2(\mathbb{R}^n)$ since the form $h(\mathbf{a})$ is linear in the first component. Also for every $u \in \mathcal{D}\big(H(\mathbf{a})\big)$ there is exactly one $v \in \mathrm{L}^2(\mathbb{R}^n)$ satisfying

$$h(\mathbf{a})(u,w) = \langle v, w\rangle$$

for every $w \in \mathcal{D}\big(h(\mathbf{a})\big)$ due to the density of the form domain. So we define an operator $H(\mathbf{a})$ in $\mathrm{L}^2(\mathbb{R}^n)$ by

$$\langle H(\mathbf{a})u, w\rangle = h(\mathbf{a})(u,w) \tag{2.1}$$

for $u \in \mathcal{D}\big(H(\mathbf{a})\big)$ and every $w \in \mathcal{D}\big(h(\mathbf{a})\big)$.

**Definition 2.1.2**
*The operator $H(\mathbf{a})$ defined by (2.1) is called a Schrödinger operator in $\mathrm{L}^2(\mathbb{R}^n)$. If every $a_j$ in $\mathbf{a}$ is $0$ almost everywhere in $\mathbb{R}^n$ we write $H(\mathbf{0})$.*

If the upcoming arguments hold in the general magnetic as well as in the non-magnetic case we refer to $H(\mathbf{a})$ which includes the non-magnetic case. We explicitly write $H(\mathbf{0})$ when a non-magnetic environment is necessary.

**Lemma 2.1.3**
*Schrödinger operators $H(\mathbf{a})$ are densely defined with $(0,\infty) \subseteq \rho\big(-H(\mathbf{a})\big)$.*

*Proof.* Although the proof can be found as Proposition 1.22 on page 13 in [14] we present some details to show that Lemma 2.1.1 is essential for proving that $(0,\infty) \subseteq \rho\big(-H(\mathbf{a})\big)$ holds. Also see Lemma 3.19 on page 32 in [20].

We focus on proving that 1 is contained in $\rho\big(-H(\mathbf{a})\big)$. For $\lambda > 0$ the proof works with similar arguments. Let $v \in \mathrm{L}^2(\mathbb{R}^n)$ be arbitrary but fixed. Then

$$|\langle v, w\rangle| \le \|v\|_2 \cdot \|w\|_2 \le \|v\|_2 \cdot \|w\|_{h(\mathbf{a})}$$

is true for every $w \in \mathcal{D}\big(h(\mathbf{a})\big)$ since $h(\mathbf{a})$ is positive. Hence the map

$$\{\ w \mapsto \langle v, w\rangle\ \}$$

is an element of the anti-dual space $\mathcal{D}\big(h(\mathbf{a})\big)^*$. By Lemma 2.1.1 we use the representation theorem by F. Riesz to argue that there exists an element $u \in \mathcal{D}\big(h(\mathbf{a})\big)$ such that

$$\langle u, w\rangle + h(\mathbf{a})(u, w) = \langle u, w\rangle_{h(\mathbf{a})} = \langle v, w\rangle$$

holds for every $w \in \mathcal{D}\big(h(\mathbf{a})\big)$. But then $u$ is an element of the domain $\mathcal{D}\big(H(\mathbf{a})\big)$ by definition such that

$$\langle H(\mathbf{a})u, w\rangle = \langle v - u, w\rangle$$

is true for every $w \in \mathcal{D}\big(h(\mathbf{a})\big)$. That means that $\mathrm{Id} + H(\mathbf{a})$ is surjective due to the density of $\mathcal{D}\big(h(\mathbf{a})\big)$ in $\mathrm{L}^2(\mathbb{R}^n)$. Furthermore from the postivity of the form $h(\mathbf{a})$ we can easily argue that

$$\|\big(\mathrm{Id} + H(\mathbf{a})\big)u\|_2 \ge \|u\|_2$$

holds for every $u \in \mathcal{D}\big(h(\mathbf{a})\big)$. So $\mathrm{Id} + H(\mathbf{a})$ is injective. Finally this last inequatility also gives a bounded inverse. So $1 \in \rho\big(-H(\mathbf{a})\big)$ follows.

□

**Lemma 2.1.4**
*Schrödinger operators are self-adjoint in* $\mathrm{L}^2(\mathbb{R}^n)$.

This is proved by showing that the adjoint operator of $H(\mathbf{a})$ in $\mathrm{L}^2(\mathbb{R}^n)$ is associated to the adjoint form $h(\mathbf{a})'$ which is defined by

$$h(\mathbf{a})'(u, v) := \overline{h(\mathbf{a})(v, u)}$$

for $u, v \in \mathcal{D}\big(h(\mathbf{a})'\big) = \mathcal{D}\big(h(\mathbf{a})\big)$. Hence $H(\mathbf{a})$ is self-adjoint since $h(\mathbf{a})$ is equal to its adjoint form. We skip the details and refer to Proposition 1.24 on page 15 in [14].

**Theorem 2.1.5**
*Negative Schrödinger operators $-H(\mathbf{a})$ generate $C_0$-semigroups*

$$\left\{ \ \mathrm{e}^{-tH(\mathbf{a})} \mid t \geq 0 \ \right\}$$

*of contractions in $\mathrm{L}^2(\mathbb{R}^n)$ that we call Schrödinger Semigroups in $\mathrm{L}^2(\mathbb{R}^n)$. So in particular $\|\mathrm{e}^{-tH(\mathbf{a})}u\|_2 \leq \|u\|_2$ holds for every $u \in \mathrm{L}^2(\mathbb{R}^n)$ and $t \geq 0$.*

*Proof.* The famous Lumer-Phillips theorem characterizes generators of $C_0$-semigroups of contractions in $\mathrm{L}^2(\mathbb{R}^n)$ as m-accretive operators. Please consider Theorem 3.18 on page 32 in [20] for reference if needed.

Since the form $h(\mathbf{a})$ is positive the associated operator $H(\mathbf{a})$ is accretive. Also $H(\mathbf{a})$ is densely defined and $1 \in \rho(-H(\mathbf{a}))$ holds by Lemma 2.1.3. So $H(\mathbf{a})$ is m-accretive by Lemma 3.16 and Lemma 3.19 on page 32 in [20]. □

**Remark 2.1.6**

*i.) We defined $H(\mathbf{a})$ as suggested in [13] by H. Leinfelder and C. G. Simader in 1981. But forms $h(\mathbf{a})$ including magnetic vector potentials $\mathbf{a}$ were studied before. Tosio Kato in 1978 and one year later Barry Simon both wrote articles that included magnetic Schrödinger forms in [11] and in [17] respectively.*

*ii.) In both articles was already shown that the form closure to $h(\mathbf{a})$ on $C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$ is equal to the maximal domain $\mathcal{D}\big(h(\mathbf{a})\big)$ we chose. Therefore every rational choice of a domain of $h(\mathbf{a})$ that contains $C_c^\infty(\mathbb{R}^n)$ leads to $\mathcal{D}\big(h(\mathbf{a})\big)$ in order to define a Schrödinger semigroup in $\mathrm{L}^2(\mathbb{R}^n)$ in the sense of the Lumer-Phillips theorem.*

Let us finalize this section by mentioning the operator extension method by K. Friedrichs to define Schrödinger Operators. It requires stronger assumptions on the electric potential $q$ and the magnetic vector potential $\mathbf{a}$. Additionally we assume $q \in \mathrm{L}^2_{loc}(\mathbb{R}^n, \mathbb{R})$ and $\mathbf{a} = (a_1, \ldots, a_n)$ such that

i.) $a_j \in \mathrm{L}^4_{loc}(\mathbb{R}^n, \mathbb{R})$

ii.) $\nabla \cdot \mathbf{a} \in \mathrm{L}^2_{loc}(\mathbb{R}^n, \mathbb{R})$

are satisfied where $\nabla \cdot \mathbf{a}$ denotes the divergence of $\mathbf{a}$. For $u \in C_c^\infty(\mathbb{R}^n)$

$$\tilde{H}(\mathbf{a})u = -\Delta u + |\mathbf{a}(x)|^2 u + q(x)u + \mathrm{i}\big(2\mathbf{a}(x)\nabla u + (\nabla \cdot \mathbf{a})(x)u\big)$$

is a well defined operator in $\mathrm{L}^2(\mathbb{R}^n)$ which is essentially self-adjoint by Theorem 2 on page 12 of [13]. Its closure is equal to $H(\mathbf{a})$ we defined earlier.

## 2.2 Spectral Analysis of $H(\mathbf{a})$

Schrödinger operators $H(\mathbf{a})$ are self-adjoint in $\mathrm{L}^2(\mathbb{R}^n)$ such that $(-\infty, 0)$ is contained in $\rho\big(H(\mathbf{a})\big)$. Therefore $\sigma\big(H(\mathbf{a})\big) \subseteq [0, \infty)$ is satisfied. Nevertheless an empty spectrum is possible at this point as well. For an electric potential $q$ additionally satisfying

$$q(x) \to \infty \tag{2.2}$$

for $|x| \to \infty$ we show that $\sigma\big(H(\mathbf{a})\big)$ is not empty. Furthermore the spectrum of $H(\mathbf{a})$ is discrete consisting only of eigenvalues with finite dimensional eigenspaces in this case. There even exists a basis of eigenfunctions in $\mathrm{L}^2(\mathbb{R}^n)$.

In the following let $H(\mathbf{a})$ and $H(\mathbf{0})$ be Schrödinger operators to an electric potential $q$ satisfying (2.2). We use Theorem 2.9 on page 21 in [12] to prove the lemma below.

**Lemma 2.2.1**
*The resolvents of $-H(\mathbf{0})$ are compact.*

*Proof.* Let $(v_k) \subseteq \mathrm{L}^2(\mathbb{R}^n)$ be bounded and define

$$u_k = \big(\mathrm{Id} + H(\mathbf{0})\big)^{-1} v_k \ \in \mathcal{D}\big(H(\mathbf{0})\big).$$

We want to prove that $(u_k)$ posseses a convergent subsequence in $\mathrm{L}^2(\mathbb{R}^n)$.

i.) The bounded sequence $(v_k)$ has a weakly convergent subsequence $(v_{k_j})$ due to the reflexivity of $\mathrm{L}^2(\mathbb{R}^n)$. For esthetic reasons we keep the notation. Let $v \in \mathrm{L}^2(\mathbb{R}^n)$ be the weak limit of the subsequence $(v_k)$ and define

$$u = \big(\mathrm{Id} + H(\mathbf{0})\big)^{-1} v \ \in \mathcal{D}\big(H(\mathbf{0})\big).$$

We conclude that a subsequence of $(u_k)$ converges weakly to $u$ in $\mathrm{L}^2(\mathbb{R}^n)$ by using the adjoint of the resolvent. Again we keep the notation.

ii.) Let us prove a technical statement that we use later. We argue that

$$\begin{aligned}\big\|H(\mathbf{0})(u_k - u)\big\|_2 &\leq \big\|\big(\mathrm{Id} + H(\mathbf{0})\big)(u_k - u)\big\|_2 + \big\|u_k - u\big\|_2 \\ &\leq \big(1 + \big\|(\mathrm{Id} + H(\mathbf{0}))^{-1}\big\|\big)\big(\|v_k\|_2 + \|v\|_2\big)\end{aligned}$$

is true. So there exists a constant $C > 0$ such that for every $k \in \mathbb{N}$

$$h(\mathbf{0})(u_k - u, u_k - u) \leq \big\|H(\mathbf{0})(u_k - u)\big\|_2 \big(\|u_k\|_2 + \|u\|_2\big) \leq C$$

holds since $(u_k)$ and $(v_k)$ are bounded.

iii.) Let $\varepsilon > 0$ be arbitrary but fixed. There exists $R = R(\varepsilon) > 0$ such that $q(x) \geq \varepsilon^{-1}$ holds for $|x| \geq R$ due to (2.2). Using

$$\big\|q^{\frac{1}{2}}(u_k - u)\big\|_2^2 \leq h(\mathbf{0})(u_k - u, u_k - u) \leq C$$

we conclude that

$$\int_{|x| \geq R} |u_k(x) - u(x)|^2 \mathrm{d}x \leq \varepsilon C$$

holds for every $k \in \mathbb{N}$. In the following we see for a given $R$ there is a $J = J(\varepsilon) \in \mathbb{N}$ and a subsequence $(u_{k_j})$ that satisfy

$$\int_{|x| < R} |u_{k_j}(x) - u(x)|^2 \mathrm{d}x < \varepsilon$$

for every $j \geq J$.

iv.) We use Rellich's embedding theorem for $H_0^1(B_{2R}(0))$. Please consider Theorem V.2.13 from page 216 in [19] as a reference.

Let $\psi_R \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ satisfy $0 \leq \psi_R(x) \leq 1$, $\psi_R(x) = 1$ for $x \in B_R(0)$ and $\operatorname{supp} \psi_R \subseteq B_{2R}(0)$. Then $\psi_R$ serves as a smooth cut-off function such that the product $\psi_R(u_k - u)$ is an element of

$$H_0^1(B_{2R}(0)) = \overline{C_c^\infty(B_{2R}(0))}.$$

Furthermore $\psi_R(u_k - u)$ is bounded sequence in $H_0^1(B_{2R}(0))$ since

$$\|\nabla(u_k - u)\|_2^2 \leq h(\mathbf{0})(u_k - u, u_k - u) \leq C$$

holds by ii.) Therefore there exists a strong limit $w \in \mathrm{L}^2(B_{2R}(0))$ to a subsequence $\psi_R(u_{k_j} - u)$ since $H_0^1(B_{2R}(0))$ is compactly embedded in $\mathrm{L}^2(B_{2R}(0))$ by Rellich's embedding theorem.

v.) We can easily conclude that $w$ is equal to 0 since

$$\langle w, v \rangle = \lim_{j \to \infty} \langle \psi_R(u_{k_j} - u), v \rangle = \lim_{j \to \infty} \langle u_{k_j} - u, \psi_R v \rangle = 0$$

holds for every $v \in \mathrm{L}^2(B_{2R}(0))$ since $(u_k)$ converges weakly to $u$. That means that $\psi_R(u_{k_j} - u)$ converges to 0 in $\mathrm{L}^2(B_{2R}(0))$.

vi.) Hence there exists a $J = J(\varepsilon) \in \mathbb{N}$ such that

$$\int_{|x|<R} |u_{k_j}(x) - u(x)|^2 \mathrm{d}x < \varepsilon$$

is true for every $j \geq J$ since $\psi_R(x) = 1$ holds for $x \in B_R(0)$ by choice. Finally we conclude that $(u_{k_j})$ converges strongly to $u$ in $\mathrm{L}^2(\mathbb{R}^n)$ since

$$\begin{aligned}
&\int_{\mathbb{R}^n} |u_{k_j}(x) - u(x)|^2 \mathrm{d}x \\
&= \int_{|x|<R} |u_{k_j}(x) - u(x)|^2 \mathrm{d}x + \int_{|x|\geq R} |u_{k_j}(x) - u(x)|^2 \mathrm{d}x \\
&< \varepsilon + \varepsilon C
\end{aligned}$$

holds for every $j \geq J$ which proves the claim.

□

**Remark 2.2.2**
*Remember that the operator product of a compact and bounded operator is always a compact operator. So due to the resolvent identity we only have to prove the compactness of* $\big(\mathrm{Id} + H(\mathbf{0})\big)^{-1}$ *to conclude that all resolvent operators of* $-H(\mathbf{0})$ *are compact.*

We introduce the following very important inequalities when dealing with Schrödinger Semigroups to expand the last result to the general possibly magnetic case. However we skip the proofs. For a detailed investigation of the given statements we refer to Lemma 6 on page 7 and the end of Remark 1 on page 9 in [13]. Other important references are [11] and [10].

**Lemma 2.2.3** (Diamagnetic inequalities)
*Let* $u$ *be a function of* $\mathrm{L}^2(\mathbb{R}^n)$ *and* $\lambda > 0$.

*i.) The first form of the diamagnetic inequality reads*

$$\left|\big(\lambda + H(\mathbf{a})\big)^{-1} u(x)\right| \leq \big(\lambda - \Delta\big)\big)^{-1} |u|(x) \tag{2.3}$$

*almost everywhere in* $\mathbb{R}^n$. *Then*

$$|\mathrm{e}^{-tH(\mathbf{a})} u(x)| \leq \mathrm{e}^{t\Delta} |u|(x)$$

*follows directly by the use of Euler's formula.*

*ii.) The second form of the diamagnetic inequality reads*

$$\left|\big(\lambda + H(\mathbf{a})\big)^{-1} u(x)\right| \leq \big(\lambda + H(\mathbf{0})\big)^{-1} |u|(x) \tag{2.4}$$

*almost everywhere in* $\mathbb{R}^n$. *Again we conclude to*

$$|\mathrm{e}^{-tH(\mathbf{a})} u(x)| \leq \mathrm{e}^{-tH(\mathbf{0})} |u|(x).$$

The compactness of the resolvents of $-H(\mathbf{a})$ is a direct consequnce of (2.4) which is shown in Theorem 1 on page 49 in [15]. Please mind that the posivity of the resolvents of $-H(\mathbf{0})$ which is needed in the reference is shown in Lemma 2.3.1 of this document.

**Lemma 2.2.4**
*The resolvents of* $-H(\mathbf{a})$ *are compact.*

We use the statements on the compactness of the resolvents to argue with rather simple arguments that the spectrum $\sigma\big(H(\mathbf{a})\big)$ cannot be empty.

**Theorem 2.2.5**
*The spectrum* $\sigma\big(H(\mathbf{a})\big)$ *is not empty.*

*Proof.* We prove by contradiction and assume that $\sigma\big(H(\mathbf{a})\big) = \emptyset$ is true.

Hence 0 is contained in the resolvent set of $H(\mathbf{a})$. So $H(\mathbf{a})^{-1}$ is compact. We show that $\sigma\big(H(\mathbf{a})^{-1}\big) = \{0\}$ follows.

i.) First we prove that 0 is contained in $\sigma\big(H(\mathbf{a})^{-1}\big)$. If $0 \in \rho\big(H(\mathbf{a})^{-1}\big)$ would be true, then $H(\mathbf{a})^{-1}$ would have a bounded inverse. Hence $H(\mathbf{a})$ would be a bounded operator in $\mathrm{L}^2(\mathbb{R}^n)$ which is obviously not the case. But even if this would be true then

$$\mathrm{Id} = H(\mathbf{a})H(\mathbf{a})^{-1}$$

would be a compact operator due to the compactness of $H(\mathbf{a})^{-1}$. But the identity operator can only be compact in an finite dimensional vector space which $\mathrm{L}^2(\mathbb{R}^n)$ is not.

ii.) Now we have to show that $\sigma\big(H(\mathbf{a})^{-1}\big)$ contains no element beside 0. Due to the compactness of $H(\mathbf{a})^{-1}$ we see that every element $\lambda \neq 0$ in the spectrum is an eigenvalue of $H(\mathbf{a})^{-1}$. In this case $\lambda^{-1}$ would be an eigenvalue of $H(\mathbf{a})$ which is not possible by our assumption.

At this point we have shown that our assumption gives $\sigma\big(H(\mathbf{a})^{-1}\big) = \{0\}$. Using Theorem 1.1 on page 95 in [5] we state that

$$\big(H(\mathbf{a})'\big)^{-1} = \big(H(\mathbf{a})^{-1}\big)'$$

holds where the adjoint of $H(\mathbf{a})$ is noted by $H(\mathbf{a})'$. So $H(\mathbf{a})^{-1}$ is self-adjoint and we conclude that $\|H(\mathbf{a})^{-1}\| = 0$ follows by using Proposition 6.9 on page

165 in [3]. Alternatively we can argue that either $\|H(\mathbf{a})^{-1}\|$ or $-\|H(\mathbf{a})^{-1}\|$ is an eigenvalue of $H(\mathbf{a})^{-1}$ due to compactness. So once again $\|H(\mathbf{a})^{-1}\| = 0$ follows. Either way we get

$$u = H(\mathbf{a})H(\mathbf{a})^{-1}u = 0$$

for every function $u \in \mathrm{L}^2(\mathbb{R}^n)$ which is obviously false.

□

Information on the spectrum of operators with compact resolvents is common knowledge and therefore is available in most textbooks on Spectral Theory. The following theorem relates to $H(\mathbf{a})$. We ommit any proofs.

**Theorem 2.2.6**

*The spectrum of $H(\mathbf{a})$ is equal to the pure point spectrum of $H(\mathbf{a})$. Hence it contains at most countably many eigenvalues with finite dimensional eigenspaces. Furthermore there are no accumulation points in $\sigma\big(H(\mathbf{a})\big)$ and there exists an orthonormal basis of $\mathrm{L}^2(\mathbb{R}^n)$ that consists only of eigenfunctions of $H(\mathbf{a})$.*

For an eigenvalue $\lambda$ of $H(\mathbf{a})$ with an associated eigenfunction $u$ we conclude that

$$0 \leq h(\mathbf{a})(u,u) = \langle H(\mathbf{a})u, u\rangle = \lambda \|u\|_2^2$$

is true. Therefore the spectrum of $H(\mathbf{a})$ is contained in $[0,\infty)$. Theorem 2.2.6 ensures the existence of a smallest eigenvalue in $\sigma\big(H(\mathbf{a})\big)$ that we call the ground state energy $E_0$ of $H(\mathbf{a})$. In the non-magnetic case we can show even more. In particular we are able to prove that the eigenspace of $E_0$ is one-dimensional spanned by an eigenfunction $\varphi$ being strictly positive almost everywhere in $\mathbb{R}^n$. We call $\varphi$ the ground state of the system. The proof relies on a generalization of the Perron-Frobenius theory to $\mathrm{e}^{-tH(\mathbf{0})}$ which does not hold in the general magnetic case unfortunately.

## 2.3 Perron-Frobenius Theory of $\mathrm{e}^{-tH(\mathbf{0})}$

Let us look at a classic theorem by O. Perron for square matrices $A \in \mathbb{R}^{n\times n}$ with positive entries that was later generalized further to non-negative entries by F.G. Frobenius. So the term Perron-Frobenius theory was established.

In general the spectrum of a matrix $A \in \mathbb{R}^{n\times n}$ consists of complex eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. The spectral radius $r(A)$ of $A$ is defined as

$$r(A) = \max\{|\lambda_1|, \dots, |\lambda_n|\}.$$

If $A$ has only strict postive entries Perron states that $r(A)$ itself is an eigenvalue of $A$ with a one-dimensional eigenspace. Furthermore there is an eigenvector $u \in \mathbb{R}^n$ to $r(A)$ with strict positive entries.

In finite dimensional vector spaces linear operators can be expressed as matrices. But in our case $\mathrm{L}^2(\mathbb{R}^n)$ is obviously not finite dimensional. So we are dealing with a quite different scenario here. Luckily there is an extension of the Perron-Frobenius theory to bounded and positivity improving operators in infinite dimensional vector spaces. But please mind that differential operators like $H(\mathbf{0})$ are not bounded. The resolvent operators of $-H(\mathbf{0})$ however are bounded and their spectra is connected to $H(\mathbf{0})$.

In the non-magnetic case the form $h(\mathbf{0})$ is defined as

$$h(\mathbf{0})(u,v) = \Big\{\sum_{j=1}^{n} \langle \partial_j u, \partial_j v\rangle\Big\} + \langle q^{\frac{1}{2}} u, q^{\frac{1}{2}} v\rangle = \langle \nabla u, \nabla v\rangle + \langle q^{\frac{1}{2}} u, q^{\frac{1}{2}} v\rangle$$

for $u, v \in \mathcal{D}\big(h(\mathbf{0})\big) = \{u \in H^1(\mathbb{R}^n) \mid q^{\frac{1}{2}} u \in \mathrm{L}^2(\mathbb{R}^n)\}$. As mentioned the associated operator $H(\mathbf{0})$ is unbounded in $\mathrm{L}^2(\mathbb{R}^n)$. To generalize the Perron-Frobenius Theory to resolvent operators of $-H(\mathbf{0})$ we focus on $\mathrm{e}^{-tH(\mathbf{0})}$ for $t > 0$. Remember that resolvent operators and the semigroup are connected via

$$\big(\lambda + H(\mathbf{0})\big)^{-1} u = \int_0^\infty \mathrm{e}^{-\lambda t}\ \mathrm{e}^{-tH(\mathbf{0})} u \ \mathrm{d}t$$

for $\lambda > 0$ and $u \in \mathrm{L}^2(\mathbb{R}^n)$. We will show that $\mathrm{e}^{-tH(\mathbf{0})}$ are real and positive operators for $t > 0$. Therefore the resolvents of $-H(\mathbf{0})$ given above are as well. Finally we get similar results to Perron's for the ground state $\varphi$ of $H(\mathbf{0})$ being positive almost everywhere in $\mathbb{R}^n$.

**Lemma 2.3.1**

*i.) The set* $\mathrm{L}^2(\mathbb{R}^n;\mathbb{R})$ *is invariant under* $\mathrm{e}^{-tH(\mathbf{0})}$ *for* $t > 0$. *So we call the operators* $\mathrm{e}^{-tH(\mathbf{0})}$ *real.*

*ii.) The set* $\mathrm{L}^2(\mathbb{R}^n;[0,\infty))$ *is invariant under* $\mathrm{e}^{-tH(\mathbf{0})}$ *for* $t > 0$. *Therefore we call the operators* $\mathrm{e}^{-tH(\mathbf{0})}$ *positive.*

*Proof.* We use Propostion 2.5 and Theorem 2.6 on page 50 in [14].

i.) Let $u \in \mathcal{D}\big(h(\mathbf{0})\big)$. The real part $\mathfrak{Re}(u)$ of $u$ is also an element of $H^1(\mathbb{R}^n)$ such that $\mathfrak{Re}(\partial_j u) = \partial_j \mathfrak{Re}(u)$ holds. Furthermore from $q^{\frac{1}{2}}u \in \mathrm{L}^2(\mathbb{R}^n)$ we conclude that $q^{\frac{1}{2}}\mathfrak{Re}(u) = \mathfrak{Re}(q^{\frac{1}{2}}u) \in \mathrm{L}^2(\mathbb{R}^n)$ is true. In total $\mathfrak{Re}(u)$ is contained in $\mathcal{D}\big(h(\mathbf{0})\big)$. The same is true for the imaginary part $\mathfrak{Im}(u)$ of $u$. So

$$h(\mathbf{0})(\mathfrak{Re}(u), \mathfrak{Im}(u)) \in \mathbb{R}$$

follows from the defintion of the form. The citation proves the claim.

ii.) Let $u \in \mathcal{D}\big(h(\mathbf{0})\big)$. As usual we define the positive part of $\mathfrak{Re}(u)$ by

$$\mathfrak{Re}(u)^+(x) = \max\{\mathfrak{Re}(u)(x), 0\} \geq 0$$

and the negative part by $\mathfrak{Re}(u)^- = \min\{\mathfrak{Re}(u)(x), 0\}$. So that

$$\mathfrak{Re}(u) = \mathfrak{Re}(u)^+ + \mathfrak{Re}(u)^-$$

holds. Furthermore we use Lemma 7.6 on page 152 in [6] by to conclude that $\mathfrak{Re}(u)^+$ and $\mathfrak{Re}(u)^-$ are weakly differentiable such that $\partial_j\mathfrak{Re}(u)^+$ is equal to $1_{\{\mathfrak{Re}(u)>0\}}\,\partial_j\mathfrak{Re}(u)$ and $\partial_j\mathfrak{Re}(u)^-$ to $1_{\{\mathfrak{Re}(u)<0\}}\,\partial_j\mathfrak{Re}(u)$. Therefore both functions $\mathfrak{Re}(u)^+$ and $\mathfrak{Re}(u)^-$ are elements of the domain $\mathcal{D}\big(h(\mathbf{0})\big)$ that satisfy

$$h(\mathbf{0})\big(\mathfrak{Re}(u)^+, \mathfrak{Re}(u)^-\big) = 0$$

due to their disjoint supports. The claim follows from the given citation.

□

The operators $\mathrm{e}^{-tH(\mathbf{0})}$ are positive for $t > 0$. But similar to Perron's original theorem we need them to be positivity improving as defined below.

**Definition 2.3.2**
*For $t > 0$ we call $\mathrm{e}^{-tH(\mathbf{0})}$ positivity improving if the function $\mathrm{e}^{-tH(\mathbf{0})}u$ is strictly positive almost everywhere in $\mathbb{R}^n$ for $u \in \mathrm{L}^2(\mathbb{R}^n; [0, \infty))$.*

We present a characterization for $\mathrm{e}^{-tH(\mathbf{0})}$ being positivity improving taken from page 54 in [14] as Corollary 2.11. Please mind the reference uses the term irreducibility of $\mathrm{e}^{-tH(\mathbf{0})}$ instead of positivity improving. Compare with Definition 2.8 on page 51 to ensure that both terms are interchangeable.

**Lemma 2.3.3**
*If $h(\mathbf{0})$ satisfies $h(\mathbf{0})(u, v) = 0$ for functions $u, v \in \mathcal{D}\big(h(\mathbf{0})\big)$ having disjoint supports then the following statements are equivalent:*

*i.) For every $t > 0$ the operators $\mathrm{e}^{-tH(\mathbf{0})}$ are positivity improving.*

*ii.) If $\Omega \subseteq \mathbb{R}^n$ satisfies $1_\Omega\, u \in \mathcal{D}\big(h(\mathbf{0})\big)$ for every $u \in \mathcal{D}\big(h(\mathbf{0})\big)$ then either $\Omega$ or $\mathbb{R}^n \setminus \Omega$ is a Lebesgue null set.*

To utilize this characterization to its full capacity we first have to show that the form $h(\mathbf{0})$ satisfies the demanded property.

**Proposition 2.3.4**
*Let $u, v \in \mathcal{D}\big(h(\mathbf{0})\big)$ have disjoint supports. Then $h(\mathbf{0})(u, v) = 0$ follows.*

*Proof.* We take $u, v \in \mathcal{D}\big(h(\mathbf{0})\big)$ such that $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$ is true.

Let us first deal with the special cases that the support of $u$ is either $\mathbb{R}^n$ or $\emptyset$. If $\mathrm{supp}(u) = \mathbb{R}^n$ then $\mathrm{supp}(v)$ has to be empty. That means $v$ is 0 almost everywhere in $\mathbb{R}^n$. Then $h(\mathbf{0})(u, v) = 0$ is true by simply using the defintion of the form. A similar argument holds for the case of $\mathrm{supp}(u) = \emptyset$. Because then $u$ is the function being 0 almost everywhere in $\mathbb{R}^n$. Once again $h(\mathbf{0})(u, v) = 0$ follows in this case.

Therefore let neither $\operatorname{supp}(u)$ nor $\operatorname{supp}(v)$ be $\emptyset$ or $\mathbb{R}^n$. We define the closed sets $S_1$ as $\operatorname{supp}(u)$ and $S_2$ as $\operatorname{supp}(v)$. Remember that $S_1 \cap S_2 = \emptyset$ is true. We check that the $j$-th weak derivative of $u$ is equal to 0 on $\mathbb{R}^n \setminus S_1$. For $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\psi) \subset \mathbb{R}^n \setminus S_1$ we get

$$0 = -\int_{\mathbb{R}^n} u(x)\, \overline{\partial_j \psi(x)}\, \mathrm{d}x = \int_{\mathbb{R}^n} \partial_j u(x)\, \overline{\psi(x)}\, \mathrm{d}x = \int_{\mathbb{R}^n \setminus S_1} \partial_j u(x)\, \overline{\psi(x)}\, \mathrm{d}x.$$

Hence the fundamental lemma in the calculus of variations states that $\partial_j u$ is 0 on $\mathbb{R}^n \setminus S_1$. A similar statement is true on $\mathbb{R}^n \setminus S_2$ for the $j$-th weak derivative of $v$. That means the equations $\partial_j u = 1_{S_1} \partial_j u$ and $\partial_j v = 1_{S_2} \partial_j v$ follow. So we conclude that

$$h(\mathbf{0})(u,v) = \Big\{\sum_{j=1}^{n} (\partial_j u, \partial_j v)_2\Big\} + (q^{\frac{1}{2}} u, q^{\frac{1}{2}} v)_2 = \sum_{j=1}^{n} (1_{S_1} \partial_j u, 1_{S_2} \partial_j v)_2 = 0$$

follows which proves the claim.

□

The arguments of the following theorem are taken from Lemma 11.1.1 on page 141 in [2]. However we modified the proof slightly to our needs. In subsection 11.1 of the given reference W. Arendt proves a similar result for semigroups that are generated by the Dirichlet or Neumann Laplacian in $\mathrm{L}^2(\Sigma)$ where $\Sigma \subseteq \mathbb{R}^n$ is open and connected. Our case is much simpler since we are not dealing with any boundary conditions.

**Theorem 2.3.5**
*For every $t > 0$ the operators $\mathrm{e}^{-tH(\mathbf{0})}$ are positivity improving.*

*Proof.* We prove by contradiction and assume that $\{\mathrm{e}^{-tH(\mathbf{0})} \mid t \geq 0\}$ is not positivity improving. By Lemma 2.3.3 we take $\Omega \subseteq \mathbb{R}^n$ such that $1_\Omega u$ is contained in $\mathcal{D}\big(h(\mathbf{0})\big)$ for every $u \in \mathcal{D}\big(h(\mathbf{0})\big)$ but neither $\Omega$ nor $\Omega^C = \mathbb{R}^n \setminus \Omega$ is a Lebesgue null set. We denote the Lebesgue measure in $\mathbb{R}^n$ by $\lambda$.

i.) We start by showing that there exists a point $x_0 \in \mathbb{R}^n$ such that neither $\Omega \cap B(x_0, r)$ nor $\Omega^C \cap B(x_0, r)$ is a Lebesgue null set for any radius $r > 0$.

We assume that this claim would be false. Then for every $x \in \mathbb{R}^n$ there is a radius $r = r(x) > 0$ such that either $\Omega \cap B(x, r)$ or $\Omega^C \cap B(x_0, r)$ is a Lebesgue null set. We define the sets $\Omega_1$ by

$$\{x \in \mathbb{R}^n \mid \exists r > 0 \ : \ \lambda(\Omega \cap B(x, r)) = 0\}$$

and $\Omega_2$ by $\{x \in \mathbb{R}^n \mid \exists r > 0 \ : \ \lambda(\Omega^C \cap B(x, r)) = 0\}$. Then $\mathbb{R}^n = \Omega_1 \cup \Omega_2$ is true. Furthermore mind that both sets are disjoint. If there would exists a $x \in \Omega_1 \cap \Omega_2$ then

$$0 = \lambda(\Omega \cap B(x, r)) + \lambda(\Omega^C \cap B(x, r)) = \lambda\big(B(x, r)\big) > 0$$

would follow for $r > 0$ chosen appropriately small enough. So $\Omega_1 \cap \Omega_2$ must be empty. It is also clear that $\Omega_1$ and $\Omega_2$ are both open sets in $\mathbb{R}^n$. But then either $\mathbb{R}^n = \Omega_1$ or $\mathbb{R}^n = \Omega_2$ must be true since $\mathbb{R}^n$ is connected.

Let $\mathbb{R}^n = \Omega_1$ be true. We choose a compact set $K \subseteq \Omega$ such that $\lambda(K) > 0$. This is possible since $\Omega$ is not a pure point set in $\mathbb{R}^n$. For every $x \in K$ there exists a $r = r(x) > 0$ such that

$$\lambda(B(x, r) \cap \Omega) = 0$$

since $K \subseteq \mathbb{R}^n = \Omega_1$. By the compactness of $K$ we conclude that

$$K \subseteq B(x_1, r_1) \cup \cdots \cup B(x_m, r_m)$$

holds for some $x_1, \ldots, x_m \in K$ where $r_j$ are given by $\Omega_1$. But then

$$0 < \lambda(K) = \lambda(K \cap \Omega) \leq \sum_{j=1}^{m} \lambda\big(B(x_j, r_j) \cap \Omega\big) = 0$$

is obviously a contradiction. Using the same argumentation $\mathbb{R}^n = \Omega_2$ leads to a contradiction. Hence there must exist a point $x_0 \in \mathbb{R}^n$ such that neither $\Omega \cap B(x_0, r)$ nor $\Omega^C \cap B(x_0, r)$ is a Lebesgue null set for any radius $r > 0$.

ii.) Let $u \in C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{D}\big(h(\mathbf{0})\big)$ be a function such that $u(x_0) = 1$. So $1_\Omega u$ is contained in $\mathcal{D}\big(h(\mathbf{0})\big)$ as we mentioned in the very beginning of this proof. In particular $1_\Omega u$ is an element of $H^1(\mathbb{R}^n)$ with $\partial_j(1_\Omega u) = 1_\Omega \partial_j u$.

This last equation needs further explanation. First remember that $\Omega$ is not a Lebesgue null set. Therefore we choose $v \in C_c^\infty(\Omega)$. So

$$\int_\Omega (\partial_j u)(x)\overline{v(x)}\ \mathrm{d}x = -\int_\Omega u(x)\overline{(\partial_j v)(x)}\ \mathrm{d}x = -\int_{\mathbb{R}^n} (1_\Omega u)(x)\overline{(\partial_j v)(x)}\ \mathrm{d}x$$
$$= \int_{\mathbb{R}^n} \partial_j(1_\Omega u)(x)\overline{v(x)}\ \mathrm{d}x = \int_\Omega \partial_j(1_\Omega u)(x)\overline{v(x)}\ \mathrm{d}x$$

is true where the last equation is due to the support of $v$ in $\Omega$. Hence we see that $\partial_j u$ is equal to $\partial_j(1_\Omega u)$ in $\Omega$ again due to the fundamental lemma in the calculus of variations. Now we remember that $\Omega^C$ is also not a Lebesgue null set. We use the similar arguments for $v \in C_c^\infty(\Omega^C)$. So

$$0 = -\int_{\mathbb{R}^n} (1_\Omega u)(x)\overline{(\partial_j v)(x)}\ \mathrm{d}x = \int_{\mathbb{R}^n} \partial_j(1_\Omega u)(x)\overline{v(x)}\ \mathrm{d}x$$
$$= \int_{\Omega^C} \partial_j(1_\Omega u)(x)\overline{v(x)}\ \mathrm{d}x$$

holds due to the support of $v$ in $\Omega^C$. Therefore $\partial_j(1_\Omega u)$ is 0 in $\Omega^C$. So in total we have shown that $\partial_j(1_\Omega u) = 1_\Omega \partial_j u$ is actually true.
Since $u$ was taken out of $C_c^\infty(\mathbb{R}^n)$ we infer that $1_\Omega u$ is contained in the Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ for every $p \geq 2$. So Sobolev embedding theorems state that $1_\Omega u$ has a continuous representative on $\mathbb{R}^n$. We treat $1_\Omega u$ as a continuous function without minding mathematical rigor.

Remember that for $x_0$ we have shown that neither $\Omega \cap B(x_0, \frac{1}{k})$ nor $\Omega^C \cap B(x_0, \frac{1}{k})$ are Lebesgue null sets for every $k \in \mathbb{N}$. Hence we choose two sequences by $x_k \in \Omega \cap B(x_0, \frac{1}{k})$ and $y_k \in \Omega^C \cap B(x_0, \frac{1}{k})$. Then

$$|(1_\Omega u)(x_k) - (1_\Omega u)(y_k)| = |u(x_k)|$$

converges to $|u(x_0)|$ for $k \to \infty$. But since $u(x_0)$ is equal to 1 by the choice of $u$ this contradicts the continuouity of $1_\Omega u$. Hence our assumption leads to a contradiction which proves the claim.

□

The resolvent operator $\big(\lambda + H(\mathbf{0})\big)^{-1}$ is real and positivity improving for every $\lambda > 0$ as we directly see in the following equation for $u \in \mathrm{L}^2(\mathbb{R}^n)$

$$\big(\lambda + H(\mathbf{0})\big)^{-1} u = \int_0^\infty \mathrm{e}^{-\lambda t}\ \mathrm{e}^{-tH(\mathbf{0})} u\ \mathrm{d}t.$$

Finally we prove the main theorem on the ground state $\varphi$ of $H(\mathbf{0})$ where we expand the Perron-Frobenius theory.

**Theorem 2.3.6**
*Let $E_0 = \min \sigma\big(H(\mathbf{0})\big) \in [0, \infty)$ be the ground state energy of $H(\mathbf{0})$. Then the eigenspace of $E_0$ is one-dimensional spanned by a ground state $\varphi$ being strictly positive almost everywhere in $\mathbb{R}^n$.*

*Proof.* Let $0 < \lambda$ be arbitrary but fixed. Then $\lambda$ is in $\rho\big(-H(\mathbf{0})\big)$.

i.) The resolvent operator $\big(\lambda + H(\mathbf{0})\big)^{-1}$ is compact and self-adjoint. Hence by Proposition 6.13 on page 66 in [20] we state that $\big\|\big(\lambda + H(\mathbf{0})\big)^{-1}\big\|$ is an eigenvalue of $\big(\lambda + H(\mathbf{0})\big)^{-1}$.

ii.) The resolvent operator $\big(\lambda + H(\mathbf{0})\big)^{-1}$ is real and positivity improving. Hence the eigenspace to this particular eigenvalue in i.) is one-dimensional containing a strictly positive eigenfunction by Theorem 10.11 on page 236 in [18].

iii.) We show that $\big\|\big(\lambda + H(\mathbf{0})\big)^{-1}\big\| = (\lambda + E_0)^{-1}$ is true. First mind that the spectrum of $\big(\lambda + H(\mathbf{0})\big)^{-1}$ satisfies

$$\sigma\Big(\big(\lambda + H(\mathbf{0})\big)^{-1}\Big) \setminus \{0\} = \Big\{ \frac{1}{\lambda - \mu} \, : \, \mu \in \sigma\big(-H(\mathbf{0})\big) \Big\}.$$

So for every element $\eta$ in the spectrum of $\big(\lambda + H(\mathbf{0})\big)^{-1}$ we conclude that

$$0 \leq \eta \leq \frac{1}{\lambda + E_0}$$

is true since $\sigma\big(H(\mathbf{0})\big) \subseteq [E_0, \infty)$. So we argue that

$$\big\|\big(\lambda + H(\mathbf{0})\big)^{-1}\big\| \leq \frac{1}{\lambda + E_0}$$

follows from i.). But due to Proposition 6.7 on page 163 in [3] the term $\big\|\big(\lambda + H(\mathbf{0})\big)^{-1}\big\|$ is also an upper bound of the spectrum of $\big(\lambda + H(\mathbf{0})\big)^{-1}$. Therefore the inequality

$$\frac{1}{\lambda + E_0} \leq \big\|\big(\lambda + H(\mathbf{0})\big)^{-1}\big\|$$

holds as well and so $\left\|\big(\lambda+H(\mathbf{0})\big)^{-1}\right\| = (\lambda+E_0)^{-1}$ follows.

iv.) Let $\varphi \in \mathcal{D}\big(H(\mathbf{0})\big)$ be such that $H(\mathbf{0})\varphi = E_0\varphi$. So we conclude that

$$\big(\lambda+H(\mathbf{0})\big)^{-1}\varphi = \frac{1}{\lambda+E_0}\,\varphi = \left\|\big(\lambda+H(\mathbf{0})\big)^{-1}\right\|\,\varphi$$

holds. So every eigenfunction of $H(\mathbf{0})$ to $E_0$ is also an eigenfunction of $\big(\lambda+H(\mathbf{0})\big)^{-1}$ to $\left\|\big(\lambda+H(\mathbf{0})\big)^{-1}\right\|$. Using ii.) we have proven the claim.

□

Let us end this section with a remark on the pitfalls of trying to argue in the same manner for the general magnetic case $H(\mathbf{a})$.

**Remark 2.3.7**
*One might be tempted to argue in a similar way for Schrödinger semigroups in the magnetic case. But* $\mathrm{e}^{-tH(\mathbf{a})}$ *do not have to be positivity improving. Even more the operators might not even be real.*

*Let us assume that* $\mathrm{e}^{-tH(\mathbf{a})}$ *would be a real operator. Then* $H(\mathbf{a})u$ *would be real-valued for* $u \in C_c^\infty(\mathbb{R}^n,\mathbb{R})$ *since*

$$-H(\mathbf{a})u = \lim_{t\to 0}\frac{1}{t}(\mathrm{e}^{-tH(\mathbf{a})}u - u)$$

*is true as* $C_c^\infty(\mathbb{R}^n,\mathbb{R}) \subseteq D\big(H(\mathbf{a})\big)$ *holds. But for the specific choice of* $\mathbf{a} = (a_1,\dots,a_n)$ *such that* $a_j \in \mathrm{L}^4_{loc}(\mathbb{R}^n,\mathbb{R})$ *with* $\nabla\cdot\mathbf{a} \in \mathrm{L}^2_{loc}(\mathbb{R}^n,\mathbb{R})$ *we see that*

$$H(\mathbf{a})u = -\Delta u + |\mathbf{a}(x)|^2u + q(x) + \mathrm{i}\big(2\mathbf{a}(x)\nabla u + (\nabla\cdot\mathbf{a})(x)u\big)$$

*is not real-valued. So we face a contradiction. Hence the operators* $\mathrm{e}^{-tH(\mathbf{a})}$ *are not real in the general magnetic case.*

*Most relevant cases in Physics are of this nature. Although there is no general technique individual approaches do exist. In Theorem 2.2 on page* 14 *of the doctoral thesis [8] of A. Hansson from the KTH Royal Institute of Technology in Stockholm ground states of several magnetic Schrödinger operators with radial symmetric electrical potentials were explicitly calculated. Even though the ground state is not strictly positive in general these explicit solutions are non zero almost almost everywhere in* $\mathbb{R}^n$*.*

# Chapter 3

# Intrinsic ultracontractivity of $e^{-tH(\mathbf{0})}$

In this chapter we present a condition on the electric potential $q$ in $H(\mathbf{0}) = -\Delta + q(x)$ such that for every time $t > 0$ there exists a constant $C_t > 0$ which satisfies

$$\left| e^{-tH(\mathbf{0})} u(x) \right| \leq C_t \, \|u\|_2 \, \varphi(x)$$

almost everywhere in $\mathbb{R}^n$ for every $u \in L^2(\mathbb{R}^n)$ and the ground state $\varphi$. We call $e^{-tH(\mathbf{0})}$ intrinsic ultracontractive in this case.

We follow the procedure of B. Davies in [4] by introducing a weighted Lebesgue space $L^2_\mu(\mathbb{R}^n)$ of measurable functions $u$ on $\mathbb{R}^n$ that satisfy $u\varphi \in L^2(\mathbb{R}^n)$ and define a semigroup in $L^2_\mu(\mathbb{R}^n)$ by

$$e^{-t\tilde{H}(\mathbf{0})} u(x) = \frac{1}{\varphi(x)} e^{tE_0} e^{-tH(\mathbf{0})} (\varphi u)(x)$$

for $u \in L^2_\mu(\mathbb{R}^n)$ that we call the weighted Schrödinger semigroup. For every time $t > 0$ we show that $e^{-t\tilde{H}(\mathbf{0})}$ maps from $L^2_\mu(\mathbb{R}^n)$ into $L^\infty_\mu(\mathbb{R}^n)$ continously. The weighted Schrödinger semigroup is ultracontractive in this case which results in the intrinsic ultracontractivity of $e^{-tH(\mathbf{0})}$. Our proof is based on Logarithmic Sobolev inequalities for $\tilde{H}(\mathbf{0})$ which were introduced by L. Gross in [7]. In particular these Logarithmic Sobolev inequalities for $\tilde{H}(\mathbf{0})$ are given by

$$\int_{\mathbb{R}^n} \ln\bigl(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x)\bigr)\bigl(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x)\bigr)^p \mathrm{d}\mu(x)$$
$$\leq \ \varepsilon \ \langle(\tilde{H}(\mathbf{0}) + E_0)\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u, (\ \mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\ )^{p-1}\rangle_\mu + \frac{2\beta(\varepsilon)}{p}\ \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p,\mu}^p$$
$$+ \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p,\mu}^p\ \ln\|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p,\mu}$$

for every $p \in (2,\infty)$ and $\varepsilon > 0$ with $\beta(\varepsilon) > 0$ where $u \in \mathcal{D}\bigl(\tilde{H}(\mathbf{0})\bigr) \cap \mathrm{L}_\mu^\infty(\mathbb{R}^n)$ is non-negative almost everywhere in $\mathbb{R}^n$.

If for every $\varepsilon > 0$ there is a $\gamma(\varepsilon) > 0$ satisfying

$$\int_{\mathbb{R}^n} -\ln\bigl(\varphi(x)\bigr)\cdot u(x)^2\ \mathrm{d}x\ \leq\ \varepsilon h(\mathbf{0})(u,u) + \gamma(\varepsilon)\|u\|_2^2$$

for every $u \in \mathcal{D}\bigl(h(\mathbf{0})\bigr)$ that is non-negative almost everywhere in $\mathbb{R}^n$ then J. Rosen proved in [16] that Logarithmic Sobolev inequalities for $\tilde{H}(\mathbf{0})$ are implied. Alternatively see in section 4.4 starting on page 117 in [4].

## 3.1 Transference to weighted $\mathrm{L}_\mu^2(\mathbb{R}^n)$ spaces

Following the procedure of B. Davies in Section 4.2 on page 111 in [4] we introduce weighted Schrödinger semigroups $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ and sesqulinear forms $\tilde{h}(\mathbf{0})$.

### 3.1.1 Weighted Schrödinger semigroups $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$

Let $E_0$ be the ground state energy of a Schrödinger operator $H(\mathbf{0})$ in $\mathrm{L}^2(\mathbb{R}^n)$. Furthermore let $\varphi$ be the associated ground state that satisfies $\|\varphi\|_2 = 1$. So for any Borel set $B$ in $\mathbb{R}^n$ we define a probability measure $\mu$ on $\mathbb{R}^n$ by

$$\mu(B) := \int_B \varphi(x)^2\ \mathrm{d}x.$$

**Definition 3.1.1**
*For every $p \in [1,\infty)$ we denote $\mathrm{L}_\mu^p(\mathbb{R}^n)$ as the Lebesgue function spaces of measurable functions $u\colon \mathbb{R}^n \to \mathbb{C}$ that satisfy*

$$\int_{\mathbb{R}^n} |u(x)|^p\ \mathrm{d}\mu(x) = \int_{\mathbb{R}^n} |u(x)|^p \cdot \varphi(x)^2\ \mathrm{d}x < \infty.$$

*Furthermore* $\mathrm{L}^\infty_\mu(\mathbb{R}^n)$ *is defined by* $\mathrm{L}^\infty(\mathbb{R}^n)$.

For the definition of $\mathrm{L}^\infty_\mu(\mathbb{R}^n)$ please mind that Lebesgue null sets with respect to $\mu$ coincide with the null sets with respect to the regular Lebesgue measure on $\mathbb{R}^n$ since $\varphi$ is strictly positive almost everywhere in $\mathbb{R}^n$. By

$$\|u\|_{p,\mu} = \left(\int_{\mathbb{R}^n} |u(x)|^p \,\mathrm{d}\mu(x)\right)^{\frac{1}{p}}$$

for $u \in \mathrm{L}^p_\mu(\mathbb{R}^n)$ we define a norm on $\mathrm{L}^p_\mu(\mathbb{R}^n)$ and show completeness.

**Proposition 3.1.2**
*For every* $p \in [1,\infty)$ *the weighted Lebesgue spaces* $\mathrm{L}^p_\mu(\mathbb{R}^n)$ *are complete.*

*Proof.* Let $p \in [1,\infty)$ be chosen arbitrarily but fixed and let $(u_n)$ be a Cauchy sequence in $\mathrm{L}^p_\mu(\mathbb{R}^n)$. By definition $(\varphi^{\frac{2}{p}} u_n)$ is Cauchy in $\mathrm{L}^p(\mathbb{R}^n)$. Hence there is a function $v \in \mathrm{L}^p(\mathbb{R}^n)$ as a limit of $\varphi^{\frac{2}{p}} u_n$. We define $u = \varphi^{-\frac{2}{p}} v \in \mathrm{L}^p_\mu(\mathbb{R}^n)$. So

$$\int_{\mathbb{R}^n} |u_n(x) - u(x)|^p \,\mathrm{d}\mu(x) = \int_{\mathbb{R}^n} |\varphi(x)^{\frac{2}{p}} u_n(x) - v(x)|^p \,\mathrm{d}x$$

is true and therefore $\mathrm{L}^p_\mu(\mathbb{R}^n)$ is complete.

□

In particular $\mathrm{L}^2_\mu(\mathbb{R}^n)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_\mu = \langle \varphi u, \varphi v\rangle$$

for $u, v \in \mathrm{L}^2_\mu(\mathbb{R}^n)$. We define the operator $U \colon \mathrm{L}^2_\mu(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ by $Uu = \varphi u$ which is unitary as we see. The operator $U$ is bounded and therefore has an adjoint $U'$ that maps $\mathrm{L}^2(\mathbb{R}^n)$ to $\mathrm{L}^2_\mu(\mathbb{R}^n)$. Now let $u, v \in \mathrm{L}^2(\mathbb{R}^n)$. So $\varphi^{-1} u \in \mathrm{L}^2_\mu(\mathbb{R}^n)$ is satisfied as well as

$$\langle u, v\rangle = \langle U\varphi^{-1} u, v\rangle = \langle \varphi^{-1} u, U' v\rangle_\mu = \langle u, \varphi U' v\rangle.$$

Since $\varphi U' v$ is contained in $\mathrm{L}^2(\mathbb{R}^n)$ we conclude $v = \varphi U' v$ follows. So $U' v = \varphi^{-1} v$ holds pointwise almost everywhere in $\mathbb{R}^n$ and $U$ is in fact unitary.

**Definition 3.1.3**
*We define the domain* $\mathcal{D}\big(\tilde{H}(\mathbf{0})\big) := U'\mathcal{D}\big(H(\mathbf{0})\big) \subseteq \mathrm{L}^2_\mu(\mathbb{R}^n)$ *for the operator*

$$\tilde{H}(\mathbf{0}) := U'\big(H(\mathbf{0}) - E_0\big)U.$$

*We call* $\tilde{H}(\mathbf{0})$ *the weighted Schrödinger operator in* $\mathrm{L}^2_\mu(\mathbb{R}^n)$.

We show that $-\tilde{H}(\mathbf{0})$ generates a strongly continuous semigroup

$$\{\mathrm{e}^{-t\tilde{H}(\mathbf{0})} \mid t \geq 0\}$$

of contractions in the weighted Lebesgue function space $\mathrm{L}^2_\mu(\mathbb{R}^n)$. We use a rather sloppy notation in the following by writing $\varphi u$ and $\varphi^{-1}u$ instead of $Uu$ and $U'u$ respectively.

**Theorem 3.1.4**
*The operator* $-\tilde{H}(\mathbf{0})$ *generates a* $C_0$*-semigroup of contractions in* $\mathrm{L}^2_\mu(\mathbb{R}^n)$.

*Proof.* Using the theorem of G. Lumer and R. Phillips we only have to show the weighted Schrödinger operator $\tilde{H}(\mathbf{0})$ is m-accretive. For a reference please see chapter 3.4 starting from page 32 in [20].

Theorem 2.19 on page 77 in [18] offers

$$E_0 = \inf \sigma\big(H(\mathbf{0})\big) = \inf\big\{\langle H(\mathbf{0})u, u\rangle \mid u \in \mathcal{D}\big(H(\mathbf{0})\big),\ \|u\|_2 = 1\big\}.$$

Therefore we conclude that

$$\big\langle \tilde{H}(\mathbf{0})u, u\big\rangle_\mu = \big\langle H(\mathbf{0})\varphi u, \varphi u\big\rangle - E_0\|\varphi u\|_2^2 \geq 0$$

holds for every $u \in \mathcal{D}\big(\tilde{H}(\mathbf{0})\big)$. So $\tilde{H}(\mathbf{0})$ is accretive. Furthermore we already know that $(-E_0, \infty) \subseteq \rho\big(-H(\mathbf{0})\big)$ is true. We easily conclude that $(0, \infty)$ is contained in the resolvent set of $-\tilde{H}(\mathbf{0})$. In particular the weighted Schrödinger operator $\tilde{H}(\mathbf{0})$ is m-accretive.

□

In the next lemma we characterize the semigroup generated by $-\tilde{H}(\mathbf{0})$.

**Lemma 3.1.5**
*For every $t \geq 0$ the operator $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ is equal to $\varphi^{-1}\mathrm{e}^{tE_0}\mathrm{e}^{-tH(\mathbf{0})}\varphi$.*

*Proof.* First we understand that $\{\varphi^{-1}\mathrm{e}^{tE_0}\mathrm{e}^{-tH(\mathbf{0})}\varphi \mid t \geq 0\}$ is a strongly continuous semigroup in $\mathrm{L}^2_\mu(\mathbb{R}^n)$ with a generator $B$. We show that $B$ is equal to $-\tilde{H}(\mathbf{0})$. The semigroups have to match as well.

i.) First we see that $-\tilde{H}(\mathbf{0})$ is an extension of $B$. Let $u \in \mathcal{D}(B)$. Then

$$\left\|\frac{1}{t}(\mathrm{e}^{-tH(\mathbf{0})}\varphi u - \varphi u) - (\varphi Bu - E_0\varphi u)\right\|_2$$

$$\leq \left\|\frac{1}{t}(\mathrm{e}^{tE_0}\mathrm{e}^{-tH(\mathbf{0})}\varphi u - \varphi u) - \varphi Bu\right\|_2 + \left\|\frac{1}{t}(\mathrm{e}^{tE_0} - 1)\mathrm{e}^{-tH(\mathbf{0})}\varphi u - E_0\varphi u)\right\|_2$$

$$= \left\|\frac{1}{t}(\varphi^{-1}\mathrm{e}^{tE_0}\mathrm{e}^{-tH(\mathbf{0})}\varphi u - u) - Bu\right\|_{2,\mu} + \left\|\frac{1}{t}(\mathrm{e}^{tE_0} - 1)\mathrm{e}^{-tH(\mathbf{0})}\varphi u - E_0\varphi u)\right\|_2$$

converges to 0 for $t \to 0$. So $\varphi u \in \mathcal{D}\big(H(\mathbf{0})\big)$ and

$$\varphi Bu = \big(E_0 - H(\mathbf{0})\big)\varphi u = -\big(H(\mathbf{0}) - E_0\big)\varphi u$$

both follow. Hence $B \subseteq -\tilde{H}(\mathbf{0})$ is true by using the definition of $\tilde{H}(\mathbf{0})$.

ii.) Now we see that $-\tilde{H}(\mathbf{0}) \subseteq B$ holds as well. Let $u \in \mathcal{D}\big(\tilde{H}(\mathbf{0})\big)$. Then

$$\left\|\frac{1}{t}(\varphi^{-1}\mathrm{e}^{tE_0}\mathrm{e}^{-tH(\mathbf{0})}\varphi u - u) + \tilde{H}(\mathbf{0})u\right\|_{2,\mu}$$

$$= \left\|\frac{1}{t}(\mathrm{e}^{tE_0}\mathrm{e}^{-tH(\mathbf{0})}\varphi u - \varphi u) + (H(\mathbf{0}) - E_0)\varphi u\right\|_2$$

$$\leq \left\|\frac{1}{t}(\mathrm{e}^{tE_0} - 1)\mathrm{e}^{-tH(\mathbf{0})}\varphi u - E_0\varphi u\right\|_2 + \left\|\frac{1}{t}(\mathrm{e}^{-tH(\mathbf{0})}\varphi u - \varphi u) + H(\mathbf{0})\varphi u\right\|_2$$

converges to 0 for $t \to 0$ which results in $-\tilde{H}(\mathbf{0}) \subseteq B$.

□

With this characterization of the weighted Schrödinger semigroup we conclude that every operator $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ for $t > 0$ is real and positivity improving as a direct conclusion of the Perron-Frobenius theory on $\mathrm{e}^{-tH(\mathbf{0})}$. So $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ is an operator in $\mathrm{L}^2_\mu(\mathbb{R}^n, \mathbb{R})$. Furthermore $\mathrm{L}^2_\mu(\mathbb{R}^n, \mathbb{R})$ is contained in $\mathrm{L}^1_\mu(\mathbb{R}^n, \mathbb{R})$ since $\mu$ is a probability measure. So we might be able to uniquely extend $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ to an operator in $\mathrm{L}^1_\mu(\mathbb{R}^n, \mathbb{R})$. We keep the notation.

**Lemma 3.1.6**
*For $t > 0$ the operator $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ is a contraction in $\mathrm{L}^1_\mu(\mathbb{R}^n, \mathbb{R})$.*

*Proof.* Let $t > 0$ be chosen arbitrary but fixed.

i.) Let $u \in \mathrm{L}^2_\mu(\mathbb{R}^n, \mathbb{R}) \subseteq \mathrm{L}^1_\mu(\mathbb{R}^n, \mathbb{R})$ be non-negative almost everywhere in $\mathbb{R}^n$. So we reason that

$$\int_{\mathbb{R}^n} |\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\ (x)|\ \mathrm{d}\mu(x) = \int_{\mathbb{R}^n} \mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\ (x)\ \mathrm{d}\mu(x)$$

$$= \mathrm{e}^{tE_0}\left\langle \mathrm{e}^{-tH(\mathbf{0})}\varphi u, \varphi \right\rangle = \mathrm{e}^{tE_0}\left\langle \varphi u, \mathrm{e}^{-tH(\mathbf{0})}\varphi \right\rangle$$

$$= \mathrm{e}^{tE_0}\left\langle \varphi u, \mathrm{e}^{-tE_0}\varphi \right\rangle = \left\langle \varphi u, \varphi \right\rangle = \|u\|_{1,\mu}$$

is true where we used $\mathrm{e}^{-tH(\mathbf{0})}\varphi = \mathrm{e}^{-tE_0}\varphi$.

ii.) Let $u \in \mathrm{L}^2_\mu(\mathbb{R}^n, \mathbb{R})$ we denote the positive and negative part of $u$ by $u^+ = 1_{\{u \geq 0\}}\ u$ and $u^- = -1_{\{u<0\}}\ u$ respectively. Using i.) we end up with

$$\begin{aligned} &\int_{\mathbb{R}^n} |\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\ (x)|\ \mathrm{d}\mu(x) \\ &\leq \int_{\mathbb{R}^n} |\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u^+\ (x)|\ \mathrm{d}\mu(x) + \int_{\mathbb{R}^n} |\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u^-\ (x)|\ \mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}^n} \mathrm{e}^{-t\tilde{H}(\mathbf{0})}u^+\ (x)\ \mathrm{d}\mu(x) + \int_{\mathbb{R}^n} \mathrm{e}^{-t\tilde{H}(\mathbf{0})}u^-\ (x)\ \mathrm{d}\mu(x) \\ &= \int_{\mathbb{R}^n} u^+(x)\ \mathrm{d}\mu(x) + \int_{\mathbb{R}^n} u^-(x)\ \mathrm{d}\mu(x) = \int_{\mathbb{R}^n} |u(x)|\ \mathrm{d}\mu(x) = \|u\|_{1,\mu} \end{aligned}$$

iii.) Now let $u \in \mathrm{L}^1_\mu(\mathbb{R}^n, \mathbb{R})$. We define a sequence $u_k = 1_{\{u \leq k\}}\, u$ for $k \in \mathbb{N}$. Mind that $u_k$ is a bounded function on $\mathbb{R}^n$ and therefore is contained in $\mathrm{L}^2_\mu(\mathbb{R}^n, \mathbb{R})$ since $\mu$ is a probability measure. Furthermore $(u_k)$ converges to $u$ in $\mathrm{L}^1_\mu(\mathbb{R}^n)$ by Lebesgue's theorem since $u_k$ converges pointwise almost everywhere to $u$ in $\mathbb{R}^n$ and $|u_k - u| \leq 2|u| \in \mathrm{L}^1_\mu(\mathbb{R}^n)$ is true as well. So using ii.) we conclude that $(\mathrm{e}^{-t\tilde{H}(\mathbf{0})} u_k)$ is a Cauchy sequence in $\mathrm{L}^1_\mu(\mathbb{R}^n)$. Hence there exists a unique and bounded extension of $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ to $\mathrm{L}^1_\mu(\mathbb{R}^n)$ that proves the claim.

□

Mind that $\mathrm{L}^\infty_\mu(\mathbb{R}^n, \mathbb{R})$ is dual to $\mathrm{L}^1_\mu(\mathbb{R}^n, \mathbb{R})$. Furthermore $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ as an operator in $\mathrm{L}^2_\mu(\mathbb{R}^n, \mathbb{R})$ is well defined in $\mathrm{L}^\infty_\mu(\mathbb{R}^n, \mathbb{R})$ since it is a subset of $\mathrm{L}^2_\mu(\mathbb{R}^n, \mathbb{R})$. We can easily see that $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ is a contraction in $\mathrm{L}^\infty_\mu(\mathbb{R}^n, \mathbb{R})$.

**Corollary 3.1.7**
*For every $t > 0$ the operator $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ is a contraction in $\mathrm{L}^\infty_\mu(\mathbb{R}^n, \mathbb{R})$.*

*Proof.* Let $u \in \mathrm{L}^\infty_\mu(\mathbb{R}^n, \mathbb{R})$ be chosen arbitrarily. We denote the adjoint operator of $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ in $\mathrm{L}^1_\mu(\mathbb{R}^n, \mathbb{R})$ by $(\mathrm{e}^{-t\tilde{H}(\mathbf{0})})'$ which is bounded in $\mathrm{L}^\infty_\mu(\mathbb{R}^n, \mathbb{R})$ and satisfies

$$\left\|(\mathrm{e}^{-t\tilde{H}(\mathbf{0})})'\right\|_{\mathrm{L}^\infty_\mu \to \mathrm{L}^\infty_\mu} = \left\|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}\right\|_{\mathrm{L}^1_\mu \to \mathrm{L}^1_\mu} \leq 1.$$

For every $v \in \mathrm{L}^2_\mu(\mathbb{R}^n, \mathbb{R}) \subseteq \mathrm{L}^1_\mu(\mathbb{R}^n, \mathbb{R})$ we conclude that

$$\left\langle \mathrm{e}^{-t\tilde{H}(\mathbf{0})} u, v \right\rangle_\mu = \left\langle u, \mathrm{e}^{-t\tilde{H}(\mathbf{0})} v \right\rangle_\mu = \left\langle (\mathrm{e}^{-t\tilde{H}(\mathbf{0})})' u, v \right\rangle_\mu$$

where $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ is taken as a self-adjoint operator in $\mathrm{L}^2_\mu(\mathbb{R}^n, \mathbb{R})$ in the first equality and as an operator in $\mathrm{L}^1_\mu(\mathbb{R}^n, \mathbb{R})$ with its adjoint in $\mathrm{L}^\infty_\mu(\mathbb{R}^n, \mathbb{R})$ in the second. Hence $\mathrm{e}^{-t\tilde{H}(\mathbf{0})} u$ is equal to $(\mathrm{e}^{-t\tilde{H}(\mathbf{0})})' u$ which proves the claim.

□

### 3.1.2 The weighted Schrödinger form $\tilde{h}(\mathbf{0})$

We introduce a sesquilinear form $\tilde{h}(\mathbf{0})$ associated to $\tilde{H}(\mathbf{0}) + E_0$ that is called the weighted Schrödinger form. By

$$\mathcal{D}\big(\tilde{h}(\mathbf{0})\big) = \varphi^{-1} \mathcal{D}\big(h(\mathbf{0})\big) \subseteq \mathrm{L}^2_\mu(\mathbb{R}^n)$$

the domain of $\tilde{h}(\mathbf{0})$ is given. For $u, v \in \mathcal{D}\big(\tilde{h}(\mathbf{0})\big)$ the form is defined by

$$\tilde{h}(\mathbf{0})(u, v) = h(\mathbf{0})(\varphi u, \varphi v).$$

Then $\langle u, v\rangle_{\tilde{h}(\mathbf{0})} = \tilde{h}(\mathbf{0})(u, v) + \langle u, v\rangle_\mu$ defines an inner product and

$$\|u\|_{\tilde{h}(\mathbf{0})} = \sqrt{\langle u, u\rangle_{\tilde{h}(\mathbf{0})}}$$

a norm on $\mathcal{D}\big(\tilde{h}(\mathbf{0})\big)$. Again we prove completeness.

**Lemma 3.1.8**
*The function space $\mathcal{D}\big(\tilde{h}(\mathbf{0})\big)$ is complete.*

*Proof.* Let $(u_n) \subseteq \mathcal{D}\big(\tilde{h}(\mathbf{0})\big)$ be a Cauchy sequence with respect to $\|\cdot\|_{\tilde{h}(\mathbf{0})}$. So $(\varphi u_n) \subseteq \mathcal{D}\big(h(\mathbf{0})\big)$ is a Cauchy sequence with respect to $\|\cdot\|_{h(\mathbf{0})}$. Hence there exists a limit $v$ in $\mathcal{D}\big(h(\mathbf{0})\big)$ of $(\varphi u_n)$. Then $\varphi^{-1}v$ is the limit of $(u_n)$ in $\mathcal{D}\big(\tilde{h}(\mathbf{0})\big)$ which proves the claim.

□

We prove that the self-adjoint operator $\tilde{H}(\mathbf{0})+E_0$ in the weighted Lebesgue function space $\mathrm{L}^2_\mu(\mathbb{R}^n)$ is the associated operator to the weighted form $\tilde{h}(\mathbf{0})$.

**Lemma 3.1.9**
*The operator $\tilde{H}(\mathbf{0}) + E_0$ is associated to $\tilde{h}(\mathbf{0})$.*

*Proof.* Using the argumentation given in Section 2.1 there exists an associated self-adjoint operator $A$ in $\mathrm{L}^2_\mu(\mathbb{R}^n)$ with a domain $\mathcal{D}(A)$ given by

$$\Big\{u \in \mathcal{D}\big(\tilde{h}(\mathbf{0})\big) \mid \exists v \in \mathrm{L}^2_\mu(\mathbb{R}^n) :\ \tilde{h}(\mathbf{0})(u, w) = \langle v, w\rangle_\mu \text{ for every } w \in \mathcal{D}\big(\tilde{h}(\mathbf{0})\big)\Big\}.$$

Let $u \in \mathcal{D}\big(\tilde{H}(\mathbf{0})\big)$ and $w \in \mathcal{D}\big(\tilde{h}(\mathbf{0})\big)$. Then

$$\langle(\tilde{H}(\mathbf{0}) + E_0)u, w\rangle_\mu = \langle H(\mathbf{0})\varphi u, \varphi w\rangle = h(\mathbf{0})(\varphi u, \varphi w) = \tilde{h}(\mathbf{0})(u, w)$$

follows and therefore $A$ is an extension of $\tilde{H}(\mathbf{0})+E_0$. But since both operators are self-adjoint they coincide.

□

## 3.2 Rosen's lemma

In this section we use Rosen inequalities that read

$$-\ln(\varphi(x)) \ \leq \ \varepsilon q(x) + \gamma(\varepsilon)$$

for every $\varepsilon > 0$ with an associated $\gamma(\varepsilon) > 0$ to prove a pre version of Logarithmic Sobolev inequalities for the weighted Schrödinger operator $\tilde{H}(\mathbf{0})$. These inequalities are of the form

$$\int_{\mathbb{R}^n} \ln\big(u(x)\big)\big(u(x)\big)^2 \ \mathrm{d}\mu(x) \ \leq \ \varepsilon\tilde{h}(\mathbf{0})(u,u) + \beta(\varepsilon)\|u\|_{2,\mu}^2 + \|u\|_{2,\mu}^2 \ln\|u\|_{2,\mu}$$

for every $\varepsilon > 0$ with an associated $\beta(\varepsilon) \in \mathbb{R}$ where $u \in \mathcal{D}\big(\tilde{h}(\mathbf{0})\big) \cap \mathrm{L}_\mu^\infty(\mathbb{R}^n)$ is non-negative almost everywhere in $\mathbb{R}^n$.

Let $Q\colon [0,\infty) \to (0,\infty)$ be a continuous auxiliary function such that

$$\left(\int_{r_0}^r Q(t)^{\frac{1}{2}} \ \mathrm{d}t\right) P\left[\ln\left(\int_{r_0}^r Q(t)^{\frac{1}{2}} \ \mathrm{d}t\right)\right] \ \leq \ Q(r)$$

holds for $r_0 \leq r$ for a given $r_0 \in [0,\infty)$ is given and $P\colon \mathbb{R} \to (0,\infty)$ being strictly monotone increasing such that $P(\mathbb{R}) = (0,\infty)$ and

$$\int_0^\infty P(t)^{-1} \ \mathrm{d}t \ < \ \infty$$

hold. Also let $Q$ satisfy

$$\int_{r_0}^\infty |Q'(t)|^\gamma \ Q(t)^{\frac{1}{2}-\frac{3}{2}\gamma} \ \mathrm{d}t \ < \ \infty$$

for a constant $\gamma \in (1,2]$. By Proposition 2.2 a) on page 4102 in [1]

$$\int_{r_0}^\infty Q(t)^{-\frac{1}{2}} \ \mathrm{d}t < \infty$$

is implied. Typical examples of such auxiliary functions $Q$ are

$$Q(r) = r^{2+\delta}, \ r^2(\ln r)^{2+\delta}, \ r^2(\ln r)^2(\ln(\ln r))^{2+\delta}$$

for $r_0 \leq r$ or more generally $Q(r) = r^2(\ln r)^2(\ln\ln r)^2 \ldots (\ln^m(r))^{2+\delta}$ where

$$\ln^m(r) = \underbrace{\ln\ln\ln\ldots\ln}_{m \text{ times}}(r).$$

For details please consider Example 2.5 on page 4105 and Proposition 2.2 on page 4102 in [1]. In the following we always consider $q\colon \mathbb{R}^n \to [0,\infty)$ as a continuous function satisfying

$$\left(\int_{r_0}^{|x|_2} Q(t)^{\frac{1}{2}}\ \mathrm{d}t\right) P\left[\ln\left(\int_{r_0}^{|x|_2} Q(t)^{\frac{1}{2}}\ \mathrm{d}t\right)\right] \ \leq\ q(x)\ \leq\ Q(|x|_2). \tag{3.1}$$

In that case Proposition 5.1 on page 4116 in [1] implies

$$-\ln\left(\varphi(x)\right)\ \leq\ \varepsilon q(x) + \underbrace{C_1\tilde{P}\left(\frac{C_1}{\varepsilon}\right) + C_2}_{=\gamma(\varepsilon)} \tag{3.2}$$

for constants $C_1, C_2 > 0$ where $\tilde{P}$ is the inverse function of $P \circ \ln$ on $(0,\infty)$.

**Remark 3.2.1**
*Please consider the appendix A for a detailed presentation of the arguments in [1] which we cited here.*

Before we infer a pre version of Logarithmic Sobolev inequalities for $\tilde{H}(\mathbf{0})$ we need to prove a rather technical proposition. Due to Sobolev embedding theorems we let $n > 2$ for the dimension of $\mathbb{R}^n$ from now on.

**Proposition 3.2.2**
*There exists a constant $C > 0$ such that*

$$\big\langle vu, u\big\rangle \leq C\ \|v\|_{\frac{n}{2}}\ \big(\ h(\mathbf{0})(u,u) + \|u\|_2^2\ \big)$$

*holds for every real-valued $u \in \mathcal{D}\big(h(\mathbf{0})\big)$ and every $v \in \mathrm{L}^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R})$.*

*Proof.* Let $u \in \mathcal{D}\big(h(\mathbf{0})\big)$ be real-valued and $v \in \mathrm{L}^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R})$.
By definition $u$ is contained in the Sobolev space $H^1(\mathbb{R}^n)$. Using Sobolev embedding theorems $H^1(\mathbb{R}^n)$ is continuously embedded in $\mathrm{L}^{\frac{2n}{n-2}}(\mathbb{R}^n)$. So there exists a constant $C > 0$ independent of $u$ such that

$$\|u^2\|_{\frac{n}{n-2}}\ =\ \|u\|^2_{\frac{2n}{n-2}}\ \leq C\big(\ \|\nabla u\|_2^2 + \|u\|_2^2\ \big)$$

holds. The Hölder inequality ensures that $vu^2 \in \mathrm{L}^1(\mathbb{R}^n)$ follows from

$$\left|\langle vu, u\rangle\right| \;\leq\; \|v\|_{\frac{n}{2}}\;\|u^2\|_{\frac{n}{n-2}}.$$

Therefore the claimed inequality holds since

$$\langle vu, u\rangle \;\leq\; C\;\|v\|_{\frac{n}{2}}\;\left(\;\|\nabla u\|_2^2 + \|u\|_2^2\;\right) \leq C\;\|v\|_{\frac{n}{2}}\;\left(\;h(\mathbf{0})(u,u) + \|u\|_2^2\;\right)$$

follows by the definition of $h(\mathbf{0})$.

□

Now we prove the following pre version of Logarithmic Sobolev inequalities for $\tilde{H}(\mathbf{0})$.

**Lemma 3.2.3**
*Let $u \in \mathcal{D}\big(\tilde{h}(\mathbf{0})\big) \cap \mathrm{L}^\infty_\mu(\mathbb{R}^n)$ be non-negative almost everywhere in $\mathbb{R}^n$. Then for every $\varepsilon > 0$ we claim that*

$$\int_{\mathbb{R}^n} \ln\big(u(x)\big)\big(u(x)\big)^2\;\mathrm{d}\mu(x) \;\leq\; \varepsilon\tilde{h}(\mathbf{0})(u,u) + \beta(\varepsilon)\|u\|_{2,\mu}^2 + \|u\|_{2,\mu}^2 \ln\|u\|_{2,\mu}$$

*holds for $\beta(\varepsilon) = \frac{\varepsilon}{2} - \frac{n}{4}\ln(\frac{\varepsilon}{2}) + \gamma(\frac{\varepsilon}{2}) + C$ with a constant $C \in \mathbb{R}$.*

*Proof.* Let $\xi > 0$ be arbitrary but fixed and additionally let $\|u\|_{2,\mu} = 1$.

i.) We define an auxiliary function $\mathbf{1}\colon \mathbb{R}^n \to \mathbb{R}$ by

$$\mathbf{1}(x) = \begin{cases} 1, & \text{if } \xi u(x)\varphi(x) \geq 1 \\ 0, & \text{else} \end{cases}$$

Furthermore there is an upper bound $b > 0$ such that

$$\ln(s)^{\frac{n}{2}} \leq bs^2$$

holds for every $s \in [1,\infty)$. We easily see that since $s^{-2}\ln(s)^{\frac{n}{2}}$ converges to 0 for $s \to \infty$ by l'Hospital's theorem.

ii.) So

$$\int_{\mathbb{R}^n} \mathbf{1}(x)\big(\ln(\xi u(x)\varphi(x))\big)^{\frac{n}{2}}\;\mathrm{d}x \leq \int_{\mathbb{R}^n} \mathbf{1}(x) b\xi^2 u(x)^2\varphi(x)^2\;\mathrm{d}x \leq b\xi^2$$

holds. Then $\mathbf{1}\ln(\xi u\varphi) \in \mathrm{L}^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R})$ with $\|\mathbf{1}\ln(\xi u\varphi)\|_{\frac{n}{2}} \leq b^{\frac{2}{n}}\xi^{\frac{4}{n}}$ is implied. We use Propostion 3.2.2 to conclude that

$$\begin{aligned}&\big\langle \mathbf{1}\ln(\xi u\varphi)u, u\big\rangle_\mu = \big\langle \mathbf{1}\ln(\xi u\varphi)\varphi u, \varphi u\big\rangle\\ &\leq Cb^{\frac{2}{n}}\xi^{\frac{4}{n}}\big(h(\mathbf{0})(\varphi u, \varphi u) + \|\varphi u\|_2^2\big)\\ &= Cb^{\frac{2}{n}}\xi^{\frac{4}{n}}\big(\tilde{h}(\mathbf{0})(u, u) + 1\big)\end{aligned}$$

follows where we used $\|\varphi u\|_2 = \|u\|_{2,\mu} = 1$.

iii.) Furthermore

$$\begin{aligned}&\ln(\xi) + \big\langle \ln(u)u, u\big\rangle_\mu + \big\langle \ln(\varphi)u, u\big\rangle_\mu = \big\langle \ln(\xi u\varphi)u, u\big\rangle_\mu\\ &\leq \big\langle \mathbf{1}\ln(\xi u\varphi)u, u\big\rangle_\mu \leq Cb^{\frac{2}{n}}\xi^{\frac{4}{n}}\big(\tilde{h}(\mathbf{0})(u, u) + 1\big)\end{aligned}$$

holds.

iv.) We define $\varepsilon = Cb^{\frac{2}{n}}\xi^{\frac{4}{n}}$. Then

$$\ln(\xi) = \frac{n}{4}\ln(\varepsilon) - \tilde{C}$$

holds for an appropriate constant $\tilde{C} \in \mathbb{R}$. We use (3.2) for

$$-\int_{\mathbb{R}^n} \ln\big(\varphi(x)\big)\big(u(x)\big)^2 \,\mathrm{d}\mu(x) \;\leq\; \varepsilon\tilde{h}(\mathbf{0})(u, u) + \gamma(\varepsilon)\|u\|_{2,\mu}^2$$

to finally conclude that

$$\begin{aligned}&\int_{\mathbb{R}^n} \ln\big(u(x)\big)\big(u(x)\big)^2 \,\mathrm{d}\mu(x)\\ &\leq\; \varepsilon\tilde{h}(\mathbf{0})(u, u) + \varepsilon - \frac{n}{4}\ln(\varepsilon) + \tilde{C} - \big\langle \ln(\varphi)u, u\big\rangle_\mu\\ &\leq\; 2\varepsilon\tilde{h}(\mathbf{0})(u, u) + \varepsilon - \frac{n}{4}\ln(\varepsilon) + \tilde{C} + \gamma(\varepsilon)\\ &= 2\varepsilon\tilde{h}(\mathbf{0})(u, u) + \beta(2\varepsilon).\end{aligned}$$

v.) For $\|u\|_{2,\mu} \neq 1$ the claim follows by using $u\|u\|_{2,\mu}^{-1}$ instead of $u$ in the inequality above.

□

## 3.3 Logarithmic Sobolev inequalities for $\tilde{H}(0)$

In this section the pre version of Logarithmic Sobolev inequalities for $\tilde{H}(0)$ is given. So every $u \in \mathcal{D}\big(\tilde{h}(\mathbf{0})\big) \cap \mathrm{L}^\infty_\mu(\mathbb{R}^n)$ being non-negative almost everywhere in $\mathbb{R}^n$ satisfies

$$\int_{\mathbb{R}^n} \ln\big(u(x)\big)\big(u(x)\big)^2 \,\mathrm{d}\mu(x) \ \le\ \varepsilon \tilde{h}(\mathbf{0})(u,u) + \beta(\varepsilon)\|u\|^2_{2,\mu} + \|u\|^2_{2,\mu} \ln \|u\|_{2,\mu}$$

for every $\varepsilon > 0$ and $\beta(\varepsilon) = \frac{\varepsilon}{2} - \frac{n}{4}\ln(\frac{\varepsilon}{2}) + \gamma(\frac{\varepsilon}{2}) + C$ with a constant $C \in \mathbb{R}$. We prove Logarithmic Sobolev inequalities for $\tilde{H}(\mathbf{0})$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \ln\big(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x)\big)\big(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x)\big)^p \,\mathrm{d}\mu(x) \\
&\le\ \varepsilon \,\langle (\tilde{H}(\mathbf{0}) + E_0)\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u, (\ \mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\ )^{p-1}\rangle_\mu + \frac{2\beta(\varepsilon)}{p}\ \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|^p_{p,\mu} \\
&+ \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|^p_{p,\mu} \ln \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p,\mu}
\end{aligned}$$

for every $p \in (2, \infty)$. We use Lemma 2.2.6 on page 67 in [4] but present an alternative proof. Since we are dealing with a probability measure $\mu$ on $\mathbb{R}^n$ the arguments can be simplified quite a bit. Nonetheless we have to prove a technical proposition first.

**Proposition 3.3.1**
*Let $p \in (2, \infty)$ and $v \in \mathcal{D}\big(\tilde{H}(\mathbf{0})\big) \cap \mathrm{L}^\infty_\mu(\mathbb{R}^n)$ be non-negative almost everywhere in $\mathbb{R}^n$. Then $v^{\frac{p}{2}}$ is an element of $\mathcal{D}\big(\tilde{h}(\mathbf{0})\big)$ which satisfies*

$$\tilde{h}(\mathbf{0})(v^{\frac{p}{2}}, v^{\frac{p}{2}}) \le \frac{p}{2}\langle (\tilde{H}(\mathbf{0}) + E_0)v, v^{p-1}\rangle_\mu.$$

*Proof.* Lemma 1.56 on page 38 in [14] states that $v^{\frac{p}{2}}$ is an element of $\mathcal{D}\big(\tilde{h}(\mathbf{0})\big)$ if and only if

$$\sup_{t>0} \frac{1}{t}\langle v^{\frac{p}{2}} - \mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}v^{\frac{p}{2}}, v^{\frac{p}{2}}\rangle_\mu$$

exists in $\mathbb{R}$. In that case

$$\tilde{h}(\mathbf{0})(v^{\frac{p}{2}}, v^{\frac{p}{2}}) = \lim_{t\to 0} \frac{1}{t}\langle v^{\frac{p}{2}} - \mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}v^{\frac{p}{2}}, v^{\frac{p}{2}}\rangle_\mu$$

follows. Please remember that $\tilde{H}(\mathbf{0}) + E_0$ is associated to $\tilde{h}(\mathbf{0})$.

i.) First we prove that $\alpha^{\frac{p}{2}}(\alpha^{\frac{p}{2}} - \beta^{\frac{p}{2}}) \leq \frac{p}{2}\ \alpha^{p-1}(\alpha - \beta)$ holds for every $\alpha, \beta \geq 0$. Without loss of generality we assume that $0 \leq \alpha < \beta$ is true and conclude that

$$\alpha^{\frac{p}{2}} - \beta^{\frac{p}{2}} = -\int_{\alpha}^{\beta} \frac{p}{2}\xi^{\frac{p}{2}-1}\mathrm{d}\xi \leq -\frac{p}{2}\alpha^{\frac{p}{2}-1}(\beta - \alpha) = \frac{p}{2}\alpha^{\frac{p}{2}-1}(\alpha - \beta).$$

The case $0 \leq \beta < \alpha$ is treated similar.

ii.) For $x \in \mathbb{R}^n$ fixed the constant function $v(x)^{\frac{p}{2}}$ is contained in $\mathrm{L}^2_\mu(\mathbb{R}^n)$ since $\mu(\mathbb{R}^n) = 1$ holds. Also the linearity of $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ gives

$$\mathrm{e}^{-t\tilde{H}(\mathbf{0})}(v(x)^{\frac{p}{2}}) = v(x)^{\frac{p}{2}}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}\mathbf{1} = v(x)^{\frac{p}{2}}$$

where we used that $\mathbf{1} \in \mathrm{L}^2_\mu(\mathbb{R}^n)$ with $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}\mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ maps $\mathbb{R}^n$ to 1. Furthermore

$$\big(\mathrm{e}^{\frac{2tE_0}{p}}v(x)\big)^{\frac{p}{2}}\big((\mathrm{e}^{\frac{2tE_0}{p}}v(x))^{\frac{p}{2}} - v^{\frac{p}{2}}\big) \leq \frac{p}{2}\ \big(\mathrm{e}^{\frac{2tE_0}{p}}v(x)\big)^{p-1}\big(\mathrm{e}^{\frac{2tE_0}{p}}v(x) - v\big)$$

is true almost everywhere in $\mathbb{R}^n$ by i.) since $v$ is non-negative. Then $v^{\frac{p}{2}} \in \mathrm{L}^2_\mu(\mathbb{R}^n)$ is true since $v \in \mathrm{L}^\infty_\mu(\mathbb{R}^n)$. So $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}v^{\frac{p}{2}}$ is well defined. Using the positivity of the operator $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ we conclude that

$$\begin{aligned}
&\big(\mathrm{e}^{\frac{2tE_0}{p}}v(x)\big)^{\frac{p}{2}}\big((\mathrm{e}^{\frac{2tE_0}{p}}v(x))^{\frac{p}{2}} - \mathrm{e}^{-t\tilde{H}(\mathbf{0})}v^{\frac{p}{2}}\big)\\
&= \big(\mathrm{e}^{\frac{2tE_0}{p}}v(x)\big)^{\frac{p}{2}}\big(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}(\mathrm{e}^{\frac{2tE_0}{p}}v(x))^{\frac{p}{2}} - \mathrm{e}^{-t\tilde{H}(\mathbf{0})}v^{\frac{p}{2}}\big)\\
&= \mathrm{e}^{-t\tilde{H}(\mathbf{0})}\Big(\big(\mathrm{e}^{\frac{2tE_0}{p}}v(x)\big)^{\frac{p}{2}}\big((\mathrm{e}^{\frac{2tE_0}{p}}v(x))^{\frac{p}{2}} - v^{\frac{p}{2}}\big)\Big)\\
&\leq \mathrm{e}^{-t\tilde{H}(\mathbf{0})}\Big(\frac{p}{2}\ \big(\mathrm{e}^{\frac{2tE_0}{p}}v(x)\big)^{p-1}\big(\mathrm{e}^{\frac{2tE_0}{p}}v(x) - v\big)\Big)\\
&= \frac{p}{2}\ \big(\mathrm{e}^{\frac{2tE_0}{p}}v(x)\big)^{p-1}\big(\mathrm{e}^{\frac{2tE_0}{p}}v(x) - \mathrm{e}^{-t\tilde{H}(\mathbf{0})}v\big)\\
&= \frac{p}{2}\ \mathrm{e}^{2tE_0}\ v(x)^{p-1}\ \big(v(x) - \mathrm{e}^{-\frac{2tE_0}{p}}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}v\big)
\end{aligned}$$

holds almost everywhere in $\mathbb{R}^n$. Then $\mathrm{e}^{-tE_0} < \mathrm{e}^{-\frac{2tE_0}{p}}$ follows from $2 < p$. Therefore we argue that

$$\big(v(x) - \mathrm{e}^{-\frac{2tE_0}{p}} \mathrm{e}^{-t\tilde{H}(\mathbf{0})} v\big) \leq \big(v(x) - \mathrm{e}^{-tE_0} \mathrm{e}^{-t\tilde{H}(\mathbf{0})} v\big)$$

since $v$ is non-negative almost everywhere in $\mathbb{R}^n$ and $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ is a positive operator. We summarize that

$$\begin{aligned}&\big(\mathrm{e}^{\frac{2tE_0}{p}} v(x)\big)^{\frac{p}{2}} \big((\mathrm{e}^{\frac{2tE_0}{p}} v(x))^{\frac{p}{2}} - \mathrm{e}^{-t\tilde{H}(\mathbf{0})} v^{\frac{p}{2}}\big)\\ &\leq \frac{p}{2}\ \mathrm{e}^{2tE_0}\ v(x)^{p-1}\ \big(v(x) - \mathrm{e}^{-tE_0} \mathrm{e}^{-t\tilde{H}(\mathbf{0})} v\big)\end{aligned}$$

is true almost everywhere in $\mathbb{R}^n$.

iii.) So we conclude with ii.) that

$$\begin{aligned}&\mathrm{e}^{2tE_0} \langle (\mathrm{Id} - \mathrm{e}^{-tE_0} \mathrm{e}^{-t\tilde{H}(\mathbf{0})}) v^{\frac{p}{2}}, v^{\frac{p}{2}} \rangle_\mu \\
&= \langle (\mathrm{e}^{tE_0} - \mathrm{e}^{-t\tilde{H}(\mathbf{0})}) v^{\frac{p}{2}}, \mathrm{e}^{tE_0} v^{\frac{p}{2}} \rangle_\mu \\
&= \langle (\mathrm{e}^{\frac{2tE_0}{p}} v)^{\frac{p}{2}} - \mathrm{e}^{-t\tilde{H}(\mathbf{0})} v^{\frac{p}{2}}, (\mathrm{e}^{\frac{2tE_0}{p}} v)^{\frac{p}{2}} \rangle_\mu \\
&= \int_{\mathbb{R}^n} \big(\mathrm{e}^{\frac{2tE_0}{p}} v(x)\big)^{\frac{p}{2}} \big((\mathrm{e}^{\frac{2tE_0}{p}} v(x))^{\frac{p}{2}} - (\mathrm{e}^{-t\tilde{H}(\mathbf{0})} v^{\frac{p}{2}})(x)\big)\ \mathrm{d}\mu(x) \\
&\leq \frac{p}{2}\ \mathrm{e}^{2tE_0} \int_{\mathbb{R}^n} v(x)^{p-1}\ \big(v(x) - (\mathrm{e}^{-tE_0} \mathrm{e}^{-t\tilde{H}(\mathbf{0})} v)(x)\big)\ \mathrm{d}\mu(x) \\
&= \frac{p}{2}\ \mathrm{e}^{2tE_0}\ \langle (\mathrm{Id} - \mathrm{e}^{-tE_0} \mathrm{e}^{-t\tilde{H}(\mathbf{0})}) v, v^{p-1} \rangle_\mu\end{aligned}$$

iv.) For $t \to 0$ the terms $\frac{1}{t}\ \langle (\mathrm{Id} - \mathrm{e}^{-tE_0} \mathrm{e}^{-t\tilde{H}(\mathbf{0})}) v, v^{p-1} \rangle_\mu$ converge to

$$\langle (\tilde{H}(\mathbf{0}) + E_0) v, v^{p-1} \rangle_\mu$$

since $v$ is an element of $\mathcal{D}\big(\tilde{H}(\mathbf{0})\big)$. Hence we conclude that

$$\sup_{t>0}\ \frac{1}{t}\ \langle (\mathrm{Id} - \mathrm{e}^{-tE_0} \mathrm{e}^{-t\tilde{H}(\mathbf{0})}) v^{\frac{p}{2}}, v^{\frac{p}{2}} \rangle_\mu$$

exists and that $v^{\frac{p}{2}}$ is an element of $\mathcal{D}\big(\tilde{h}(\mathbf{0})\big)$ with

$$\tilde{h}(\mathbf{0})(v^{\frac{p}{2}}, v^{\frac{p}{2}}) \;\leq \frac{p}{2}\langle(\tilde{H}(\mathbf{0}) + E_0)v, v^{p-1}\rangle_\mu$$

by the mentioned reference at the beginning of this proof.

□

Finally we prove Logarithmic Sobolev inequalities for $\tilde{H}(\mathbf{0})$.

**Lemma 3.3.2**
*Let $u \in \mathcal{D}\big(\tilde{H}(\mathbf{0})\big) \cap \mathrm{L}^\infty_\mu(\mathbb{R}^n)$ be non-negative almost everywhere in $\mathbb{R}^n$. Then for every $\varepsilon > 0$ and $t > 0$ we claim that*

$$\int_{\mathbb{R}^n} \ln\big(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x)\big)\big(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x)\big)^p \,\mathrm{d}\mu(x)$$
$$\leq \;\varepsilon\; \langle(\tilde{H}(\mathbf{0}) + E_0)\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u, (\;\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\;)^{p-1}\rangle_\mu + \frac{2\beta(\varepsilon)}{p}\; \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|^p_{p,\mu}$$
$$+ \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|^p_{p,\mu}\; \ln \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p,\mu}$$

*is true for every $p \in (2, \infty)$.*

*Proof.* Let $u \in \mathcal{D}\big(\tilde{H}(\mathbf{0})\big) \cap \mathrm{L}^\infty_\mu(\mathbb{R}^n)$ be non-negative almost everywhere in $\mathbb{R}^n$. Furthermore let $\varepsilon > 0$, $t > 0$ and $p \in (2, \infty)$.

i.) Then $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u \in \mathcal{D}\big(\tilde{H}(\mathbf{0})\big) \cap \mathrm{L}^\infty_\mu(\mathbb{R}^n)$ is strictly positive almost everywhere in $\mathbb{R}^n$ since $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ is positivity improving. By Proposition 3.3.1 we conclude that $(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{\frac{p}{2}} \in \mathcal{D}\big(\tilde{h}(\mathbf{0})\big)$ is true and satisfies

$$\tilde{h}(\mathbf{0})\big((\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{\frac{p}{2}}, (\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{\frac{p}{2}}\big) \;\leq\; \frac{p}{2}\,\langle(\tilde{H}(\mathbf{0}){+}E_0)\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u, (\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{p-1}\rangle_\mu$$

ii.) We use Lemma 3.2.3 for

$$\int_{\mathbb{R}^n} \ln\Big(\big(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x)\big)^{\frac{p}{2}}\Big)\; \big(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x)\big)^p \mathrm{d}\mu(x)$$

$$\leq \varepsilon\; \tilde{h}(\mathbf{0})\big((\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{\frac{p}{2}}, (\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{\frac{p}{2}}\big) + \beta(\varepsilon)\; \|(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{\frac{p}{2}}\|^2_{2,\mu}$$
$$+ \|(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{\frac{p}{2}}\|^2_{2,\mu}\; \ln \|(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{\frac{p}{2}}\|_{2,\mu}.$$

Finally using i.) we argue that

$$\int_{\mathbb{R}^n} \ln\big(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x)\big)\ \big(\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x)\big)^p \mathrm{d}\mu(x)$$

$$\leq \frac{2\varepsilon}{p}\ \tilde{h}(\mathbf{0})\big((\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{\frac{p}{2}}, (\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{\frac{p}{2}}\big) + \frac{2\beta(\varepsilon)}{p}\ \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p,\mu}^p$$
$$+ \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p,\mu}^p\ \ln \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p,\mu}$$

$$\leq \varepsilon\ \langle(\tilde{H}(\mathbf{0}) + E_0)\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u, (\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{p-1}\rangle_\mu + \frac{2\beta(\varepsilon)}{p}\ \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p,\mu}^p$$
$$+ \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p,\mu}^p\ \ln \|\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p,\mu}$$

follows which proves the claim.

□

## 3.4 The core argumentation

For proving the intrinsic ultracontractivity of $\mathrm{e}^{-tH(\mathbf{0})}$ we cite a crucial theorem from section 2.2 in [4]. Due to its technical nature we neglect the proof in this document and only give a reference instead. Then we use all the gathered information to prove the intrinsic ultracontractivity of $\mathrm{e}^{-tH(\mathbf{0})}$.

### 3.4.1 Preliminary

Let $u \in \mathcal{D}\big(\tilde{H}(\mathbf{0})\big) \cap \mathrm{L}_\mu^\infty(\mathbb{R}^n)$ be non-negative almost everywhere in $\mathbb{R}^n$ and let $p\colon [0,T) \to [2,\infty)$ be continuously differentiable for a given $T > 0$. Then $\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u$ is strictly positive and bounded almost everywhere in $\mathbb{R}^n$ for every $t \in [0,T)$. So $\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u \in \mathrm{L}_\mu^{p(t)}(\mathbb{R}^n)$ follows as $\mu(\mathbb{R}^n) = 1$. Remember that $\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ is a contraction in $\mathrm{L}_\mu^1(\mathbb{R}^n,\mathbb{R})$ as well as in $\mathrm{L}_\mu^\infty(\mathbb{R}^n,\mathbb{R})$ and therefore meets the criteria of Lemma 2.2.2 on page 64 in [4].

**Theorem 3.4.1**
*The function* $\Big\{\ t \mapsto \big\|\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\big\|_{p(t),\mu}^{p(t)}\ \Big\}$ *is differentiable in* $(0,T)$ *with a*

*derivative equal to*

$$p(t)\big\langle -(\tilde{H}(\mathbf{0})+E_0)\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u, (\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u)^{p(t)-1}\big\rangle_\mu$$
$$+\,p'(t)\int_{\mathbb{R}^n}(\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x))^{p(t)}\ \ln\big(\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x)\big)\ \mathrm{d}\mu(x)$$

### 3.4.2 Intrinsic ultracontractivity of $\mathrm{e}^{-tH(\mathbf{0})}$

Let $Q$, $P$, $\tilde{P}$ and $q$ be as in Section 3.2. First we define

$$T = 4\int_{\frac{1}{2}\ln(2)}^{\infty} P(r)^{-1}\ \mathrm{d}r\ \in (0,\infty). \tag{3.3}$$

Using the intermediate value theorem for every $t \in (0,T]$ there exists a $\xi(t) \geq 0$ such that

$$4\int_{\frac{1}{2}\ln(2)+\xi(t)}^{\infty} P(r)^{-1}\ \mathrm{d}r\ =\ t$$

holds. Mind that $\xi$ is decreasing with $\xi(t)\to\infty$ for $t \to 0$.

Let $t \in (0,T]$ be arbirary but fixed. In contrast to equation (2.2.11) on page 70 in [4] we define

$$\varepsilon_t(p) = \frac{2}{P(\frac{1}{2}\ln(p)+\xi(t))} \tag{3.4}$$

for $p \in [2,\infty)$. Compare with equation (59) on page 4120 in [1]. Then

$$\int_2^\infty \frac{\varepsilon_t(p)}{p}\ \mathrm{d}p = 2\int_2^\infty P\big(\frac{1}{2}\ln(p)+\xi(t)\big)^{-1}p^{-1}\ \mathrm{d}p = 4\int_{\frac{1}{2}\ln(2)+\xi(t)}^{\infty} P(r)^{-1}\ \mathrm{d}r = t$$

follows. Let us prove the following proposition for later use.

**Proposition 3.4.2**
*The improper integral* $\int_2^\infty \beta\big(\varepsilon_t(p)\big)p^{-2}\ \mathrm{d}p$ *exists in* $\mathbb{R}$.

*Proof.* So first we state that $\beta\big(\varepsilon_t(p)\big) = \frac{\varepsilon_t(p)}{2} - \frac{n}{4}\ln(\frac{\varepsilon_t(p)}{2}) + \tilde{P}(\frac{2}{\varepsilon_t(p)}) + C$ is given by the definition of $\beta$.

i.) It is easy to see that $0 \leq \int_2^\infty \frac{\varepsilon_t(p)}{p^2}\ \mathrm{d}p \leq \int_2^\infty \frac{\varepsilon_t(p)}{p}\ \mathrm{d}p = t$ holds.

ii.) Furthermore $\int_2^\infty Cp^{-2}\ \mathrm{d}p = \frac{C}{2}$ is true.

iii.) Now we focus on $\ln(\frac{\varepsilon_t(p)}{2})$. Using the definition of $\varepsilon_t$ we state that

$$\ln(\frac{\varepsilon_t(p)}{2}) = -\ln\big(P(\frac{1}{2}\ln(p) + \xi(t))\big)$$

is true. We use $P(r) \leq K\exp(\mathrm{e}^r)$ for $r \geq \delta$. Then

$$\ln\big(P(\frac{1}{2}\ln(2) + \xi(t))\big) \leq \ln\big(P(\frac{1}{2}\ln(p) + \xi(t))\big) \leq \ln(K) + \mathrm{e}^{\xi(t)}p^{\frac{1}{2}}$$

holds for $\max\{2, \exp\big(2(\delta - \xi(t))\big)\} \leq p$ by using the monotonicity of $P$. Without loss of generality we assume that $\exp\big(2(\delta - \xi(t))\big) \leq 2$ is satisfied. The improper integral of the upper bounding function is

$$\int_2^\infty \frac{\ln(K)}{p^2} + \mathrm{e}^{\xi(t)}p^{-\frac{3}{2}}\ \mathrm{d}p = \frac{\ln(K)}{2} + \sqrt{2}\mathrm{e}^{\xi(t)}.$$

For the lower bounding function

$$\int_2^\infty \frac{\ln\big(P(\frac{1}{2}\ln(2) + \xi(t))\big)}{p^2}\ \mathrm{d}p = \frac{\ln\big(P(\frac{1}{2}\ln(2) + \xi(t))\big)}{2}$$

is true. Hence the improper integral of $\ln(\frac{\varepsilon_t(p)}{2})p^{-2}$ satisfies

$$-\frac{\ln(K)}{2} - \sqrt{2}\mathrm{e}^{\xi(t)} \leq \int_2^\infty \ln(\frac{\varepsilon_t(p)}{2})p^{-2}\ \mathrm{d}p \leq -\frac{\ln\big(P(\frac{1}{2}\ln(2) + \xi(t))\big)}{2}.$$

iv.) Finally we focus on the improper integral of $\tilde{P}(\frac{2}{\varepsilon_t(p)})p^{-2}$ over $(2, \infty)$. The equations

$$\begin{aligned}
&\int_2^\infty \tilde{P}(\frac{2}{\varepsilon_t(p)})p^{-2}\ \mathrm{d}p \\
&= \int_2^\infty \tilde{P}\big(P(\frac{1}{2}\ln(p) + \xi(t))\big)p^{-2}\ \mathrm{d}p \\
&= \int_2^\infty \exp(\frac{1}{2}\ln(p) + \xi(t))p^{-2}\ \mathrm{d}p \\
&= \mathrm{e}^{\xi(t)} \int_2^\infty p^{-\frac{3}{2}}\ \mathrm{d}p = \sqrt{2}\mathrm{e}^{\xi(t)}
\end{aligned}$$

hold where we used $\tilde{P} \circ P = \exp$ on $\mathbb{R}$ which follows from $\tilde{P}$ being the inverse function of $P \circ \ln$ on $(0, \infty)$.

□

So far we have seen that $\int_2^\infty \beta\big(\varepsilon_t(p)\big)p^{-2}\,\mathrm{d}p$ exist in $\mathbb{R}$ for every $t \in (0,T]$. Using the estimates in the proof of Proposition 3.4.2 we show that this family of integrals is bounded from below.

**Corollary 3.4.3**
*There is a constant $C \in \mathbb{R}$ with $C \le \int_2^\infty \beta\big(\varepsilon_t(p)\big)p^{-2}\,\mathrm{d}p$ for every $t \in (0,T]$.*

*Proof.* The claim is a direct consequence of

$$-\int_2^\infty \ln\Big(\frac{\varepsilon_t(p)}{2}\Big)p^{-2}\,\mathrm{d}p = \int_2^\infty \ln\big(P(\frac{1}{2}\ln(p)+\xi_0)\big)p^{-2}\,\mathrm{d}p \ge \frac{\ln\big(P(\frac{1}{2}\ln(2))\big)}{2}$$

from the proof of Proposition 3.4.2.

□

Finally we are able to prove intrinsic ultracontractivity of $\mathrm{e}^{-tH(\mathbf{0})}$.

**Theorem 3.4.4**
*Let $n \ge 3$ and $q\colon \mathbb{R}^n \to \mathbb{R}$ satisfy (3.1). Furthermore we assume $\varphi$ to be continuous. Then $\mathrm{e}^{-tH(\mathbf{0})}$ is intrinsic ultracontractive in $\mathrm{L}^2(\mathbb{R}^n)$. In particular for every time $t > 0$ there exists a constant $C_t > 0$ satisfying*

$$\big|\mathrm{e}^{-tH(\mathbf{0})}u(x)\big| \;\le\; C_t\|u\|_2\,\varphi(x)$$

*almost everywhere in $\mathbb{R}^n$ for every $u \in \mathrm{L}^2(\mathbb{R}^n)$.*

*Proof.* Let $t \in (0,T]$ arbitrary but fixed for $T > 0$ from (3.3) and let $\varepsilon_t$ be defined by (3.4).

i.) We define $p\colon [0,t) \to [2,\infty)$ as a solution of the initial value problem

$$\begin{cases} p'(s) &= \frac{p(s)}{\varepsilon_t(p(s))} \text{ for } s \in (0,t) \\ p(0) &= 2 \end{cases}$$

by a separation of variables approach. Define $G\colon [2,\infty) \to \mathbb{R}$ as

$$G(r) = \int_2^r \frac{\varepsilon_t(s)}{s}\mathrm{d}s.$$

Then $G'(r) = \frac{\varepsilon_t(r)}{r} > 0$ is true for every $r \geq 2$. Hence $G$ is a strictly monotone increasing and continuous on $[2, \infty)$ such that $G(2) = 0$ and $G(r) \to t$ for $r \to \infty$. In particular $G$ is a bijection from $[2, \infty)$ to $[0, t)$ by the intermediate value theorem. We define $p$ as the inverse function $G^{-1}$ of $G$ on $[0, t)$. Then $p$ satisfies $p(0) = 2$ and

$$p'(s) = \frac{1}{G'\big(p(s)\big)} = \frac{p(s)}{\varepsilon_t(p(s))}$$

for every $s \in [0, t)$ by the inverse function theorem.

ii.) We define a function $N$ as

$$N(s) \;=\; 2\int_0^s \frac{\beta\big(\varepsilon_t\big(p(r)\big)\big)}{\varepsilon_t\big(p(r)\big)p(r)}\,\mathrm{d}r \;=\; 2\int_2^{p(s)} \frac{\beta\big(\varepsilon_t(q)\big)}{q^2}\,\mathrm{d}q$$

for $s \in [0, t)$ and $M(t) = \lim_{s \to t} N(s)$. Please mind Proposition 3.4.2.

iii.) Let $u \in \mathcal{D}\big(\tilde{H}(0)\big) \cap \mathrm{L}^\infty_\mu(\mathbb{R}^n, \mathbb{R})$ be non-negative almost everywhere in $\mathbb{R}^n$. By Theorem 3.4.1

$$\left\{\; s \mapsto \big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|^{p(s)}_{p(s),\mu} \;\right\}$$

is differentiable in $(0, t)$ with a derivative given by

$$\begin{aligned} &p(s)\big\langle -(\tilde{H}(\mathbf{0}) + E_0)\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u, (\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u)^{p(s)-1}\big\rangle_\mu \\ &\quad + p'(s)\int_{\mathbb{R}^n} (\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u(x))^{p(s)}\;\ln\!\big(\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u(x)\big)\,\mathrm{d}\mu(x). \end{aligned}$$

From the following equation

$$\ln\!\big(\,\mathrm{e}^{-N(s)}\big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|_{p(s),\mu}\,\big) = -N(s) + \frac{1}{p(s)}\ln\big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|^{p(s)}_{p(s),\mu}$$

we infer that its derivative is given by

$$\begin{aligned} &-\frac{2\beta\big(\varepsilon_t\big(p(s)\big)\big)}{\varepsilon_t\big(p(s)\big)p(s)} - \frac{1}{p(s)\varepsilon_t\big(p(s)\big)}\;\ln\big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|^{p(s)}_{p(s),\mu} \\ &+ \frac{1}{p(s)}\;\big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|^{-p(s)}_{p(s),\mu}\;\frac{\mathrm{d}}{\mathrm{d}s}\big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|^{p(s)}_{p(s),\mu} \end{aligned}$$

which is equal to

$$
\begin{aligned}
&\frac{1}{\varepsilon_t\big(p(s)\big)} \ \big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|_{p(s),\mu}^{-p(s)} \\
&\bigg\{ \int_{\mathbb{R}^n} (\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u(x))^{p(s)} \ \ln\!\big(\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u(x)\big) \ \mathrm{d}\mu(x) \\
&- \varepsilon_t\big(p(s)\big) \ \big\langle(\tilde{H}(\mathbf{0}) + E_0)\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u, (\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u)^{p(s)-1}\big\rangle_\mu \\
&- \frac{2\beta\big(\varepsilon_t\big(p(s)\big)\big)}{p(s)} \ \big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|_{p(s),\mu}^{p(s)} \\
&- \big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|_{p(s),\mu}^{p(s)} \ \ln\big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|_{p(s),\mu}\bigg\}.
\end{aligned}
$$

We use Logarithmic Sobolev inequalities for $\tilde{H}(\mathbf{0})$ on $\mathrm{e}^{-sE_0}u$ instead of $u$. Please compare with Lemma 3.3.2 and mind that

$$
\mathrm{e}^{-sE_0}u \in \mathcal{D}\big(\tilde{H}(0)\big) \cap \mathrm{L}^\infty_\mu(\mathbb{R}^n, \mathbb{R})
$$

is true. Therefore

$$
\begin{aligned}
&\bigg\{ \int_{\mathbb{R}^n} (\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u(x))^{p(s)} \ \ln\!\big(\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u(x)\big) \ \mathrm{d}\mu(x) \\
&- \varepsilon_t\big(p(s)\big) \ \big\langle(\tilde{H}(\mathbf{0}) + E_0)\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u, (\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u)^{p(s)-1}\big\rangle_\mu \\
&- \frac{2\beta\big(\varepsilon_t\big(p(s)\big)\big)}{p(s)} \ \big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|_{p(s),\mu}^{p(s)} \\
&- \big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|_{p(s),\mu}^{p(s)} \ \ln\big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|_{p(s),\mu}\bigg\} \leq 0
\end{aligned}
$$

follows and hence we conclude that

$$
\frac{\mathrm{d}}{\mathrm{d}s} \ln\big( \ \mathrm{e}^{-N(s)}\big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|_{p(s),\mu} \ \big) \leq 0
$$

is true on $[0, t)$. So

$$
\mathrm{e}^{-N(s)}\big\|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\big\|_{p(s),\mu} \ \leq \ \mathrm{e}^{-N(0)}\|u\|_{p(0),\mu} = \|u\|_{2,\mu}
$$

follows for every $s \in [0, t)$ due to monotonicity.

iv.) Mind that for $r > 0$ the operator $\mathrm{e}^{-rE_0}\mathrm{e}^{-r\tilde{H}(\mathbf{0})}$ is a contraction in $\mathrm{L}^1_\mu(\mathbb{R}^n,\mathbb{R})$ as well as in $\mathrm{L}^\infty_\mu(\mathbb{R}^n,\mathbb{R})$. For every $p \in [1,\infty]$ we use the Riesz-Thorin interpolation theorem to uniquely extend $\mathrm{e}^{-rE_0}\mathrm{e}^{-r\tilde{H}(\mathbf{0})}$ to a contraction in $\mathrm{L}^p_\mu(\mathbb{R}^n,\mathbb{R})$. For $s \in [0,t)$ we argue that

$$\begin{aligned}\|\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p(s),\mu} &= \|\mathrm{e}^{-(t-s)E_0}\mathrm{e}^{-(t-s)\tilde{H}(\mathbf{0})}\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\|_{p(s),\mu} \\ &\leq \|\mathrm{e}^{-(t-s)E_0}\mathrm{e}^{-(t-s)\tilde{H}(\mathbf{0})}\|_{p(s)\to p(s),\mu}\ \|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\|_{p(s),\mu} \\ &\leq \|\mathrm{e}^{-sE_0}\mathrm{e}^{-s\tilde{H}(\mathbf{0})}u\|_{p(s),\mu} \leq \mathrm{e}^{N(s)}\ \|u\|_{2,\mu}\end{aligned}$$

is true where $\|\mathrm{e}^{-(t-s)E_0}\mathrm{e}^{-(t-s)\tilde{H}(\mathbf{0})}\|_{p(s)\to p(s),\mu} \leq 1$ holds for the operator norm in $\mathrm{L}^{p(s)}_\mu(\mathbb{R}^n,\mathbb{R})$.

v.) Since $u$ is bounded almost everywhere in $\mathbb{R}^n$ so is $\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u$. Hence for every $t \in (0,T]$ we infer

$$\begin{aligned}\|\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{\infty,\mu} &= \lim_{s\to t}\|\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{p(s),\mu} \\ &\leq \lim_{s\to t}\mathrm{e}^{N(s)}\ \|u\|_{2,\mu} = \mathrm{e}^{M(t)}\ \|u\|_{2,\mu}\end{aligned}$$

is true. For $t > T$ we argue that

$$\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x) = \mathrm{e}^{-TE_0}\mathrm{e}^{-T\tilde{H}(\mathbf{0})}\Big(\mathrm{e}^{-(t-T)E_0}\mathrm{e}^{-(t-T)\tilde{H}(\mathbf{0})}u\Big)(x)$$

holds almost everywhere in $\mathbb{R}^n$ and so

$$\left|\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u(x)\right| \leq \mathrm{e}^{M(T)}\ \|\mathrm{e}^{-(t-T)E_0}\mathrm{e}^{-(t-T)\tilde{H}(\mathbf{0})}u\|_{2,\mu} \leq \mathrm{e}^{M(T)}\ \|u\|_{2,\mu}$$

follows almost everywhere in $\mathbb{R}^n$ in this case as well. So in general we conclude that for every $t > 0$ there exists a constant $C_t > 0$ that

$$\|\mathrm{e}^{-tE_0}\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\|_{\infty,\mu} \leq\ C_t\ \|u\|_{2,\mu}$$

is true. Using the definition of $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ we get

$$\left|\mathrm{e}^{-tH(\mathbf{0})}(\varphi u)(x)\right|\ \leq\ C_t\ \varphi(x)\ \|\varphi u\|_2 \tag{3.5}$$

almost everywhere in $\mathbb{R}^n$.

vi.) In v.) we have proven

$$|\mathrm{e}^{-tH}(\varphi u)(x)| \;\leq\; C_t\varphi(x)\|\varphi u\|_2$$

for every $u \in D(\tilde{H}) \cap \mathrm{L}^\infty_\mu(\mathbb{R}^n, \mathbb{R})$ being non-negative almost everywhere in $\mathbb{R}^n$. Using the definition of $D(\tilde{H})$ which is $D(\tilde{H}) = \varphi^{-1}D(H)$ we rewrite to

$$|\mathrm{e}^{-tH}(v)(x)| \;\leq\; C_t\varphi(x)\|v\|_2$$

for every $v \in D(H)$ being non-negative such that $\frac{1}{\varphi}v$ is bounded almost everywhere. Using the continuity and positivity of $\varphi$ we state $\varphi^{-1}$ is bounded on every compact set $M$ in $\mathbb{R}^n$. So for every non-negative function $v$ in $C_c^\infty(\mathbb{R}^n) \subseteq D(H)$ we conclude that $\frac{1}{\varphi}v$ is bounded. Hence

$$|\mathrm{e}^{-tH}(v)(x)| \leq C_t\varphi(x)\|v\|_2$$

holds for every non-negative $v \in C_c^\infty(\mathbb{R}^n)$. The inequality follows for every non-negative function in $\mathrm{L}^2(\mathbb{R}^n)$ as well by density argumentation.

vii.) Now let $u \in \mathrm{L}^2(\mathbb{R}^n, \mathbb{R})$ but not necessarily non-negative almost everywhere in $\mathbb{R}^n$. But since $u$ is real-valued we use the positive and negative parts $u^+$ and $u^-$ of $u$. These are functions in $\mathrm{L}^2(\mathbb{R}^n, \mathbb{R})$ which are non-negative almost everywhere in $\mathbb{R}^n$. Using the linearity and positivity of the operator $\mathrm{e}^{-tH(\mathbf{0})}$ we see that $\left(\mathrm{e}^{-tH(\mathbf{0})}u\right)^+ = \mathrm{e}^{-tH(\mathbf{0})}u^+$ is true which means that the positive part of $\mathrm{e}^{-tH(\mathbf{0})}u$ is given by $\mathrm{e}^{-tH(\mathbf{0})}u^+$. A similar argument can be made for the negative part of $\mathrm{e}^{-tH(\mathbf{0})}u$. So the claimed inequality holds also in the case of a general $u \in \mathrm{L}^2(\mathbb{R}^n, \mathbb{R})$.

viii.) Finally we let $u \in \mathrm{L}^2(\mathbb{R}^n)$ but not necessarily be real-valued. Then the argumentation is similar to x.) since the real and imaginary parts of $u$ are functions in $\mathrm{L}^2(\mathbb{R}^n, \mathbb{R})$. This time we use the fact that $\mathrm{e}^{-tH(\mathbf{0})}$ is linear and real to argue that the claim follows also in the most general case. Therefore the theorem is proven.

□

Let us prove a consequence of Theorem 3.4.4 for eigenfunctions of $H(\mathbf{0})$.

**Corollary 3.4.5**
*Let $H(\mathbf{0})$ be defined as in Theorem 3.4.4 and $u \in \mathrm{L}^2(\mathbb{R}^n)$ be a normed eigenfunction of $H(\mathbf{0})$ to an eigenvalue $\lambda \geq 0$. Then there is a $C(\lambda) > 0$ depending only on the energy level $\lambda$ such that $|u(x)| \leq C(\lambda)\varphi(x)$ holds almost everywhere in $\mathbb{R}^n$.*

*Proof.* For a fixed $t \in (0, T]$ we use v.) in the proof of Theorem 3.4.4 to conclude that $\left|\mathrm{e}^{-tH(\mathbf{0})}u(x)\right| \;\leq\; \mathrm{e}^{M(t)}\varphi(x)$ holds almost everyhwhere in $\mathbb{R}^n$. Then

$$\frac{1}{\varphi(x)}|u(x)| \;\leq\; \mathrm{e}^{M(t)+\lambda t} \;\leq\; \mathrm{e}^{M(t)}\mathrm{e}^{\lambda T}$$

is true almost everyhwhere in $\mathbb{R}^n$ since $u$ is an eigenfunction of $H(\mathbf{0})$ to $\lambda$. Mind that the left hand side does not depend on $t$ anymore. For an arbitrarily chosen but fixed $x$ not in a null set we infer that

$$\frac{1}{\varphi(x)}|u(x)| \;\leq\; \inf_{t\in(0,T]} \mathrm{e}^{M(t)}\mathrm{e}^{\lambda T} = \mathrm{e}^{\lambda T}\exp(\inf_{t\in(0,T]} M(t))$$

is true where we used that $\inf_{t\in(0,T]} M(t)$ exists due to Corollary 3.4.3. The right hand side is completely independent of $x$ so it serves as an upper bound of the function $\varphi^{-1}|u|$ almost everywhere on $\mathbb{R}^n$ which proves the claim.

□

# Chapter 4

# Some pitfalls in the magnetic case

In the case of a magnetic vector potential $\mathbf{a}$ being non-zero on $\mathbb{R}^n$ we defined magnetic Schrödinger operators $H(\mathbf{a})$. These operators are self-adjoint in $\mathrm{L}^2(\mathbb{R}^n)$ for real-valued $q$ with a spectrum that contains at most countably many eigenvalues with a ground state energy $E_0$ if $q$ satisfies a given growth condition. Unfortunately in the magnetic case we no longer have a generalization of the theory of O. Perron and F.G. Frobenius at our hands. Hence we might not have a one-dimensional eigenspace of $E_0$ with a non-negative ground state $\varphi$. This creates problems when referring to a possible intrinsic ultracontractivity of $\mathrm{e}^{-tH(\mathbf{a})}$ since the ground state might not be unique any more. There might exist eigenfunctions $\varphi$ to $E_0$ that satisfy

$$|\mathrm{e}^{-tH(\mathbf{a})}u(x)| \ \leq C_t\|u\|_2\ |\varphi(x)|$$

almost everywhere in $\mathbb{R}^n$ and some that don't. Furthermore we used weighted Schrödinger semigroups which required $\varphi$ to be non-zero almost everywhere in $\mathbb{R}^n$. There is no general theory that guarantees the existence of such $\varphi$ in the magnetic case but specific examples are known as mentioned in Remark 2.3.7. We proceed assuming there exists at least one ground state $\varphi$ to $H(\mathbf{a})$ that is non-zero almost everywhere in $\mathbb{R}^n$.

We define weighted $\mathrm{L}^2_\mu(\mathbb{R}^n)$ spaces as we did in the non-magnetic case. Also the definition of weighted Schrödinger semigroups $\mathrm{e}^{-t\tilde{H}(\mathbf{a})}$ and forms $\tilde{h}(\mathbf{a})$ are done in the same way. One major pitfall in the magnetic case is the proof of $\mathrm{e}^{-t\tilde{H}(\mathbf{a})}$ being a contraction in $\mathrm{L}^1_\mu(\mathbb{R}^n)$. First of all let us understand that these operators might not be real anymore since neither $\mathrm{e}^{-tH(\mathbf{a})}$ needs to be real as we mention in Remark 2.3.7 nor $\varphi$ has to be a real-valued function. Therefore we are dealing with contractions in $\mathrm{L}^1_\mu(\mathbb{R}^n)$ instead of $\mathrm{L}^1_\mu(\mathbb{R}^n,\mathbb{R})$. But this is only a minor concern. The much bigger problem occurs in the proof of Lemma 3.1.6 where

$$\int_{\mathbb{R}^n} |\mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\ (x)|\ \mathrm{d}\mu(x) = \int_{\mathbb{R}^n} \mathrm{e}^{-t\tilde{H}(\mathbf{0})}u\ (x)\ \mathrm{d}\mu(x)$$

$$= \mathrm{e}^{tE_0}\big\langle \mathrm{e}^{-tH(\mathbf{0})}\varphi u, \varphi\big\rangle = \mathrm{e}^{tE_0}\big\langle \varphi u, \mathrm{e}^{-tH(\mathbf{0})}\varphi\big\rangle$$

$$= \mathrm{e}^{tE_0}\big\langle \varphi u, \mathrm{e}^{-tE_0}\varphi\big\rangle = \big\langle \varphi u, \varphi\big\rangle = \|u\|_{1,\mu}$$

holds for $u \in \mathrm{L}^2_\mu(\mathbb{R}^n,\mathbb{R}) \subseteq \mathrm{L}^1_\mu(\mathbb{R}^n,\mathbb{R})$ being non-negative almost everywhere in $\mathbb{R}^n$. We directly see that $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ being a positivite operator is needed to get the desired result. In the magnetic case we do not have this property in general. But Lemma 3.1.6 is crucial for the proof of the intrinsic ultra-contractivity of $\mathrm{e}^{-tH(\mathbf{a})}$. In the proof of the Theorem 3.4.4 we used Theorem 3.4.1 as well as the Riesz-Thorin interpolation theorem for $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$. For both of these arguments $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ has to be at least bounded in $\mathrm{L}^1_\mu(\mathbb{R}^n,\mathbb{R})$. The diamagnetic inequalities in Lemma 2.2.3 might be helpful to prove Lemma 3.1.6 in the magnetic case since

$$|\mathrm{e}^{-t\tilde{H}(\mathbf{a})}\varphi u(x)| \leq (\mathrm{e}^{-t\tilde{H}(\mathbf{0})}|\varphi u|)(x)$$

holds almost everywhere in $\mathbb{R}^n$. Let us also define the probability measure $\mu$ by $|\varphi|^2$ instead of $\varphi^2$. Then we argue that

$$\int_{\mathbb{R}^n} |\mathrm{e}^{-t\tilde{H}(\mathbf{a})}u\ (x)|\ \mathrm{d}\mu(x) = \int_{\mathbb{R}^n} \mathrm{e}^{tE_0}|(\mathrm{e}^{-tH(\mathbf{a})}\varphi u)(x)|\ |\varphi(x)|\ \mathrm{d}x$$

$$\leq \mathrm{e}^{tE_0}\int_{\mathbb{R}^n} (\mathrm{e}^{-tH(\mathbf{0})}|\varphi u|)(x)\ |\varphi(x)|\ \mathrm{d}x$$

$$= \mathrm{e}^{tE_0}\big\langle \mathrm{e}^{-tH(\mathbf{0})}|\varphi u|, |\varphi|\big\rangle = \mathrm{e}^{tE_0}\big\langle |\varphi u|, \mathrm{e}^{-tH(\mathbf{0})}|\varphi|\big\rangle$$

is true. An inequality such that $\mathrm{e}^{-tH(\mathbf{0})}|\varphi| \leq C_t|\varphi|$ holds almost everywhere in $\mathbb{R}^n$ for a constant $C_t > 0$ is desired but missing. In the non-magnetic case this was almost trivial since $\varphi$ was positive and an eigenfunction of $H(\mathbf{0})$ to $E_0$. But here we cannot assume that $|\varphi|$ is an eigenfunction of $H(\mathbf{0})$.

But it might be soothing to hear that our new version of Rosen's lemma works for the magentic case as well. Therefore we get a pre version of Logarithmic Sobolev inequalities for $\tilde{H}(\mathbf{a})$. Unfortunately Proposition 3.3.1 requires $\mathrm{e}^{-t\tilde{H}(\mathbf{0})}$ to be a positive operator. Hence we cannot expand Lemma 3.3.2 to the magnetic case using the same arguments as in the non-magnetic case. But there is another issue with Logarithmic Sobolev inequalities for $\tilde{H}(\mathbf{a})$. We have to give meaning to the term $\ln(\mathrm{e}^{-t\tilde{H}(\mathbf{a})}u)$ when $\mathrm{e}^{-t\tilde{H}(\mathbf{a})}u$ is no longer positive but complex-valued. Here the complex logarithm might offer an alternative but we have to argue that $(\mathrm{e}^{-t\tilde{H}(\mathbf{a})}u)(x) \neq 0$ holds almost everywhere in $\mathbb{R}^n$. The problem of $\mathrm{e}^{-t\tilde{H}(\mathbf{a})}u$ being no longer positive reoccurs in Theorem 3.4.1.

We conclude that the proof of intrinsic ultracontractivity of $\mathrm{e}^{-tH(\mathbf{a})}$ might need a different strategy after all. Nevertheless we offer a trivial result using the diamagnetic inequalities in Lemma 2.2.3 and call this property a quasi intrinsic ultracontractivity of $\mathrm{e}^{-tH(\mathbf{a})}$.

**Lemma 4.0.1**
*Let $n \geq 3$ and $q\colon \mathbb{R}^n \to \mathbb{R}$ be continuous and satisfy (3.1). Furthermore let $\varphi$ be the ground state of $H(\mathbf{0})$ instead of $H(\mathbf{a})$.*
*Then there exists a constant $C_t > 0$ that satisfies*

$$\left|\mathrm{e}^{-tH(\mathbf{a})}u(x)\right| \;\leq\; C_t \|u\|_2 \, \varphi(x)$$

*almost everywhere in $\mathbb{R}^n$ for every $u \in \mathrm{L}^2(\mathbb{R}^n)$ and $t > 0$.*

# Appendix A

# Rosen's Lemma in [1]

In Section 3.2 Rosen inequalities are taken from Proposition 5.1 in [1]. Therefore this appendix is purely informative.

We will define a class of electric potentials $q$ such that the ground state $\varphi$ of the operator $H = -\Delta + q(x)$ in $\mathrm{L}^2(\mathbb{R}^n)$ satisfies the Rosen inequalities

$$-\ln(\varphi(x)) \ \leq \ \varepsilon q(x) + \gamma(\varepsilon)$$

for every $x \in \mathbb{R}^n$ and an arbitrarily chosen $\varepsilon > 0$ with an associated $\gamma(\varepsilon) > 0$.

## A.1 The class of electric potentials $q$

In [1] the potentials $q$ that offer Rosen inequalities are characterized by an auxiliary function $Q\colon [0,\infty) \to (0,\infty)$ being continuous such that

$$\left(\int_{r_0}^{r} Q(t)^{\frac{1}{2}} \, \mathrm{d}t\right) P\left[\ln\left(\int_{r_0}^{r} Q(t)^{\frac{1}{2}} \, \mathrm{d}t\right)\right] \ \leq \ Q(r)$$

holds for $r_0 \leq r$ where $r_0 \in [0,\infty)$ is given and $P\colon \mathbb{R} \to (0,\infty)$ is a strictly monotone increasing function satisfying $P(\mathbb{R}) = (0,\infty)$ and

$$\int_0^\infty P(t)^{-1} \, \mathrm{d}t \ < \ \infty.$$

Also let $Q$ satisfy

$$\int_{r_0}^{\infty} |Q'(t)|^{\gamma}\ Q(t)^{\frac{1}{2}-\frac{3}{2}\gamma}\ \mathrm{d}t\ <\ \infty$$

for a constant $\gamma \in (1,2]$. Please mind the comment given in the paragraph on the top of page 4101 in [1] regarding the exponent $\frac{1}{2}-\frac{3}{2}\gamma$. By Proposition 2.2 a) on page 4102 in [1]

$$\int_{r_0}^{\infty} Q(t)^{-\frac{1}{2}}\ \mathrm{d}t < \infty$$

is implied. Typical examples of such auxiliary functions $Q$ are

$$Q(r) = r^{2+\delta},\ r^2(\ln r)^{2+\delta},\ r^2(\ln r)^2(\ln(\ln r))^{2+\delta}$$

for $r_0 \leq r$ or more generally $Q(r) = r^2(\ln r)^2(\ln\ln r)^2 \ldots (\ln^m(r))^{2+\delta}$ where

$$\ln^m(r) = \underbrace{\ln\ln\ln\ldots\ln}_{m\ \text{times}}(r).$$

For details please consider Example 2.5 on page 4105 and Proposition 2.2 on page 4102 in [1]. In the following we always consider $q\colon \mathbb{R}^n \to [0,\infty)$ as a continuous function satisfying

$$\left(\int_{r_0}^{|x|_2} Q(t)^{\frac{1}{2}}\ \mathrm{d}t\right) P\left[\ln\left(\int_{r_0}^{|x|_2} Q(t)^{\frac{1}{2}}\ \mathrm{d}t\right)\right]\ \leq\ q(x)\ \leq\ Q(|x|_2). \quad \text{(A.1)}$$

**Remark A.1.1**
*Please compare with Remark A.3.2 as well. While the argument for the Hartman-Wintner asymptotic formula works for any $\gamma \in (1,2]$ we might need to demand $\gamma = 2$ to argue that the function $\psi$ in the following subsections is contained in $H^1(\Omega_R)$ where the set*

$$\Omega_R = \{x \in \mathbb{R}^n :\ |x|_\infty \geq R\}$$

*and $R$ are chosen in the Hartman-Wintner context. In particular $\psi \in H^1(\Omega_R)$ is needed in the final Theorem A.3.1 to argue that $v^-$ is in the domain $D(h_{2Q})$. For that reason the class admissible potentials might even be smaller than cited above.*

## A.2 The Hartman-Wintner asymptotic formula

For the auxiliary bounding function $Q$ we define the operator

$$H_Q = -\Delta + Q(|x|_2)$$

in $L^2(\mathbb{R}^n)$ where $E_Q \geq 0$ is the ground state energy of $H_Q$. Ideally we are looking for a radial symmetric solution $\psi$ to the following eigenvalue equation

$$(H_Q - E_Q)\psi = 0.$$

Using the Laplace operator in spherical coordinates we end up with the ordinary differential equation

$$-\psi''(r) - \frac{n-1}{r}\psi'(r) + (Q(r) - E_Q)\psi(r) = 0 \tag{A.2}$$

for $r > 0$. With this ODE in mind we want to use the Hartman-Wintner formula ($\delta$) on page 80 in [9]. We need to transform (A.2) to the form of

$$f''(r) - V(r)f(r) = 0$$

where $V\colon [0,\infty) \to (0,\infty)$ is continuous such that

$$\int_0^\infty |V'(t)|^\gamma \; V(t)^{\frac{1}{2}-\frac{3}{2}\gamma} \,\mathrm{d}t \;<\; \infty$$

holds. For a solution $\psi$ of (A.2) we take $f(r) = r^\alpha \psi(r)$ to get

$$\begin{aligned} f''(r) &= r^\alpha \left(\psi''(r) + 2\frac{\alpha}{r}\psi'(r)\right) + \alpha(\alpha-1)r^{\alpha-2}\psi(r) \\ &= r^\alpha \left((Q(r) - E_Q)\psi(r) - \frac{n-1}{r}\psi'(r) + 2\frac{\alpha}{r}\psi'(r)\right) + \alpha(\alpha-1)r^{\alpha-2}\psi(r). \end{aligned}$$

We choose $\alpha = \frac{n-1}{2}$ to end up with

$$f''(r) = \left(Q(r) - E_Q + \frac{(n-1)(n-3)}{4r^2}\right) f(r).$$

Now we need to make further adjustments such that the term

$$Q(r) - E_Q + \frac{(n-1)(n-3)}{4r^2}$$

meets all the required properties of $V$. From

$$\int_{r_0}^{\infty} Q(t)^{-\frac{1}{2}} \, \mathrm{d}t < \infty$$

we conclude that $Q(t) \to \infty$ follows for $t \to \infty$. So we argue that

$$\exists R > 0 \; \forall r \in [0, \infty): \; \frac{1}{2} Q(r+R) \geq E_Q \text{ and } Q(r+R) \geq 1$$

is true. We define the auxiliary potential $V$ by

$$V(r) = Q(r+R) - E_Q + \frac{(n-1)(n-3)}{4(r+R)^2}$$

for $r \geq 0$. So $V$ is continuous and strictly positive on $[0, \infty)$. We have to prove that

$$\int_0^{\infty} |V'(t)|^{\gamma} \, V(t)^{\frac{1}{2} - \frac{3}{2}\gamma} \, \mathrm{d}t \; < \; \infty$$

holds. Therefore we start by stating that

$$V(r)^{\frac{1}{2} - \frac{3}{2}\gamma} \leq 2^{\frac{3}{2}\gamma - \frac{1}{2}} \; Q(r+R)^{\frac{1}{2} - \frac{3}{2}\gamma}$$

follows for $r \geq 0$ from $V(r) \geq \frac{1}{2} Q(r+R)$ and $\gamma \geq 1$. Then

$$\begin{aligned} |V'(r)|^{\gamma} &\leq \left( |Q'(r+R)| + \frac{(n-1)(n-3)}{(r+R)^3} \right)^{\gamma} \\ &\leq 2^{\gamma} \max \left\{ |Q'(r+R)|, \frac{(n-1)(n-3)}{(r+R)^3} \right\}^{\gamma} \\ &\leq 2^{\gamma} \left( |Q'(r+R)|^{\gamma} + \left( \frac{(n-1)(n-3)}{(r+R)^3} \right)^{\gamma} \right). \end{aligned}$$

Hence

$$\int_0^{\infty} |V'(t)|^{\gamma} \, V(t)^{\frac{1}{2} - \frac{3}{2}\gamma} \, \mathrm{d}t \leq \; C(\gamma) \left( \int_R^{\infty} |Q'(r)|^{\gamma} Q(r)^{\frac{1}{2} - \frac{3}{2}\gamma} + \int_R^{\infty} r^{-3\gamma} \, \mathrm{d}r \right) < \infty$$

is true. Now having found a proper definition of $V$ we forget about $\psi$ and (A.2) for a moment and focus on a solution $f$ of

$$f''(r) - V(r) f(r) = 0. \tag{A.3}$$

We use ($\delta$) on page 80 in [9] to conclude that there exists a solution

$$f \in C^2([0,\infty),\mathbb{R})$$

to (A.3) such that

$$f(r) = cV(r)^{-\frac{1}{4}} \exp\left(h(r) - \int_0^r V(t)^{\frac{1}{2}} \, \mathrm{d}t\right)$$

holds for $r \geq 0$ where $c \in \mathbb{R}$ is a constant and $h$ is a function with $h(r) \to 0$ for $r \to \infty$. We define a function

$$\psi(x) = |x|_2^{\frac{1-n}{2}} f(|x|_2 - R)$$

for every $x \in \mathbb{R}^n$ such that $|x|_2 \geq R$. Then

$$-\Delta\psi(x) + Q(|x|_2)\psi(x) - E_Q\psi(x) = 0$$

holds pointwise for every $x \in \mathbb{R}^n$ such that $|x|_2 \geq R$ (see Proposition B.0.1). Due to the explicit form of the solution $\psi$ we can use Corollary 4.6 on page 4113 in [1] and state that there are constants $c_1, c_2 > 0$ such that

$$-\ln(\psi(x)) \;\leq\; c_1 \int_{r_0}^{|x|_2} Q(t)^{\frac{1}{2}} \, \mathrm{d}t + c_2$$

is true for $|x|_2 \geq R$. Furthermore using equation (49) on page 4118 we end up with

$$-\ln(\psi(x)) \;\leq\; \varepsilon q(x) + c_1 \tilde{P}\left(\frac{c_1}{\varepsilon}\right) + c_2 \tag{A.4}$$

for $|x|_2 \geq R$. So $\psi$ satisfies Rosen inequalities. But be careful as $\psi$ is not the ground state $\varphi$. We need a comparison argument of the two functions.

## A.3 A comparison argument

Next we focus on showing that there exists a constant $C > 0$ such that

$$-\ln(\varphi(x)) \;\leq\; -\ln(\psi(x)) + C$$

holds for $|x|_\infty \geq R$. Please mind some of the following argumentation is put in the appendix. Also the following is taken from Lemma 4.1 on page 4110 in

[1]. However more details are presented here. Let us consider $\psi$ as a solution of

$$-\Delta\psi(x) + 2Q(|x|_2)\psi(x) - E_{2Q}\psi(x) = 0$$

for $|x|_2 \geq R$ in the context of the last subsection. Please mind that

$$q(x) \leq Q(|x|_2) \leq 2Q(|x|_2)$$

is obviously true as well. We understand the reason for this minor change from $Q$ to $2Q$ later. We start by defining the set

$$\Omega_R = \{x \in \mathbb{R}^n : \ |x|_\infty > R\}$$

and $v(x) = \gamma\varphi(x) - \psi(x)$ for $x \in \overline{\Omega_R}$ where $\gamma$ is given by

$$\gamma = \sup_{|x|_\infty = R} \frac{\psi(x)}{\varphi(x)} \in (0, \infty).$$

Please mind that $\varphi$ is strictly positive and contained $C(\mathbb{R})$ due to Remark 4.3 in [1]. Also $\gamma$ is defined a little different to the proof of Lemma 4.1 in [1] for technical reasons only.

**Theorem A.3.1**
*For every $x \in \Omega_R$ we conclude*

$$-\ln(\varphi(x)) \ \leq \ -\ln(\psi(x)) + \ln(\gamma).$$

*Proof.* First let us write $v = v^+ - v^-$ where

$$0 \leq v^-(x) = \max\{-v(x), 0\}$$

for $x \in \overline{\Omega_R}$. Mind that $v^-$ is 0 for every $x \in \mathbb{R}^n$ such that $|x|_\infty = R$. This is due to $v(x) = \gamma\varphi(x) - \psi(x) \geq 0$ being true for every $|x|_\infty = R$ using the definition of $\gamma$. Setting $v^-$ to 0 in $[-R, R]^n$ we conclude that $v^- \in H^1(\mathbb{R}^n)$ by Proposition B.0.3. Using Lemma 7.6 on page 152 in [6] we argue that

$$\langle \nabla v, \nabla v^- \rangle = \underbrace{\langle \nabla v^+, \nabla v^- \rangle}_{=0} - \langle \nabla v^-, \nabla v^- \rangle = -\|\nabla v^-\|^2$$

is true. From $\gamma\varphi > 0$ we infer that $0 \leq v^- \leq \psi$ holds in $\Omega_R$. From Proposition B.0.5 we state that $Q(|x|_2)^{\frac{1}{2}}\psi$ is contained in $\mathrm{L}^2(\Omega_R)$ which implies

$$Q(|x|_2)^{\frac{1}{2}} v^- \in \mathrm{L}^2(\mathbb{R}^n).$$

So

$$\langle (2Q(|x|_2))^{\frac{1}{2}} v, (2Q(|x|_2))^{\frac{1}{2}} v^- \rangle = -\langle (2Q(|x|_2))^{\frac{1}{2}} v^-, (2Q(|x|_2))^{\frac{1}{2}} v^- \rangle.$$

Also $v^-$ is contained in the domain $D(h_{2Q})$ of the form $h_{2Q}$ associated to the operator $H_{2Q} = -\Delta + 2Q(|x|_2)$. We conclude that

$$\begin{aligned}
&\langle \nabla v, \nabla v^- \rangle + \langle (2Q(|x|_2))^{\frac{1}{2}} v, (2Q(|x|_2))^{\frac{1}{2}} v^- \rangle - E_{2Q}\langle v, v^- \rangle \\
&= \gamma \left( \langle \nabla \varphi, \nabla v^- \rangle + \langle \underbrace{(Q(|x|_2) - E_{2Q})}_{\geq 0} \varphi, v^- \rangle + \langle \underbrace{Q(|x|_2)}_{\geq q(x)} \varphi, v^- \rangle \right) \\
&- \left( \langle \nabla \psi, \nabla v^- \rangle + \langle 2Q(|x|_2)\psi, v^- \rangle - E_{2Q}\langle \psi, v^- \rangle \right)
\end{aligned}$$

holds where we choose $R > 0$ large such that $Q(|x|_2) \geq E_{2Q}$ holds for every $|x|_2 \geq R$. Here we also see the reason for the slight technical change going from $Q$ to $2Q$. Since $v^-$ is equal to 0 in $[-R, R]^n$ we conclude that

$$\begin{aligned}
&\langle \nabla v, \nabla v^- \rangle + \langle (2Q(|x|_2))^{\frac{1}{2}} v, (2Q(|x|_2))^{\frac{1}{2}} v^- \rangle - E_{2Q}\langle v, v^- \rangle \\
&\geq \gamma \left( \langle \nabla \varphi, \nabla v^- \rangle + \langle q(x)\varphi, v^- \rangle \right) - \langle \underbrace{\Delta \psi + (2Q(|x|_2) - E_{2Q})\psi}_{=0}, v^- \rangle \\
&= \gamma \langle H(\mathbf{0})\varphi, v^- \rangle = \gamma E_0 \langle \varphi, v^- \rangle \geq 0
\end{aligned}$$

is true. But also

$$\begin{aligned}
&\langle \nabla v, \nabla v^- \rangle + \langle (2Q(|x|_2))^{\frac{1}{2}} v, (2Q(|x|_2))^{\frac{1}{2}} v^- \rangle - E_{2Q}\langle v, v^- \rangle \\
&= -h_{2Q}(v^-, v^-) + E_{2Q}\|v^-\|_2^2.
\end{aligned}$$

Then $h_{2Q}(v^-, v^-) \leq E_{2Q}\|v^-\|_2^2$ follows. On page 4112 in [1] in the proof of Lemma 4.1 is argued that the ground state $\varphi_{2Q}$ is the unique minimizer to the form $h_{2Q}$ up to a scalar multiple. So there exists a scalar $c \in \mathbb{C}$ such that $v^- = c\varphi_{2Q}$. Since $\varphi_{2Q} > 0$ holds almost everywhere in $\mathbb{R}^n$ and $v^- = 0$ in $[-R, R]^n$ it is concluded that $c = 0$ and therefore $v^- = 0$ in $\mathbb{R}^n$. So $v \geq 0$ holds almost everywhere in $\Omega_R$. Hence we conclude that $\psi \leq \gamma\varphi$ follows in $\Omega_R$ which results in

$$-\ln(\varphi(x)) \leq -\ln(\psi(x)) + \ln(\gamma)$$

for every $x \in \Omega_R$. □

So using the continuity of $\varphi$ and (A.4) we conclude that there exist constants $c_1, \tilde{c}_2 > 0$ such that

$$-\ln(\varphi(x)) \leq \varepsilon q(x) + c_1 \tilde{P}\left(\frac{c_1}{\varepsilon}\right) + \tilde{c}_2$$

holds for almost every $x \in \mathbb{R}^n$ and every $\varepsilon > 0$.

**Remark A.3.2**

*a) Please mind that in Proposition B.0.3 the choice of $\gamma = 2$ is needed to conclude that $\psi$ is contained in $H^1(\Omega_R)$ which is used to conclude that $v^-$ is in the domain of the form $h_{2Q}$.*

*b) In the proof Theorem A.3.1 we used an argument taken from [1] that the ground state $\varphi_{2Q}$ is the unique minimizer of the form $h_{2Q}$ up to a scalar multiple. For the numerical range of the operator $H_{2Q} = -\Delta + 2Q(|x|_2)$ this is well known. Here it is used for the form $h_{2Q}$ of $H_{2Q}$. We give reason to this argument in the following that is not included in the original article [1].*

Let us understand the final argument in the proof of Theorem A.3.1 namely that there exists a constant $c \in \mathbb{C}$ such that

$$v^- = c\varphi_{2Q}$$

holds as an equation in $\mathrm{L}^2(\mathbb{R}^n)$. We start by using a well known equation for the numerical range of a self-adjoint operator in a Hilbert space

$$\inf\{\langle H_{2Q}u, u\rangle \ : \ u \in D(H_{2Q}),\ \|u\|_2 = 1\} \ = \ E_{2Q}$$

where $E_{2Q}$ is of course the ground state energy of $H_{2Q}$. This equation is found in most textbooks such as [19] on page 411. The ground state $\varphi_{2Q}$ is up to a scalar multiple the unique minimizer of the numerical range of $H_{2Q}$. Furthermore Lemma 1.28 on page 16 in [14] gives

$$\begin{aligned}&\inf\{h_{2Q}(u, u) \ : \ u \in D(H_{2Q}),\ \|u\|_2 = 1\}\\ &= \inf\{\langle H_{2Q}u, u\rangle \ : \ u \in D(H_{2Q}),\ \|u\|_2 = 1\}.\end{aligned}$$

Therefore we know that

$$\inf\{h_{2Q}(u,u) \;:\; u \in D(H_{2Q}),\ \|u\|_2 = 1\} \;=\; E_{2Q}$$

is true. Hence we argue that $h_{2Q}(v^-, v^-) = E_{2Q}\|v^-\|_2^2$ holds in the proof of Theorem A.3.1. So $v^-$ is a minimizer of the numerical range of $h_{2Q}$. If we can show that $v^-$ is contained not only the domain of the form $h_{2Q}$ but in $D(H_{2Q})$ then $v^-$ would be another minimizer of the numerical range of $H_{2Q}$. Hence we argue that

$$v^- = c\varphi_{2Q}$$

has to follow in this case for a constant $c \in \mathbb{C}$.

**Lemma A.3.3**
*We claim that $v^- \in D(H_{2Q})$ is true.*

*Proof.* We prove by using the Lagrange multiplier theorem for Banach spaces. In the following we are focussing on $\mathrm{L}^2(\mathbb{R}^n, \mathbb{R})$ and real-valued functions in $D(h_{2Q})$ and $D(H_{2Q})$. We keep the notation for aesthetic reasons.

We already know that $v^-$ is indeed a real-valued minimizer of the numerical range of $h_{2Q}$. By

$$f(u) = \frac{1}{2} h_{2Q}(u,u)$$

for $u \in D(h_{2Q})$ we define a function on the domain $D(h_{2Q})$. Then

$$\|u\|_{h_{2Q}} = (h_{2Q}(u,u) + \langle u, u\rangle)^{\frac{1}{2}}$$

for $u \in D(h_{2Q})$ defines a norm that makes $D(h_{2Q})$ a Hilbert space. The function $f$ is Fréchet differentiable with the derivative $(Df)(u) = h_{2Q}(u, \cdot)$ since

$$\frac{1}{\|w\|_{h_{2Q}}} \; |f(u+w) - f(u) - h_{2Q}(u,w)|$$

$$= \frac{1}{\|w\|_{h_{2Q}}} \; \left| \frac{1}{2} h_{2Q}(u+w, u+w) - \frac{1}{2} h_{2Q}(u,u) - h_{2Q}(u,w) \right|$$

$$= \frac{1}{2} \; \frac{h_{2Q}(w,w)}{\|w\|_{h_{2Q}}} \leq \frac{1}{2} \; \frac{\|w\|_{h_{2Q}}^2}{\|w\|_{h_{2Q}}} = \frac{1}{2} \; \|w\|_{h_{2Q}} \to 0$$

follows for $\|w\|_{h_{2Q}} \to 0$ and every $u \in D(h_{2Q})$. Please mind that

$$h_{2Q}(u, w) = h_{2Q}(w, u)$$

holds since we are dealing only with real-valued functions as mentioned above. We define another function $g\colon D(h_{2Q}) \to \mathbb{R}$ by

$$g(u) = \frac{1}{2}\ \|u\|_2^2 - \frac{1}{2}$$

for $u \in D(h_{2Q})$. Mind that

$$\{u \in D(h_{2Q}) \mid g(u) = 0\} = \{u \in D(h_{2Q}) \mid \|u\|_2 = 1\}$$

characterizes the constraint in terms of the Lagrange multiplier theorem. Then $g$ is also Fréchet differentiable with the derivative $(Dg)(u) = \langle u, \cdot\rangle$ since

$$\frac{1}{\|w\|_2}\ |g(u+w) - g(u) - \langle u, w\rangle| \to 0$$

follows for $\|w\|_2 \to 0$. Furthermore the map $(Dg)(v^-)$ is surjective from $D(h_{2Q})$ to $\mathbb{R}$ since

$$(Dg)(v^-)(\alpha v^-) = \langle v^-, \alpha v^-\rangle = \alpha \|v^-\|_2^2$$

holds for every $\alpha \in \mathbb{R}$. We have shown all the requirements to use the Lagrange multiplier theorem for the Banach spaces $D(h_{2Q})$ and $\mathbb{R}$. So there exists a constant $\lambda \in \mathbb{R}$ such that

$$(Df)(v^-)\ =\ \lambda\ (Dg)(v^-)$$

holds as an equation in $D(h_{2Q})^*$. Hence for every $w \in D(h_{2Q})$ we conclude that

$$h_{2Q}(v^-, w) = (Df)(v^-)(w) = \lambda\ (Dg)(v^-)(w) = \langle \lambda v^-, w\rangle$$

holds. But here we have to be careful. We can't directly infer that $v^-$ is contained in $D(H_{2Q})$ since we only focus on real-valued functions $w$ in $D(h_{2Q})$ as stated in the above. But this is not an issue since for every complex-valued function $w \in D(h_{2Q})$ the real part $\mathfrak{Re}(w)$ as well as the imaginary part $\mathfrak{Im}(w)$ are both real-valued functions in $D(h_{2Q})$ such that

$$\begin{aligned} h_{2Q}(v^-, w) &= h_{2Q}(v^-, \mathfrak{Re}(w)) - \mathrm{i} h_{2Q}(v^-, \mathfrak{Im}(w)) \\ &= \langle \lambda v^-, \mathfrak{Re}(w)\rangle + \langle \lambda v^-, \mathrm{i}\mathfrak{Im}(w)\rangle \\ &= \langle \lambda v^-, \mathfrak{Re}(w) + \mathrm{i}\mathfrak{Im}(w)\rangle = \langle \lambda v^-, w\rangle \end{aligned}$$

holds as well. Now we conclude that $v^- \in D(H_{2Q})$ is true by the definition of $D(H_{2Q})$.

$\square$

# Appendix B

# Additional proofs

First let $f \in C^2([0,\infty),\mathbb{R})$ such that $f''(r) = V(r)f(r)$ is satisfied for $r \geq 0$. Define a function

$$\psi(x) = |x|_2^{\frac{1-n}{2}} f(|x|_2 - R)$$

for $|x|_2 \geq R > 0$. Then we prove the following proposition regarding $\psi$ as a solution to a certain PDE.

**Proposition B.0.1**
*For $|x|_2 \geq R$ we have $0 = -\Delta\psi(x) + Q(|x|_2)\psi(x) - E_Q\psi(x)$.*

*Proof.* i)

$$\partial_k \psi(x) = \frac{1-n}{2}\ |x|_2^{-\frac{(n+1)}{2}}\ \frac{x_k}{|x|_2} f(|x|_2 - R) + |x|_2^{\frac{1-n}{2}}\ f'(|x|_2 - R)\ \frac{x_k}{|x|_2}$$

$$= \frac{1-n}{2}\ |x|_2^{-\frac{(n+3)}{2}}\ x_k\ f(|x|_2 - R) + |x|_2^{-\frac{(n+1)}{2}}\ x_k\ f'(|x|_2 - R)$$

ii)

$$
\begin{aligned}
\partial_k^2\psi(x) =& -\frac{(1-n)(n+3)}{4}\ |x|_2^{-\frac{(n+5)}{2}}\ \frac{x_k}{|x|_2}\ x_k\ f(|x|_2-R) \\
&+\frac{1-n}{2}\ |x|_2^{-\frac{(n+3)}{2}}\ f(|x|_2-R)+\frac{1-n}{2}\ |x|_2^{-\frac{(n+3)}{2}}\ x_k\ f'(|x|_2-R)\ \frac{x_k}{|x|_2} \\
&-\frac{(n+1)}{2}\ |x|_2^{-\frac{(n+3)}{2}}\ \frac{x_k}{|x|_2}\ x_k\ f'(|x|_2-R)+|x|_2^{-\frac{(n+1)}{2}}\ f'(|x|_2-R) \\
&+|x|_2^{-\frac{(n+1)}{2}}\ x_k\ f''(|x|_2-R)\ \frac{x_k}{|x|_2} \\
&= f(|x|_2-R)\left(\frac{1-n}{2}|x|_2^{-\frac{(n+3)}{2}}-\frac{(1-n)(n+3)}{4}\frac{x_k^2}{|x|_2^2}|x|_2^{-\frac{(n+3)}{2}}\right) \\
&+f'(|x|_2-R)\left(\frac{1-n}{2}|x|_2^{-\frac{(n+1)}{2}}\frac{x_k^2}{|x|_2^2}-\frac{n+1}{2}|x|_2^{-\frac{(n+1)}{2}}\frac{x_k^2}{|x|_2^2}+|x|_2^{-\frac{(n+1)}{2}}\right) \\
&+|x|_2^{\frac{1-n}{2}}\frac{x_k^2}{|x|_2^2}\ f''(|x|_2-R)
\end{aligned}
$$

iii)

$$
\begin{aligned}
-\Delta\psi(x) =& f(|x|_2-R)\left(\frac{n(n-1)}{2}|x|_2^{-\frac{(n+3)}{2}}-\frac{(n-1)(n+3)}{4}|x|_2^{-\frac{(n+3)}{2}}\right) \\
&+f'(|x|_2-R)\left(\underbrace{\frac{n-1}{2}|x|_2^{-\frac{(n+1)}{2}}+\frac{n+1}{2}|x|_2^{-\frac{(n+1)}{2}}-n|x|_2^{-\frac{(n+1)}{2}}}_{=0}\right) \\
&-|x|_2^{\frac{1-n}{2}}\ f''(|x|_2-R)
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&-\Delta\psi(x)+Q(|x|_2)\psi(x)-E_Q\psi(x)\\
&=|x|_2^{\frac{1-n}{2}}\left(f(|x|_2-R)\left(\frac{n(n-1)}{2|x|_2^2}-\frac{(n-1)(n+3)}{4|x|_2^2}+Q(|x|_2)-E_Q\right)\;-\;f''(|x|_2-R)\right)\\
&=|x|_2^{\frac{1-n}{2}}\left(f(|x|_2-R)\underbrace{\left(\frac{(n-1)(n-3)}{4|x|_2^2}+Q(|x|_2)-E_Q\right)}_{=V(|x|_2-R)}-f''(|x|_2-R)\right)\\
&=|x|_2^{\frac{1-n}{2}}\left(V(|x|_2-R)f(|x|_2-R)\;-\;f''(|x|_2-R)\right)=0
\end{aligned}
$$

holds for every $|x|_2 \geq R$.

□

We define $\Omega_R = \{x \in \mathbb{R}^n \;:\; |x|_\infty > R\}$ and

$$v(x) = \gamma\varphi(x) - \psi(x)$$

for $x \in \overline{\Omega_R}$ where $\gamma = \sup_{|x|_\infty = R} \frac{\psi(x)}{\varphi(x)} \in (0,\infty)$.

**Proposition B.0.2**
*We claim that $v \in \mathrm{L}^2(\Omega_R)$ is true.*

*Proof.* First let us understand that $\psi$ is contained in $\mathrm{L}^2(\Omega_R)$. We see that

$$
\begin{aligned}
\psi(x)^2 &= |x|_2^{1-n} f(|x|_2-R)^2\\
&= c^2|x|_2^{1-n}V(|x|_2-R)^{-\frac{1}{2}}\exp\left(2h(|x|_2-R)-2\int_0^{|x|_2-R}V(t)^{\frac{1}{2}}\;\mathrm{d}t\right)\\
&\leq (\tilde{c})^2|x|_2^{1-n}Q(|x|_2)^{-\frac{1}{2}}
\end{aligned}
$$

holds for $|x|_2 \geq R$ where $\tilde{c} > 0$ is a constant. Please mind that $R$ was chosen big enough such that

$$V(r) \geq \frac{1}{2}Q(r+R)$$

holds for every $r \geq 0$ and that

$$\int_{\Omega_R} |x|_2^{1-n} Q(|x|_2)^{-\frac{1}{2}} \, \mathrm{d}x \;\leq\; C \int_R^\infty Q(r)^{-\frac{1}{2}} \, \mathrm{d}r \;<\; \infty$$

holds by Proposition 2.2 a) on page 4102 in [1] for a constant $C > 0$. We conclude that $\psi$ is contained in $\mathrm{L}^2(\Omega_R)$ and therefore $v$ is as well.

□

Furthermore $v$ is in $C^1(\overline{\Omega_R})$. Next we argue that $v \in H^1(\Omega_R)$ is true.

**Proposition B.0.3**
*The function $v$ is contained in $H^1(\Omega_R)$.*

*Proof.* We only need to prove that $\psi \in H^1(\Omega_R)$ is true. For $x \in \Omega_R$ we have

$$\partial_k \psi(x) = \frac{1-n}{2} \, |x|_2^{-\frac{(n+1)}{2}} \, \frac{x_k}{|x|_2} \, f(|x|_2 - R) + |x|_2^{\frac{1-n}{2}} \, \frac{x_k}{|x|_2} \, f'(|x|_2 - R)$$

where the derivative is meant in the strong sense. We show that both functions of the sum are contained in $\mathrm{L}^2(\Omega_R)$ to conclude that $\partial_k \psi$ is as well.

i)

$$\begin{aligned}
&\int_{\Omega_R} |x|_2^{-n-1} \, \frac{x_k^2}{|x|_2^2} \, f(|x|_2 - R)^2 \, \mathrm{d}x \\
&\leq \int_{\Omega_R} |x|_2^{-n-1} \, f(|x|_2 - R)^2 \, \mathrm{d}x \leq c \int_R^\infty r^{-n-1} \, r^{n-1} \, f(r-R)^2 \, \mathrm{d}r \\
&\leq \frac{\tilde{c}}{R^2} \int_R^\infty V(r-R)^{-\frac{1}{2}} \, \mathrm{d}r \leq \sqrt{2} \, \frac{\tilde{c}}{R^2} \int_R^\infty Q(r)^{-\frac{1}{2}} \, \mathrm{d}r < \infty
\end{aligned}$$

for constants $c, \tilde{c} > 0$. We used that $R$ was chosen large enough such that $V(r) \geq \frac{1}{2} Q(r+R)$ holds for every $r \geq 0$. So we see that

$$\frac{1-n}{2} \, |x|_2^{-\frac{(n+1)}{2}} \, \frac{x_k}{|x|_2} \, f(|x|_2 - R)$$

is contained in $\mathrm{L}^2(\Omega_R)$.

ii) Now let us consider the more complicated case of

$$|x|_2^{\frac{1-n}{2}} \; \frac{x_k}{|x|_2} \; f'(|x|_2 - R).$$

Consider that

$$f'(r) =$$
$$c\exp\left(h(r) - \int_0^r V(t)^{\frac{1}{2}}\ \mathrm{d}t\right)\left(V(r)^{-\frac{1}{4}}(h'(r) - V(r)^{\frac{1}{2}}) - \frac{1}{4}V(r)^{-\frac{5}{4}}\ V'(r)\right).$$

Mind that $\exp\left(h(r) - \int_0^r V(t)^{\frac{1}{2}}\ \mathrm{d}t\right)$ is bounded. We show that both terms of the sum of functions in $f'$ are in $\mathrm{L}^2(0,\infty)$.

a) First we see that

$$\int_0^\infty \left(V(r)^{-\frac{5}{4}}V'(r)\right)^2\ \mathrm{d}r = \int_0^\infty (V'(r))^2 V(r)^{-\frac{5}{2}}\ \mathrm{d}r.$$

Exactly at this point we see that the choice of $\gamma = 2$ in the definition of $Q$ is needed such that

$$\int_0^\infty (V'(r))^2 V(r)^{-\frac{5}{2}}\ \mathrm{d}r = \int_0^\infty (V'(r))^\gamma V(r)^{\frac{1}{2}-\frac{3}{2}\gamma}\ \mathrm{d}r$$

holds. By definition of $Q$ and $V$ we argue that

$$\int_0^\infty (V'(r))^\gamma V(r)^{\frac{1}{2}-\frac{3}{2}\gamma}\ \mathrm{d}r \ < \ \infty.$$

b) Let us now focus on the second term in $f'$. Here the situation is a bit more complicated. We show that

$$\left|\ \exp\left(h(r) - \int_0^r V(t)^{\frac{1}{2}}\ \mathrm{d}t\right)(h'(r) - V(r)^{\frac{1}{2}})\ \right|$$

is bounded in $(0,\infty)$. Due to the properties of $h$ we only need to show that

$$\exp\left(-\int_0^r V(t)^{\frac{1}{2}}\ \mathrm{d}t\right)V(r)^{\frac{1}{2}}$$

is bounded in $(0,\infty)$. From $\int_0^\infty V(r)^{-\frac{5}{2}}V'(r)^2 \ \mathrm{d}r < \infty$ we conclude that

$$\left(\frac{V'(r)}{V(r)^{\frac{5}{4}}}\right)^2 = V(r)^{-\frac{5}{2}}V'(r)^2 \ \to \ 0$$

follows for $r \to \infty$. Hence there exists a $r_0 > 0$ such that

$$V'(r) \le V(r)^{\frac{5}{4}} = V(r)^{\frac{1}{2}}V(r)^{\frac{3}{4}}$$

holds for every $r \in [r_0,\infty)$. Then

$$-V(r)^{\frac{1}{2}} \le -V(r)^{-\frac{3}{4}}V'(r)$$

follows for $r \in [r_0,\infty)$. So

$$\begin{aligned}
\exp\left(-\int_{r_0}^r V(t)^{\frac{1}{2}} \ \mathrm{d}t\right) &\le \exp\left(-\int_{r_0}^r V(t)^{-\frac{3}{4}}V'(t) \ \mathrm{d}t\right) \\
&= \exp\left(-\int_{V(r_0)}^{V(r)} t^{-\frac{3}{4}} \ \mathrm{d}t\right) \\
&= C\exp\left(-4V(r)^{\frac{1}{4}}\right)
\end{aligned}$$

holds for $r \in [r_0,\infty)$ where $C > 0$ is a constant. Then

$$\begin{aligned}
&\exp\left(-\int_0^r V(t)^{\frac{1}{2}} \ \mathrm{d}t\right)V(r)^{\frac{1}{2}} \le D\exp\left(-\int_{r_0}^r V(t)^{\frac{1}{2}} \ \mathrm{d}t\right)V(r)^{\frac{1}{2}} \\
&\le CD\exp\left(-4V(r)^{\frac{1}{4}}\right)V(r)^{\frac{1}{2}} \le K < \infty
\end{aligned}$$

holds for every $r \in [r_0,\infty)$ where $D$ and $K$ are positive constants. Furthermore the boundedness of $\exp\left(-4V(r)^{\frac{1}{4}}\right)V(r)^{\frac{1}{2}}$ is checked by using l'Hospital's theorem for $r \to \infty$. So

$$\left| \ \exp\left(h(r) - \int_0^r V(t)^{\frac{1}{2}} \ \mathrm{d}t\right)(h'(r) - V(r)^{\frac{1}{2}}) \ \right|$$

is bounded. Hence there exists a cosntant $C > 0$ such that

$$\begin{aligned}
&\int_0^\infty V(r)^{-\frac{1}{2}}\exp\left(2h(r) - 2\int_0^r V(t)^{\frac{1}{2}} \ \mathrm{d}t\right)(h'(r) - V(r)^{\frac{1}{2}})^2 \ \mathrm{d}r \\
&\le C\int_0^\infty V(r)^{-\frac{1}{2}} \ \mathrm{d}r < \infty.
\end{aligned}$$

In total we conclude that $f'$ is contained in $\mathrm{L}^2(0,\infty)$. So

$$\begin{aligned}
&\int_{\Omega_R} |x|_2^{1-n}\ \frac{x_k^2}{|x|_2^2}\ (f'(|x|_2 - R))^2\ \mathrm{d}x \\
&\leq \int_{\Omega_R} |x|_2^{1-n}\ (f'(|x|_2 - R))^2\ \mathrm{d}x \leq C \int_R^\infty r^{1-n}\ r^{n-1}\ (f'(r-R))^2\ \mathrm{d}r \\
&= C \int_R^\infty (f'(r-R))^2\ \mathrm{d}r < \infty.
\end{aligned}$$

We have shown that $\psi$ is contained in $H^1(\Omega_R)$ and so is $v$. □

**Proposition B.0.4**
*We claim that*

$$v^- \in H^1(\mathbb{R}^n)$$

*is satisfied where* $v^-(x) = \max\{-v(x), 0\}$ *for* $x \in \Omega_R$ *and* $v^-(x) = 0$ *for* $x \in [-R,R]^n$.

*Proof.* For $|x|_\infty = R$ we infer that

$$v(x) = \gamma\varphi(x) - \psi(x)\ \geq\ 0.$$

Hence $v^-$ is continuous in $\mathbb{R}^n$. Furthermore $v^-$ is piecewise differentiable in $\mathbb{R}^n$ by Lemma 7.6 on page 152 in [6] and Remark 4.3 on page 4111 in [1]. So we determine the weak derivative of $v^-$ by integration by parts.

Let $k \in \{1,\dots,n\}$ be arbitrary but fixed and let $w \in C_c^\infty(\mathbb{R}^n)$. Then there exists a $r > 0$ such that

$$\operatorname{supp} w \in [-r,r]^n$$

is satisfied. Then

$$\begin{aligned}
&\int_{[-r,r]^n} v^-(x)(\partial_k w)(x)\ \mathrm{d}x \\
&= \int_{[-r,r]^{n-1}} \int_{-r}^r v^-(x)(\partial_k w)(x)\ \mathrm{d}x_k \mathrm{d}(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n) \\
&= -\int_{[-r,r]^{n-1}} \int_{-r}^r (\partial_k v^-)(x) w(x)\ \mathrm{d}x_k \mathrm{d}(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n) \\
&= -\int_{[-r,r]^n} (\partial_k v^-) w(x)\ \mathrm{d}x
\end{aligned}$$

is satisfied due to the continuity and piecewise differentiability of $v^-$. Of course the integration by parts of the inner integrals only needs to be done on the intervals where $v^-$ is not 0. So $v \in H^1(\mathbb{R}^n)$. □

**Proposition B.0.5**
*We state that $Q(|x|_2)^{\frac{1}{2}}\psi \in \mathrm{L}^2(\Omega_R)$ is true.*

*Proof.* Let $x \in \Omega_R$. Then

$$\begin{aligned} &Q(|x|_2)\psi(x)^2 = |x|_2^{1-n}\ Q(|x|_2)\ f(|x|_2 - R)^2 \\ &= c^2\ |x|_2^{1-n}\ V(|x|_2 - R)^{-\frac{1}{2}}\ Q(|x|_2)\exp\left(2h(|x|_2 - R) - 2\int_0^{|x|_2-R} V(t)^{\frac{1}{2}}\ \mathrm{d}t\right). \end{aligned}$$

Since $|x|_2^{1-n}V(|x|_2 - R)^{-\frac{1}{2}}$ is contained in $\mathrm{L}^1(\Omega_R)$ we need to show that $Q(|x|_2)\exp(-2\int_0^{|x|_2-R} V(t)^{\frac{1}{2}}\ \mathrm{d}t)$ is bounded in $\Omega_R$. Mind that

$$Q(|x|_2)\exp\left(-2\int_0^{|x|_2-R} V(t)^{\frac{1}{2}}\ \mathrm{d}t\right) \leq 2V(|x|_2-R)\exp\left(-2\int_0^{|x|_2-R} V(t)^{\frac{1}{2}}\ \mathrm{d}t\right)$$

follows from $V(r) \geq \frac{1}{2}Q(r+R)$ for $r \geq 0$. Now we use

$$\exp\left(-\int_0^{|x|_2-R} V(t)^{\frac{1}{2}}\ \mathrm{d}t\right) \leq C\exp\left(-4V(|x|_2 - R)^{\frac{1}{4}}\right)$$

from the proof of Proposition B.0.3 above to conclude that

$$\begin{aligned} &Q(|x|_2)\exp\left(-2\int_0^{|x|_2-R} V(t)^{\frac{1}{2}}\ \mathrm{d}t\right) \leq 2V(|x|_2 - R)\exp\left(-2\int_0^{|x|_2-R} V(t)^{\frac{1}{2}}\ \mathrm{d}t\right) \\ &\leq 2CV(|x|_2 - R)\exp\left(-8V(|x|_2 - R)^{\frac{1}{4}}\right) \end{aligned}$$

holds for every $x \in \Omega_R$. Using l'Hospital's theorem multiple times on $V(r)\exp(-V(r)^{\frac{1}{4}})$ for $r \to \infty$ we infer that $V(|x|_2 - R)\exp(-8V(|x|_2 - R)^{\frac{1}{4}})$ is bounded on $\Omega_R$. Hence $Q(|x|_2)\psi^2 \in \mathrm{L}^1(\Omega_R)$ is implied. □

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**Talks:**

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"INTRINSIC ULTRACONTRACTIVITY OF SCHROEDINGER SEMIGROUPS IN $L^2(\mathbb{R}^n)$"

Bergische Universität Wuppertal
"INTRINSIC ULTRACONTRACTIVITY OF SCHROEDINGER SEMIGROUPS IN $L^2(\mathbb{R}^n)$"

27th Internet seminar at CIRM
"CHARACTERIZATION OF SECTORIAL OPERATORS ON $L^p$-SPACES POSESSING A BOUNDED $H^\infty$-CALCULUS "
Advisors: Sahiba Arora, Jochen Glück and Felix Schwenninger

2022 25th Internet seminar
"SPECTRA OF SELF-ADJOINT OPERATORS GENERATING SEMIGROUPS THAT SATISFY GAUSSIAN ESTIMATES "
Advisor: Delio Mugnolo

**Undergraduate theses:**

2024: Ann-Kathrin Jirsch, Rostock (supervised)
"Study of Gleick-Lorenz systems"

2023: Raphael Demko, Rostock (co-supervised)
"Basis properties of Fucik eigenfunctions"

2022: Jan Raßfeld, Rostock (co-supervised)
"Spectral properties of the quantum harmonic oscillator"

**Teaching Activities:**

| | | |
|---|---|---|
| 2025 | Spring | Analysis |
| | | Analysis (tutorial) |
| | | Analysis: ODE and Fourier transforms (tutorial) |
| 2024 | Fall | Analysis I (tutorial) |
| | Spring | Analysis II (tutorial) |
| | | Ordinary Differential Equations (tutorial) |
| 2023 | Fall | Analysis I (tutorial) |
| | | Functional Analysis (tutorial) |
| | Spring | Analysis II (tutorial) |
| 2022 | Fall | Analysis I (tutorial) |
| | | Mathematics I for Informatics (tutorial) |
| | Spring | Analysis II (tutorial) |
| | | Ordinary Differential Equations (tutorial) |

| | | |
|---|---|---|
| 2021 | Fall | Partial Differential Equations (tutorial) |
| | | Mathematics I for Informatics (tutorial) |
| | Spring | Analysis II (tutorial) |
| 2020 | Fall | Analysis I (tutorial) |
| | Spring | Ordinary Differential Equations (tutorial) |
| | | Complex Analysis (tutorial) |