

Data-Driven Control of Periodic Orbits

Cumulative Dissertation

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Abstract

It is a common theme in multiple branches of sciences to detect and control genuine unstable states periodic in time, as illustrated by e.g. dynamical properties in neuroscience, chemical catalytic surface reactions, or optimizing atomic force microscopy in surface physics, to mention just a few. When underlying equations of motion are known, such as satellite dynamics in celestial motion, the task reduces to solving boundary value problems. A considerable challenge occurs when no suitable accurate mathematical model is available, such as in areas mentioned above. Then the control and detection tasks have to be based predominantly on experimental observations. This work proposes methods for the stabilization of unstable periodic orbits by dynamically adjusting a system parameter. In particular, the case where the equations of motion of the system that possesses periodic orbits are unknown is discussed. First, a stabilization scheme based on a linear state feedback of a Poincaré map is presented. Furthermore, a nonlinear control scheme that leads to a large basin of attraction of the stabilized periodic orbit is proposed. We also develop a strategy that enables nonlinear control directly from input-state time series of a Poincaré map. Thus neither the equations of motion have to be known nor arbitrary choices for the control gains are necessary. The presented methods are numerically validated on both autonomous and non-autonomous nonlinear dynamical systems.

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Chapter 1

Summary

Mechanical, electrical, and industrial systems are all examples of systems that must be brought into a desired state by appropriately adjusting their inputs [6]. This is typically achieved by adjusting the input according to the deviation of the current state from the desired state, the central idea of feedback control [37, 16]. In this work, we focus on the case where the desired behavior of the system is given as a periodic state which could either be stable or unstable [25]. This refers to a state that changes periodically over time. We aim to stabilize unstable periodic orbits with the help of feedback control. This even allows observation of such objects in laboratory experiments.

This dissertation contains the results achieved during my time as a doctoral student. The present chapter serves to summarize these results. In section 1.1 we provide an introduction to the concepts from dynamical systems theory that are necessary for the subsequent discussion of the results. Section 1.2 presents the methods developed for stabilizing periodic states and puts them in relation to the current state of research. Finally, section 1.3 concludes this work and section 1.4 contains a statement regarding my own contribution to the manuscripts contained in this dissertation. The paper [20] of chapter 2 proposes a stabilization scheme for periodic orbits via linear feedback control of a Poincaré map. Enlargement of the basins of attraction and therefore increased robustness of the control scheme is achieved in the published work [32] of chapter 3. These ideas are also applicable in a data-based setup where the stabilizing feedback is constructed from recorded input-state time series of a Poincaré map, see the publication [30] of chapter 4. The paper [31] of chapter 5 applies similar ideas of data-driven control based on the Koopman operator to a simulation of a nonlinear ball-on-a-wheel system. Experimental results obtained from stabilizing unstable periodic orbits of a scanning

force microscope are presented in the publication [10] of chapter 6. A method that also obtains stability information during the application of a proportional-integral control in an experiment of the Zeeman catastrophe machine, is presented in the paper [19] of chapter 7.

1.1 Dynamical systems and Poincaré maps

First, we assume that the change of state with time of the given system can be described by a set of autonomous ordinary differential equations

$$\dot{x} = f(x, \mu). \quad (1.1)$$

Here, x denotes the N -dimensional state vector, μ is an M -dimensional parameter, and a dot over a letter denotes the derivative with respect to time. In this work, the parameter μ is also referred to as the control parameter. We further assume that for each $x_0 \in \mathbb{R}^N$ and $\mu \in \mathbb{R}^M$ there exists a function $\phi(\cdot, x_0, \mu) : \mathbb{R} \rightarrow \mathbb{R}^N$ that satisfies

$$\dot{\phi}(t, x_0, \mu) = f(\phi(t, x_0, \mu), \mu)$$

for all $t \in \mathbb{R}$ and $\phi(0, x_0, \mu) = x_0$. That is, $\phi(\cdot, x_0, \mu)$ solves the initial value problem given by (1.1) and $x(0) = x_0$. With a continuously differentiable f , for example, the function $\phi(\cdot, x_0, \mu)$ may only be defined on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. This follows from the Picard-Lindelöf theorem [2]. But since we have applications of our methods for real experimental systems in mind, we may as well assume global existence of $\phi(\cdot, x_0, \mu)$.

Second, we also consider non-autonomous systems

$$\dot{x} = f(x, t, \mu) = f(x, t + \tau(\mu), \mu). \quad (1.2)$$

Here, the function f is periodic in time with period $\tau(\mu)$. That is, the period depends on the parameter μ .

As mentioned in the introductory remarks, we are interested in periodic orbits of (1.1) and (1.2). These are defined as follows. Suppose that for a fixed initial condition x_0 and parameter value μ_* , the solution of (1.1) or (1.2) satisfies $\phi(t, x_0, \mu_*) = \phi(t + T, x_0, \mu_*)$ for some $T > 0$ and all $t \in \mathbb{R}$. Then the set $\gamma = \{\phi(t, x_0, \mu_*) : 0 \leq t < T\} \subset \mathbb{R}^N$ is called a periodic orbit. The smallest number T with this property is called the period of the orbit.

The analysis of a given periodic orbit can be based on the following construction. Suppose that for a fixed parameter value μ_* , the system (1.1) possesses a periodic orbit γ with period T . Let $\Sigma \subset \mathbb{R}^N$ denote an $(N - 1)$ -dimensional submanifold and let $x_* \in \Sigma$ be the unique point of intersection of γ with Σ . We further assume that the standard inner product $(f(x, \mu_*), n(x)) \neq 0$ for all $x \in \Sigma$. Here, $n(x)$ denotes a normal vector to Σ at x . Geometrically, this means that the vector field $x \mapsto f(x, \mu_*)$ is transverse to Σ . Under these assumptions, there exists an open set $U \subset \Sigma$ containing x_* and an open set $V \subset \mathbb{R}^M$ containing μ_* such that the map $P : U \times V \rightarrow \Sigma$ given by

$$P(x, \mu) = \phi(\tau(x, \mu), x, \mu) \quad (1.3)$$

exists. Here, $\tau(x, \mu) \in \mathbb{R}$ denotes the time it takes for the orbit starting at x with a fixed parameter μ to first return to the submanifold Σ . The map P is called the Poincaré map [25], see Figure 1.1.

Note that by definition $\tau(x_*, \mu_*) = T$ and therefore

$$P(x_*, \mu_*) = x_*,$$

i.e., x_* is a fixed point of the map $P(\cdot, \mu_*)$. Note that, in practice, one often works with a coordinate representation of P , see e.g. [33]. That is, if $\varphi : \mathbb{R}^{N-1} \supset W \mapsto \Sigma$, $(\xi_1, \xi_2, \dots, \xi_{N-1}) \mapsto x$ is a parametrization of Σ , see e.g. [35], the coordinate representation of P is given as the map

$$\hat{P}(\xi_1, \xi_2, \dots, \xi_{N-1}, \mu) = (\varphi^{-1} \circ P)(\varphi(\xi_1, \xi_2, \dots, \xi_{N-1}), \mu)$$

with $\hat{P} : \varphi^{-1}(U) \times V \rightarrow \mathbb{R}^{N-1}$. If $\xi_* = \varphi^{-1}(x_*)$ are the coordinates of the fixed point x_* , it follows that $\hat{P}(\xi_*, \mu_*) = \xi_*$. Now, the local stability of the periodic orbit γ is characterized by the local stability of the fixed point ξ_* of the Poincaré map. The stability of the latter can be examined by computing the eigenvalues of the $(N - 1) \times (N - 1)$ jacobian matrix

$$\left(\frac{\partial \hat{P}_i}{\partial \xi_j}(\xi_*, \mu_*) \right), \quad i, j = 1, \dots, N - 1.$$

If all eigenvalues have an absolute value less than one, then it follows that the fixed point and the periodic orbit are asymptotically stable [25].

For the class of non-autonomous systems described by (1.2), the construction of a Poincaré map is more straightforward. Suppose that for a fixed parameter

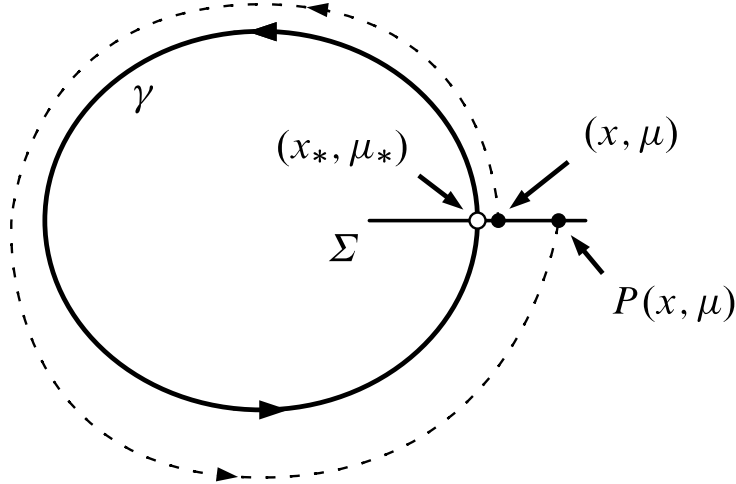


Figure 1.1: An illustration of a Poincaré map. For a given periodic orbit γ of the vector field $x \mapsto f(x, \mu_*)$, $x \in \mathbb{R}^N$, we construct an $(N - 1)$ -dimensional submanifold Σ which intersects the orbit at the unique point x_* . In a neighborhood of the point (x_*, μ_*) , the Poincaré map is defined as follows. For a given point (x, μ) in this neighborhood, we integrate the vector field $f(\cdot, \mu)$ starting at the initial value x until the resulting trajectory first returns to the submanifold Σ , intersecting in a point $P(x, \mu)$. This map defines a Poincaré map where $P(x_*, \mu_*) = x_*$ holds.

value μ_* , the system (1.2) possesses a periodic orbit γ of period $T = \tau(\mu_*)$. We now define the Poincaré map $P : \mathbb{R}^N \times \mathbb{R}^M \rightarrow \mathbb{R}^N$ as

$$P(x, \mu) = \phi(\tau(\mu), x, \mu). \quad (1.4)$$

If $P(x_*, \mu_*) = x_*$ for some x_* , then it follows that $x_* \in \gamma$. Similar to the reasoning above, the local stability of the periodic orbit can then be examined by computing the eigenvalues of the $N \times N$ jacobian matrix

$$\left(\frac{\partial P_i}{\partial x_j}(x_*, \mu_*) \right), \quad i, j = 1, \dots, N.$$

In summary, for autonomous systems (1.1) and non-autonomous systems (1.2), one obtains Poincaré maps that can be used for the analysis of periodic orbits. Regardless of whether the obtained map is constructed from (1.1) or (1.2), we

denote it by $(x, \mu) \mapsto P(x, \mu)$. Here x and P denote either the coordinate vector ξ and the corresponding coordinate representation $\hat{P}$ in the case where the map is constructed from (1.1), or they denote the state vector of (1.2) and the map (1.4). Thus, the analysis of periodic orbits boils down to the study of the parameter-dependent time-discrete system

$$x_{n+1} = P(x_n, \mu), \quad n = 0, 1, 2, \dots \quad (1.5)$$

Here, $x_n \in \mathbb{R}^d$. Note that due to the submanifold construction described above, $d = N - 1$ for (1.1), while starting with system (1.2) implies $d = N$.

As an example of (1.2), consider the two-dimensional differential equation

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\omega^2 x + \varepsilon \left(-\gamma v - U'(x) - h^{(c)} \cos(\omega t) - h^{(s)} \sin(\omega t) \right), \end{aligned}$$

which describes a forced one degree of freedom oscillator. Here, (x, v) is the state vector, $U : \mathbb{R} \rightarrow \mathbb{R}$ denotes the potential energy, and $\omega, \varepsilon, \gamma$ are fixed real parameters. The control parameter μ is given as the vector $(h^{(c)}, h^{(s)})$ and the period is the constant function $\tau(\mu) = 2\pi/\omega$. This system forms the basis of the publication [32] from chapter 3. In fact, it is shown there that using a first order perturbation expansion in ε , one obtains the Poincaré map

$$\begin{aligned} x_{n+1} &= x_n + \varepsilon \frac{\pi}{\omega^2} \left(-\gamma \omega x_n + 2 \frac{v_n}{\omega} w(x_n^2 + (v_n^2/\omega^2)) + h^{(s)} \right) + \mathcal{O}(\varepsilon^2) \\ v_{n+1} &= v_n + \varepsilon \frac{\pi}{\omega} \left(-\gamma v_n - 2x_n w(x_n^2 + (v_n^2/\omega^2)) - h^{(c)} \right) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

with a suitably chosen function $w : \mathbb{R} \rightarrow \mathbb{R}$.

In general, computing an analytic expression for the Poincaré map (1.5) is not possible since this requires an analytic expression for the solution of the corresponding differential equation. This can be seen from the definitions (1.3) and (1.4) since the solution ϕ appears explicitly in both expressions. However, an explicit expression for ϕ is only available in a few cases. Consequently, one must resort to numerical computations of ϕ for the evaluation of the Poincaré map.

Let x_* be a fixed point of the Poincaré map (1.5) for a fixed parameter value μ_* . That is, x_* satisfies the equation

$$x_* - P(x_*, \mu_*) = 0. \quad (1.6)$$

We compute the $d \times d$ jacobian matrix of the function $x \mapsto x - P(x, \mu)$ with respect to the state x and evaluate it at (x_*, μ_*) . This matrix is given by

$$I - \frac{\partial P}{\partial x}(x_*, \mu_*) \quad (1.7)$$

and if it is nonsingular, then the implicit function theorem implies the existence of a smooth function $\mu \mapsto g(\mu)$ that is defined on an open set containing μ_* , where it satisfies $g(\mu) - P(g(\mu), \mu) = 0$. In particular, $g(\mu_*) = x_*$. Consequently, the dependence of the fixed point on local variations of the parameter μ is described by the function $g(\mu)$. If the matrix (1.7) turns out to be singular, the point (x_*, μ_*) is called a bifurcation point [25]. To illustrate in an example what can happen in this case, we consider a Poincaré map given by

$$x_{n+1} = P(x_n, \mu) = x_n - x_n^2 + \mu. \quad (1.8)$$

The fixed point condition (1.6) reads $x_*^2 = \mu_*$. Consequently, for a negative μ_* , there exists no real fixed point. If μ_* is chosen positive, we have the fixed points $\sqrt{\mu_*}$ and $-\sqrt{\mu_*}$, which coalesce for $\mu_* = 0$. The derivative of P with respect to the state evaluated at the points $(\pm\sqrt{\mu_*}, \mu_*)$ is given by

$$\frac{\partial P}{\partial x}(\pm\sqrt{\mu_*}, \mu_*) = 1 \mp 2\sqrt{\mu_*}.$$

Thus, for $0 < \mu_* < 1$ the points $\sqrt{\mu_*}$ are asymptotically stable and the points $-\sqrt{\mu_*}$ are unstable. Therefore, the graph of the function $\sqrt{\mu_*}$ over the interval $0 < \mu_* < 1$ is called the stable branch and the graph of $-\sqrt{\mu_*}$ over the same interval is referred to as the unstable branch. At the point $(x_*, \mu_*) = (0, 0)$ we have $(\partial P / \partial x)(0, 0) = 1$ and therefore $(0, 0)$ is a bifurcation point, at which the two branches coincide, see Figure 1.2.

Such a bifurcation analysis is possible for much more general systems, where stable and unstable branches typically coexist [25, 33]. In general, if we fix a point (x_*, μ_*) on the stable branch and initialize the system $x_{n+1} = P(x_n, \mu_*)$ sufficiently close to x_* , the stability of x_* implies the convergence of the sequence $x_0, x_1, x_2, \dots$ to the fixed point. Thus, if the Poincaré map results from an experimental system, one observes convergence of the state to the stable periodic orbit corresponding to the fixed point x_* . Note that the parameter μ is kept constant while the experiment is running. In contrast, for a point (x_*, μ_*) on an unstable branch, every initialization at points different from the fixed point x_* results in the divergence of the state sequence from it. But since experimental setups

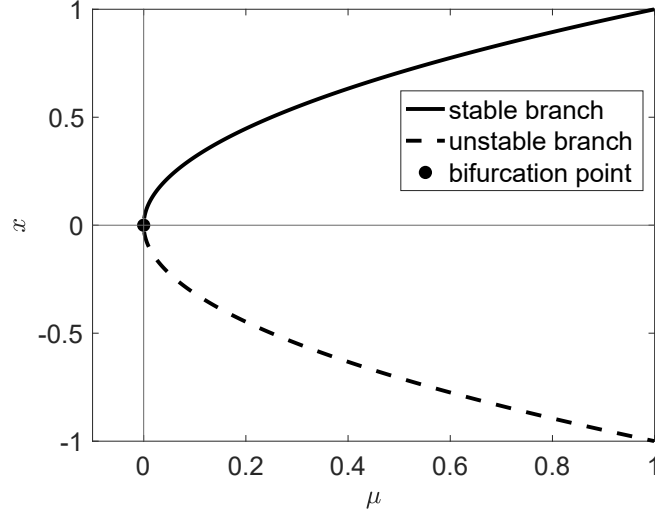


Figure 1.2: Bifurcation diagram of the system $x_{n+1} = P(x_n, \mu) = x_n - x_n^2 + \mu$. The diagram is obtained from the solution of the fixed point condition $x_*^2 = \mu_*$, which implies $x_* = \pm\sqrt{\mu_*}$. Since $(\partial P / \partial x)(\pm\sqrt{\mu_*}, \mu_*) = 1 \mp 2\sqrt{\mu_*}$, it consists of a stable branch $\sqrt{\mu_*}$ (black solid line), an unstable branch $-\sqrt{\mu_*}$ (black dashed line), and the bifurcation point $(0, 0)$ (black dot), where we have $(\partial P / \partial x)(0, 0) = 1$.

never allow for an exact initialization, fixed points on the unstable branch are not observed in experiments. However, if such an unstable state fulfills a given optimality property, for example, one would still want to drive the system to that state. This work aims to show how this goal can be achieved by dynamically adjusting the parameter μ . The results obtained are summarized in the following section.

1.2 Control of periodic orbits

In the previous section 1.1 it has been shown that the analysis of given periodic orbit boils down to the study of a corresponding Poincaré map

$$x_{n+1} = P(x_n, \mu), \quad n = 0, 1, 2, \dots \quad (1.9)$$

We assume that this parameter-dependent time-discrete system possesses coexisting stable and unstable branches, as explained in the example (1.8). Furthermore,

we work with the restriction of not knowing the equations of motion of the given experimental system. Thus, an analytic expression of the Poincaré map (1.9) is not available. However, we assume that one is able to record time series. That is, for a given initial condition x_0 and a fixed parameter μ , it is possible to record $x_0, x_1, x_2, \dots$. If the experimental system is described by an autonomous differential equation, this amounts to taking a state measurement whenever the orbit crosses the constructed submanifold. In the non-autonomous case, one records the state at times $0, \tau(\mu), 2\tau(\mu), \dots$ resulting in $x_0, x_1, x_2, \dots$. Finally, we assume that the parameter μ can be dynamically adjusted in the following way. Suppose that the system is in the initial state x_0 . We now take μ_0 as our initial control parameter. The state at the next time instant is given by $x_1 = P(x_0, \mu_0)$. At this stage, we choose μ_1 , resulting in the state $x_2 = P(x_1, \mu_1)$. Continuing in this manner, a new value for the parameter μ is chosen at each time instant [7]. The key idea of feedback control is to base the decision μ_n at the time instant n on the measurement x_n [37]. This feedback law is described by a function $h : \mathbb{R}^d \rightarrow \mathbb{R}^M$. Consequently, the Poincaré map under the effect of such a feedback is given by

$$x_{n+1} = P(x_n, h(x_n)), \quad n = 0, 1, 2, \dots \quad (1.10)$$

We want to choose the function h so that the sequence of states defined by (1.10) and the corresponding sequence of decisions $\mu_0 = h(x_0), \mu_1 = h(x_1), \mu_2 = h(x_2), \dots$ converges to some point (x_*, μ_*) on an unknown unstable branch of (1.9). That is, the feedback control tracks and stabilizes a formerly unstable periodic orbit of the uncontrolled system (1.9). See Figure 1.3 for an illustration of this concept.

By stabilizing different such orbits, one reveals parts of the unknown unstable branches of (1.9). In the publication [20] of chapter 2, we apply an affine linear feedback law to a Poincaré map and the obtained results are summarized in the following section.

1.2.1 Linear control

The idea of using a Poincaré map in combination with linear feedback control to stabilize an unstable periodic orbit was proposed in [38]. It has been applied to a simulation model of an atomic force microscope [36], for example. Also, a data-driven identification method [12, 13] has been used to identify a model for a Poincaré map. This model is then used in combination with linear feedback

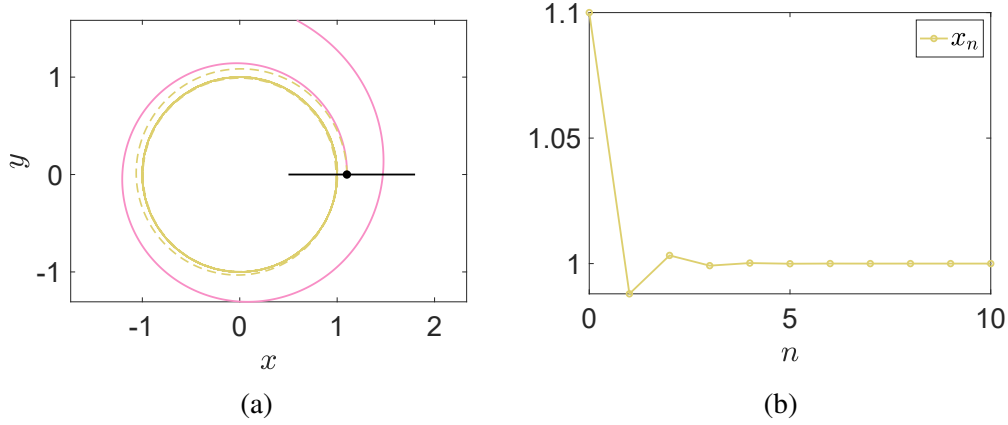


Figure 1.3: An illustration of successful control of an unstable periodic orbit which is given as the unit circle in the two-dimensional plane. (a) The constructed cross section (black line) is a subset of the x -axis. The given system is integrated from the initial condition (black dot) for both the uncontrolled (pink line) and the controlled case (yellow dashed line). The uncontrolled case shows the divergence of the state away from the unstable periodic orbit. In the controlled case, the trajectory converges to the periodic orbit. (b) Corresponding points of intersection $x_0, x_1, \dots, x_{10}$ with the cross section in the controlled case, i.e., a state sequence x_n of the controlled Poincaré map. The sequence converges to the point $x_* = 1$ that corresponds to the periodic orbit.

control for the stabilization of a periodic orbit [11]. There exists also alternative ways of stabilizing periodic orbits. Here we mention the idea of time-delayed feedback control [40, 28, 44, 43], and feedback methods that are based on an approximation by a truncated Fourier series [15, 14, 42].

We assume that the control parameter μ is scalar and propose the feedback law

$$\mu_n = \mu_R + k^T (x_n - x_R), \quad \mu_R \in \mathbb{R} \quad x_n, x_R \in \mathbb{R}^d$$

where x_n denotes the state of the system at time n , $k^T = (k_1 \ k_2 \ \dots \ k_d)$ is the vector of control gains, and μ_R, x_R denote reference values. A state x_* is a fixed point of the controlled Poincaré map if and only if

$$x_* = P(x_*, \mu_R + k^T (x_* - x_R)).$$

This holds if and only if the pair (x_*, μ_*) , where $\mu_* = \mu_R + k^T(x_* - x_R)$, is an intersection point of the manifold given by $\{(x, \mu) \in \mathbb{R}^d \times \mathbb{R} : x = P(x, \mu)\}$ and the hyperplane defined by $\{(x, \mu) \in \mathbb{R}^d \times \mathbb{R} : \mu = \mu_R + k^T(x - x_R)\}$. Thus, by adjusting the reference values and the control gains, one can select points on the stable and unstable branches of the Poincaré map. If we define $\delta x_n = x_n - x_*$, the linearized dynamics of the controlled Poincaré map $x_{n+1} = P(x_n, \mu_R + k^T(x_n - x_R))$ in the neighborhood of the fixed point is given by

$$\delta x_{n+1} = \frac{\partial P}{\partial x}(x_*, \mu_*) \delta x_n + \frac{\partial P}{\partial \mu}(x_*, \mu_*) k^T \delta x_n. \quad (1.11)$$

With the abbreviations

$$A = (\partial P / \partial x)(x_*, \mu_*) \quad (1.12)$$

$$b = (\partial P / \partial \mu)(x_*, \mu_*) k^T \quad (1.13)$$

it holds that

$$\delta x_{n+1} = (A + b k^T) \delta x_n. \quad (1.14)$$

If all eigenvalues of the $d \times d$ matrix $A + b k^T$ have an absolute value less than one, the fixed point x_* is asymptotically stable. This means, that if the controlled Poincaré map is initialized sufficiently close to x_* , the iterates $x_0, x_1, x_2, \dots$ converge to the fixed point and the corresponding chosen parameter values $\mu_0, \mu_1, \mu_2, \dots$ converge to μ_* .

If the $d \times d$ matrix $(b \quad Ab \quad \dots \quad A^{d-1}b)$ is nonsingular, it follows that by adjusting the control gains, the eigenvalues of $A + b k^T$ can be freely placed in the complex plane. In particular, it is then possible to place the eigenvalues within the unit circle to ensure stability. The condition on the matrix $(b \quad Ab \quad \dots \quad A^{d-1}b)$ is called controllability [37, 16]. If the gains are chosen such that all eigenvalues of the matrix $A + b k^T$ are zero, then for every initial condition δx_0 , the linear dynamics (1.14) converge in at most d steps to the fixed point $\delta x_* = 0$ [37, 16]. Therefore, such a choice of control gains is referred to as optimal.

In the publication [20] of chapter 2, we apply the above-mentioned method to a two-dimensional non-autonomous system

$$\dot{x} = f(x, t, \mu) = f(x, t + \tau(\mu), \mu). \quad (1.15)$$

If the equations of motion (1.15) are known, the derivatives that appear in (1.11) can be calculated numerically in the following way. By definition of the Poincaré

map, we have

$$\frac{\partial P}{\partial x}(x_*, \mu_*) = \frac{\partial \phi}{\partial x}(T, x_*, \mu_*) \quad (1.16)$$

$$\frac{\partial P}{\partial \mu}(x_*, \mu_*) = \frac{\partial \phi}{\partial \mu}(T, x_*, \mu_*) + \dot{\phi}(T, x_*, \mu_*)\tau'(\mu_*) \quad (1.17)$$

$$= \frac{\partial \phi}{\partial \mu}(T, x_*, \mu_*) + f(x_*, T, \mu_*)\tau'(\mu_*), \quad (1.18)$$

where ϕ denotes the solution of the differential equation (1.15), $T = \tau(\mu_*)$ is the period of the given periodic orbit, and $\tau'(\mu_*)$ denotes the derivative of τ with respect to the parameter μ evaluated at μ_* . The relevant derivatives $X(t) = (\partial\phi/\partial x)(t, x_*, \mu_*)$ and $y(t) = (\partial\phi/\partial\mu)(t, x_*, \mu_*)$ satisfy the initial value problems

$$\dot{X}(t) = \frac{\partial f}{\partial x}(\phi(t, x_*, \mu_*), t, \mu_*)X(t), \quad X(0) = I \quad (1.19)$$

$$\dot{y}(t) = \frac{\partial f}{\partial x}(\phi(t, x_*, \mu_*), t, \mu_*)y(t) + \frac{\partial f}{\partial \mu}(\phi(t, x_*, \mu_*), t, \mu_*), \quad y(0) = 0, \quad (1.20)$$

where I denotes the 2×2 identity matrix. This can be seen by differentiating the equations

$$\dot{\phi}(t, x, \mu) = f(\phi(t, x, \mu), t, \mu), \quad \phi(0, x, \mu) = x$$

with respect to the state and the parameter. Thus, by integrating (1.15), (1.19), and (1.20) numerically, one obtains the derivatives of the Poincaré map via equations (1.16) and (1.18). For the two-dimensional system (1.15) it follows by requiring vanishing eigenvalues, see [20], that the optimal control gains are given by

$$k^T = \frac{u^T \det(A) + v^T \text{tr}(A)}{v^T b},$$

where

$$\begin{aligned} u^T &= (b_2 \quad -b_1) \\ v^T &= (-(\text{adj}(A)b)_2 \quad (\text{adj}(A)b)_1). \end{aligned}$$

Here, A and b denote the partial derivatives of the Poincaré map with respect to the state x and the control parameter μ , see equations (1.12) and (1.13), $\det(A)$

is the determinant, $\text{tr}(A)$ refers to the trace, and $\text{adj}(A)$ denotes the adjugate [23] of the matrix A . Furthermore, b_1, b_2 denote the components of the vector b and $(\text{adj}(A)b)_1, (\text{adj}(A)b)_2$ are the components of the vector $\text{adj}(A)b$.

The proposed feedback scheme can be used to track and stabilize multiple points on the fixed point curve $\{(x, \mu) \in \mathbb{R}^d \times \mathbb{R} : x = P(x, \mu)\}$ to obtain the bifurcation diagram with bifurcation parameter μ . Suppose (x_*, μ_*) denotes a stabilized point on this one-dimensional manifold. We select the new reference values μ_R, x_R along the tangent vector of the curve at the point (x_*, μ_*) , refer to e.g. [21, 22, 33, 3, 43]. This is done by computing the jacobian matrix of the map $F : (x, \mu) \mapsto P(x, \mu) - x$ at the point (x_*, μ_*) . With (1.12) and (1.13), this matrix is given by $DF(x_*, \mu_*) = \begin{pmatrix} A - I & b \end{pmatrix}$ and the tangent space of the manifold at the point (x_*, μ_*) is the kernel of $DF(x_*, \mu_*)$. The vector

$$\begin{pmatrix} -\text{adj}(A - I)b \\ \det(A - I) \end{pmatrix}$$

spans this kernel. Thus, we set the new reference values to $(x_R, \mu_R) = (x_* + dx, \mu_* + d\mu)$, where $(dx, d\mu)$ is any vector that is parallel to the basis vector of the tangent space. Continuing in this way, the feedback control stabilizes unstable states along the fixed point curve of the uncontrolled system resulting in a parameter-dependent bifurcation diagram. For a numerical illustration of this method applied to a Toda oscillator, we refer the reader to the paper [20] in chapter 2. There, we also discuss the case where only some components of the state x are measurable. In this case, a feedback law $\mu = h(x)$ cannot be implemented but it is possible, for example, to use delayed observations of the measurable components.

Without an analytically given form of the equations of motion (1.15), the required derivatives needed for computing stabilizing control gains are not directly accessible. Also, even in the case where a stabilizing controller is used, the system has to be initialized sufficiently close to the fixed point to be stabilized. The set of initial values that lead to convergence of the controlled system to the fixed point is called the basin of attraction of the fixed point. We aim for a control design that leads to a large basin of attraction. The results achieved with respect to this goal are summarized in the following section.

1.2.2 Nonlinear control

Linear control design is based solely on the linearization of the system at the fixed point to be stabilized. However, the resulting basin of attraction is determined by

higher-order terms, i.e., the nonlinearities. Therefore, a promising strategy is to design a control law that minimizes the effect of these nonlinearities. In the best case scenario, the nonlinearities are cancelled and the resulting controlled system is linear and stabilized, resulting in a global basin of attraction. This strategy is referred to as feedback linearization. Although feedback linearization is well known for stabilizing stationary states of systems [26, 27], it has, to the best of our knowledge, not yet been used to stabilize periodic states via a Poincaré map.

For an illustration of this strategy, we consider a one-dimensional Poincaré map given by

$$x_{n+1} = P(x_n, \mu) = \mu + f(x_n).$$

With the feedback $\mu = \mu_n = \mu_R + g(x_n)$, the controlled system reads

$$x_{n+1} = \mu_R + g(x_n) + f(x_n). \quad (1.21)$$

A state x_* is a fixed point of (1.21) if and only if

$$x_* = \mu_R + g(x_*) + f(x_*).$$

But this holds if and only if the pair (x_*, μ_*) , where $\mu_* = \mu_R + g(x_*)$, is an intersection point of the manifolds given by $\{(x, \mu) \in \mathbb{R} \times \mathbb{R} : x = \mu + f(x)\}$ and $\{(x, \mu) \in \mathbb{R} \times \mathbb{R} : \mu = \mu_R + g(x)\}$. Hence, by adjusting the reference value μ_R , the fixed point x_* can be selected. The corresponding parameter value μ_* is therefore given by $\mu_* = \mu_R + g(x_*)$. A suitable choice for the function g is to set $g(x) = -f(x)$, since this leads to the controlled system

$$x_{n+1} = \mu_R,$$

that converges in one step to the fixed point given by μ_R , regardless of the initial condition x_0 . In the paper [32] of chapter 3 we apply this strategy to a two-dimensional non-autonomous differential equation describing a periodically forced nonlinear oscillator. We summarize the results below.

Consider the differential equation

$$\dot{x} = v \quad (1.22)$$

$$\dot{v} = -\omega^2 x + \varepsilon \left(-\gamma v - U'(x) - h^{(c)} \cos(\omega t) - h^{(s)} \sin(\omega t) \right) \quad (1.23)$$

already discussed in section 1.1. As also mentioned in this section, by using a first order perturbation expansion in ε , one obtains the Poincaré map

$$\begin{aligned}x_{n+1} &= x_n + \varepsilon \frac{\pi}{\omega^2} \left(-\gamma \omega x_n + 2 \frac{v_n}{\omega} w(x_n^2 + (v_n^2/\omega^2)) + h^{(s)} \right) + \mathcal{O}(\varepsilon^2) \\v_{n+1} &= v_n + \varepsilon \frac{\pi}{\omega} \left(-\gamma v_n - 2x_n w(x_n^2 + (v_n^2/\omega^2)) - h^{(c)} \right) + \mathcal{O}(\varepsilon^2),\end{aligned}$$

where $w : \mathbb{R} \rightarrow \mathbb{R}$ denotes a suitably chosen function. Since the control parameters $h^{(c)}, h^{(s)}$, the amplitudes of the driving force in (1.22) and (1.23), appear in both equations of the perturbed system, we are able to cancel the nonlinearities by choosing the feedback law

$$h_n^{(c)} = h_R^{(c)} - 2x_n w(x_n^2 + (v_n^2/\omega^2)) \quad (1.24)$$

$$h_n^{(s)} = h_R^{(s)} - 2 \frac{v_n}{\omega} w(x_n^2 + (v_n^2/\omega^2)), \quad (1.25)$$

where $h_R^{(c)}$ and $h_R^{(s)}$ denote reference values. This results in the controlled system

$$\begin{aligned}x_{n+1} &= x_n + \varepsilon \frac{\pi}{\omega^2} \left(-\gamma \omega x_n + h_R^{(s)} \right) \\v_{n+1} &= v_n + \varepsilon \frac{\pi}{\omega} \left(-\gamma v_n - h_R^{(c)} \right),\end{aligned}$$

where we have omitted the Landau symbol $\mathcal{O}(\varepsilon^2)$. Consequently, the fixed point of the controlled system is given by

$$x_* = \frac{h_R^{(s)}}{\gamma \omega} \quad (1.26)$$

$$v_* = -\frac{h_R^{(c)}}{\gamma}. \quad (1.27)$$

Hence, by adjusting the reference values $h_R^{(s)}$ and $h_R^{(c)}$, a fixed point can be selected. The stability of the fixed point can be investigated by the linear system

$$\begin{pmatrix} \delta x_{n+1} \\ \delta v_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - \varepsilon(\pi/\omega)\gamma & 0 \\ 0 & 1 - \varepsilon(\pi/\omega)\gamma \end{pmatrix} \begin{pmatrix} \delta x_n \\ \delta v_n \end{pmatrix},$$

where $\delta x_n = x_n - x_*$ and $\delta v_n = v_n - v_*$. Thus, as long as $|1 - \varepsilon(\pi/\omega)\gamma| < 1$, the fixed point is asymptotically stable. For small ε , the perturbation expansion and the resulting linear description of the controlled system are accurate, implying

convergence to the fixed point for a large set of initial conditions. Even for large ε , the feedback law (1.24) – (1.25) performs well. For a detailed discussion of these results, we refer the reader to the paper [32] of chapter 3.

The proposed control scheme can be used to track and stabilize the unstable branch of the fixed point curve $\{(x, v, h^{(c)}) \in \mathbb{R}^2 \times \mathbb{R} : P(x, v, h^{(c)}, 0) = (x, v)\}$, where $P(x, v, h^{(c)}, 0)$ denotes the Poincaré map with $h^{(s)} = 0$. This fixed point curve represents both stable and unstable periodic orbits of the driven nonlinear oscillator (1.22) and (1.23). Using the perturbation expansion of the Poincaré map, one can show that this curve can be parameterized as

$$x(s) = -2w(s)s/h^{(c)}(s) \quad (1.28)$$

$$v(s) = -\gamma\omega^2 s/h^{(c)}(s) \quad (1.29)$$

$$h^{(c)}(s) = \sqrt{s} \sqrt{(\gamma\omega)^2 + (2w(s))^2}, \quad (1.30)$$

where s is a scalar parameter. If $x_* = x(s_*)$, $v_* = v(s_*)$ denotes the fixed point to be stabilized, the needed reference values for doing so are obtained from the equations (1.26) and (1.27). For a numerical illustration of this strategy for both small and large ε , we refer the reader once again to chapter 3.

In order to implement the feedback scheme (1.24) – (1.25), the function w has to be determined. In an experimental setup, one can compute an estimate for w from the stable part of $\{(x, v, h^{(c)}) \in \mathbb{R}^2 \times \mathbb{R} : P(x, v, h^{(c)}, 0) = (x, v)\}$. Points on this stable part can be obtained by fixing the parameter $h^{(c)}$ and letting the experiment run until convergence of the state is observed. The parameter is then slightly modified and the experiment is run again until convergence. This process is repeated until sufficiently many points on the stable branches are obtained. This results in a data set of the form $(h_1^{(c)}, x_1, v_1), (h_2^{(c)}, x_2, v_2), \dots$. For a given fixed point (x_i, v_i) , we compute the corresponding curve parameter s_i by using the identity $x(s)^2 + (v(s)^2/\omega^2) = s$, which follows from equations (1.28) – (1.30). Finally, we perform a least squares fit of equation (1.30) to the data set $(h_1^{(c)}, s_1), (h_2^{(c)}, s_2), \dots$, where $w(s) = w_0 + w_1 s$. The paper [32] of chapter 3 contains an illustration of the resulting basin of attraction, when the control is implemented in this data-based fashion. We reproduce it here in Figure 1.4.

1.2.3 Data-based global control strategy

We have shown that for systems of the form (1.22) – (1.23), the strategy of cancelling the nonlinearities via feedback leads to large basins of attraction. Furthermore, by using the described least squares fit, the control is data-driven, meaning

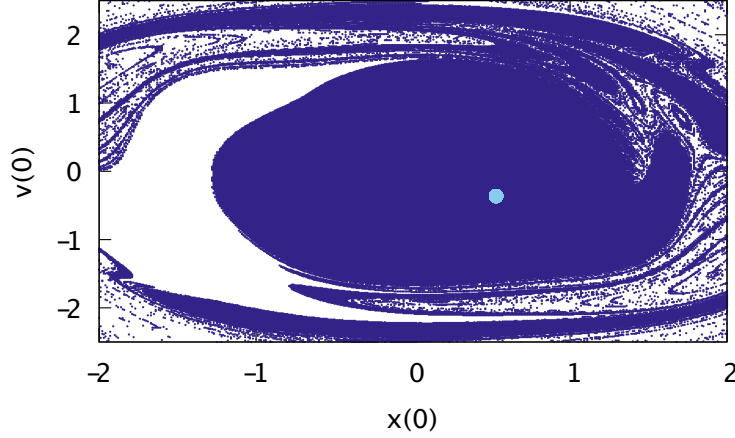


Figure 1.4: Resulting basin of attraction of a stabilized periodic orbit corresponding to the fixed point given by the filled circle (cyan) (after Fig. 12 in the paper [32] of chapter 3). The controlled system is the forced one degree of freedom oscillator model (1.22) – (1.23) in the highly nonlinear regime ($\varepsilon = 1$). The control is based on the nonlinearity cancellation defined by equations (1.24) and (1.25), where the function w is obtained from a least squares fit using data points on the stable branches of the Poincaré map. We kindly thank the journal for permission to reproduce this figure.

that the equations of motion (1.22) – (1.23) are not needed for the implementation of the control scheme. It is only necessary to be able to record time series of the Poincaré map of the system. The paper [30] of chapter 4 extends this data-driven strategy to general parameter-dependent Poincaré maps

$$x_{n+1} = P(x_n, \mu), \quad n = 0, 1, 2, \dots \quad (1.31)$$

without relying on a perturbation expansion. We summarize the results below. Consider the one-dimensional Poincaré map

$$x_{n+1} = x_n - x_n^2 + \mu.$$

Note that each control law $\mu_n = k_1 x_n + k_2 x_n^2$ with $|1 + k_1| < 1$ and $k_2 = 1$ leads to the controlled system

$$x_{n+1} = (1 + k_1)x_n,$$

which converges to zero for every initial condition x_0 . We now aim for a method, that computes such a globally stabilizing set of control gains solely from recorded

time series of general Poincaré maps (1.31). When stabilizing gains are computed directly from time series data, the underlying method is referred to as trajectory-based. These trajectory-based schemes have been used, for example, to stabilize stationary states [18, 17, 8, 9, 1]. In particular, it was shown in [17] that the trajectory-based approach can be combined with nonlinearity cancellation to obtain large basins of attraction for the stabilized stationary states. We extend this idea to the stabilization of periodic orbits. Note, that a key feature of such a method is that there is no explicit model identification step involved, as is the case, for example, with the control-based dynamic mode decomposition [39].

Explanation of the main ideas for a linear control system

Trajectory-based methods are usually formulated within the framework of linear control theory. Therefore, we assume for the moment that the Poincaré map is given as the linear system

$$x_{n+1} = Ax_n + b\mu_n, \quad x_n \in \mathbb{R}^d, \mu_n \in \mathbb{R}, \quad (1.32)$$

where A is a $d \times d$ matrix and $b \in \mathbb{R}^d$. For a fixed initial condition x_0 and a sequence of N parameter values $\mu_0, \mu_1, \dots, \mu_{N-1}$, a simulation of (1.32) results in a sequence of $N + 1$ states $x_0, x_1, \dots, x_N$. Including a linear feedback law $\mu_n = k^T x_n$ in (1.32) results in the controlled system

$$x_{n+1} = (A + bk^T) x_n,$$

where $k^T = (k_1 \ k_2 \ \dots \ k_d)$ is the vector of control gains. If all eigenvalues of the matrix $A + bk^T$ have an absolute value less than one, the origin is asymptotically stable. Normally, we require to know A and b in order to check this condition for a given k . However, in [18] it was shown that the matrix $A + bk^T$ can be computed directly from the obtained time series data. In fact, if the matrix

$$\begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_{N-1} \\ x_0 & x_1 & \dots & x_{N-1} \end{pmatrix} \quad (1.33)$$

has rank $d + 1$, the equation

$$\begin{pmatrix} k^T \\ I \end{pmatrix} = \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_{N-1} \\ x_0 & x_1 & \dots & x_{N-1} \end{pmatrix} G,$$

with the $d \times d$ identity matrix I , always has a solution $G \in \mathbb{R}^{N \times d}$. Then it follows with (1.32) that

$$\begin{aligned} A + bk^T &= (b \ A) \begin{pmatrix} k^T \\ I \end{pmatrix} = (b \ A) \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_{N-1} \\ x_0 & x_1 & \dots & x_{N-1} \end{pmatrix} G \\ &= (x_1 \ x_2 \ \dots \ x_N) G. \end{aligned}$$

Thus, we can check in a purely data-driven fashion whether a given k is stabilizing.

On the other hand, if we choose an $N \times d$ matrix G with

$$I = (x_0 \ x_1 \ \dots \ x_{N-1}) G \quad (1.34)$$

such that all eigenvalues of $(x_1 \ x_2 \ \dots \ x_N) G$ have an absolute value less than one, then a stabilizing k is given by $k^T = (\mu_0 \ \mu_1 \ \dots \ \mu_{N-1}) G$. This gives an alternative formulation of finding stabilizing control gains when only times series are at hand. Note that the rank condition on the matrix (1.33) also implies that the equation (1.34) always has a solution. In the following, we use the abbreviations $X = (x_0 \ x_1 \ \dots \ x_{N-1})$, $Y = (x_1 \ x_2 \ \dots \ x_N)$, and $U = (\mu_0 \ \mu_1 \ \dots \ \mu_{N-1})$.

Now, we reformulate the condition on the eigenvalues in terms of the Lyapunov function $V(x) = x^T P x$, where P is symmetric and $x^T P x > 0$ holds for all nonzero $x \in \mathbb{R}^d$. We have $V((YG)^T x) - V(x) = x^T (YGP G^T Y^T - P) x$. Consequently, if

$$x^T (YGP G^T Y^T - P) x < 0 \quad (1.35)$$

for all nonzero $x \in \mathbb{R}^d$, then the origin is asymptotically stable for the system $x_{n+1} = (YG)^T x_n$. However, YG has the same eigenvalues as $(YG)^T$, so the same holds for the original system $x_{n+1} = YG x_n$. If we define $Q = GP$, the condition (1.35) reads

$$x^T (YQP^{-1} Q^T Y^T - P) x < 0. \quad (1.36)$$

Thus, if we find a positive definite symmetric matrix P and a matrix Q with $XQ = P$ such that $YQP^{-1} Q^T Y^T - P$ is negative definite, the gain $k^T = UQP^{-1}$ is stabilizing [18]. But $YQP^{-1} Q^T Y^T - P$ is negative definite with a positive definite P if and only if the symmetric matrix

$$\begin{pmatrix} P & Q^T Y^T \\ YQ & P \end{pmatrix}$$

is positive definite, as follows by using the Schur complement [41]. This is an example of a linear matrix inequality. Efficient solvers such as CVX [24] can be used for the numerical computation of P and Q that satisfy the conditions

$$P = P^T, \quad XQ = P, \quad \begin{pmatrix} P & Q^T Y^T \\ YQ & P \end{pmatrix} \text{ is positive definite.}$$

The computed matrices can then be used to calculate a stabilizing control gain via $k^T = UQP^{-1}$.

Extension to nonlinear control

For an illustration of how this method can be combined with nonlinearity cancellation, we consider the one-dimensional Poincaré map

$$x_{n+1} = f(x_n) + b\mu_n, \quad f(0) = 0$$

and the corresponding time series data. Assuming that the function is sufficiently well approximated by the first M coefficients of its Taylor series at $x = 0$, we write

$$x_{n+1} = a_1 x_n + a_2 x_n^2 + \dots + a_M x_n^M + b\mu_n + \mathcal{O}(x_n^{M+1}).$$

With the nonlinear feedback law given by $\mu_n = k^T h(x_n)$, the controlled system reads

$$x_{n+1} = a^T h(x_n) + b k^T h(x_n) + \mathcal{O}(x_n^{M+1}) = (b \quad a^T) \begin{pmatrix} k^T \\ I \end{pmatrix} h(x_n) + \mathcal{O}(x_n^{M+1}),$$

where $a^T = (a_1 \ a_2 \ \dots \ a_M)$ contains the unknown Taylor coefficients of f , $k^T = (k_1 \ k_2 \ \dots \ k_M)$ is the vector of control gains, and $h(x) = (x, x^2, \dots, x^M)$. Similar to equation (1.34), we choose an $N \times M$ matrix G that satisfies

$$I = (h(x_0) \ h(x_1) \ \dots \ h(x_{N-1})) G, \quad (1.37)$$

which is always possible if the matrix $(h(x_0) \ h(x_1) \ \dots \ h(x_{N-1}))$ has rank M . With the control gain defined by $k^T = (\mu_0 \ \mu_1 \ \dots \ \mu_{N-1}) G$, the controlled system reads

$$x_{n+1} = (x_1 \ x_2 \ \dots \ x_N) G h(x_n) + \mathcal{O}(x_n^{M+1}).$$

Using the partitioning of $G = (G_1 \ G_2)$, $G_1 \in \mathbb{R}^{N \times 1}$, $G_2 \in \mathbb{R}^{N \times (M-1)}$, we can also write

$$x_{n+1} = (x_1 \ x_2 \ \dots \ x_N) G_1 x_n + (x_1 \ x_2 \ \dots \ x_N) G_2 \begin{pmatrix} x_n^2 \\ x_n^3 \\ \vdots \\ x_n^M \end{pmatrix} + \mathcal{O}(x_n^{M+1}). \quad (1.38)$$

Now, the matrix G is chosen such that $|(x_1 \ x_2 \ \dots \ x_N) G_1| < 1$ and the norm $\|(x_1 \ x_2 \ \dots \ x_N) G_2\|_2$ corresponding to the weight of the nonlinearities is minimized. Again, this can be done by solving an optimization problem

$$\underset{P \in \mathbb{R}, Q \in \mathbb{R}^{N \times 1}, G_2 \in \mathbb{R}^{N \times (M-1)}}{\text{minimize}} \quad \|Y G_2\|_2$$

subject to

$$H Q = \begin{pmatrix} P \\ 0_{(M-1) \times 1} \end{pmatrix} \quad (1.39)$$

$$H G_2 = \begin{pmatrix} 0_{1 \times (M-1)} \\ I_{M-1} \end{pmatrix} \quad (1.40)$$

$$\begin{pmatrix} P & Q^T Y^T \\ Y Q & P \end{pmatrix} \text{ is positive definite,} \quad (1.41)$$

where $H = (h(x_0) \ h(x_1) \ \dots \ h(x_{N-1})) \in \mathbb{R}^{M \times N}$, $Y = (x_1 \ x_2 \ \dots \ x_N) \in \mathbb{R}^{1 \times N}$, $0_{k \times l}$ denotes the $k \times l$ zero matrix, and I_k is the $k \times k$ identity matrix. The CVX solver can also be used for this purpose [17, 24]. Suppose that P, Q, G_2 denotes a solution to this optimization problem. With the definition $G_1 = Q P^{-1}$, it follows from the equations (1.39) and (1.40) that $H G = I$, which is equation (1.37). Consequently, setting $k^T = (\mu_0 \ \mu_1 \ \dots \ \mu_{N-1}) G$ leads to the controlled system (1.38). But since the condition (1.41) holds, it follows that $|(x_1 \ x_2 \ \dots \ x_N) G_1| < 1$, as was explained in the linear setting. In addition, the controller minimizes the nonlinearities. Thus, if the function f is sufficiently well approximated by the first M coefficients of its Taylor series, it can be expected that the controller will lead to a large basin of attraction of the stabilized fixed point $x_* = 0$.

Instead of using a Taylor series for approximation, it is also possible to choose different basis functions. Furthermore, what has just been said can be immediately

extended to higher-dimensional Poincaré maps, where even a multidimensional parameter μ can be taken into account. For a numerical illustration of the method, we refer the reader to the paper [30] of chapter 4.

While the above discussion was based on the assumption that the Poincaré map is linear in the parameter μ , the method can also be applied to general Poincaré maps of the form

$$x_{n+1} = f(x_n, \mu_n).$$

By defining the extended state vector (x_n, μ_n) and the new control input v_n , the extended system reads

$$\begin{aligned} x_{n+1} &= f(x_n, \mu_n) \\ \mu_{n+1} &= v_n, \end{aligned}$$

which is linear in the new control input.

We have shown that if the given Poincaré map is sufficiently well approximated by a set of basis functions, a stabilizing controller with a large basin of attraction can be obtained from time series data. So far, we have restricted our discussion to the case of stabilizing the pair $(x_*, \mu_*) = (0, 0)$. If the fixed point to be stabilized x_* and the corresponding parameter μ_* are known but not both are zero, we transform the system by defining new variables $y_n = x_n - x_*$, $v_n = \mu_n - \mu_*$, and apply the proposed method to the system

$$y_{n+1} = \tilde{P}(y_n, v_n) = P(y_n + x_*, v_n + \mu_*) - x_*$$

for which $\tilde{P}(0, 0) = 0$ holds.

Now, if the obtained controller $\mu_n = g(x_n)$ stabilizes a known pair (x_*, μ_*) on the fixed point manifold $\{(x, \mu) \in \mathbb{R}^d \times \mathbb{R} : P(x, \mu) = x\}$ of the uncontrolled Poincaré map, further, but now unknown, points on this manifold can be selected by including reference values μ_R, x_R in the control law. Thus, as explained at the beginning of this section, we use the control law $\mu_n = \mu_R + g(x_n - x_R)$. Note, that it is typically possible to find a pair (x_*, μ_*) on the stable branches of the fixed point manifold by initialization, which can serve as a starting point for this strategy.

Tracking reference values

In general, the reference value x_R differs from the actual fixed point x_* that is stabilized by the control. In the case of a one-dimensional Poincaré map, it is

typically possible to obtain a good estimate of an interval in which unstable fixed points lie. That is, one knows that for any reference x_R in that interval, there exists a pair (x_*, μ_*) such that $P(x_*, \mu_*) = x_*$ and $x_R = x_*$. However, the corresponding parameter μ_* is unknown. In this case, the proposed trajectory-based method can be applied to the extended system

$$x_{n+1} = P(x_n, \mu_n), \quad P(0, 0) = 0 \quad (1.42)$$

$$v_{n+1} = v_n - P(x_n, \mu_n), \quad (1.43)$$

where $x_n, \mu_n, v_n \in \mathbb{R}$. Under the assumption that the obtained controller $\mu_n = g(x_n, v_n)$ cancels the nonlinearities of the system (1.42) – (1.43), the controlled system reads

$$\begin{pmatrix} x_{n+1} \\ v_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ v_n \end{pmatrix},$$

where all eigenvalues of $A \in \mathbb{R}^{2 \times 2}$ have an absolute value less than one. The controller is now included in the system

$$x_{n+1} = P(x_n, \mu_n), \quad (1.44)$$

$$v_{n+1} = v_n + x_R - P(x_n, \mu_n). \quad (1.45)$$

Thus, one obtains the controlled system

$$\begin{pmatrix} x_{n+1} \\ v_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ v_n \end{pmatrix} + \begin{pmatrix} 0 \\ x_R \end{pmatrix},$$

which converges to the point

$$\begin{aligned} \begin{pmatrix} \bar{x} \\ \bar{v} \end{pmatrix} &= \lim_{n \rightarrow \infty} \begin{pmatrix} x_n \\ v_n \end{pmatrix} = \lim_{n \rightarrow \infty} \left(A^n \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \right) + \left(\lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} A^i \right) \begin{pmatrix} 0 \\ x_R \end{pmatrix} \\ &= \left(\lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} A^i \right) \begin{pmatrix} 0 \\ x_R \end{pmatrix} = (I - A)^{-1} \begin{pmatrix} 0 \\ x_R \end{pmatrix} \end{aligned}$$

for every initial condition (x_0, v_0) . The corresponding control converges to

$$\bar{\mu} = \lim_{n \rightarrow \infty} g(x_n, v_n) = g(\bar{x}, \bar{v})$$

and from equation (1.42) it follows that

$$\bar{x} = \lim_{n \rightarrow \infty} x_{n+1} = P(\bar{x}, \bar{\mu}). \quad (1.46)$$

But equation (1.45) implies that $\bar{x} = x_R$ and if the unknown parameter μ_* is the unique value such that $x_R = P(x_R, \mu_*)$, it follows from equation (1.46) that $\bar{\mu} = \mu_*$. Thus, the controller stabilizes the pair given by the reference value x_R and the corresponding parameter μ_* . The strategy of augmenting the system with an equation of the form (1.45) and designing a controller for the extended system in order to track a given reference is referred to as integral control [37]. The paper [30] of chapter 4 contains a numerical illustration of this method applied to a two-dimensional autonomous differential equation and the corresponding one-dimensional Poincaré map. We reproduce these numerical illustrations here in Figures 1.5 and 1.6.

1.2.4 Applications of control in experimental setups

Before concluding this summary, we would like to comment on the papers contained in chapters 5, 6, and 7. These have not been mentioned so far, but are also closely related to the methods mentioned above. The paper [31] of chapter 5 considers a time-discrete system

$$\begin{aligned} x_{n+1} &= f(x_n, \mu_n) \\ y_n &= h(x_n), \end{aligned}$$

where y_n denotes the measurement at time n . Here, f describes the discretized equations of motion of a nonlinear ball-on-a-wheel system, but both f and h are assumed to be unknown, see Figure 1.7.

For a given initial condition x_0 and a sequence of parameter values $\mu_0, \mu_1, \dots$, we record the sequence of measurements $y_0, y_1, \dots$. This data is then used for the identification of a Koopman predictor [34], which is included in a model predictive control scheme, as was proposed in [29]. The resulting output feedback control $\mu_n = g(y_n)$ drives the system to a prescribed reference y_R . Since a Poincaré map is also given as a time-discrete dynamical system, this method could be applied directly for the stabilization of periodic states. In particular, it covers the more realistic case where, for example, only some components of the state x_n can be measured. This provides an outlook for further research in this area.

The paper [10] of chapter 6 contains the experimental results obtained from stabilizing unstable periodic orbits of a scanning force microscope. Although we

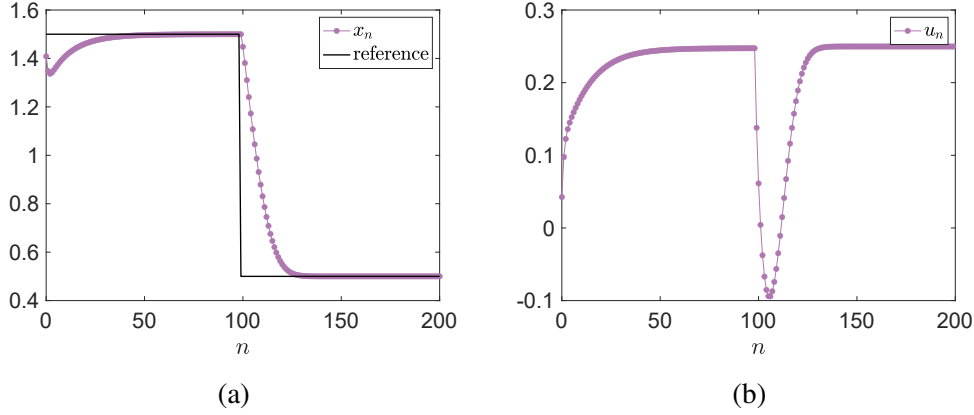


Figure 1.5: Simulated time series of a controlled Poincaré map $x_{n+1} = P(x_n, u_n)$ corresponding to a two-dimensional autonomous differential equation (after Figure 6 in the paper [30] of chapter 4). The control signal is set to $u_n = k^T h(x_n, v_n)$ in the extended system $x_{n+1} = P(x_n, u_n)$, $v_{n+1} = v_n + x_R - P(x_n, u_n)$, where k^T is obtained from the solution of the optimization problem discussed above and h is defined as $h(x, v) = (x, v, x^2, xv, v^2, x^3, x^2v, xv^2, v^3)$. For $0 \leq n \leq 98$, the reference is set to $x_R = 1.5$, while for $99 \leq n \leq 200$ we choose $x_R = 0.5$. This induces a switching from the stable periodic orbit given by the fixed point 1.5 to the unstable periodic orbit corresponding to the fixed point 0.5. (a) Time series of the state x_n together with the reference signal x_R . (b) Time series of the feedback control signal u_n . We kindly thank the journal for permission to reproduce this figure.

used a Fourier discretization in combination with a proportional-integral control for this purpose, a control scheme based on a Poincaré map could possibly also be applied in future work.

Finally, the paper [19] of chapter 7 contains further experimental results. In particular, we propose a control scheme that tracks and stabilizes unstable branches of stationary states including the bifurcation points. Furthermore, the stability information about the uncontrolled stationary states is also obtained in a data-driven manner. The proposed method is applied to a physical experiment of the Zeeman catastrophe machine.

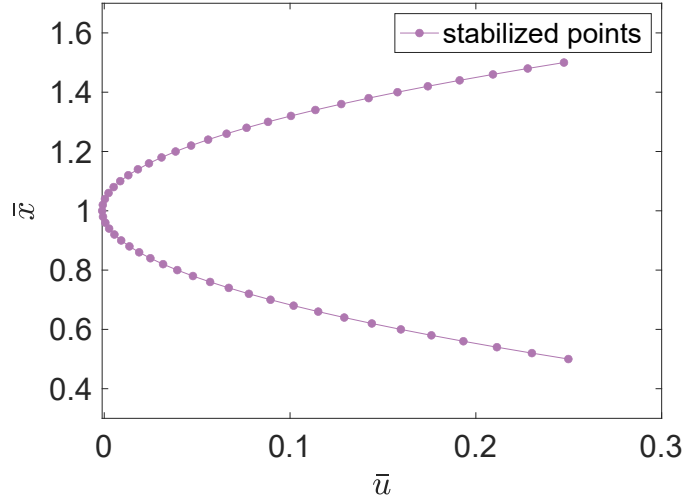


Figure 1.6: Bifurcation diagram with parameter $\bar{u}$: A set of stabilized points on the stable and unstable branch of a Poincaré map $x_{n+1} = P(x_n, u_n)$ corresponding to a two-dimensional autonomous differential equation (after Figure 5 in the paper [30] of chapter 4). Here, the controller $u_n = k^T h(x_n, v_n)$ is used in the extended system $x_{n+1} = P(x_n, u_n)$, $v_{n+1} = v_n + x_R - P(x_n, u_n)$, where k^T is obtained from the solution of the optimization problem discussed above. The function h is defined as $h(x, v) = (x, v, x^2, xv, v^2, x^3, x^2v, xv^2, v^3)$. In order to drive the system to different points on the branches, the reference x_R is varied. We kindly thank the journal for permission to reproduce this figure.

1.3 Conclusion

In this work, we proposed multiple strategies to stabilize unstable periodic orbits with a suitable dynamic adjustment of a parameter. In particular, we have dealt with the situation where the equations of motion of the given system are unknown. In this case, we only assume that time series of a Poincaré map are accessible. That is, one is able to measure the state of the system at times determined by the constructed Poincaré map. The parameter-based control at a given time instant n is computed based on the state measurement at time n . First, we considered the scenario where the parameter depends linearly on the state, resulting in a linear state feedback control scheme. By including reference values in this control law, different periodic orbits of the uncontrolled system can be selected and sta-

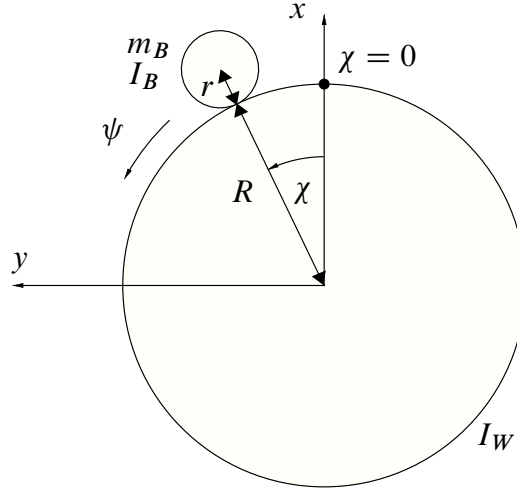


Figure 1.7: Illustration of a ball-on-a-wheel system (after Fig. 1 in the paper [31] of chapter 5). A ball with mass m_B and mass moment of inertia I_B is balanced on a driven wheel. For further details see [31]. We kindly thank for permission to reproduce this figure.

bilized. In particular, if the underlying equations of motion are available, optimal control gains can be computed. These lead to convergence of the linearized controlled system to the selected fixed point in at most d steps, where d denotes the state dimension. The derivatives used for the computation of the optimal control gains can also be utilized to update the reference values, if the method is applied to track and stabilize an unstable branch of periodic orbits of the uncontrolled system. However, since linear feedback control is based exclusively on the linearization of the system at the fixed point to be stabilized, the resulting basin of attraction can be very small. This means that for successful stabilization, the system must be initialized extremely close to the fixed point, even in the case when optimal control gains are used. After all, the basin of attraction is determined by the nonlinearities of the given system. This motivates the use of a nonlinear feedback control that minimizes the intrinsic nonlinearities of the system and at the same time stabilizes the linear part. In the best case scenario, all nonlinearities are cancelled and the controlled system behaves as a stable linear system, resulting in a global basin of attraction. First, we proposed a design of such a nonlinear feedback control for systems that can be described by a one degree of freedom oscillator model. While for the initial perturbation-theory-based construction of the controller, we assumed that the model is weakly nonlinear, numerical studies

showed that the control scheme also works in cases of strong nonlinearity. In particular, large basins of attraction are obtained for the controlled system. We have also showed how the feedback law can be obtained in a data-driven way using a least squares fit to data points on the stable branches of the fixed point curve corresponding to the uncontrolled system. Thus, even if a perturbation analysis of the Poincaré map is not possible, the control scheme can still be implemented in this data-based fashion. Finally, we proposed a data-based method that applies the idea of nonlinearity cancellation to general Poincaré maps. This strategy is based on an input-state time series of the given Poincaré map. That is, for a given initial condition and a sequence of parameter values, one computes the resulting sequence of states. Furthermore, one has to select a suitable set of basis functions that are used to minimize the nonlinearities of the given system. Suitable control gains are then determined by solving a convex optimization problem, that is based on linear matrix inequalities. In particular, there is no model identification step involved and the control gains are computed directly from the time series data. Also in this case, numerical studies showed that the obtained controller leads to large basins of attraction. For the case where a good estimate for the fixed point to be stabilized is available but the corresponding parameter is unknown, the proposed strategy can be embedded in an integral control scheme. The resulting controlled system converges automatically to the pair given by the estimate of the fixed point and the corresponding parameter. This allows, for example, to switch between two periodic orbits without requiring them to be close to each other, as is the case with linear controllers.

We have a number of ideas for further research in this area. First of all, the presented control methods are based on state feedback. As was already mentioned in the summary, typically only a few components of the state can be measured. While the paper [20] of chapter 2 already discusses this restriction and uses a delayed observation of the measurable component to compensate for this, the impact of using such delay coordinates on a nonlinear feedback controller could be studied. Furthermore, the method of stabilizing periodic orbits by minimizing the nonlinearities could be tested in a real experimental setup, where large basins of attraction are in particular important to deal with possible fluctuations. In this case especially, it will be crucial to choose adequate basis functions. Different such choices could be compared. To circumvent the problem of choosing basis functions, one could apply the theory of dynamic programming to obtain a nonlinear controller [4, 6, 5]. In this case, a cost function is defined that penalizes the distance to a reference state. Then, based on a sequence of low-dimensional optimization problems, a control law is computed that minimizes this cost function.

In fact, since dynamic programming can be naturally formulated in the setup of time-discrete systems, an application to Poincaré maps seems promising.

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1.4 Declaration regarding personal contribution to manuscripts

This dissertation contains six publications and I will outline my own contribution to these below. Regarding the paper [20] of chapter 2, I contributed to the drafting, writing, and implementation of ideas for applying linear time-discrete control to a Poincaré map, including establishing links to control theory concepts such as controllability, optimal control gains and time-delayed observation when only a partial state can be measured. Building on this idea in the paper [32] of chapter 3, I suggested applying the concept of feedback linearization, as used in control engineering, to a Poincaré map resulting from a forced oscillator model in order to obtain large basins of attraction and thus increasing the robustness of the control. Here, I also contributed in the numerical implementation, drafting and writing. For the manuscript [30] in chapter 4, I proposed applying the method of feedback linearization to general Poincaré maps and implemented this using the theory of trajectory-based control of time-discrete systems, including the complete written preparation of the manuscript and the necessary simulations and illustrations. I also contributed the idea of applying Koopman-based control to a ball-on-a-wheel system for the paper [31] of chapter 5, including the written preparation of the manuscript, simulations, and figures. As part of the manuscript [10] of chapter 6, I was involved in the development of control-based methods for the stabilization of unstable periodic orbits in a real atomic force microscope. Finally, for the paper [19] of chapter 7, I played a central role in developing the feedback law, which allows for the control-based continuation of an experiment of the Zeeman catastrophe machine. Furthermore, I build up the laboratory experiment of the Zeeman catastrophe machine and carried out the first experimental verifications of the control scheme.

Chapter 2

Stroboscopic control and tracking of periodic states



Stroboscopic control and tracking of periodic states

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Abstract Numerical continuation tools are nowadays standard methods for the bifurcation analysis of dynamical systems. Unfortunately, the full power of these methods is still unavailable in experiments, in particular, if no underlying mathematical model is at hand. We here aim to narrow this gap by providing control based continuation of periodic states which can be ultimately implemented in real-world experimental setups. Taking inspiration from atomic force microscopy, we develop experimentally relevant control and tracking tools for time periodic solutions in driven nonlinear oscillator systems based on stroboscopic maps.

Keywords Nonlinear dynamics · Control · Experimental bifurcation analysis · Equation-free analysis · Atomic force microscopy

Mathematics Subject Classification 37N35 · 34C15 · 34C55 · 93B52 · 93A30

Anna Dittus, Niklas Kruse, Hannes Wallner, Lukas Böttcher, Ingo Barke, Sylvia Speller and Jens Starke have contributed equally to this work.

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1 Introduction and experimental context

Tools to analyse the phase space structure of dynamical systems are probably at the heart of most subjects in science and engineering. Approaches which exploit linear dynamical properties have a well established tradition, which to some extent dates back more than a century. Linear tools still provide some of the most powerful methods to analyse dynamical behaviour, as most vividly illustrated by the success of linear response theory, which underpins among others large branches of physics and chemistry (see, for example, textbooks such as [1]). Within the context of data science, one should probably also mention the seminal idea of [2] which did plant the seed for a whole universe of successful data analysis tools exploiting the concept of linearity.

With the advent of nonlinear dynamics and chaotic behaviour at about four decades ago, the need to supplement these approaches by suitable nonlinear techniques became apparent (see, for example, [3]). From a plain theoretical perspective, bifurcations as well as unstable solutions are a key to understand the dynamics in systems with a finite number of degrees of freedom. The importance of unstable solutions is also emphasised by the fact that these orbits may provide a skeleton for an otherwise observable stable chaotic behaviour [4]. Powerful numerical tools have been developed to explore the bifurcation and phase space structure of low dimensional dynamical systems, whenever the mathematical equations of motion are accessible [5]. We ultimately

mately aim at taking this idea to the next level and make bifurcation analysis and tracking of unstable solutions available in an experimental context, when no underlying equations of motion are at hand. Equation-free analysis can be viewed as a first step in such a direction since the ideas proposed in [6] have the potential to reduce a microscopic model with a huge number of degrees of freedom to an effective low dimensional macroscopic dynamics where bifurcation analysis becomes a meaningful approach. But contrary to a theoretical analysis the detection and tracking of bifurcations in an experimental setting becomes a challenge as unstable states are not directly accessible. For this reason, one ultimately relies on control techniques which on the one hand are able to stabilise unknown periodic states and which on the other hand have no impact on the target state so that the control force finally vanishes. We ultimately aim at proposing a true model-free approach by developing concepts for phase space reductions and bifurcation analysis to analyse real-world experiments for which no underlying mathematical model is available (see, for example, [7] for a recent application in the context of crowd management), and which cannot be tackled by state of the art methods. In this context, experimental control techniques are a key for stabilising and tracking experimentally unstable states, and these techniques can then even be considered as a kind of novel spectroscopic tool for systems analysis.

There exists a considerable number of non-invasive feedback control methods in the literature which are able to stabilise unstable equilibria or unstable periodic orbits and where in principle no a priori knowledge of the unstable state is required. These methods may be suitable for stabilising physical systems and to implement a model-free approach by identifying effective phase space structures based on plain observations. The class of washout filter-aided feedback control methods introduced in [8–10] can be understood as a proportional-plus-integral control to stabilise unstable stationary points. Linear stability theory can be utilised to prove linear stabilisation of equilibria or periodic orbits of the closed-loop system by washout filter-aided feedback control under certain conditions [10]. The seminal work of Ott, Grebogi and Yorke [11] was based on a similar reasoning where, however, some a priori knowledge of the Poincaré map is required. These essential details, such as an estimate of the target state, can be obtained from data, normally by a suitable preprocessing of experimental time series, and

[12] presents generalisations of these ideas covering stabilisation and tracking of unstable orbits as well as a successful application of the method in a model of atomic force microscopy. Stabilisation of unstable periodic orbits with a feedback control using a time-delayed signal has been introduced in [13] and applications can be found, for example, in [14]. In the same spirit, Sieber and Krauskopf [15] proposed a method based on time-delayed feedback control to stabilise periodic orbits. The authors used pseudo-arclength continuation for tracking of periodic orbits and this method succeeds in experiments even when fold bifurcations occur. An experimental application of this approach for pendula in the vicinity of saddle-node bifurcations is presented in [16]. Pyragas et al. suggested the simplest instalment of a state observer and proportional feedback to deal with the stabilisation of unknown equilibria in experiments [17, 18]. The method has much in common with the previously mentioned washout filter-aided feedback control [9, 19]. These methods have the limitation that tracking around a fold bifurcation fails at the bifurcation point. A numerical implementation of an equation-free approach where a bifurcation diagram at a macroscopic scale is obtained by applying pseudo-arclength continuation with a feedback control law to equations on a microscopic scale can be found in [20]. Application of feedback control with a suitable prediction of target states has been proposed in [21] to explore bifurcation diagrams for laboratory experiments. Along similar lines, an implementation of an experimental continuation scheme based on local proportional–derivative control for forced nonlinear oscillators is presented in [22–25].

While our subsequent analysis will be entirely theoretical, our approach is inspired by real-world experiments on fast time scales where a model-free approach has so far never been put to a test. As a paradigmatic experimental set-up, we consider atomic force microscopy which is a quite versatile method to inspect surfaces and attached structures or objects thereon, down to the nanoscale [26, 27]. The sensor element is a bending micro-cantilever of crystalline silicon or amorphous Si_3N_4 with a tip at one end. Most popular is the dynamic mode, where the cantilever is driven into oscillation via a dither or shaker piezo. A registration laser spot is reflected from the top end of the metal-coated cantilever onto a segmented photodiode. The photocurrents are dynamic, and they serve as a measure of the vertical and lateral displace-

ment of the tip upon cantilever bending or torsion. Many attempts have been undertaken to model force microscopy, using driven, damped harmonic oscillator models with more or less complicated tip-sample contact models. A theory-reality gap persists, mainly due to poorly controllable effects such as dissipation, capillary forces, patch charges, soft crashes, and others. Nonlinear effects are already caused by anharmonicities of the bending cantilever. In frequency scans of dynamic force microscopy close to a surface, a low- and a high-amplitude branch with an unstable branch at intermediate amplitude does exist (cf. Fig. 1b). Jumping between the two stable branches during acquisition of topography is a common distortion, particularly with mobile molecular species, requiring manual intervention. Particularly beyond the resonance frequency of the cantilever a rather broad range of tip-sample separations is prone to such imaging instabilities. In [28], stabilisation of a cantilever oscillation has been managed, leading to reduced noise. For suspended two-dimensional materials, the analysis of nonlinear dynamics enabled determination of elastic and heat transport properties [29,30]. In dynamic force microscopy, two bistabilities can be accessed, one in the attractive (soft interaction) and another in the repulsive (strong interaction) regime [31]. In experiments with control electronics, the unstable branches cannot be reproduced, and the resonance curves take on shark

fin signatures, see Fig. 1b. In addition, chaotic motion of micro-cantilevers has been reported in [32–35].

Atomic force microscopes may therefore serve as a paradigmatic model system to test a model-free approach properly for systems at the nanoscale and on the kHz frequency scale. The device clearly displays instabilities and bifurcations known from dynamical systems theory. Hence, it looks promising to apply suitable control techniques to track unstable periodic orbits and to continue related unstable branches and bifurcation points. We aim at proposing a control scheme which can stabilise unknown periodic states and which does not require any kind of data processing, i.e. the control is based on the measurement of a single data point per period. In addition, the scheme needs to be non-invasive, i.e. control forces have to vanish asymptotically so that the final state is a periodic solution of the system without control. These constraints rule out various of the control techniques mentioned previously; for instance, no time-consuming data processing can be done, e.g. to reconstruct potential target states from observed data. We want to stress again that we see here atomic force microscopy just as a motivation. We do not claim that the control scheme developed here can be instantly implemented in experiments. Nevertheless, we consider the theoretical challenge of designing a control scheme within the limitations described above to be of interest from the plain theoretical point of view. Of course that does not rule out that our ideas may ulti-

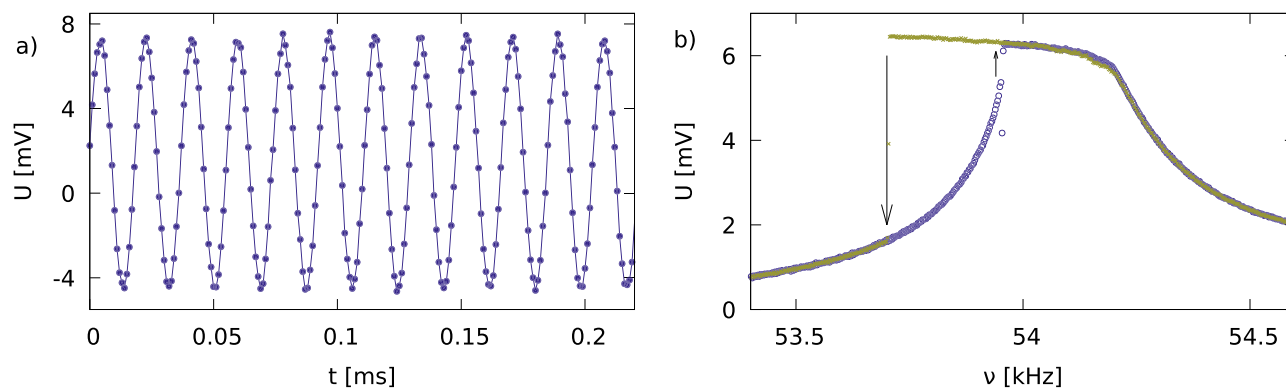


Fig. 1 **a** Experimental time series (voltage at the photodiode vs time) for an atomic force microscope micro-cantilever in dynamic mode with driving frequency of 53.7 kHz. **b** Nonlinear resonance curve (amplitude of the oscillation vs. driving frequency) for an atomic force microscope in dynamic mode. Experimental frequency sweep (shark fin) under ambient conditions, in the vicinity of a Si surface (attractive interaction), expos-

ing a clearly visible jump (left arrow) from a high-amplitude branch to a low-amplitude branch when sweeping from high to low frequencies (cross, bronze). A jump, though much smaller in size (right arrow), from the low-amplitude to the high-amplitude branch appears as well when sweeping from low to high frequencies (circle, blue). The amplitude has been estimated from the Fourier coefficient of the first harmonic (cf. as well a)

mately find an experimental realisation within atomic force microscopy.

We introduce in Sect. 2 a simple driven oscillator model which will be used to illustrate our control design, and we set up the required notation for the formal analysis of the Poincaré map. Linear proportional full-state feedback on a system parameter in the Poincaré map will be used to achieve stabilisation of unstable periodic orbits, and the details are outlined in Sect. 3. The selection of optimal control gains and the analytic procedure for tracking unstable branches are introduced in Sect. 4 where we also cover the limitations of the control design which are determined by controllability conditions known from classical control theory. Various generalisations of the concept, such as the impact of a ceiling function for the control feedback, or the use of delayed observations in cases where full-state feedback is not feasible is covered in Sect. 5. Finally, we also discuss the impact of our theoretical considerations for the real-world application outlined above.

2 Stroboscopic map and periodic states

As a motivation to develop the theoretical concepts for tracking unstable states with a view to perform a data-based experimental bifurcation analysis, let us consider the one-dimensional motion of a particle in an external potential subjected to damping and a periodic harmonic drive with amplitude h

$$\dot{x} = v, \quad \dot{v} = -\gamma v - U'(x) - h \cos(\omega t). \quad (1)$$

For the purpose of a numerical illustration, we will choose later a Toda oscillator [36] in non-dimensional units with

$$U(x) = \exp(-x) + x - 1 \quad (2)$$

and parameter values

$$\gamma = 10^{-2}, \quad h = 8 \times 10^{-3}, \quad \omega = \frac{2\pi}{T}. \quad (3)$$

Equation (1) is an example for a two-dimensional non-autonomous differential equations with T -periodic

time-dependence which can, in general, be written as

$$\dot{z} = f_\mu(z, t), \quad f_\mu(z, t) = f_\mu(z, t + T). \quad (4)$$

Here, the two-dimensional vector z describes the state of the system, f_μ the right-hand sides, and the subscript μ takes the dependence of the system on a static real-valued external parameter into account. We will base our theoretical considerations on the fairly general class of equations given by Eq. (4), and that clearly covers the concrete physical example Eq. (1).

As we are finally interested in stabilising and tracking time-periodic solutions of Eq. (4), it will turn out to be convenient to capture the dynamics in terms of a stroboscopic map. If $z(t) = \xi_\mu(t, z_0)$ denotes the solution of the system Eq. (4) with initial condition $z(0) = z_0$, that means

$$\partial_t \xi_\mu(t, z_0) = f_\mu(\xi_\mu(t, z_0), t), \quad \xi_\mu(0, z_0) = z_0 \quad (5)$$

the stroboscopic map F_μ is given by

$$z_{n+1} = F_\mu(z_n) = \xi_\mu(T, z_n). \quad (6)$$

Here, $z_n = z(T_n)$ denotes the solution at times T_n , where T_n stands for discrete time points with fixed phase of the driving field, e.g. $T_n = nT$ is the end of the n -th period of the drive. In experimental terms, a stroboscopic map is easily realised by recording the output at integer multiples of the period only. While it is obvious how to obtain the stroboscopic map, Eq. (6), based on numerical integration over one period, we finally aim at a scheme which does not require the precise form of the map F_μ and thus can be implemented as well in experiments. For the time being, let us assume that we have access to the dynamics and to the stroboscopic map, say via Eq. (5).

A T -periodic solution of Eq. (4) turns out to correspond to a fixed point of the stroboscopic map, i.e.

$$z_* = F_\mu(z_*) \quad (7)$$

with $z(t) = \xi_\mu(t, z_*) = \xi_\mu(t + T, z_*)$ being the periodic time-dependent state. The condition Eq. (7) determines the fixed point manifold $z_* = z_*(\mu)$ in dependence on the system parameter. Using $z_n = z_* + \delta z_n$, a linear stability analysis of Eq. (6) yields the so-called variational equation

$$\delta z_{n+1} = D F_\mu(z_*) \delta z_n \quad (8)$$

where $DF_\mu(z_*)$ denotes the Jacobian matrix at the fixed point. Stability requires all eigenvalues of the Jacobian to be contained in the unit circle. The so-called Jury criterion [37] gives necessary and sufficient conditions for the stability in time discrete dynamical systems. For a two-dimensional map $F_\mu(z)$, these conditions can be expressed in terms of the trace and the determinant of the two-dimensional Jacobian matrix $DF_\mu(z_*)$ evaluated at the fixed point z_* . The stability criteria read

$$\det(DF_\mu(z_*)) < 1, |\operatorname{tr}(DF_\mu(z_*))| < 1 + \det(DF_\mu(z_*)). \quad (9)$$

To illustrate the considerations so far, we resort to the example defined in Eqs. (1–3). The state vector is given by $z = (x, v)$, and we chose the frequency ω as our external static parameter μ . Integration over a period simply gives the stroboscopic map, and iteration gives a discrete time series $(z_0, z_1, \dots)$ which contains position and velocity at discrete times $T_n = nT$. Numerical simulations demonstrate that this series ultimately may converge to a fixed point $z_* = (x_*, v_*)$. Skipping the transient we record the fixed point, while we change the parameter μ in a quasi-stationary way. While we could simply record each component of the state vector $z_* = (x_*, v_*)$, we chose here the combination $r_*^2 = x_*^2 + v_*^2/\omega^2$ as this quantity would give the actual amplitude of the oscillation if the solution were a plain harmonic function. The result displayed in Fig. 2 clearly shows the usual tilted nonlinear resonance curve with a region of bistability which contains as well an invisible unstable state. This unstable state normally determines the basin boundaries of the two coexisting stable branches via a separatrix. We aim to complete this bifurcation scenario by supplementing the unstable state with a method which can then also be implemented in experiments.

3 Instantaneous full-state feedback

A bifurcation analysis and the corresponding numerical continuation tools can easily supplement the data shown in Fig. 2 with the unstable branch. To achieve such a goal in experiments, one needs to design control tools to stabilise and track unstable states based on observations. In addition, such a control has to be non-

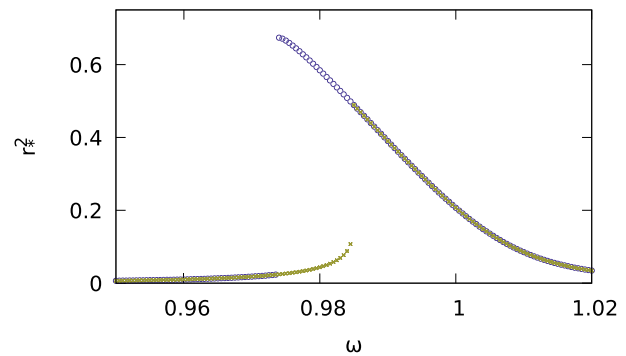


Fig. 2 Nonlinear resonance in the driven Toda oscillator, Eqs. (1) and (2), with parameters as in Eq. (3). Stable fixed points of the stroboscopic map, $r_*^2 = x_*^2 + v_*^2/\omega^2$, in dependence on the driving frequency, ω . Circle (blue): adiabatic parameter downswEEP, cross (bronze) adiabatic parameter upswEEP

invasive as the control should not alter the position of the final unstable state. As we deal here with time periodic solutions of time continuous dynamical systems, we base our approach on the stroboscopic map, Eq. (6), since just the stabilisation of unstable fixed points is at stake. We chose the parameter μ as our control input, so that μ becomes a dynamical quantity μ_n . The simplest way to install a closed-loop control is by linear state feedback. That means on each time step we update the parameter μ by a linear function of the current state, so that

$$\mu_n = \mu_R + \alpha^T \cdot (z_n - z_R). \quad (10)$$

Here, $\alpha^T = (\alpha_1, \alpha_2)$ are the two control gains and μ_R and z_R denote reference values which result in a single offset. The closed-loop dynamics then reads

$$z_{n+1} = F_{\mu_n}(z_n). \quad (11)$$

The fixed point of Eq. (11) is given by (see as well Eq. (10))

$$z_* = F_{\mu_*}(z_*), \quad \mu_* = \mu_R + \alpha^T \cdot (z_* - z_R). \quad (12)$$

Hence, the fixed point z_* and the corresponding parameter value μ_* are determined by the intersection of the fixed point manifold given in Eq. (7) and a hyperplane which is determined by the reference values (μ_R, z_R) . By making the system parameter μ a dynamical quantity, we aim at stabilising fixed points which have been previously unstable, by a suitable choice of the control

gains α^T . Stability of the fixed point is governed again by the variational equation of Eq. (11) which reads

$$\begin{aligned}\delta z_{n+1} &= D\mathbf{F}_{\mu_*}(z_*)\delta z_n + \partial_{\mu_*}\mathbf{F}_{\mu_*}(z_*)\left(\alpha^T \cdot \delta z_n\right) \\ &= A\delta z_n\end{aligned}\quad (13)$$

where the 2×2 square matrix A is given by

$$A = D\mathbf{F}_* + \partial\mathbf{F}_* \otimes \alpha^T. \quad (14)$$

and the contribution caused by the control $\partial\mathbf{F}_* \otimes \alpha^T$ is the dyadic product of the control gains and the coupling of the control action to the dynamical system. Here, and in what follows, we use the shorthand notation

$$D\mathbf{F}_* = D\mathbf{F}_{\mu_*}(z_*), \quad \partial\mathbf{F}_* = \partial_{\mu_*}\mathbf{F}_{\mu_*}(z_*) \quad (15)$$

to abbreviate the Jacobian matrix and the parameter derivative of the system at the fixed point. Stability of the fixed point subjected to control is again determined by the Jury criterion for the matrix A , cf. Eq. (9). To evaluate these conditions, we need the trace and the determinant of the expression Eq. (14). They are simply given by

$$\begin{aligned}\text{tr}(A) &= \text{tr}(D\mathbf{F}_*) + \alpha^T \cdot \partial\mathbf{F}_* \\ \det(A) &= \det(D\mathbf{F}_*) + \alpha^T \cdot \text{adj}(D\mathbf{F}_*)\partial\mathbf{F}_*.\end{aligned}\quad (16)$$

Here, $\text{adj}(D\mathbf{F}_*)$ denotes the adjugate matrix which has the property that the matrix product $D\mathbf{F}_*\text{adj}(D\mathbf{F}_*)$ results in the determinant $\det(D\mathbf{F}_*)$. That means the adjugate is a kind of scaled inverse matrix. Even if the Jacobian $D\mathbf{F}_*$ becomes singular, the adjugate matrix is still well defined.

The Jury criterion for the matrix A (see Eq. (9)) gives three linear inequalities for the control gains α^T . These inequalities determine a simplex, that means a triangle, in the parameter space of control gains. As long as the vectors $\partial\mathbf{F}_*$ and $\text{adj}(D\mathbf{F}_*)\partial\mathbf{F}_*$ are linearly independent, the two equations Eq. (16) can be solved for the control gains α^T for any values of $\text{tr}(A)$ and $\det(A)$. That means we can always find control gains so that the matrix A has eigenvalues which guarantee linear stability, i.e. so-called pole placement is possible. Thus, the region for successful control in the parameter plane of control gains becomes a nonempty triangle. To perform pole placement and to ensure a stable state, one needs to solve Eq. (16) for given left-hand side for the control gains α^T . That requires the coefficient matrix to be non-singular, i.e. that $\{\partial\mathbf{F}_*, \text{adj}(D\mathbf{F}_*)\partial\mathbf{F}_*\}$ is a set

of linearly independent vectors. This condition of linear independence is equivalent to the linear independence of $\{\partial\mathbf{F}_*, D\mathbf{F}_*\partial\mathbf{F}_*\}$, i.e. to the classical condition of controllability. The equivalence is obvious for a non-singular matrix $D\mathbf{F}_*$ but it can also be shown to be valid for singular matrices with a little bit of linear algebra. Hence, our condition for successful control coincides, of course, with the classical engineering controllability condition for the matrix defined in Eq. (14).

Even if the controllability condition is violated, that means if the set $\{\partial\mathbf{F}_*, D\mathbf{F}_*\partial\mathbf{F}_*\}$ is linearly dependent so that $\partial\mathbf{F}_*$ is an eigenvector of $D\mathbf{F}_*$, one can show by a short calculation, using for instance a coordinate system where $D\mathbf{F}_*$ is diagonal, that one can still find control gains so that the system becomes stable, as long as the eigenvector of $D\mathbf{F}_*$ which differs from $\partial\mathbf{F}_*$ corresponds to a stable direction of the system without control. This case corresponds in the following sense to the seminal control algorithm proposed by Ott, Grebogi and Yorke [11] more than three decades ago. The original OGY control algorithm has been essentially motivated by a geometric construction whereby the control force in phase space is applied in the direction of the unstable manifold to push the phase space point on the stable manifold in phase space. The internal dynamics of the system then moves the phase space point towards the desired target state at a rate determined by the stable eigenvalue of the target state. Formally, this successful algorithm violates the condition of controllability. The vector which determines how the control force couples to the system, $\partial\mathbf{F}_*$, is an eigenvector of the Jacobian matrix of the system, $D\mathbf{F}_*$. Thus, the two expressions $\partial\mathbf{F}_*$ and $D\mathbf{F}_*\partial\mathbf{F}_*$ are linearly dependent and the formal condition of controllability is violated. Nevertheless stabilisation works. Controllability is in fact a stronger condition as controllability ensures that by proper control gains eigenvalues of the system with control can be placed at will, a feature which the original OGY algorithm does not share (see as well [38]). Thus, violation of controllability does not necessarily mean that stabilisation fails, as briefly outlined above.

To demonstrate the feasibility of the proposed approach, we illustrate the control again for the driven Toda oscillator, Eqs. (1) and (2). For the control input, we use the parameter ω , i.e. the frequency. This quantity is often easily accessible and adjustable in experiments, but the dynamics normally depends in a very sensitive way on the frequency of the drive. At the end of each period, we record the state of the system given by the

values of $x(T_n)$ and $v(T_n)$. One then uses the linear law, Eq. (10), to determine a new value for the frequency which is then kept constant during the following period. This process is repeated after completion of each period, so we apply the control in a stroboscopic way. The time T_n for the end of the n -th period is now not any longer an integer multiple of a base period. For the control law, Eq. (10), we made up reference values $\mu_R = \omega_R$ and $\mathbf{z}_R^T = (x_R, v_R)$ as well as control gains α^T out of thin air. We demonstrate successful control of the unstable state in the region of bistability, see Fig. 2, simply by time traces, see Fig. 3. Frequency ω_n and amplitude r_n^2 converge to values which are on the unstable branch of the nonlinear resonance curve, cf. Fig. 2. We should, however, remark that control may fail if initial conditions are set far away from the unstable target state, i.e. the scheme so far has occasionally bad global properties.

The basic idea of the stroboscopic control algorithm can be summarised in non-technical terms in the following way. One adjusts an external parameter at the beginning of each period to a new value so that the parameter becomes a time-dependent piecewise constant quantity. The control gains are adjusted in such a way that the time-dependent parameter converges to a constant value, while at the same time the state of the system converges to a periodic solution. By design, this periodic solution is an unstable state of the original dynamics, with a parameter value which is given by the limit obtained during the control process. The offset

in the control law enables one to tune this limit value and to select the target state. Above all, the scheme is able to stabilise an unknown periodic state without the need of any data-based estimation or reconstruction in phase space, and the scheme is non-invasive, even though there is no quantity in the control loop which tends to zero.

4 Optimal control gains and tracking

From a linear perspective optimal control gains result in transients as short as possible, i.e. these gains cause the fixed point to become superstable with both eigenvalues of $\mathbf{A}$ vanishing (see Fig. 3 for an illustration). Solving Eqs. (16) with $\text{tr}(\mathbf{A}) = \det(\mathbf{A}) = 0$ for α^T gives

$$\alpha^T = -\frac{\mathbf{u}^\perp \det(\mathbf{D}\mathbf{F}_*) + \mathbf{v}^\perp \text{tr}(\mathbf{D}\mathbf{F}_*)}{\mathbf{v}^\perp \cdot \partial \mathbf{F}_*} \quad (17)$$

where

$$\begin{aligned} \mathbf{u}^\perp &= ((\partial \mathbf{F}_*)_2, -(\partial \mathbf{F}_*)_1) \\ \mathbf{v}^\perp &= (-(\text{adj}(\mathbf{D}\mathbf{F}_*)\partial \mathbf{F}_*)_2, (\text{adj}(\mathbf{D}\mathbf{F}_*)\partial \mathbf{F}_*)_1) \end{aligned} \quad (18)$$

are orthogonal vectors to the set $\{\partial \mathbf{F}_*, \text{adj}(\mathbf{D}\mathbf{F}_*)\partial \mathbf{F}_*\}$. The analytic computation of these optimal control gains requires the knowledge of $\mathbf{D}\mathbf{F}_*$ and $\partial \mathbf{F}_*$. It is not surprising that without any knowledge about the underlying dynamics an a priori determination of optimal control gains cannot be achieved. It is in fact already

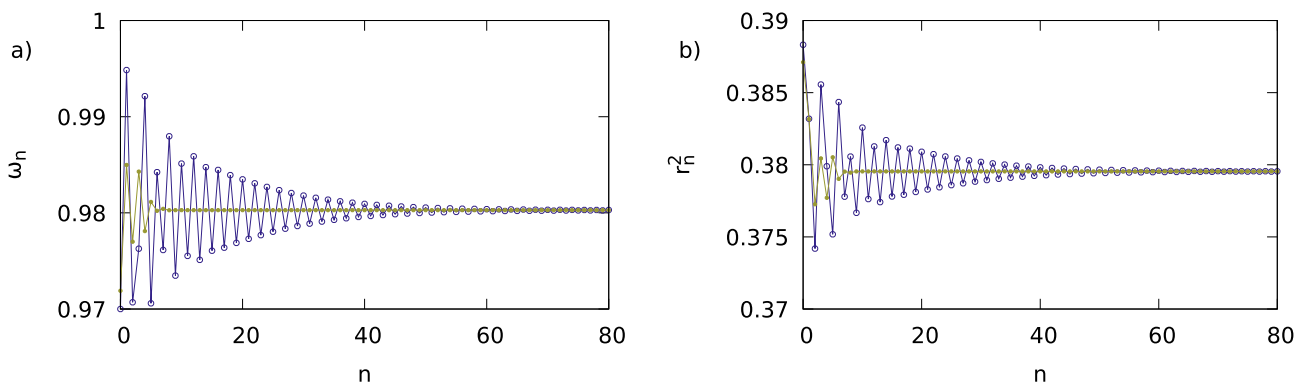


Fig. 3 Successful control of the unstable state in the driven Toda oscillator, Eqs. (1) and (2), with parameter setting according to Eq. (3), see Fig. 2 for data without control. The frequency ω has been used as control input μ . Reference values in the control law Eq. (10) are $\omega_R = 0.97$, $x_R = (z_R)_1 = -0.28$, $v_R = (z_R)_2 = -0.54$, while gains have been chosen as $\alpha_1 = 4.9$

and $\alpha_2 = 8.5$. Time series of the stroboscopic map with control, Eq. (11), and initial condition $x(0) = x_R$, $v(0) = v_R$ (blue, open symbols, lines are a guide to the eye); **a** frequency value ω_n , **b** amplitude estimate $r_n^2 = x_n^2 + v_n^2/\omega_n^2$. For comparison, the corresponding time series for optimal control gains, see Sect. 4, are shown as well (bronze, full symbols)

quite remarkable that we were able to show that for a generic dynamical system there exists a range of gains where stabilisation of the target state is possible. Based on this observation, it is then possible to find such gains in experiments and optimise their values when looking, e.g. at the transient dynamics or at the spectral width of residual signals, as the latter gives an estimate of the eigenvalues of the system subjected to control (see, for example, [39] for an application of such a general idea in a different control setting). Hence, maximising such linewidths by tuning the gains will result in optimal control gains as well.

Let us here study in more detail the analytic impact of Eq. (17). At least in numerical simulations the required derivatives can be quite easily computed from the variational equation of Eq. (5), along the lines of the stroboscopic map. In fact, the Jacobian matrix

$$DF_{\mu}(z_0) = \partial_{z_0} \xi_{\mu}(T, z_0) = X_{\mu}(T, z_0) \quad (19)$$

follows from the derivative of the time-dependent solution with respect to the initial condition and thus results in the so-called fundamental matrix X_{μ} . Taking the derivative of Eq. (5) with respect to the initial condition, we obtain

$$\begin{aligned} \partial_t X_{\mu}(t, z_0) \\ = Df_{\mu}(\xi_{\mu}(t, z_0), t) X_{\mu}(t, z_0), \quad X_{\mu}(0, z_0) = \mathbf{1}. \end{aligned} \quad (20)$$

Similarly, we have for the derivative of the stroboscopic map with respect to the parameter

$$\begin{aligned} \partial_{\mu} F_{\mu}(z_0) &= \partial_{\mu} \xi_{\mu}(T, z_0) \\ &+ \partial_T \xi_{\mu}(T, z_0) \partial_{\mu} T = \eta_{\mu}(T, z_0) \\ &+ f_{\mu}(F_{\mu}(z_0), T) \partial_{\mu} T \end{aligned} \quad (21)$$

where the derivative $\partial_{\mu} \xi_{\mu}$ of the solution ξ_{μ} with respect to the parameter obeys (see Eq. (5))

$$\begin{aligned} \partial_t \eta_{\mu}(t, z_0) &= Df_{\mu}(\xi_{\mu}(t, z_0), t) \eta_{\mu}(t, z_0) \\ &+ \partial_{\mu} f_{\mu}(\xi_{\mu}(t, z_0), t), \quad \eta_{\mu}(0, z_0) = 0. \end{aligned} \quad (22)$$

If we integrate Eqs. (20) and (22) together with Eq. (5), we obtain the stroboscopic map, Eq. (6), and the relevant derivatives, Eqs. (19) and (21). Hence, the optimal control gains, Eq. (17), are easily accessible.

The reference values (μ_R, z_R) in Eq. (10) determine the actual fixed point z_* to be stabilised and the corresponding parameter value μ_* . By a slow quasi-stationary change of the reference values, it is possible

to track unstable states. While not required, it is suitable to set reference values close to the desired target state (μ_*, z_*) . When control is achieved, one may then increase the reference values (μ_R, z_R) in the direction of the curve of fixed points. Suitable increments can thus be obtained from the tangent vector of this manifold, which is given in terms of the differential of the fixed point condition $z_* = F_{\mu_*}(z_*)$, that means

$$0 = (DF_{\mu_*}(z_*) - \mathbf{1}) dz + \partial_{\mu_*} F_{\mu_*}(z_*) d\mu. \quad (23)$$

Hence,

$$\left(\frac{d\mu}{dz} \right) \parallel \begin{pmatrix} \det(DF_* - \mathbf{1}) \\ -\text{adj}(DF_* - \mathbf{1}) \partial F_* \end{pmatrix} \quad (24)$$

so that the vector of increments is parallel to a vector determined by the Jacobian matrix and derivatives of the stroboscopic map. The increments can be computed explicitly with the quantities that have been already used to determine the optimal control gains, Eq. (17). Thus, no additional input is needed to implement a quasi-stationary change of the reference values, $(\mu_R, z_R) = (\mu_* + d\mu, z_* + dz)$, which allows to perform a control-based tracking of unstable states.

Again we illustrate the tracking with the driven Toda oscillator, Eqs. (1) and (2), with parameter setting as in Eq. (3). We implement control, Eq. (10), with the frequency ω being the control input μ . The control gains are adjusted to their linear optimal values, Eq. (17), where the relevant quantities can be obtained, as mentioned, from Eqs. (20) and (22), when control has been successful. The reference values are changed by the tracking procedure based on the tangent of the fixed point manifold, Eq. (24), as described above. Results obtained from a manifold up-sweep and down-sweep are shown in Fig. 4a. We clearly recover the full resonance line including the unstable branch (cf. Fig. 2 for the data without control). Near the tip of the resonance curve the tracking fails. Here, control gains start to diverge as the controllability condition is violated. That feature is reflected by a vanishing denominator in Eq. (17), see Fig. 4b for the value of the denominator along the parameter up-sweep and down-sweep.

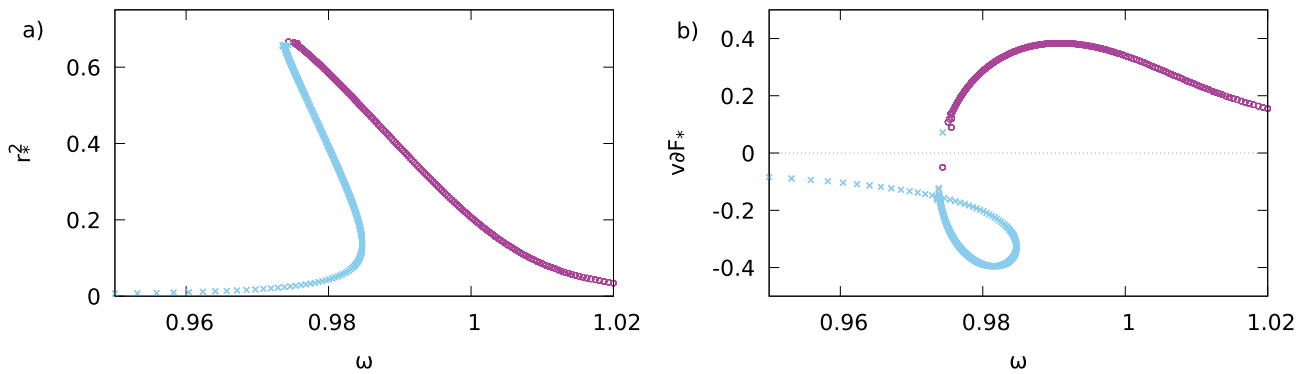


Fig. 4 Control-based tracking of the driven Toda oscillator, Eqs. (1) and (2) with parameter setting specified in Eq. (3). For the control input, the frequency is used and control gains have been adjusted to their optimal values, Eq. (17). Tracking is done by incrementing the reference values in the control law, Eq. (10), along the tangent of the control manifold, see Eq. (24). **a** Station-

ary amplitude $r_*^2 = x_*^2 + v_*^2/\omega^2$ as a function of the frequency (see Fig. 2 for the data without control), **b** value of the denominator of the optimal control gains, see Eq. (10). The denominator vanishes near the tip of the resonance curve. Red (circles): manifold downsweep, cyan (crosses): manifold upsweep

5 Practical challenges and extensions

The linear full-state feedback along the lines of Eq. (10) is just a special case of a more general instantaneous law defined by a function h

$$\mu_n = h(z_n, \alpha, v_R) \quad (25)$$

where v_R stands for some reference value used, e.g. for selecting fixed points or tracking. The fixed point is now determined by the intersection of two manifolds

$$z_* = F_{\mu_*}(z_*), \quad \mu_* = h(z_*, \alpha, v_R). \quad (26)$$

As far as linear stability is concerned the previous analysis by and large still stands as the variational equation of Eq. (25) reads

$$\delta\mu_n = \partial_{z_*} h(z_*, \alpha, v_R) \cdot \delta z_n. \quad (27)$$

Hence, stability is now governed by some renormalised control gains $\partial_{z_*} h(z_*, \alpha, v_R)$ but otherwise the analysis remains unchanged. In particular, the constraint caused by controllability still stands, even for the fairly general scheme of Eq. (25). The original linear scheme, Eq. (10), may have bad global properties, e.g. with regard to the size of the basin of attraction or the structural stability of the control loop. One may aim at optimising some of those features by suitable choices of the feedback h .

5.1 Nonlinear observation and ceiling function

It is worth to consider a few special cases of Eq. (25) more explicitly. If one observes the state of the system through two observables $g_1(z)$ and $g_2(z)$ instead of the state vector, a corresponding linear feedback reads

$$\mu_n = \mu_R + \alpha^T \cdot (g(z_n) - g_R) \quad (28)$$

where g_R are some reference values. The variational equation

$$\delta\mu_n = \alpha^T \cdot Dg(z_*) \delta z_n \quad (29)$$

tells us that stability is governed by rescaled control gains $\alpha^T Dg(z_*)$ which are still independent quantities as long as the Jacobian $Dg(z_*)$ is not singular (i.e. the two observables are in this sense independent).

The system may depend very sensitively on the parameter μ . That is in particular the case if the frequency of the drive is used as control input. Then, the control tends to overshoot, in particular for the linear law Eq. (10) which is defined by an unbounded function. One can try to cure this problem by using a ceiling function. A simple version where the control feedback stays bounded is given by

$$\mu_n = \mu_R + \eta \tanh \left(\frac{\alpha^T \cdot (z_n - z_R)}{\eta} \right) \quad (30)$$

where $\eta > 0$ determines the cut-off of the control feedback. By this choice, we ensure on the one hand that the parameter stays in a neighbourhood of the reference value μ_R , while on the other hand the linear stability is unaffected as the variational equation coincides with the case of linear feedback. It may, however, happen that the system settles on coexisting stable branches as for large deviations from the reference value the control feedback along the lines of Eq. (30) becomes constant.

5.2 Incomplete information and time-delayed observation

An important variant of the scheme outlined in Eq. (28) occurs if instead of the full-state vector $\mathbf{z}$ only partial information, say one component, is accessible. To analyse this case in slightly more detail, let us use the notation $\mathbf{z} = (x, v)$ for the two components, and assume that the first component $x(t)$ is accessible. The stroboscopic measurement at times T_n , i.e. at the end of the n -th period of the driving field, gives us $x_n = x(T_n)$. Since we assume the second component to be inaccessible, we are left with the option to base our feedback on $x(t)$ evaluated at a different time during the period of the drive. Following, for instance, the reasoning of delay embedding [40], it is instructive to choose the value of $\xi_n = x(T_n - \tau)$ as our second observation, where τ denotes a fixed time delay. For a theoretical analysis of a scheme which uses such a delayed observation, we recall that ξ_n can be expressed in terms of the solution of the equations of motion, Eq. (5), as

$$\xi_n = \zeta_{\mu_{n-1}}^{(1)}(-\tau, \mathbf{z}_n) = h_{\mu_{n-1}}(\mathbf{z}_n) \quad (31)$$

since μ_{n-1} is the actual value of the control input during the period when the delayed observation is performed. In some sense, we are aiming at a control feedback along the lines of Eq. (28) with the choice $g_1(\mathbf{z}) = x$ and $g_2(\mathbf{z}) = h_{\mu_{n-1}}(\mathbf{z})$. Since the delayed observation already depends on the parameter value μ_{n-1} of the past period, it looks consistent to take that also explicitly into account for the feedback law of μ_n . Thus, we arrive at

$$\mu_n = \mu_R + \alpha_1(x_n - x_R) + \alpha_2(\xi_n - \xi_R) + \beta(\mu_{n-1} - \mu_R)$$

$$= \mu_R + \alpha_1(x_n - x_R) + \alpha_2(h_{\mu_{n-1}}(\mathbf{z}_n) - \xi_R) + \beta(\mu_{n-1} - \mu_R) \quad (32)$$

with some fixed reference values μ_R , x_R , and ξ_R . The theoretical analysis now closely parallels the steps of the previous sections. The fixed point of the closed-loop dynamics, Eqs. (11) and (32) is determined by

$$\mathbf{z}_* = \mathbf{F}_{\mu_*}(\mathbf{z}_*), \quad \mu_* = \mu_R + \alpha_1(x_* - x_R) + \alpha_2(h_{\mu_*}(\mathbf{z}_*) - \xi_R) + \beta(\mu_* - \mu_R), \quad (33)$$

while the variational equation with $\mathbf{z}_n = \mathbf{z}_* + \delta\mathbf{z}_n$, $\mu_n = \mu_* + \delta\mu_n$ (cf. as well Eq. (13))

$$\begin{aligned} \delta\mathbf{z}_{n+1} &= \mathbf{D}\mathbf{F}_* \delta\mathbf{z}_n + \partial\mathbf{F}_* \delta\mu_n \\ \delta\mu_n &= \alpha_1 \delta x_n + \alpha_2 (\mathbf{d}h_* \cdot \delta\mathbf{z}_n + \partial h_* \delta\mu_n) + \beta \delta\mu_{n-1} \end{aligned} \quad (34)$$

provides the details about the stability of the fixed point. Here,

$$\mathbf{d}h_* = \nabla_{\mathbf{z}_*} h_{\mu_*}(\mathbf{z}_*), \quad \partial h_* = \partial_{\mu_*} h_{\mu_*}(\mathbf{z}_*) \quad (35)$$

abbreviate the gradient of the delayed observation and its derivative with respect to the external parameter at the fixed point. Introducing rescaled gains by

$$\hat{\alpha}_1 = \alpha_1 + \alpha_2 (\mathbf{d}h_*)_1, \quad \hat{\alpha}_2 = \alpha_2 (\mathbf{d}h_*)_2, \quad \hat{\beta} = \beta + \alpha_2 \partial h_* \quad (36)$$

the variational equation, Eq. (34) can be conveniently written as

$$\begin{pmatrix} \delta\mathbf{z}_{n+1} \\ \delta\mu_n \end{pmatrix} = \mathbf{B} \begin{pmatrix} \delta\mathbf{z}_n \\ \delta\mu_{n-1} \end{pmatrix} \quad (37)$$

with

$$\mathbf{B} = \begin{pmatrix} \mathbf{D}\mathbf{F}_* & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix} + \begin{pmatrix} \partial\mathbf{F}_* \\ 1 \end{pmatrix} \otimes (\hat{\alpha}^T, \hat{\beta}). \quad (38)$$

It is quite easy to check that, as expected, the conditions for controllability of the matrix $\mathbf{B}$ coincide with the simpler case of matrix $\mathbf{A}$ discussed in Sect. 3, as long as $\mathbf{D}\mathbf{F}_*$ is non-singular. The same statement essentially holds in the singular case, as we will see shortly. To assess the stability properties, let us focus on the characteristic polynomial of the matrix $\mathbf{B}$, Eq. (38), which evaluates as

$$\begin{aligned}
\det(\mathbf{B} - \lambda \mathbf{1}) &= -\lambda^3 \\
&+ \lambda^2 \left(\text{tr}(\mathbf{D}\mathbf{F}_*) + \hat{\alpha}^T \cdot \partial \mathbf{F}_* + \hat{\beta} \right) \\
&- \lambda \left(\det(\mathbf{D}\mathbf{F}_*) + \hat{\alpha}^T \cdot \text{adj}(\mathbf{D}\mathbf{F}_*) \partial \mathbf{F}_* + \hat{\beta} \text{tr}(\mathbf{D}\mathbf{F}_*) \right) \\
&+ \hat{\beta} \det(\mathbf{D}\mathbf{F}_*). \quad (39)
\end{aligned}$$

The Jury criterion [37] gives again necessary and sufficient conditions for stability, but the criterion is slightly involved for a polynomial of degree three, see, for example, [41] for a simple application on delayed feedback control. Here, we resort to the simpler concept of pole placement which requires to find gains $\hat{\alpha}^T$ and $\hat{\beta}$ so that the subleading coefficients in Eq. (39) can take any prescribed numerical values. Thanks to the linear dependence of the coefficients on the control gains that is indeed the case if $\det(\mathbf{D}\mathbf{F}_*) \neq 0$ and if the set $\{\partial \mathbf{F}_*, \text{adj}(\mathbf{D}\mathbf{F}_*) \partial \mathbf{F}_*\}$ is linearly independent. The last condition coincides in fact with the controllability condition encountered for full-state feedback discussed in Sect. 3. Even if $\mathbf{D}\mathbf{F}_*$ becomes singular, the condition of linear independence still guarantees successful control since the third eigenvalue of the matrix Eq. (38) just vanishes. As for the optimal control gains, i.e. for control gains so that the stabilised orbit becomes superstable, all nontrivial coefficients in Eq. (39) have to vanish. That means these optimal control gains are given by $\hat{\beta} = 0$, while $\hat{\alpha}^T$ obeys the conditions of optimality discussed in Sect. 4, see Eqs. (16) and (17). In particular, the inclusion of the damping term $\beta \mu_{n-1}$ in the control law Eq. (32) has turned out to be crucial, as otherwise stabilisation or optimal gains could be inac-

cessible, see Eq. (36). Finally, we need to ensure that the rescaled gains $\hat{\alpha}^T$ and $\hat{\beta}$ translate into actual control gains α^T and β via Eq. (36). That requires $(dh_*)_2$ to be nonzero, a condition which means that the second unobserved component of the state vector has a finite impact on the delayed observation, see Eq. (31). Such a constraint can thus be considered as a trivial installment of what is called observability in the engineering context.

In summary, we conclude that control with incomplete state information is possible when delayed observations are employed. If one has access to the underlying equations of motion, the required control gains can be computed in the same way as before, since Eq. (31) and the required derivatives, Eq. (35), can be obtained again from the solution of Eqs. (5), (20), and (22). For the experimental context, our considerations tell us that a control law along the lines of Eq. (32) will be successful for a suitable set of control gains.

For the purpose of an illustration, we again resort to the Toda oscillator with the parameter setting already employed in Fig. 3. Now, however, we base the control on the observation of the position $x(t)$ only, Eq. (32), with the frequency being the control input, $\mu_n = \omega_n$. Measuring the position at an intermediate time, $\xi_n = x(T_n - \tau)$, and at the end of the period, $x_n = x(T_n)$, we update the frequency after each period. For the choice of a suitable time delay τ , we recall that for harmonic solutions the observation of the position with a delay of a quarter of the period results in a value which is proportional to the velocity. Therefore, we take $\tau =$

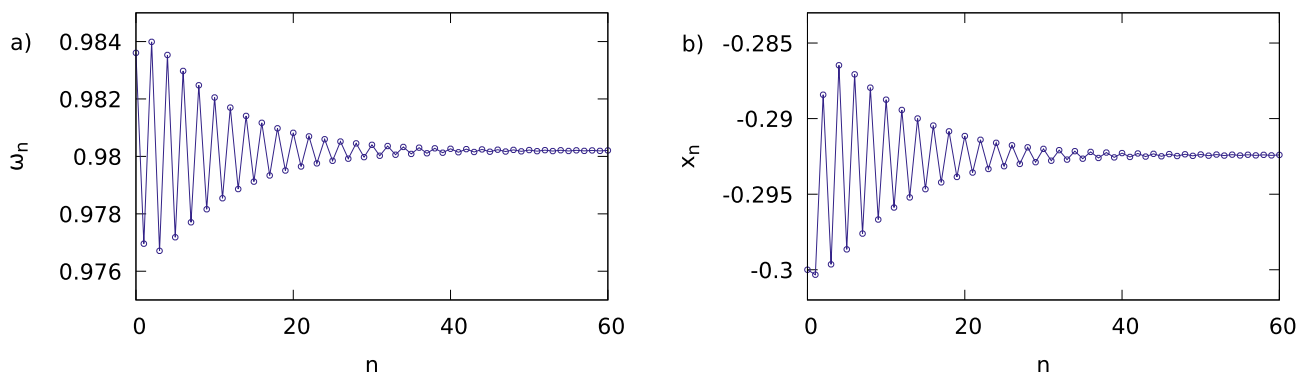


Fig. 5 Control based on measurements of the position in the driven Toda oscillator, Eqs. (1) and (2), with parameter setting according to Eq. (3), see Fig. 2 for data without control. The frequency ω has been used as control input μ in the control law Eq. (32) with delay time $\tau = 1.5$. Reference values in the con-

trol law are $\omega_R = 0.98$, $x_R = -0.3$, $g_R = 0.527$, while gains have been chosen as $\alpha_1 = 5.0$, $\alpha_2 = -8.0$, and $\beta = 0.1$. Time series of the stroboscopic map with control, Eq. (11), and initial condition $x(0) = x_R$, $v(0) = -0.53$. **a** Frequency value ω_n , **b** position x_n

1.5. While optimal control gains can be computed as outlined above, here we make up some suitable values out of thin air. Figure 5 shows time traces of successful stabilisation of an unstable state.

6 Conclusion

We have introduced a design for the non-invasive control of an unknown periodic orbit. Our approach is based on a stroboscopic observation of a time series and an implementation of a full-state proportional feedback for a parameter of the system. During the stabilisation, the parameter and the time-dependent state simultaneously adjust in such a way that a proper periodic orbit of the plain system at a target parameter value is obtained. Our approach relies on simple measurements of the state of the system and does not require any further data processing. Our design is thus able to deal with systems on fast time scales, like the example of amplitude modulation dynamic force microscopy described as a motivation in the introduction, and where no accurate mathematical model is available.

We have evaluated the performance of the control in terms of linear stability analysis. That enabled us to determine optimal control gains by and large by analytic means, to implement a tracking algorithm for unstable branches, and to clarify the role and the limitations caused by controllability conditions, which are well established in the engineering context. By definition, a linear stability approach cannot give detailed information about the global performance of control, such as the basin of attraction of the stabilised state, or the invariant manifolds which cause basin boundaries. Numerical findings indicate that the basin is occasionally quite small, and any analytic estimate turns out to be quite challenging. We will address this important issue elsewhere.

In typical situations, information about the full state of the system is hardly available as one often just relies on the measurement of a single scalar time series. Even in such cases our design can succeed by employing delayed observations, as shown in the example above. With access to the underlying dynamics, one can even determine suitable control gains a priori, and essentially by analytic means. In addition, it is worth to explore whether the control performance of periodic orbits can be improved by using multiple delayed observations for a larger set of time delays and for a larger number

of control gains, following in some sense the spirit of extended time-delayed feedback control [42].

While the implementation of the control design outlined above is based on the stroboscopic observation of an experimental time series, we have used some properties of the underlying equations of motion for the theoretical analysis. While in numerical simulations one has full access to the details of the dynamical system, so that the optimal control gains, Eq. (17), can be evaluated, such information is hardly available in actual real-world experiments, since a data-based accurate estimate of the required derivatives normally turns out to be a delicate problem. Therefore, finding suitable control gains in experimental set-ups may turn out to be a considerable challenge. It is therefore still a highly nontrivial task to design a feedback, Eq. (25), in fast real-world experiments so that stability estimates and good global properties can be obtained a priori if only a scalar experimental time series is at hand.

Our analysis has been based on linear stability and thus cannot address the questions mentioned above in full detail. In fact, numerical simulations with successful control may show basins in phase space which are small, even if we employ dead beat control with vanishing eigenvalues. The mechanism of stroboscopic control and the conditions for controllability do not rely on any time scale separation. Control is successful even if the internal dynamics is fast compared to the period of the orbit if one discounts for the well-known impact of control loop latency. Eigenvalues of the target states involved do not have to be a small perturbation of unity so that one can deal with situations where one is either far apart or in the vicinity of a bifurcation point. While all these aspects may turn out to be useful in applications, they make an intuitive understanding of the control performance quite challenging, if not impossible. To make some progress, it could be sensible to take a step back and to focus on models and situations where the dynamics is a perturbation of harmonic motion, and where time scale separation and averaging can be applied. In such cases, one would be able to gain analytic and intuitive insight into global properties of the control mechanism such as basins of attraction or the impact of the parameter that is used to couple the control force. In experiments, the frequency and the amplitude of the drive are usually easily accessible for control input. It would be interesting to figure out which of these variables is more suited to optimise the global control performance such as basins of attraction

or removing obstacles caused by controllability. In that respect looking at the seemingly simple case of weakly nonlinear systems would be a promising next step, even from the point of view of a potential application, and details about the corresponding global analysis will be published elsewhere.

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Data availability No data sets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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Chapter 3

Large basins of attraction for control-based continuation of unstable periodic states



RESEARCH

Large basins of attraction for control-based continuation of unstable periodic states

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Abstract Numerical continuation tools are nowadays standard to analyse nonlinear dynamical systems by numerical means. These powerful methods are unfortunately not available in real experiments without having access to an accurate mathematical model. Implementing such a concept in real world experiments using control and data processing to track unstable states and their bifurcations, requires robust control techniques with large basins and good global properties. Here we propose design principles for control techniques for periodic states which lead to large basins and which are robust, without the need to have access to a detailed mathematical model. Our analytic considerations for the control design will be based on weakly nonlinear analysis of periodically driven oscillator systems. We then demonstrate by numerical means that in strong nonlinear regimes successful control with large basins

of attraction can be achieved when only plain time series data are available.

Keywords Nonlinear dynamics · Control · Experimental bifurcation analysis · Equation-free analysis · Atomic force microscopy

Mathematics subject classification 37N35 · 34C15 · 34C55 · 93B52 · 93A30

1 Introduction and experimental context

Dynamical systems theory, in particular the investigation of instabilities and of chaotic motion in nonlinear systems is one of the key themes in theoretical and experimental sciences of the last decades [1–3]. Numerous tools have been developed to study the complex motion in such systems. From a theoretical perspective continuation of stable as well as unstable solutions and of bifurcations play a prominent role [4] since global structures in phase space and a skeleton of unstable states for the dynamics can be discovered which gives insight into details of the dynamics [5]. Equation free analysis has been proposed as a practical tool to derive effective equations of motion from microscopic models by numerical means [6]. These effective models can then be studied by analytic and numerical bifurcation analysis to understand the low-dimensional nonlinear structures and pattern formation in complex real world systems.

Niklas Kruse, Hannes Wallner, Anna Dittus, Lukas Böttcher, Ingo Barke, Sylvia Speller and Jens Starke are contributed equally to this work.

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The idea of a model free approach has taken this concept a step further by proposing an approach where one avoids the intermediate derivation of effective equations of motion, making the idea directly applicable to experiments. In this context the required continuation of stable or unstable dynamical states relies on non-invasive control methods which are directly applicable to real world experiments. Non-invasive control, sometimes called orbit control in the engineering context, has the crucial property that the control forces ultimately tend to zero, so that the stabilised state is a genuine unstable orbit of the original system without control (see as well [7], where this idea has been popularised within the physics community). In addition to numerical implementations of an equation-free approach to obtain bifurcation diagrams at a macroscopic scale, see for instance [8], a successful implementation of this concept has been already demonstrated in mechanical and electro-mechanical hybrid experiments [9–11], in electrochemical setups [12], or even in experimental studies of pedestrian flows [13]. Key to these experimental implementations of continuation techniques is the availability of suitable control methods which can deal with quite diverse experimental conditions. Control problems have of course a long standing tradition in engineering. They took a large boost during and after world war two, where linear control theory was formalised in a systematic manner. Subsequently these ideas have been extended to nonlinear systems, see e.g. [14, 15]. Putting the emphasis on non-invasive methods the relevance of control techniques for the purpose of system analysis has been rediscovered in the context of chaotic dynamics [7]. Unlike in engineering, control is considered here merely as a kind of spectroscopic tool to identify structures in the phase space of the system.

To illustrate some of the challenges one faces when one aims at implementing control based experimental continuation of unstable states we refer to atomic force microscopy as a paradigm. A comprehensive theoretical study of control-based continuation in a model of atomic force microscopy can be found in [16]. In experimental terms atomic force microscopy is a quite versatile method to inspect surfaces and attached structures or objects thereon, down to the nanoscale [17, 18]. During the last decades scanning platforms have been substantially augmented, so that frame sizes of $80\text{ }\mu\text{m}$ down to 1 nm are possible without any drawback in lateral and vertical linearity. During topography acqui-

sition under ambient conditions the relative amplitude drop can serve as setpoint variable while the control variable is the piezo actuator height. To a good degree, thereby the tip-sample separation is kept constant during lateral scanning and a topography is acquired as a discrete data set. Typically, physical lateral resolution is at about 1 nm while vertical resolution is at about 1 Å . It is quite common to operate atomic force microscopes in the dynamic mode where the sensor element, a bending micro-cantilever with a tip, is driven into oscillation via a dither or shaker piezo. The nonlinear features in the interaction with the surface turn atomic force microscopy into a complex nonlinear dynamical systems which shows bistability, a variety of bifurcations, and chaotic motion [19–22]. In dynamic force microscopy stable low and high amplitude branches with an unstable branch in between do coexist. Jumping between the two stable branches during acquisition of topography is a common distortion, particularly with molecular species, requiring manual intervention (see Fig. 1 for experimental images). Especially beyond the resonance frequency of the cantilever a rather broad range of tip sample separations is prone to such imaging instabilities. By applying ideas from time-delayed feedback control to atomic force microscopy noise reduction in the imaging process has been reported [23].

We have mentioned atomic force microscopy just to motivate our plain theoretical considerations. Control challenges which may occur in such an experimental context, such as the lack of a proper mathematical model, fast time scales like those encountered in nanosystems which prevent extensive online data processing, parameter drifts and non-stationary behaviour which preclude a preliminary data based modelling, or the dynamical impact of noise are the topic of our interest. Above all we want to design a control scheme which is robust and comes with a large basin of attraction, and which finally can be set up if just plain measurements of the system are available. In a previous theoretical study [24] we have outlined a scheme to deal in principle with these issues by linear control schemes applied to stroboscopic maps. However, such linear schemes have severe limitations when it comes to global properties of the control in nonlinear setups. In our work we focus on the design of non-invasive control schemes with large basins of attraction. We will demonstrate their success by numerical simulations. Actual implementations in experiments will be addressed elsewhere.

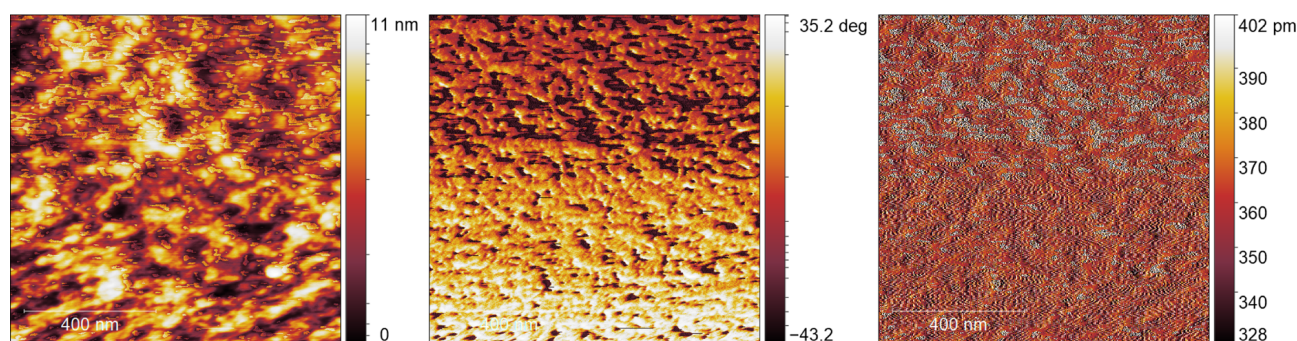


Fig. 1 Dynamic force microscopy of a conductive polymer blend (PEDOT:PSS) thin film deposited by spin coating on a glass substrate (rotational frequency is 5000 revolutions per minute, film thickness ≈ 32 nm). Cantilever type: Nanosensors SSS-NCHR with a force constant in the regime of 10–13 N/m and a

nominal tip radius of 2 nm. AFM instrument: Park Systems NX-20 (in ambient air). AM amplitude drop $A/A_{free} \approx 0.75$. Left: topography, middle: phase between excitation and cantilever bending, right: amplitude. The phase exhibits a clear bimodal distribution peaking at -33° and $+14^\circ$ (not shown)

The design of control schemes with good global properties is of course a standard theme in engineering. For instance, the problem of globally stable control design has been solved by exact state-space linearisation for single-input systems, see [14]. Having said that, the implementation of such ideas requires some knowledge about the underlying dynamics. In very basic terms the issue of globally stable control has been revisited in the context of control of chaos [25] and has been popularised beyond the remit of engineering problems. To keep our presentation self-contained Sect. 2 will give a brief sketch of the basic ideas and of the related challenges in a theoretical setting. Design of control schemes with good global properties requires access to some properties of the dynamics. Since we have ultimately applications for atomic force microscopy in mind we focus here on general nonlinear driven oscillators and the corresponding nonlinear resonance behaviour. We will outline in Sect. 3 how weakly nonlinear perturbation expansions will give us analytic access to the dynamics, and in particular to the stroboscopic map of the equations of motion. Based on these analytic estimates we will show in Sect. 4 how to design a globally stable control scheme to stabilise periodic orbits in a non-invasive way. In particular, our design utilises the phase of the driving field as a key component of a globally stable control scheme. While our approach has been based on weakly nonlinear analysis and on the analytic expression of the stroboscopic map, we will show by numerical means that our control scheme also works well beyond the perturbative regime and can thus be applied to generic nonlinear oscilla-

tor systems. We demonstrate in Sect. 5 that the control scheme can be used for a data based tracking of unstable orbits and for the generation of a complete bifurcation scenario. While the design of the control scheme has been based on a perturbative treatment of model equations, we will show in Sect. 6 that all the elements of the control scheme can be obtained from data, in particular from scanning a bistable nonlinear resonance curve, as long as higher order harmonic components do not dominate the dynamics of the system. Above all, the control scheme can be implemented without any a priori access to a mathematical model. Finally, we briefly discuss in the conclusion limitations and merits of the proposed approach, in particular in the context of atomic force microscopy and related experimental setups.

2 Globally stable control in a nutshell

As we are aiming at controlling periodic states of a dynamical system it seems promising to focus on Poincaré maps or stroboscopic maps since periodic states will become fixed points of the time discrete dynamical systems. To illustrate the basic idea how to design a globally stable control scheme consider a time-discrete dynamical system given by a one-dimensional map f_μ

$$x_{n+1} = f_\mu(x_n) = \mu + g(x_n) \quad (1)$$

where the right hand side depends on a parameter μ which will serve as control input. For simplicity and for the purpose of illustration we assume here that the map

depends on the parameter μ in an additive way. Such an assumption is by no means essential for the subsequent considerations. We aim at controlling fixed points of the dynamical system, Eq. (1), where the fixed point manifold, that means the fixed point x_* in dependence on the parameter μ , is determined by

$$x_* = f_{\mu_*}(x_*) = \mu_* + g(x_*) . \quad (2)$$

In order to stabilise such a fixed point we make the parameter μ a dynamical time-dependent quantity. In its simplest instalment the time-dependence of μ is just given by a static relation with the state variable, say

$$\mu_n = \mu_R + h(x_n) \quad (3)$$

where h specifies the control law, i.e., the dependence of the parameter μ_n on the state of the system x_n . Then the closed loop dynamics reads

$$x_{n+1} = f_{\mu_n}(x_n) = \mu_R + h(x_n) + g(x_n) . \quad (4)$$

The actual fixed point to be stabilised is determined by eq. (2) together with

$$\mu_* = \mu_R + h(x_*) . \quad (5)$$

In geometric terms Eqs. (2) and (5) amount to the intersection of two manifolds. The static offset μ_R which has been included in the control law, Eq. (3), serves as an external parameter by which the fixed point to be controlled, x_* , and the corresponding actual parameter value μ_* can be selected.

We aim to choose the control feedback $h(x_n)$ in such a way so that the fixed point x_* becomes a globally stable fixed point of the closed loop dynamics, Eq. (4). The obvious choice is of course $h(x) = -g(x)$ since the fixed point of the closed loop dynamics becomes super-stable and convergence happens in one iteration step. The offset μ_R of the control loop becomes in fact the fixed point value to be stabilised and the corresponding parameter value is given by Eq. (5). Global convergence can be obtained as well under less stringent conditions. For instance it is sufficient that the closed loop dynamics, Eq. (4), gives rise to a contraction map. Hence a rough estimate of the full internal dynamics may be sufficient to design a successful control scheme. As a simple illustration consider the logistic map on the interval $[0, 1]$ which reads

$$x_{n+1} = f_{\mu}(x_n) = \mu x_n(1 - x_n) . \quad (6)$$

To stabilise the fixed point we employ the control scheme $\mu = \mu_n$ with the choice

$$\mu_n = \mu_R h(x_n) = \mu_R / \sin(\pi x_n) . \quad (7)$$

Then the closed loop dynamics

$$x_{n+1} = f_{\mu_n}(x_n) = \mu_R \frac{x_n(1 - x_n)}{\sin(\pi x_n)} \quad (8)$$

yields a contraction on the interval $[0, 1]$ and the dynamics converges globally to a unique fixed point x_* . The actual value of the fixed point is determined by the control gain μ_R , and the corresponding actual parameter value is given by $\mu_* = \mu_R h(x_*)$, see Eq. (7). Figure 2 shows time traces of successful control of the unstable fixed point for some typical parameter values. It is worth to mention, that by construction, the control scheme outlined above is non-invasive, even though a control force does not seem to tend to zero. By making the system parameter μ a dynamical quantity μ_n , which is adjusted by an instantaneous feedback law such as Eqs. (3) or (7), successful control is signalled by the dynamical parameter tending to a limit value μ_{∞} (see Fig. 2). In such a case the resulting dynamics for x , in our case a fixed point (see Fig. 2), coincides by construction with an orbit of the system without control at fixed parameter value μ_{∞} . Hence, we have a non-invasive control scheme which stabilises a proper unstable orbit of the original equations of motion. In fact, the fixed point which is stabilised in Fig. 2 (red symbols), coincides with the unstable fixed point of the logistic map for the limit parameter value μ_{∞} (Fig. 2, cyan symbols). We also note that the reference value μ_R in the feedback law tunes the limit value μ_{∞} and the control target, see Eq. (5), but is by no means per se the target value of the control scheme.

To design a globally stable control law one needs the full knowledge or at least a sufficiently accurate estimate of the underlying equations of motion. That is of course far from surprising as a globally stable solution requires the control law to remove all potentially occurring basin boundaries of the stabilised state. Such boundaries are normally caused by unstable saddles and their stable manifolds which, as said, have to be removed from the equations of motion. The reasoning summarised in this section is far from novel. Such considerations are at the heart of any sensible control design and they are normally covered in any basic course in control engineering. The main idea is also well established for, say, almost a century and simplified versions have also popped up in the physics context, see e.g. [25]. Nevertheless we felt it useful to recall these basic considerations as we will use such reason-

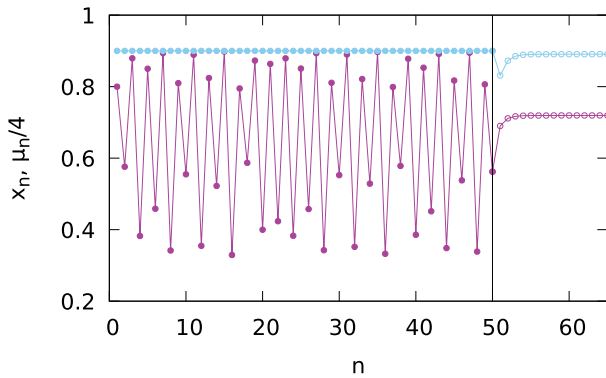


Fig. 2 Red: Time series of the logistic map, Eq. (6), for $\mu = 3.6$ with initial condition $x_0 = 1/3$, without control for $n \leq 50$ (full symbols) and with control, Eq. (8), with $\mu_R = 2.75$, for $n > 50$ (open symbols). Cyan: corresponding time evolution of the parameter μ_n without control for $n \leq 50$ (full symbols), and with control, Eq. (7), with $\mu_R = 2.75$, for $n > 50$ (open symbols)

ing to design globally stable control of periodic states in oscillator systems in the next sections.

3 Stroboscopic map for weakly nonlinear oscillators

We will mainly deal with control of unstable periodic solutions in two-dimensional driven oscillator systems, which are described by the set of differential equations

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -\omega^2 x \\ &+ \varepsilon \left(-\gamma v - U'(x) - h^{(c)} \cos(\omega t) - h^{(s)} \sin(\omega t) \right). \end{aligned} \quad (9)$$

Here γ denotes the viscous damping constant, $U(x)$ the anharmonic part of the potential, and $h^{(s/c)}$ the amplitudes of a harmonic periodic driving force. As pointed out in the previous section we will require some degree of access to the full dynamics, hence we adopt a system with a small parameter ε which can be employed to perform an analytic perturbation expansion.

The purpose of the model, Eq. (9), is twofold. For $\varepsilon = 1$ the model constitutes a general nonlinear driven oscillator which will be used to test our control design (see Sects. 5 and 6), even to the level where we base control only on a recorded time series of the model. On the other hand we will use the regime of small ε

in Eq. (9) to develop, inspired by analytic perturbation expansion, a globally stable control design in Sect. 4. In particular, we will show that access to the amplitude and the phase of the driving field is sufficient to set up the control scheme. While this design will work by construction in the perturbative regime we will also demonstrate that good global properties persist for typical oscillators.

As far as dynamical phenomena of Eq. (9) are concerned we can simply resort to numerical computations of stroboscopic maps, that means computing $x_n = x(n2\pi/\omega)$ and $v_n = v(n2\pi/\omega)$, and study the properties of the map $(x_n, v_n) \mapsto (x_{n+1}, v_{n+1})$ by numerical means. While in our analytical studies we will keep a general potential we use for numerical purposes a simple Duffing oscillator with potential given by

$$U(x) = \frac{a}{2}x^2 + \frac{b}{4}x^4 \quad (10)$$

and we chose in numerical simulations a standard set of parameter values

$$\gamma = 0.1, \quad a = -0.5, \quad b = 1.0, \quad \omega = 1.0. \quad (11)$$

A dominant dynamical signature of such oscillator systems is a nonlinear resonance which occurs even beyond the perturbative regime of small ε values, where a bistability between a large and a small amplitude branch appears. That can be vividly illustrated by a numerical computation of time traces of the stroboscopic map which tend toward fixed points, and where the bistability occurs when the amplitude of the driving field is changed in a quasi-stationary manner, see Fig. 3. This bistability will be at our centre of interest as we aim to stabilise the unstable branch which separates the two stable states in the bistable region, with a view towards performing finally a control based data driven continuation of bifurcations in experiments such as atomic force microscopy.

One of the two amplitudes $h^{(c)}$ or $h^{(s)}$ in eq. (9) seem to be redundant because

$$\begin{aligned} h^{(c)} \cos(\omega t) + h^{(s)} \sin(\omega t) &= h \cos(\omega t - \theta), \\ h &= \sqrt{(h^{(c)})^2 + (h^{(s)})^2}. \end{aligned} \quad (12)$$

Here θ denotes the phase of the driving field relative to the cross section of the stroboscopic map, i.e., relative to the times $n2\pi/\omega$ of the observation. The phase θ can be easily eliminated by choosing an appropriate cross section for the stroboscopic map, i.e., by shifting the

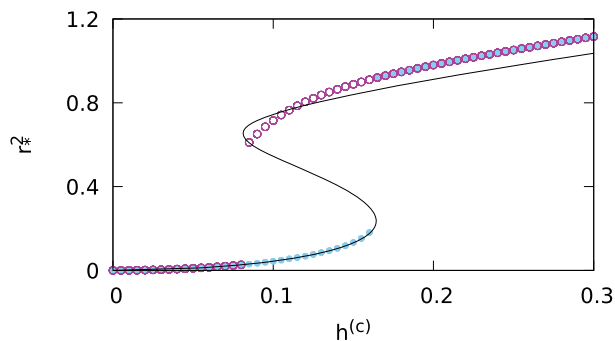


Fig. 3 Bifurcation diagram of Eqs. (9) and (10) with $\varepsilon = 1$, $h^{(s)} = 0$, and other parameters as in Eq. (11). Data have been obtained from a numerical simulation of the stroboscopic map, i.e., from time traces evaluated at integer multiples of the period of the driving field. Skipping a transient of 50 iterations the orbit settles on a stable fixed point (x_*, v_*) . The dependence of $r_*^2 = x_*^2 + v_*^2/\omega^2$ on $h^{(c)}$ is shown for a quasi-stationary parameter upswing (cyan, full symbols) and a parameter downswing (red, open symbols). The black line shows the analytic result obtained from a first order perturbation theory, see Eq. (24)

time of observation by a constant amount. However, we will see soon that having access to the phase of the driving field will be a key element to construct a globally stable control scheme. In fact using the phase of the driving field for the purpose of control is by no means novel, see e.g. [26]. Such feedback has been proposed in experimental contexts, for instance, for noise reduction at the microscale [27, 28]. However, neither non-invasive control, nor control with good global properties has been at the centre of interest in these studies.

As we have seen in the previous section a successful design of a suitable control feedback requires some basic understanding of the dynamics of the underlying system. That means some analytic access to the stroboscopic map is beneficial. While a closed analytic expression for the stroboscopic map is never available in non-trivial situations, we can get some insight if we constrain to the perturbative regime of small ε values. If $x(t, x_n, v_n, \varepsilon)$ and $v(t, x_n, v_n, \varepsilon)$ denote the time dependent solution of Eq. (9) with initial condition (x_n, v_n) the exact stroboscopic map is given by $(x_n, v_n) \mapsto (x_{n+1}, v_{n+1}) = (x(2\pi/\omega, x_n, v_n, \varepsilon), v(2\pi/\omega, x_n, v_n, \varepsilon))$. Using a straightforward series expansion of the solution of Eq. (9) in terms of ε we are able to derive an analytic approximation of the stroboscopic map. At lowest order $\mathcal{O}(\varepsilon^0)$, Eq. (9) reduces to a harmonic oscillator and the solution with initial

condition (x_n, v_n) reads

$$\begin{aligned} x^{(0)}(t) &= x_n \cos(\omega t) + \frac{v_n}{\omega} \sin(\omega t), \\ v^{(0)}(t) &= v_n \cos(\omega t) - \omega x_n \sin(\omega t). \end{aligned} \quad (13)$$

At first order $\mathcal{O}(\varepsilon)$ we obtain the linear inhomogeneous system

$$\begin{aligned} \dot{x}^{(1)}(t) &= v^{(1)}(t) \\ \dot{v}^{(1)}(t) &= -\omega^2 x^{(1)}(t) - \gamma v^{(0)}(t) - U'(x^{(0)}(t)) \\ &\quad - h^{(c)} \cos(\omega t) - h^{(s)} \sin(\omega t) \end{aligned} \quad (14)$$

with initial condition $x^{(1)}(0) = 0$, $v^{(1)}(0) = 0$. The solution can be easily computed and we finally obtain for the solution after a period $2\pi/\omega$, using the expression Eq. (13)

$$\begin{aligned} x^{(1)}(2\pi/\omega) &= \frac{1}{\omega} \int_0^{2\pi/\omega} \sin(\omega t) \left(\gamma v^{(0)}(t) + U'(x^{(0)}(t)) \right. \\ &\quad \left. + h^{(c)} \cos(\omega t) + h^{(s)} \sin(\omega t) \right) dt \\ &= -\gamma x_n \frac{\pi}{\omega} + \frac{1}{\omega} \int_0^{2\pi/\omega} \sin(\omega t) U'(x^{(0)}(t)) dt \\ &\quad + h^{(s)} \frac{\pi}{\omega^2} \\ v^{(1)}(2\pi/\omega) &= - \int_0^{2\pi/\omega} \cos(\omega t) \left(\gamma v^{(0)}(t) + U'(x^{(0)}(t)) \right. \\ &\quad \left. + h^{(c)} \cos(\omega t) + h^{(s)} \sin(\omega t) \right) dt \\ &= -\gamma v_n \frac{\pi}{\omega} - \int_0^{2\pi/\omega} \cos(\omega t) U'(x^{(0)}(t)) dt \\ &\quad - h^{(c)} \frac{\pi}{\omega}. \end{aligned} \quad (15)$$

To evaluate the remaining integrals we introduce the amplitude r_n and the phase φ_n of the solution eq. (13) by

$$x_n + i \frac{v_n}{\omega} = r_n e^{i\varphi_n}. \quad (16)$$

Then we have

$$\begin{aligned} \int_0^{2\pi/\omega} e^{i\omega t} U'(x^{(0)}(t)) dt &= \int_0^{2\pi/\omega} e^{i\omega t} \\ &\quad U' \left(\frac{r_n}{2} \left(e^{i\omega t - \varphi_n} + e^{-(i\omega t - \varphi_n)} \right) \right) dt \\ &= \frac{e^{i\varphi_n}}{\omega} \int_0^{2\pi} e^{is} U' \left(\frac{r_n}{2} \left(e^{is} + e^{-is} \right) \right) ds. \end{aligned} \quad (17)$$

A simple substitution shows that the last integral is an odd function of r_n , that means the integral can be written as r_n times an even function. Therefore we can define

$$r_n w(r_n^2) = \frac{1}{2\pi} \int_0^{2\pi} e^{is} U' \left(\frac{r_n}{2} \left(e^{is} + e^{-is} \right) \right) ds \quad (18)$$

where the quantity w can be viewed as an effective suitably averaged force. Then Eq. (17) reads

$$\int_0^{2\pi/\omega} e^{i\omega t} U'(x^{(0)}(t)) dt = \left(x_n + i \frac{v_n}{\omega}\right) \frac{2\pi}{\omega} w(r_n^2) \quad (19)$$

if we take the abbreviations (16) and (18) into account. The real and imaginary parts of this expression yield the remaining integrals contained in Eq. (15). Eqs. (13) and (15) result in the stroboscopic map at first order in the expansion parameter

$$\begin{aligned} x_{n+1} &= x_n + \varepsilon \frac{\pi}{\omega^2} \left(-\gamma \omega x_n + 2 \frac{v_n}{\omega} w(r_n^2) + h^{(s)}\right) + \mathcal{O}(\varepsilon^2) \\ v_{n+1} &= v_n + \varepsilon \frac{\pi}{\omega} \left(-\gamma v_n - 2x_n w(r_n^2) - h^{(c)}\right) + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (20)$$

This result of the perturbation expansion is valid for any type of potential $U(x)$, as long as the integral in Eq. (18) does not vanish and defines a meaningful effective force w . In particular, the result, Eq. (20), does not rely on any symmetry properties of the potential.¹ For the particular case of the Duffing oscillator, Eq. (10), Eq. (18) readily gives the explicit expression

$$w(r_n^2) = \frac{a}{2} + \frac{3b}{8} r_n^2. \quad (21)$$

It is rather straightforward to compute the fixed points of the map, Eq. (20) in closed analytic form. In fact, for the fixed point (x_*, v_*) Eq. (20) results in the linear system of equations

$$\begin{pmatrix} -\gamma\omega & 2w(r_*^2)/\omega \\ -2w(r_*^2) & -\gamma \end{pmatrix} \begin{pmatrix} x_* \\ v_* \end{pmatrix} = \begin{pmatrix} -h^{(s)} \\ h^{(c)} \end{pmatrix}. \quad (22)$$

Solving this system for x_* and v_* we get

$$\begin{aligned} x_* &= \frac{-2w(r_*^2)h^{(c)} + \gamma\omega h^{(s)}}{(\gamma\omega)^2 + (2w(r_*^2))^2}, \\ v_* &= -\omega \frac{\gamma\omega h^{(c)} + 2w(r_*^2)h^{(s)}}{(\gamma\omega)^2 + (2w(r_*^2))^2} \end{aligned} \quad (23)$$

and using the definition of the stationary amplitude, $r_*^2 = x_*^2 + v_*^2/\omega^2$ (see eq. (16)) we finally obtain an implicit equation for r_*

$$\left(h^{(c)}\right)^2 + \left(h^{(s)}\right)^2 = r_*^2 \left((\gamma\omega)^2 + (2w(r_*^2))^2\right). \quad (24)$$

¹ For the special case of the Helmholtz oscillator, a potential which contains only quadratic and cubic terms, the analytic perturbation expansion has to be carried out at second order [29] to obtain meaningful equations of motion and a bistable resonance by perturbation theory.

To demonstrate the accuracy of the perturbation expansion we compare bifurcation diagrams obtained directly from the numerical integration of the equations of motion with those computed from the first order perturbation result, Eq. (20) or (24). Figure 4 shows the stationary amplitude in dependence of the driving amplitude for quasi-stationary parameter upsweeps and downsweeps. The results obtained from the analytic first order expression are surprisingly accurate when compared with numerical simulations for small values of ε . Actually deviations turn out to be so small that at the scale used in Fig. 4 no difference between the simulation of the differential equation, the iteration of the analytic map Eqs. (20) and (21), and the analytic expression Eq. (24) is discernible. For large values of ε the first order truncation in Eq. (20) fails when iterations of this map are considered, as the sequence of iterates tends to diverge. Nevertheless, the bifurcation diagram of stationary states for large values of ε , see Fig. 3, is still in qualitative and to some extent even in quantitative agreement with the data obtained analytically for small ε , cf. Figs. 3 and 4. While time scales and transients for small and large values of ε vastly differ there seems to be little change in the stationary states, i.e., in the location of fixed points. This coincidence is surprising but not totally unexpected as our perturbation scheme is essentially equivalent to the averaging principle or the principle of harmonic balance. While those approaches are formally first order perturbation schemes they sometimes perform well even beyond the perturbative regime as these approaches can also be viewed as a non-systematic mean field type expansion.

4 Design of globally stable control

Our main goal is to construct a control feedback so that stabilisation of an unstable state occurs for a large set of initial conditions, if possible even globally in the entire phase space. Following the basic reasoning outlined in Sect. 2 we would need some information about the underlying stroboscopic map. At least in a perturbative regime of small values of ε such information is available as demonstrated in Sect. 3. Hence we will base our control design on the expression Eq. (20) for the stroboscopic map. Thanks to two driving field amplitudes, i.e. thanks to the fact that we have taken explicitly the phase of the driving field into account, a field amplitude occurs as an additive part in each of the

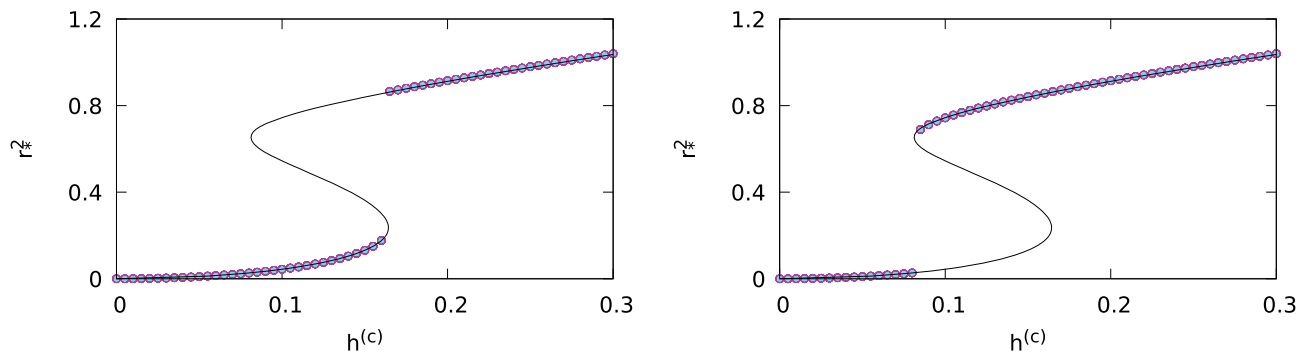


Fig. 4 Bifurcation diagram of Eq. (9) for $\varepsilon = 0.05$ (other parameters are as in Fig. 3). Data are obtained from a numerical simulation of the stroboscopic map (red, open symbols), or using iterates of the analytic map derived by first order perturbation theory Eqs. (20) and (21) (cyan, full symbols). Skipping a transient of 800 iterations the orbit settles on a stable fixed point

(x_*, v_*) . The dependence of $r_*^2 = x_*^2 + v_*^2/\omega^2$ on $h^{(c)}$ is shown for a quasi-stationary parameter upswEEP (left) and a parameter downswEEP (right). The black line shows the analytic expression obtained for the fixed point at first order perturbation theory, see Eq. (24)

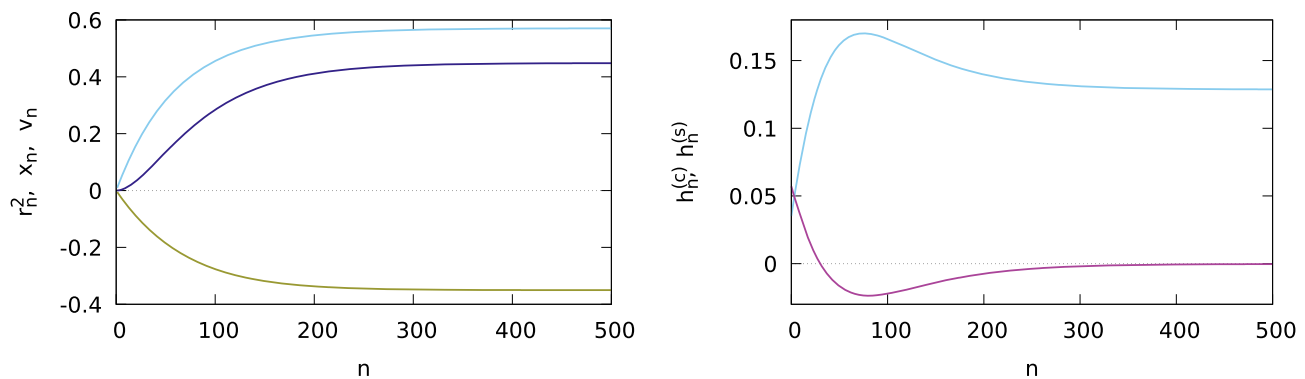


Fig. 5 Time series of the stroboscopic map, for the driven Duffing oscillator, Eqs. (9) and (10), subjected to stroboscopic control, Eq. (25), for parameter values as in Eqs. (11) and $\varepsilon = 0.05$. Control offsets have been chosen as $h_R^{(c)} = 0.03515$

and $h_R^{(s)} = 0.05713$. Left: Time dependence of x_n (cyan), v_n (amber) and $r_n^2 = x_n^2 + v_n^2/\omega^2$ (blue). Right: Time dependence of the driving amplitudes, $h_n^{(c)}$ (cyan) and $h_n^{(s)}$ (red)

components of the stroboscopic map, and we can use these amplitudes for control purposes. Following the reasoning outlined in Sect. 2 it looks tempting to implement a control feedback which removes the nonlinear part, so that the remaining dissipative linear contribution ensures global convergence towards a stabilised fixed point. Such a reasoning leads us to the design

$$h_n^{(c)} = h_R^{(c)} - 2x_n w(r_n^2), \quad h_n^{(s)} = h_R^{(s)} - 2\frac{v_n}{\omega} w(r_n^2) \quad (25)$$

where $r_n^2 = x_n^2 + v_n^2/\omega^2$, see Eq. (16). The reasoning which lead us to the design, Eq. (25), follows the idea of feedback linearisation which is well established in the engineering context, see e.g. [30]. However, we want to emphasise that the control scheme can be implemented

easily by experimentally accessible parameters such as the amplitude and the phase of the driving field. Evaluating the fixed point condition under control (see Eq. (20) with $h^{(c/s)}$ replaced by $h_n^{(c/s)}$) we have

$$-\gamma v_* - h_R^{(c)} = 0, \quad -\gamma \omega x_* + h_R^{(s)} = 0 \quad (26)$$

so that the two offset values $h_R^{(c/s)}$ in our control design select the components of the fixed point to be stabilised. The amplitudes $h_n^{(c)}$ and $h_n^{(s)}$ of the driving field will then converge towards values $h_\infty^{(c)}$ and $h_\infty^{(s)}$ which correspond to the stabilised fixed point (x_*, v_*) , as already illustrated in Fig. 2 for our toy model. Having access to the phase of the driving field, i.e., having thereby access to both terms of the driving field has turned out to be crucial for our design.

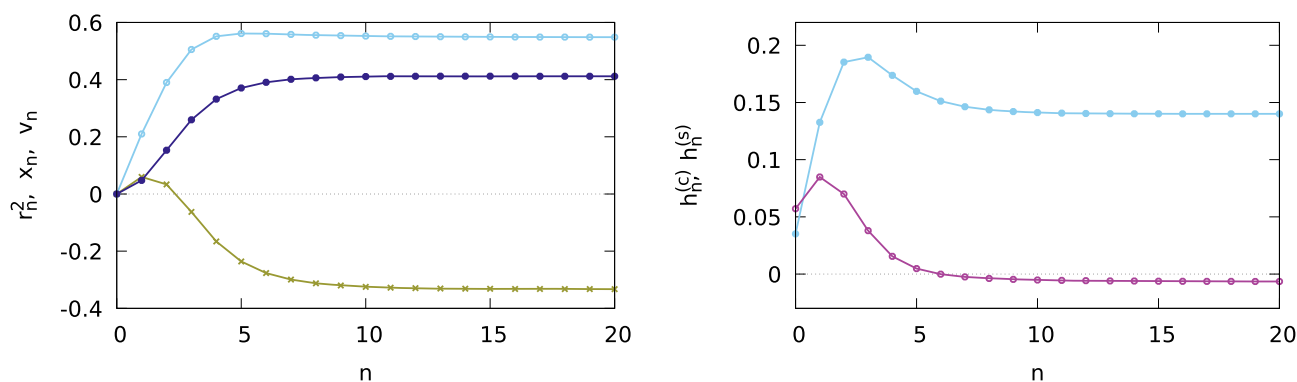


Fig. 6 Time series of the stroboscopic map, for the driven Duffing oscillator, Eqs. (9) and (10), subjected to stroboscopic control, Eq. (25), for parameter values as in Eqs. (11) and $\varepsilon = 1.0$. The control offsets have been chosen as $h_R^{(c)} = 0.03515$ and $h_R^{(s)} = 0.05713$. Left: Time dependence of x_n (cyan, open sym-

bols), v_n (amber, cross) and $r_n^2 = x_n^2 + v_n^2 / \omega^2$ (blue, full symbols). Right: Time dependence of the driving amplitudes, $h_n^{(c)}$ (cyan, full symbols) and $h_n^{(s)}$ (red, open symbols). Cf. Fig. 5 for the analogous result for small values of ε

While the setup defined in Eq. (25) will by definition work for the stroboscopic map in first order of ε we still need to confirm whether the scheme has good convergence properties when applied to the full equations of motion. For that purpose let us first consider numerical simulations for small values of ε where first order perturbation theory has turned out to be quite accurate even at a quantitative level, see Sect. 3. Implementing the control means that at the beginning of each period of the drive we readjust the amplitudes of the driving field using Eq. (25), i.e., using the current values of $x_n = x(n2\pi/\omega)$ and $v_n = v(n2\pi/\omega)$. We could use in principle any values for the offsets $h_R^{(c)}$ and $h_R^{(s)}$. However we are aiming at stabilising unstable states in the bistable regime and for driving fields which have no non-trivial phase (i.e. driving fields with ultimate value $h_\infty^{(s)} = 0$). In practice one could tune such offsets when stabilisation has been achieved, but there is also a way to determine suitable offsets a priori, which we will address in the next section. For the moment we just make up “suitable” values out of thin air. Figure 5 shows time traces obtained for the driven Duffing oscillator with the control scheme along the lines of Eq. (25). The time traces of the phase space coordinates prove successful stabilisation with a limiting value of $r_\infty^2 = 0.448 \dots$, while the time traces of the control forces $h_n^{(c/s)}$ settle on values of the driving amplitude $h_\infty^{(c)} = 0.128 \dots$ and $h_\infty^{(s)} = -0.0002 \dots$, which correspond to the unstable branch right in the middle of the bistable region, cf. Fig. 4. We have also checked how

the control performs for different initial conditions. For initial conditions in the range $-5 \leq x(0) \leq 5$, $-5 \leq v(0) \leq 5$ we always find successful stabilisation of the unstable periodic state, in line with what we expect from the first order perturbation treatment. For larger values of the initial condition occasionally solutions seem to diverge, which is far from surprising given that the equations of motion and the control scheme contain cubic terms. Hence, even if the basin for successful control is finite, its size is so large that the scheme can be considered as sufficiently robust.

It is not so surprising that the control design works quite successfully for small values of ε since the analysis of the previous section has shown that this case is quite well covered by the perturbative analysis. As a test for our approach we address parameter settings beyond the perturbative regime and apply the control defined by Eq. (25) to the Duffing oscillator, Eqs. (9) and (10), for larger values of ε , say $\varepsilon = 1$. Time traces of the oscillator subjected to control with the parameter setup used in Fig. 3 are shown in Fig. 6. The performance in the strongly nonlinear regime, $\varepsilon = 1$, is in fact comparable to the perturbative regime, cf. Fig. 5, even though the transients are now much shorter thanks to the larger dissipation $\varepsilon\gamma$. For large times the amplitude tends towards the limit $r_\infty^2 = 0.411 \dots$ and the amplitude of the driving field has the limiting value $h_\infty^{(c)} = 0.140 \dots$, so that the asymptotic state is in fact on the unstable branch of the strongly nonlinear Duffing oscillator, see Fig. 3. In addition there is as well a small component $h_\infty^{(s)} = -0.0066 \dots$ giving rise to a

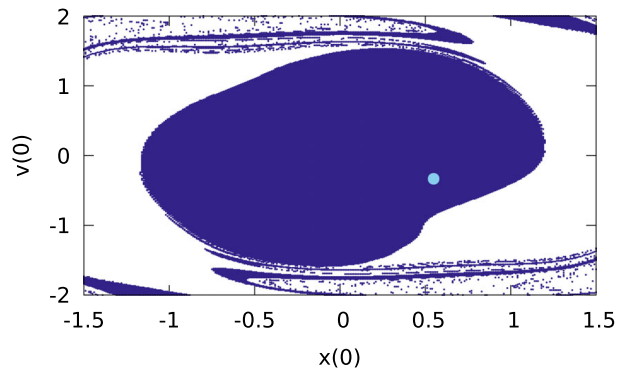


Fig. 7 Basin for stroboscopic control, Eq. (25), with control parameter setting $h_R^{(c)} = 0.03515$ and $h_R^{(s)} = 0.05713$ (see Fig. 6) applied to the driven Duffing oscillator, Eqs. (9) and (10) with parameter values as in Eq. (11) and $\varepsilon = 1.0$. The filled circle (cyan) indicates the stabilised orbit

non-vanishing phase in the drive, see Eq. (12), which could be removed by a slight adjustment of the offsets $h_R^{(c)}$ or $h_R^{(s)}$. Overall our control design performs quite well even beyond the perturbative regime.

To judge the overall performance we investigate by numerical means the basin of the control with parameter settings used in Fig. 6, i.e., $h_R^{(c)} = 0.03515$ and $h_R^{(s)} = 0.05713$. We are no longer within the perturbative regime and the basin is finite, see Fig. 7. However, the basin is still quite large and covers all the states which occur in plain parameter sweeps, cf. Fig. 3. Hence the control design can be considered as a success even from an experimental point of view. The boundary of the basin resembles the fractal structure caused by homoclinic tangles. Hence, the basin boundary in our case may be caused by the stable manifold of a saddle.

5 Control parameter setting and tracking

The control offsets $h_R^{(c)}$ and $h_R^{(s)}$ determine the final amplitudes of the driving field and the fixed point to be stabilised. Since the control design has good global properties one could start with any values for the offsets. After successful control a continuation of unstable states could be done, as usual, by small quasi-stationary changes of the offsets.

If one employs the knowledge from the weakly nonlinear analysis one can do better and determine a priori estimates for suitable offsets to stabilise an unstable state with an approximate amplitude r_*^2 . Given a value for r_*^2 , Eq. (24) tells us within the limits of first order

perturbation theory that the corresponding amplitudes of the driving field read

$$h^{(c)} = r_* \sqrt{(\gamma\omega)^2 + (2w(r_*^2))^2}, \quad h^{(s)} = 0 \quad (27)$$

when we impose a constraint on the phase of the driving field. Using these values to estimate the actual fixed point coordinates via Eq. (23) we have

$$x_* = -2w(r_*^2)r_*^2/h^{(c)}, \quad v_* = -\gamma\omega^2 r_*^2/h^{(c)}. \quad (28)$$

Finally Eq. (26) yields the estimates for suitable control offsets as

$$h_R^{(c)} = -\gamma v_* = \frac{(\gamma\omega r_*)^2}{h^{(c)}}, \quad h_R^{(s)} = \gamma\omega x_* = -\frac{2\gamma\omega w(r_*^2)r_*^2}{h^{(c)}}. \quad (29)$$

We have in fact used Eq. (29) to determine offsets for the control in the previous section (with $r_* = 0.4$). The actual fixed point controlled, see Fig. 5, then differs slightly from the estimate r_*^2 , in particular, if the parameters of the equations of motion are not within the range of the first order perturbative treatment.

We use Eq. (29) for tracking of stable and unstable states without resorting to a quasi-stationary change of the control offsets $h_R^{(c)}$ and $h_R^{(s)}$. Figure 8 shows the result for the Duffing oscillator in a strongly nonlinear regime, $\varepsilon = 1$, with the parameter setup as in Fig. 3. Even though we are well beyond the validity of the perturbative regime our control design performs reasonably well, allowing to track the unstable state within the region of bistability.

There are visible deviations in Fig. 8 between the fixed point without control and the fixed point subjected to control, in particular, close to the fold instabilities which bound the region of bistability. In addition, the unstable branch shows a seemingly subtle structure. These deviations are caused by the control action resulting in a non-vanishing value for $h^{(s)}$, see Fig. 8, that means in a non-vanishing phase for the driving field, see Eq. (12). Hence our observation for the amplitude r_*^2 under control does not correspond to the setup used in the system without control, see Fig. 3, where we have $h^{(s)} = 0$. If the orbit were a plain harmonic such an effect would not matter. Since the orbit in phase space is not a perfect ellipse the measured amplitude depends on the actual phase of the driving field.

There are a couple of ways to compensate for this effect. On the one hand we can slowly tune $h_R^{(s)}$ so that $h_\infty^{(s)}$ becomes zero, which essentially amounts to

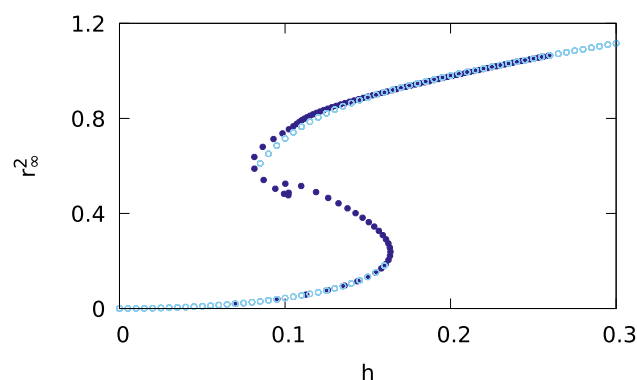
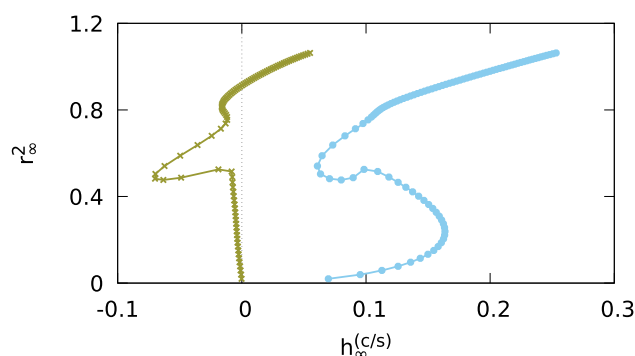


Fig. 8 Control based continuation of the stationary state in the Duffing oscillator, Eqs. (9) and (10), for $\varepsilon = 1$, with parameter settings as in Eq. (11), and with the control scheme defined in Eq. (25) and (21). Control offsets for the tracking have been taken from Eq. (29) (with r_∞^2 serving as the curve parameter). Left: Successful tracking of the stationary state r_∞^2 (blue, full symbols) in



dependence on the amplitude of the driving field, see Eq. (12). For comparison the corresponding data of the oscillator without control are shown as well (open symbols, cyan), see Fig. 3. Right: resulting amplitudes of the driving field, Eq. (25), when control has been successful. Cyan $h_\infty^{(c)}$, amber $h_\infty^{(s)}$

an experimental root finding problem. One has to keep in mind, however, that beyond the perturbative regime the rotational symmetry shared by the low order perturbation expansion Eq. (20) is not valid any longer. Therefore the stability of the unstable state may get lost during this adjustment. It may in fact happen that within our design successful control for a non-vanishing phase of the driving field does not per se translate into successful control for vanishing phase of the driving field when the offsets are readjusted accordingly. On the other hand, there is no need to perform such an adjustment. The phase of the driving field, Eq. (12), can be compensated for if we consider the stroboscopic map at a different time, i.e., if we formally chose a different cross section. If we record data at times $n2\pi/\omega + \theta/\omega$, that means if we compute a renormalised amplitude r_n^2 based on $x(n2\pi/\omega + \theta/\omega)$ and $v(n2\pi/\omega + \theta/\omega)$ then such a value of r_n^2 is identical to the value one obtains from an ordinary stroboscopic map in a Duffing oscillator where the driving field has vanishing phase $\theta = 0$. That means, while we still base control on discrete time points $n2\pi/\omega$ we use a suitable time lag in the data recording to compensate for the non-vanishing final value of $h_\infty^{(s)}$. By this trivial change we obtain a perfect match of the controlled and the uncontrolled data, see Fig. 9, where even subtle details of the unstable branch are detected with considerable accuracy. It is again worth to stress that the control, the data processing, and the control based continuation can be easily

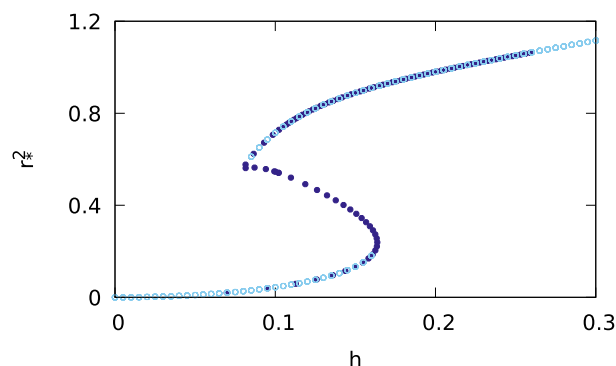


Fig. 9 Control based tracking of the stationary state in the Duffing oscillator with parameters and control design as in Fig. 8. To compensate for the non-vanishing phase of the driving field, see Eq. (12), the estimate of the stationary amplitude r_∞^2 has been computed from time series data $x(n2\pi/\omega + \theta/\omega)$ and $v(n2\pi/\omega + \theta/\omega)$ with a time lag given by the phase of the driving field. Stationary amplitude as a function of the amplitude of the driving field with control (full symbols, blue). For comparison the corresponding data without control, see Fig. 3, are shown as well (open symbols cyan)

implemented in experiments as only a plain time series is required.

6 Data-driven control design

Our control design, Eq. (25), was inspired by the first order perturbation expansion of the stroboscopic map, Eq. (20). This design has been quite successful even beyond the perturbative regime, where Eq. (20) fails

to model the dynamics properly, as exemplified by the results of Sects. 4 and 5. But the approach so far has not been really data driven and thus may have limited applicability. Having said that, any control scheme which claims to have good global properties needs to take some features of the underlying nonlinear dynamics into account.

We consider here driven oscillator systems such as Eq. (9), where now ε is not any longer a small quantity, say $\varepsilon = 1$. As data input for our control design we require that recorded time series' are available, in particular, that we have measured the nonlinear resonance curve by sweeping a system parameter and measuring the amplitude of the stationary state, see Fig. 10. We now assume that the shape of this graph is described as well by Eq. (24), so that we can compute the effective force $w(r_*^2)$ from data. The validity of such an approach, i.e. using Eq. (24), can be justified even at a quantitative level if the relevant periodic orbits, are predominantly harmonic solutions, i.e., that higher order Fourier modes are only of limited importance. Then we can apply in some sense the principle of harmonic balance and capture main aspects of the resulting periodic states in terms of an effective force $w(r^2)$. That does not mean that Eq. (20) gives an accurate quantitative description of the dynamics, it is just sufficient that the resulting periodic solutions, in particular the relation between the amplitude and the driving field is well modelled by Eq. (24) with a suitable effective force $w(r^2)$. Given the data from the frequency sweep of the nonlinear resonance, see Fig. 10, we can then use a least square fit to determine a suitable shape for the effective force.

Here, we illustrate this idea with data of the Duffing oscillator, Eqs. (9) and (10) with $\varepsilon = 1$ and other parameters as in Eq. (11). The measured nonlinear resonance curve, the dependence of the amplitude of the oscillation on the amplitude of the driving field is shown in Fig. 10. We now use a data fit of Eq. (24), that means $h^2 = r_*^2(\gamma^2\omega^2 + 4w^2(r_*^2))$, to obtain an expression for the effective force w . For that purpose we use a simple expression linear in r^2 and perform a least square fit of Eq. (24) to the data shown in Fig. 10 and obtain

$$w(r^2) = w_0 + w_1 r^2, \\ w_0 = -0.2338 \dots, \quad w_1 = 0.3290 \dots \quad (30)$$

The result of this least square fit shows a reasonable quantitative agreement with the data apart from small

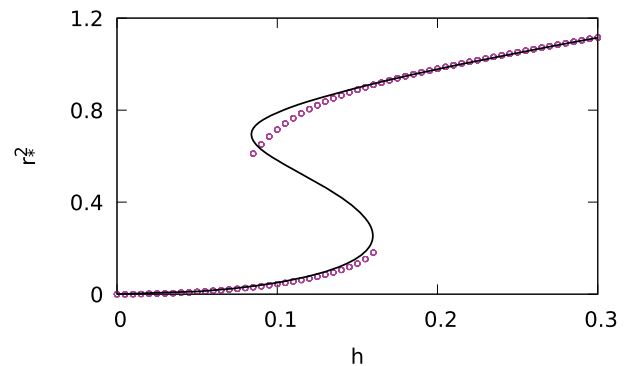


Fig. 10 Least square fit of the analytic expression Eqs. (24) and (30) to the data shown in Fig. 3, i.e., least square fit for the nonlinear resonance line of the driven Duffing oscillator, Eqs. (9) and (10) with parameters as in Eqs. (11), $h^{(s)} = 0$, and $\varepsilon = 1$. Line: Eq. (24) with eq. (30) and $w_0 = -0.2338$, $w_1 = 0.3290$. Symbols: data obtained from a parameter upsweep and parameter downsweep (cf. Fig. 3)

regions close to the fold bifurcations where the fine structure is not reproduced accurately, see Fig. 10.

The numerical values for w_0 and w_1 which were obtained from the least square fit differ by about 10% from those which correspond to the first order perturbation expansion given in Eq. (21). That again signals that orbits are not plain harmonics and differ from an elliptic shape in phase space, as already seen in the discussion in Sect. 5, see Figs. 8 and 9. Nevertheless, the effective force, eq. (30), captures quite well the periodic stationary states of the driven oscillator. It is tempting to use the effective force, Eq. (30), to implement the control law, Eq. (25), for the driven Duffing oscillator in the parameter setting specified above. Even though there is no a priori guarantee that the approach will work, as the perturbation expansion Eq. (20) definitely does not capture the dynamics any longer, the result shown in Fig. 11 demonstrates that control works quite robust even for an initial condition which is not close to the target state.

Even though the parameter setups used to produce the data in Figs. 6 and 11 coincide, the stabilised fixed points differ slightly as the effective force $w(r^2)$, i.e., the actual control feedback is not the same, so that the final driving amplitudes $h_{\infty}^{(s/c)}$ in both cases differ slightly. In addition to time traces we have as well computed the basin of the control, obtained with the data-based control design. The results shown in Fig. 12 indicate in fact a considerable improvement as compared to Fig. 7 where the design was based on the first order perturbation scheme. Hence, the approach outlined in

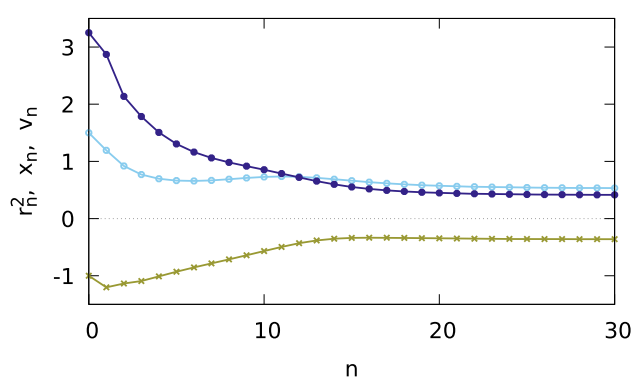
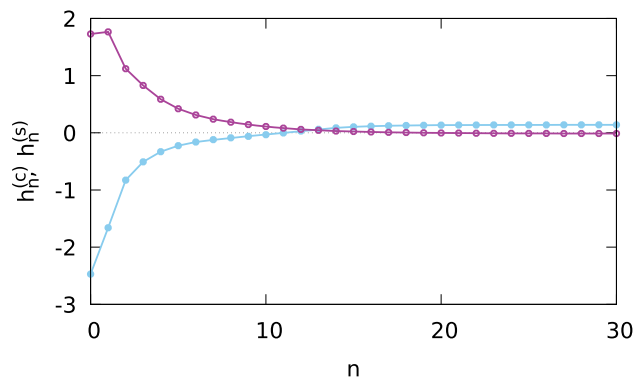


Fig. 11 Time series of the stroboscopic map, for the driven Duffing oscillator, Eqs. (9) and (10), subjected to stroboscopic control, Eq. (25) with effective force Eq. (30), for parameter values as in Eqs. (11) and $\varepsilon = 1.0$. Control offsets have been chosen as $h_R^{(c)} = 0.03515$ and $h_R^{(s)} = 0.05713$ (cf. Fig. 6). Left: Time



dependence of x_n (cyan, open symbols), v_n (amber, cross) and $r_n^2 = x_n^2 + v_n^2/\omega^2$ (blue, full symbols). Right: Time dependence of the driving amplitudes, $h_n^{(c)}$ (cyan, full symbols) and $h_n^{(s)}$ (red, open symbols)

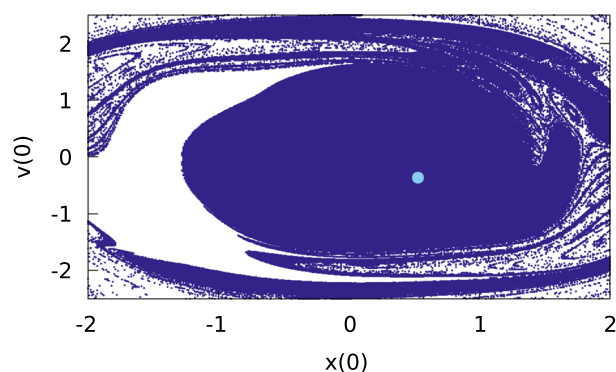


Fig. 12 Basin for stroboscopic control, Eq. (25) with effective force Eq. (30), and control parameter setting $h_R^{(c)} = 0.03515$ and $h_R^{(s)} = 0.05713$ (see Fig. 11) applied to the driven Duffing oscillator, Eqs. (9) and (10) with parameter values as in Eqs. (11) and $\varepsilon = 1.0$. The filled circle (cyan) indicates the stabilised orbit

this section, which is based on the data obtained from a simple parameter upswing and downswing provides a promising strategy for a wider class of driven oscillator systems. One may even exploit the dependence of the parameter sweeps on the driving frequency ω in conjunction with the analytic expression Eq. (24) to explore the damping mechanism in more detail. Details will be published elsewhere.

7 Conclusion

We have succeeded with our main aim to design a globally robust non-invasive control scheme for the stabil-

isation of periodic orbits, to enable control-based continuation. The control is based on the measurement of a single data point per period of the drive, so that such a scheme is applicable in fast systems like atomic force microscopy, when no extensive data processing can be done during control, and where no accurate mathematical model is available for preprocessing. The design of our control scheme was initially inspired by the weakly nonlinear analysis of oscillator systems. Having access to the phase of the driving field has turned out to be a key for the success of the control, since thereby we have been able to ensure for large basins of attraction for the stabilised state. The control feedback contains an effective averaged force, so that the design already gives vital information about the underlying dynamics. Numerical simulations indicate that the control scheme works quite well beyond the perturbative regime, and further systematic numerical studies look promising. Above all, we have shown that the required details of the control design can be obtained from a simple scan of the nonlinear resonance curve, so that in fact no underlying mathematical model is needed from the outset.

We have based our analysis on a one degree of freedom mechanical oscillator model with fairly general potential. For the theoretical analysis we have used the assumption that higher order harmonics play a limited role, even though our numerical studies show that such a constraint can be relaxed to some extent. However, it still needs to be investigated how the present approach can be generalised to higher dimensional driven dynamical systems. Again weakly nonlinear

analysis could provide a hint how to proceed in this context.

We have here considered a model with a simple viscous damping. In real world experiments, such as atomic force microscopy, the treatment of all losses by a single viscous damping constant γ may be regarded as undercomplex, even in the seemingly simple situation of a stiff, hydrophobic sample in a vacuum environment. While the dynamics of an oscillating microcantilever beam itself behaves largely linear, the complexity of the system arises from the largely unknown dynamics of the multitude of effects taking part in the nanoscopic junction between probe and sample. These unknowns and nonlinearities do pose the challenges. Currently, it is hard to say what physics or (bio)chemistry happens at the turning point of the tip in the vicinity of the sample, the point of strongest interaction which shapes contrast and stability of the measurement. While force-distance spectroscopy can help to unravel force contributions from molecule layering, visco-elasticity, plastic deformation, electrical polarisation and attraction, electro-chemical reactions, rupturing molecular bonds, entropic interaction, depletion forces, oxidation, receptor-ligand, or other (see, for instance [31,32]), this is of limited value to judge the situation of topography acquisition in the dynamic mode at cantilever frequencies in the 10 to 500 kHz regime. At a phenomenological level the complex dissipation processes can be modelled by a state dependent damping, and its impact can be analysed within a weakly nonlinear perturbation expansion as well, resulting in an additional effective damping term which along the side of the effective averaged force enters the shape of the nonlinear resonance curve. Both contributions, the effective damping and the effective force can be disentangled by monitoring in addition the frequency dependence of the nonlinear resonance. Hence, by having access to stable as well as unstable branches we can quite accurately determine potential and damping from measured data and thus contribute, for instance, to the outstanding challenge to understand the dissipative mechanisms in atomic force microscopy.

A robust control scheme which enables large basins of attraction is one of the keys to implement model free continuation of bifurcations in experiments, to make the power of continuation tools in the study of mathematical models available as a 21st century data-based spectroscopic tool. Our control design meets all

these constraints, so that we can track unstable states even when no quasi-stationary parameter sweep can be implemented. Having developed a suitably robust control scheme we have taken the next step towards finally implementing control based continuation in real world complex experiments. Details in that direction will be reported elsewhere.

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Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

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Chapter 4

Trajectory-based stabilization of periodic orbits

Trajectory-Based Stabilization of Periodic Orbits*

Niklas Kruse[†], Wolfram Just[†], and Jens Starke[†]

Abstract. We present a data-driven feedback control scheme, which stabilizes unstable periodic orbits of dynamical systems. Model equations of the underlying system are assumed to be unavailable. Our approach relies only on previously measured input-state trajectories of a Poincaré map of the system. The success of a stabilizing feedback control in experimental systems typically depends on an adequate choice of control gains. The presented scheme computes stabilizing control gains by solving an optimization problem. No explicit model is identified and the control gains are computed directly from the measured trajectories. As a demonstration, we apply the scheme to a control-based continuation problem and obtain the corresponding bifurcation diagram.

Key words. data-driven control, Poincaré map, periodic orbit, control-based continuation

MSC codes. 37N35, 34H05, 34H20

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1. Introduction. Experimental systems typically depend on parameters, which can be adjusted in order to obtain a desired behavior. This desired behavior may be represented by a stationary state or some periodic solution in the state space. If these are unstable, meaning that initialization near but not exactly at them causes the system state to diverge, the desired behavior is not observable in the experiment, which never allows for exact initialization. Even if one succeeds at initializing the experiment exactly at, for example, an unstable stationary state, the never completely avoidable external disturbances would again cause the system state to diverge. To overcome this, feedback control is often used for the stabilization of an unstable steady state of the underlying system [22, 26, 12, 29]. When the desired behavior is represented by a certain unstable periodic solution in the state space, one can again rely on feedback control to stabilize these periodic orbits, thus making them observable in the corresponding experiment. One way to extend linear feedback control to this scenario was suggested in [27]. The idea is to reduce the stabilization of the desired periodic solution to the stabilization of the corresponding fixed point of a Poincaré map [19]. Consequently, feedback control techniques for stabilizing fixed points of discrete-time systems can be applied. However, keeping the application of these techniques in real experiments in mind, we want to point out the following two problems.

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Feedback control is often formulated in a framework, which assumes that the experimental system is modeled by a system of differential equations. While in some situations, first principles allow for deriving such an analytic model, this is not always the case. To overcome this, data-based control strategies are used [31, 24]. These are based on a data-driven identification of a linear control system, which are amenable for linear feedback control [26, 12]. Furthermore, we want to point out that feedback control relies on an adequate choice of control gain parameters. Having an analytic model at hand, which was either derived or identified with the aid of the above-mentioned data-based identification methods, there exist methods for computing stabilizing control gains [26, 12]. Also, model-based methods for designing feedback control laws of nonlinear systems are available, which include, for example, linearization via feedback, sliding mode control, backstepping, and passivity-based control [22, 23]. However, motivated by the fact that experimental systems are naturally observed through time series of some number of measurement variables, it was shown in [14, 13, 6, 2], that one can directly compute stabilizing control gains from collected time series. This means, that a model identification is not required, which sometimes is referred to as a trajectory-based approach. These trajectory-based methods are used to stabilize known fixed points of discrete- and continuous-time control systems and have also been used in a model predictive control scheme [7]. In this work, we extend this framework to the stabilization of periodic orbits of continuous-time systems. In [9], the above-mentioned model identification methods were used to derive analytic expressions of Poincaré maps, which allows for stabilizing unstable periodic orbits via linear feedback control. Although we ultimately have the same goal, our approach is not based on this explicit model identification and computes stabilizing control gains directly from observed time series of a Poincaré map. We apply the proposed scheme by tracking an unstable branch of periodic orbits, which appears in a system that is defined from the normal form of the saddle-node bifurcation. This is referred to as control-based continuation [37, 35, 36, 4, 5, 11, 10, 34, 32, 3, 1, 29, 15, 30, 16, 8]. While some of the mentioned articles already use feedback control for stabilizing unstable steady states or periodic orbits in laboratory experiments, finding adequate control gains remains a challenging problem. With the proposed scheme, we are taking a step towards a systematic calculation of control gains.

The remainder of this article is structured as follows. In section 2, we give an overview of trajectory-based data-driven feedback control of discrete-time systems. Section 3 includes an extension to the stabilization of unstable periodic orbits of continuous-time systems via a Poincaré map. We embed the proposed method into an integral control setup, which is used to solve a control-based continuation problem. Section 4 concludes this article.

2. Data-driven feedback control of discrete-time systems. Trajectory-based control methods allow for constructing stabilizing feedback controllers without having to identify an explicit model. Instead, an input-state time series of the given system is recorded. Subsequently, stabilizing control gains are computed by solving an optimization problem with convex constraints. This method is based on data-driven linear control theory, to which we now turn.

2.1. Trajectory-based methods for linear systems. First, we review some of the results that are included in [14, 31]. For given matrices $A \in \mathbb{R}^{N \times N}$ and $B \in \mathbb{R}^N$, consider a discrete-time linear system

$$(2.1) \quad x_{n+1} = Ax_n + Bu_n,$$

where $x_n \in \mathbb{R}^N$ denotes the state and $u_n \in \mathbb{R}$ refers to a control parameter at time n .

Choosing an initial state x_0 and a sequence $u_0, u_1, \dots, u_{T-1}$ of $T > 0$ control parameter values, we compute via (2.1) a corresponding sequence $x_0, x_1, \dots, x_T$ of states. The first question is, whether it is possible to compute A and B only from the obtained simulation data

$$(2.2) \quad \begin{aligned} U &= [u_0 \ u_1 \ \dots \ u_{T-1}], \\ X &= [x_0 \ x_1 \ \dots \ x_{T-1}], \\ Y &= [x_1 \ x_2 \ \dots \ x_T]. \end{aligned}$$

This is answered by the following theorem.

Theorem 2.1 (see [14]). *Suppose that*

$$\text{rank} \begin{bmatrix} U \\ X \end{bmatrix} = N + 1.$$

Then it follows that

$$\begin{bmatrix} B & A \end{bmatrix} = Y \begin{bmatrix} U \\ X \end{bmatrix}^\dagger,$$

where

$$\begin{bmatrix} U \\ X \end{bmatrix}^\dagger = [U^T \ X^T] \left(\begin{bmatrix} U \\ X \end{bmatrix} [U^T \ X^T] \right)^{-1}$$

denotes the right inverse, which exists due to the rank condition.

In the general case, starting with experimentally obtained data of the form (2.2), one is interested in solving the least squares problem

$$(2.3) \quad \min_C \left\| Y - C \begin{bmatrix} U \\ X \end{bmatrix} \right\|_F,$$

where $\|Z\|_F = \sqrt{\sum_{i,j} |z_{ij}|^2}$ denotes the Frobenius norm. Note that the square of the Frobenius norm is a quadratic function in the absolute values of the matrix elements, and therefore the minimization leads to a linear zero problem. If the rank condition of Theorem 2.1 holds, it follows that

$$C = Y \begin{bmatrix} U \\ X \end{bmatrix}^\dagger$$

is the unique minimizer of (2.3), where $\begin{bmatrix} U \\ X \end{bmatrix}^\dagger$ denotes the right inverse from Theorem 2.1. Defining matrices $A \in \mathbb{R}^{N \times N}$ and $B \in \mathbb{R}^N$ by setting

$$\begin{bmatrix} B & A \end{bmatrix} = Y \begin{bmatrix} U \\ X \end{bmatrix}^\dagger$$

gives a linear approximation $x_{n+1} \approx Ax_n + Bu_n$ of the experimental system subject to control inputs. This identification method is referred to as dynamic mode decomposition with control [31].

If the matrix $C = [B \ AB \ \dots \ A^{N-1}B]$ has full rank, the linear system of the form (2.1) is called controllable. It follows from this that one can compute a matrix $K \in \mathbb{R}^{1 \times N}$, such that the eigenvalues of the matrix $A + BK$ lie at predetermined locations. In particular, one can place all eigenvalues inside the unit circle. If this is the case, we say that the controller $u_n = Kx_n$ is stabilizing. This is equivalent to the fact that the trajectories of the closed-loop system

$$(2.4) \quad x_{n+1} = (A + BK)x_n$$

converge to the origin for every initial condition; see, e.g., [26]. Thus, after identification of a controllable linear system from data, one can design a stabilizing feedback controller. While for given matrices A and B there exist efficient algorithms for doing so, we aim for a stabilizing feedback control, which is computable directly from the data (2.2). This means no explicit model is given, nor does one have to identify one during the application of the method. A key result in this direction is given as follows.

Theorem 2.2 ([14]). *Consider the discrete-time linear system (2.1). Suppose that*

$$(2.5) \quad \text{rank} \begin{bmatrix} U \\ X \end{bmatrix} = N + 1.$$

Then, for a given $K \in \mathbb{R}^{1 \times N}$, there exists a $G \in \mathbb{R}^{T \times N}$ with

$$\begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U \\ X \end{bmatrix} G$$

and

$$A + BK = YG,$$

where I denotes the $N \times N$ identity matrix.

Conversely, if (2.5) holds, choosing a $G \in \mathbb{R}^{T \times N}$ with $I = XG$ and setting $u_n = UGx_n$ implies the closed-loop representation

$$A + BUG = YG.$$

Note that the first part of Theorem 2.2 provides for a fixed control gain matrix K a way of computing the matrix $A + BK$ of the closed-loop system (2.4) directly from the data (2.2). This makes it possible to check whether, for a given K , the feedback control $u_n = Kx_n$ is stabilizing. On the other hand, due to the second part, one can search for a G with $I = XG$, such that the feedback control $u_n = UGx_n$ becomes stabilizing without an explicit identification of A and B . This forms the basis of the following theorem.

Theorem 2.3 (see [14]). Suppose that

$$(2.6) \quad \text{rank} \begin{bmatrix} U \\ X \end{bmatrix} = N + 1.$$

Then, given a matrix $Q \in \mathbb{R}^{T \times N}$ such that XQ is symmetric and

$$(2.7) \quad \begin{bmatrix} XQ & YQ \\ Q^T Y^T & XQ \end{bmatrix}$$

is positive definite, the matrix $(XQ)^{-1}$ exists and the feedback controller $u_n = Kx_n$ with

$$(2.8) \quad K = UQ(XQ)^{-1}$$

is stabilizing.

Conversely, if (2.6) holds and K is stabilizing, then there exists a Q such that (2.7) is positive definite and (2.8) holds.

Thus, as long as the rank condition is satisfied, the second part of Theorem 2.3 parametrizes all stabilizing gain matrices K by the data matrices (2.2), which are obtained by computing one input-state trajectory of the system (2.1). If, in addition, the system is controllable, it follows that there exists at least one solution Q with the desired properties. In fact, finding a Q that renders (2.7) positive definite can be cast as a standard problem in the framework of linear matrix inequalities; see, e.g., [33]. Linear matrix inequalities are convex constraints, and thus the stated problem is convex. For a numerical computation of the matrix Q , efficient solvers such as CVX are available [18, 17].

2.2. Extension to nonlinear systems via nonlinearity cancellation. One can extend the previous considerations of linear control theory to certain classes of nonlinear systems. So far, we have seen that designing a stabilizing controller for a linear system of the form (2.1) does not require an a priori identification of the matrices A and B . Although there exists a direct extension of this idea to the case of nonlinear systems, it is restricted in the following sense. While a controller, which renders the linearization around an equilibrium point of a given nonlinear system stable, also locally stabilizes the nonlinear system, there is no further information on the basin of attraction of the stabilized point. In practice, one might have to initialize the system extremely close to the point to be stabilized, which is not always possible. To circumvent this, trajectory-based feedback linearization can be used [13, 23, 21, 25]. The key idea of this concept can be explained as follows.

Consider the control system

$$(2.9) \quad x_{n+1} = x_n - x_n^2 + u_n,$$

where $x_n \in \mathbb{R}$ and $u_n \in \mathbb{R}$. Observe that any feedback law $u_n = k_1 x_n + k_2 x_n^2$ with $|1 + k_1| < 1$ and $k_2 = 1$ globally stabilizes the origin. To see this, note that the closed-loop dynamics read

$$x_{n+1} = (1 + k_1)x_n.$$

Thus, $x_n \rightarrow 0$ for $n \rightarrow \infty$ regardless of the initial condition x_0 .

Returning to the trajectory-based setup of the stated theorems, one would like to compute such a globally stabilizing $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ directly from data. For a special class of nonlinear systems, this is indeed possible. We restrict ourselves to systems of the form

$$(2.10) \quad x_{n+1} = Ag(x_n) + Bu_n.$$

Here, x_n , u_n , and B are as in (2.1), while $A \in \mathbb{R}^{N \times R}$. The nonlinearities are included in the map $g : \mathbb{R}^N \rightarrow \mathbb{R}^R$, which is assumed to be continuous. For example, the system (2.9) is of this form with

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 \end{bmatrix}, \\ B &= 1, \\ g(x) &= \begin{bmatrix} x \\ x^2 \end{bmatrix} \end{aligned}$$

and $R = 2$. For the purpose of illustration, we assume here that we know a continuous function $h : \mathbb{R}^N \rightarrow \mathbb{R}^S$ such that

$$(2.11) \quad g(x) = Ch(x)$$

for some matrix $C \in \mathbb{R}^{R \times S}$, where $S \geq N$. Later we will show that in applications such an assumption may not be required. In the following, we review some of the results obtained in [13]. For this, we compute data matrices of the form (2.2) via (2.10) for a fixed initial condition x_0 and a chosen sequence $u_0, u_1, \dots, u_{T-1}$ of control parameter values, but also include $h(x_0), h(x_1), \dots, h(x_{T-1})$. This results in the data set

$$(2.12) \quad \begin{aligned} U &= \begin{bmatrix} u_0 & u_1 & \dots & u_{T-1} \end{bmatrix}, \\ X &= \begin{bmatrix} x_0 & x_1 & \dots & x_{T-1} \end{bmatrix}, \\ Y &= \begin{bmatrix} x_1 & x_2 & \dots & x_T \end{bmatrix}, \\ H &= \begin{bmatrix} h(x_0) & h(x_1) & \dots & h(x_{T-1}) \end{bmatrix}. \end{aligned}$$

Corresponding to Theorem 2.3, one has the following.

Theorem 2.4 (see [13]). *Suppose that*

$$(2.13) \quad \text{rank} \begin{bmatrix} U \\ H \end{bmatrix} = S + 1.$$

If there exist matrices $P = P^T \in \mathbb{R}^{N \times N}$, $Z \in \mathbb{R}^{T \times N}$, and $G \in \mathbb{R}^{T \times (S-N)}$ such that

$$(2.14) \quad \begin{aligned} HZ &= \begin{bmatrix} P \\ 0_{(S-N) \times N} \end{bmatrix}, \\ \begin{bmatrix} P & (YZ)^T \\ YZ & P \end{bmatrix} &\text{ is positive definite,} \\ HG &= \begin{bmatrix} 0_{N \times (S-N)} \\ I_{S-N} \end{bmatrix}, \\ YG &= 0_{N \times (S-N)} \end{aligned}$$

hold, then the control law $u_n = Kh(x_n)$, where

$$(2.15) \quad K = U \begin{bmatrix} ZP^{-1} & G \end{bmatrix},$$

implies closed-loop dynamics of the form

$$(2.16) \quad x_{n+1} = Dx_n,$$

such that all eigenvalues of D have absolute value less than one. Here, $0_{k \times l}$ denotes the zero matrix of dimension $k \times l$ and I_k is the $k \times k$ identity matrix.

Conversely, if (2.13) holds and there exists a stabilizing K , such that the closed-loop dynamics are of the form (2.16), then K can be written as in (2.15) for some P, Z, G satisfying (2.14).

Note that, in contrast to [13], we have slightly adjusted the notation for our purposes. Theorem 2.4 gives a way of designing globally stabilizing control laws for a specific class of nonlinear system. There is no systems identification step involved and the control gains are computed directly from the data (2.12). However, for some systems of the form (2.10), a fully linearizing controller does not exist. In this case, the convex program of Theorem 2.4 can be recast as a minimization problem. In this way, one obtains a controller that minimizes the nonlinearities of the given system and locally stabilizes the origin; for details, see [13].

As an example, consider the map

$$(2.17) \quad \begin{aligned} x_{n+1} &= y_n, \\ y_{n+1} &= -x_n + 3y_n - y_n^3 + u_n \end{aligned}$$

defined on the plane, which has been suggested as an approximation of the Poincaré map corresponding to the Duffing oscillator [20]. The origin is an unstable fixed point of the map for $u_n = 0$. The given system is of the form (2.10) with

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 3 & -1 \end{bmatrix}, \\ g(x, y) &= \begin{bmatrix} x & y & y^3 \end{bmatrix}^T, \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

We do not intend to use the information about the right-hand side of (2.17) for deriving a globally stabilizing control law. Instead we choose

$$(2.18) \quad h(x, y) = \begin{bmatrix} x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 \end{bmatrix}^T$$

as our library of basis functions. Note that with this choice we include every function that appears in (2.17), and thus (2.11) holds. In general, it is not possible to check this condition, which makes the application of the described trajectory-based method somewhat heuristic.

To compute a globally stabilizing control law via the convex program (2.14), we collect data matrices of the form (2.12). For this, we choose $x_0 = 0$ and $y_0 = 0$ as an initial condition

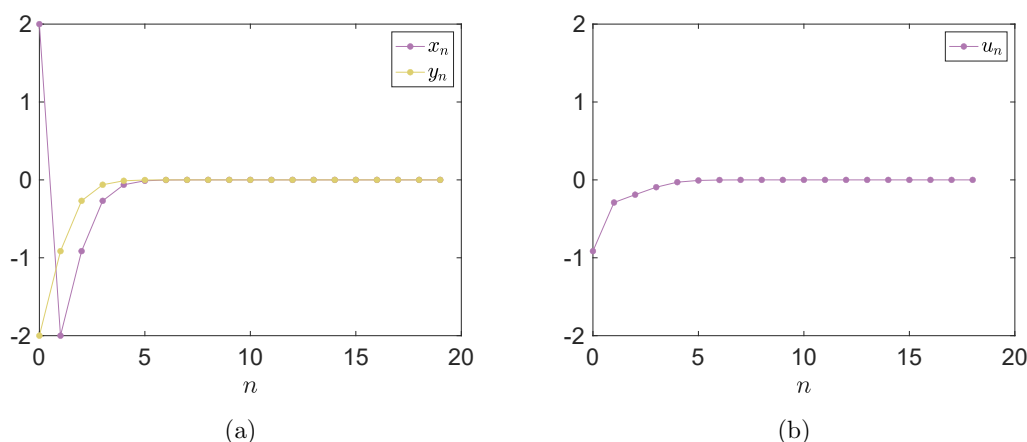


Figure 1. Simulated time series of the system (2.17) for the initial condition $(x_0, y_0) = (2, -2)$. The control signal is set to $u_n = Kh(x_n, y_n)$, where K is obtained from the solution of the convex program (2.14) and h is as in (2.18). (a) Time series of the state components. (b) Time series of the feedback control signal.

and define the control parameter values $u_0, u_1, \dots, u_{14}$ to be uniformly distributed random numbers in the interval $(-0.025, 0.025)$. Note that in this example $T = 15$ and $S = 9$ hold. Finally, we use CVX to solve the convex program (2.14). From the obtained solution we compute a gain matrix $K = [0.94 \ -2.42 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1.00]$ by using (2.15). We define the feedback control $u_n = Kh(x_n, y_n)$ and insert it into the system (2.17). A simulation with $x_0 = 2$ and $y_0 = -2$ results in the time series shown in Figure 1. The time series of the state components is shown in Figure 1(a), while Figure 1(b) displays the control signal over time. Both state components as well as the control signal rapidly converge to zero.

To illustrate the global stability of the origin under the constructed feedback control, we numerically compute its basin of attraction. By this we mean the set of initial conditions that lead to convergence of both state components to zero after a sufficiently long simulation time. A part of this basin is shown in Figure 2(a). For comparison, Figure 2(b) shows a part of a basin that is obtained by simulating the system (2.17) subject to linear feedback control of the form $u_n = k_1 x_n + k_2 y_n$. Here, the control gains k_1 and k_2 are chosen in such a way that the linearization of the system (2.17) at the origin has only eigenvalues with absolute value less than one. Note that since the chosen h given in (2.18) contains all the functions that appear in the system (2.17), Theorem 2.4 guarantees global stability of the origin.

This example shows that by solving the convex program (2.14), one may be able to construct a globally stabilizing control for certain classes of nonlinear systems. Furthermore, no explicit model is needed and the controller is derived solely from the data matrices (2.12). These are generated by one input-state trajectory of the given system as well as some suitable choice of the function h . However, so far we have only discussed the stabilization of the origin. In control-based continuation problems also nonzero fixed points are to be stabilized. Taking this into account, one has to modify the control law. This is explained in the following section.

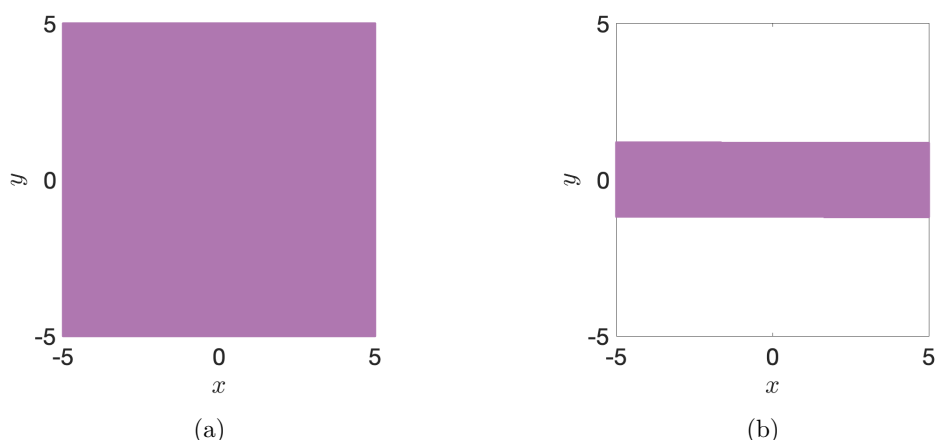


Figure 2. Numerically computed parts of the basins of attraction of the origin. These are computed by simulating the closed-loop dynamics of the system (2.17) for nonlinear as well as linear feedback control. (a) The control signal is set to $u_n = Kh(x_n, y_n)$, where K is obtained from the solution of the convex program (2.14) and h is as in (2.18). (b) The control signal is set to $u_n = k_1x_n + k_2y_n$. Here, k_1 and k_2 are chosen in such a way that the linearization of the system (2.17) at the origin has only eigenvalues with absolute value less than one.

3. Application to control-based continuation. So far we have demonstrated how to design globally stable control schemes for unstable fixed points based on plain time series. We now address the question of how to use such ideas for system analysis, in particular how to track unstable states and how to uncover bifurcation scenarios when just plain time series are at hand.

3.1. Trajectory-based tracking of unstable states. To explain the main ideas we use a simple normal form for the purpose of illustration. Consider again the one-dimensional control system

$$(3.1) \quad x_{n+1} = x_n - x_n^2 + u_n = f(x_n, u_n).$$

For a fixed value of u_n , the map $f(\cdot, u_n)$ can have multiple fixed points. In fact, if $\bar{u}$ denotes a constant parameter value, the fixed points are given as the real solutions of the equation $\bar{x}^2 = \bar{u}$. Thus, for a negative $\bar{u}$ there exists no real fixed point, while for $\bar{u} \geq 0$ we have the fixed points $\sqrt{\bar{u}}$ and $-\sqrt{\bar{u}}$, which coalesce for $\bar{u} = 0$. Furthermore, by inspecting the linearization of the map for $0 < \bar{u} < 1$, it follows that the fixed points $\sqrt{\bar{u}}$ are asymptotically stable, while the points $-\sqrt{\bar{u}}$ are unstable. One therefore refers to the graph of the function $\sqrt{\bar{u}}$ over the interval $0 < \bar{u} < 1$ as the stable branch. The graph of $-\sqrt{\bar{u}}$ over the same interval is the unstable branch. These two branches are connected by the bifurcation point $(\bar{u}, \bar{x}) = (0, 0)$. This bifurcation analysis is also possible for a more general class of parameter-dependent maps and differential equations [19].

Control-based continuation now aims for the following. If the control system is initialized for a fixed parameter value in a sufficiently small neighborhood of a stable branch, the state

is converging to the corresponding stable fixed point on the branch. Subsequently, a nearby point $(\bar{u}, \bar{x})$ on the branch is selected by the continuation scheme, and the controller acts such that both the state as well as the parameter converge to the selected coordinates $(\bar{u}, \bar{x})$. This is repeated until sufficiently many points on the stable as well as on the unstable branches are stabilized. This detects a part of a complete bifurcation diagram of the given system. In an experimental system, model equations are not always accessible. Consequently, there is no a priori information about the location of the branches. Nevertheless, the attraction of the stable branch makes it possible to locate a point on it by initialization. This can then be used as a starting point for a continuation scheme. In the following, the trajectory-based controller of section 2 is modified, so that it drives the system along its stable and unstable branches. Given the fact that the framework of section 2 is purely data-driven, model equations are not required and control gains are computed directly from an input-state trajectory of the system.

The suggested method is illustrated by the example (3.1), which serves to illustrate the effect of including a reference value directly into the control law under the assumption that we already obtained a stabilizing controller for the given system. The solution of the convex program (2.14) leads to a controller $u_n = kx_n + x_n^2$, such that in a closed-loop system we have $x_{n+1} = (1 + k)x_n$. Here, $|1 + k| < 1$ holds, so the controller globally stabilizes the origin. Including a reference parameter r in the control law yields $u_n = r + kx_n + x_n^2$ and

$$x_{n+1} = (1 + k)x_n + r.$$

The explicit solution of this system is given by

$$x_n = (1 + k)^n x_0 + r \sum_{i=0}^{n-1} (1 + k)^i, \quad n \geq 0.$$

Consequently, the modified controller globally stabilizes a fixed point $\bar{x}$ that is given by

$$(3.2) \quad \bar{x} = \lim_{n \rightarrow \infty} x_n = r \left(\frac{1}{1 - (1 + k)} \right) = r \left(\frac{1}{-k} \right).$$

Furthermore, the control parameter converges to

$$\bar{u} = \lim_{n \rightarrow \infty} u_n = r + k\bar{x} + \bar{x}^2 = r + kr \left(\frac{1}{-k} \right) + \bar{x}^2 = \bar{x}^2.$$

Thus, for a fixed nonzero value of the reference r , the system is driven to some point $(\bar{u}, \bar{x})$ on the stable or unstable branch. If one chooses $r = 0$, the system converges to the bifurcation point $(\bar{u}, \bar{x}) = (0, 0)$. Also, note that $|1 + k| < 1$ implies $-2 < k < 0$. Consequently, the relation between the reference and the stabilized fixed point is given by $\bar{x} = rc$ for some positive constant c . For the special case $k = -1$, the selected reference equals the stabilized fixed point. By using different reference values, sufficiently many points on the branches can be stabilized and a part of the full bifurcation diagram becomes visible. We further point out that the controller is computed solely from an input-state time series of the system. Finally, if a full nonlinearity cancellation is possible, the controller globally stabilizes the fixed points and a nearby initialization is not necessary. This differs from the control-based continuation methods that are based on linear feedback control, where only local stabilization is achieved.

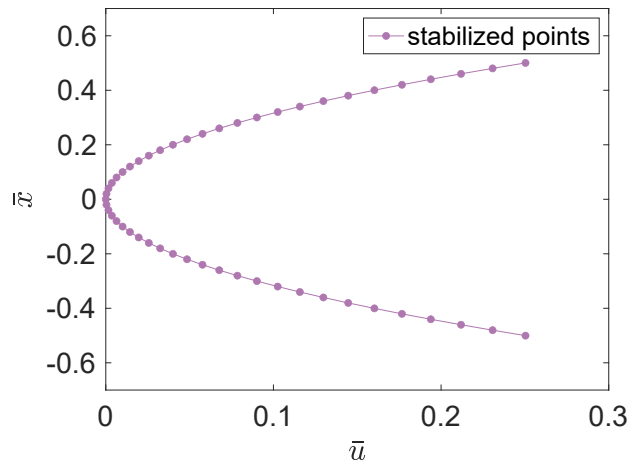


Figure 3. Bifurcation diagram with parameter $\bar{u}$: A set of stabilized points on the stable and unstable branches of the system (3.1). Here, the controller $u_n = Kh(x_n) + r$ is used, where K is obtained from the solution of the convex program (2.14). The function h is defined as $h(x) = [x \ x^2 \ x^3 \ x^4 \ x^5]^T$. In order to drive the system to different points on the branches, the reference r is varied.

Figure 3 shows a set of stabilized points on the stable as well as the unstable branch of example (3.1). These are generated by using the presented continuation scheme. The controller is obtained by numerically solving the convex program (2.14) for a recorded input-state trajectory of the system (3.1). For details of how to generate such a trajectory, we refer the reader to example (2.17) of section 2. The function h is set to $h(x) = [x \ x^2 \ x^3 \ x^4 \ x^5]^T$. In order to drive the system to different points on the branches, we vary the reference r in the control law.

As was already pointed out, if a full nonlinearity cancellation is possible, the controller globally stabilizes the points on the branches. Continuation schemes that are based on linear feedback control generally do not have this property and may depend on an initialization near the point to be stabilized. The global property can be utilized in the following situation. Suppose an experimental system has to abruptly change its state from one point on the stable branch to some point on the unstable branch, in order to maintain some optimality property [27]. The presented control scheme is able to meet such a requirement without having to assume that the points lie in a small neighborhood. Figure 4 illustrates a fast switching from one point on the stable branch of the system (3.1) to one point of its unstable branch.

3.2. Trajectory-based tracking of unstable periodic orbits with nonlinear integral control. In the following, we apply Theorem 2.4 or the mentioned minimization problem to discrete-time control systems that are induced by parameter-dependent Poincaré maps.

Continuous-time control systems of the form

$$(3.3) \quad \dot{x} = f(x, u),$$

where $x \in \mathbb{R}^N$ denotes the state and $u \in \mathbb{R}$ denotes the control parameter, can have periodic solutions [19]. These are characterized by the property that there exists a real $T > 0$ such

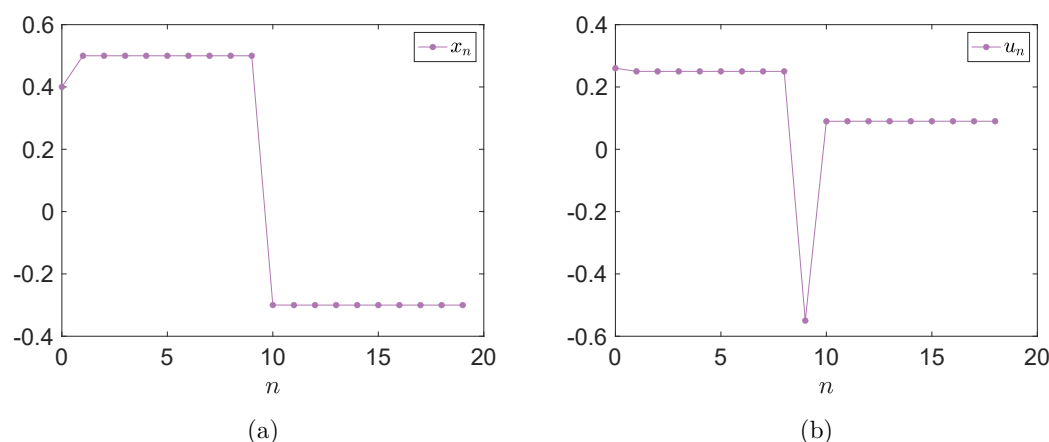


Figure 4. Simulated time series of the system (3.1) for the initial condition $x_0 = 0.4$. The control signal is set to $u_n = Kh(x_n) + r$, where K is obtained from the solution of the convex program (2.14) and the function h is defined as $h(x) = [x \ x^2 \ x^3 \ x^4 \ x^5]^T$. For $0 \leq n \leq 8$, the reference is set to $r = 0.5$, while for $9 \leq n \leq 18$ we choose $r = -0.3$. This induces a fast switching from the fixed point 0.5 on the stable branch to the fixed point -0.3 on the unstable branch. (a) Time series of the state. (b) Time series of the feedback control signal.

that $\phi(t) = \phi(t + T)$ and $\phi(t) \neq \phi(t + \tau)$ for all real τ with $0 < \tau < T$. Here, ϕ denotes the solution curve of (3.3) through a fixed initial condition and control parameter equal to zero. Such a periodic orbit γ induces a parameter-dependent Poincaré map as follows. First, we construct an $(N - 1)$ -dimensional hypersurface Σ , which is everywhere transversely intersected by the solution curves of the vector field $f(\cdot, 0)$. We assume that γ intersects Σ at a unique point p . If γ intersects Σ at multiple points, then one can shrink Σ until there is only one intersection. Finally, we define a parameter-dependent Poincaré map on a neighborhood U around $(x_n, u_n) = (p, 0)$ as

$$(3.4) \quad x_{n+1} = P(x_n, u_n),$$

where x_{n+1} denotes the first intersection of Σ by the solution curve of $f(\cdot, u_n)$ that is initialized at $x_n \in \Sigma$. Note that

$$P(p, 0) = p,$$

and thus p is a fixed point of the map $P(\cdot, 0)$. The stability of the fixed point p characterizes the stability of the periodic orbit γ . Consequently, if the system (3.4) is of the form (2.10), we can use trajectory-based feedback linearization in order to stabilize γ .

In the following, we propose a trajectory-based control scheme, such that the prescribed reference r equals the stabilized fixed point $\bar{x}$. Note that for a fully linearizing controller $u_n = Kh(x_n) + r$ in a system like (3.1), equation (3.2) holds. Consequently, apart from the special case $k = -1$, the reference differs from the actual fixed point that is stabilized. To circumvent this problem, integral control can be used [26]. We will explain in the following how it can be used in a trajectory-based manner.

The method relies on the construction of an integrator state v_n that sums up the deviation between the system state x_n and the reference r over time. In order to take this into account, the state equations of v_n are defined as

$$(3.5) \quad v_{n+1} = v_n + r - x_{n+1} = v_n + r - P(x_n, u_n), \quad x_n, v_n, r \in \mathbb{R}^N, u_n \in \mathbb{R},$$

where $P(x_n, u_n)$ denotes a parameter-dependent Poincaré map. Then the dynamics of the extended state (x_n, v_n) are governed by the system

$$(3.6) \quad \begin{aligned} x_{n+1} &= P(x_n, u_n), \\ v_{n+1} &= v_n + r - P(x_n, u_n). \end{aligned}$$

Assume that for the system (3.6) with $r = 0$, a fully linearizing controller $u_n = Kh(x_n, v_n)$ can be constructed by solving the convex program (2.14). This yields the closed-loop system

$$\begin{bmatrix} x_{n+1} \\ v_{n+1} \end{bmatrix} = A \begin{bmatrix} x_n \\ v_n \end{bmatrix},$$

such that all eigenvalues of the matrix A have absolute value less than one. Using the constructed controller in the system (3.6) for some nonzero reference r yields

$$\begin{bmatrix} x_{n+1} \\ v_{n+1} \end{bmatrix} = A \begin{bmatrix} x_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ I_N \end{bmatrix} r,$$

where $0 \in \mathbb{R}^{N \times N}$ denotes the zero matrix and $I_N \in \mathbb{R}^{N \times N}$ the identity matrix. The given system converges to

$$\begin{aligned} \begin{bmatrix} \bar{x} \\ \bar{v} \end{bmatrix} &= \lim_{n \rightarrow \infty} \begin{bmatrix} x_n \\ v_n \end{bmatrix} = \lim_{n \rightarrow \infty} \left(A^n \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \right) + \left(\lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} A^i \right) \begin{bmatrix} 0 \\ I_N \end{bmatrix} r \\ &= \left(\lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} A^i \right) \begin{bmatrix} 0 \\ I_N \end{bmatrix} r = (I_{2N} - A)^{-1} \begin{bmatrix} 0 \\ I_N \end{bmatrix} r \end{aligned}$$

for every initial condition (x_0, v_0) . Furthermore, the control signal converges to

$$\bar{u} = \lim_{n \rightarrow \infty} u_n = Kh(\bar{x}, \bar{v}).$$

Finally, the point $\bar{x}$ is a fixed point of the map $P(\cdot, \bar{u})$ since

$$\bar{x} = \lim_{n \rightarrow \infty} x_{n+1} = P(\bar{x}, \bar{u}).$$

However, by (3.5), we have $\bar{v} = \bar{v} + r - \bar{x}$, which implies that $\bar{x} = r$.

For an illustration of the presented control scheme, consider the differential equations

$$(3.7) \quad \begin{aligned} \dot{x} &= 10y + \frac{x(u - x^2 - y^2 - 1 + 2\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}, \\ \dot{y} &= -10x + \frac{y(u - x^2 - y^2 - 1 + 2\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} \end{aligned}$$

defined on $\mathbb{R}^2 \setminus \{0\}$. Here, $u \in \mathbb{R}$ denotes the control parameter. A one-dimensional hypersurface Σ is defined as $\Sigma = \{(x, y) : x > 0, y = 0\}$. Given a point $(x_n, 0) \in \Sigma$ and a fixed value u_n of the control parameter, we numerically integrate the differential equation until the orbit of the given vector field intersects Σ again. Denoting the point of intersection by $(x_{n+1}, 0) \in \Sigma$, we obtain a one-dimensional parameter-dependent Poincaré map $x_{n+1} = P(x_n, u_n)$. Note that in this example no analytical expression for the map P is given. By transforming the system (3.7) to polar coordinates, it can be shown that for a given positive value of u , there exists a stable as well as an unstable periodic orbit. In fact, qualitatively, the bifurcation diagram of the map P looks the same as the one depicted in Figure 3, where the upper branch corresponds to stable periodic orbits and the lower branch corresponds to the unstable ones. Our aim is to reconstruct the bifurcation diagram of P by using the presented trajectory-based integral control scheme.

In the first step, the convex program (2.14) is solved for the extended system (3.6) with $r = 0$. For this, data of the form (2.12) is generated, where

$$h(x, v) = [x \ v \ x^2 \ xv \ v^2 \ x^3 \ x^2v \ xv^2 \ v^3]^T.$$

Finally, the obtained controller $u_n = Kh(x_n, v_n)$ with

$$K = [-0.72 \ 0.09 \ 0.69 \ 0 \ 0 \ -0.60 \ 0 \ 0 \ 0]$$

is used in the extended system (3.6) for different nonzero reference values r . By construction, the integral controller drives the state x_n to the selected reference. The control parameter converges to $\bar{u}$ such that $P(r, \bar{u}) = r$. A number of different stabilized points corresponding to different choices of the reference are shown in Figure 5. The described scheme is summarized in Algorithm 3.1.

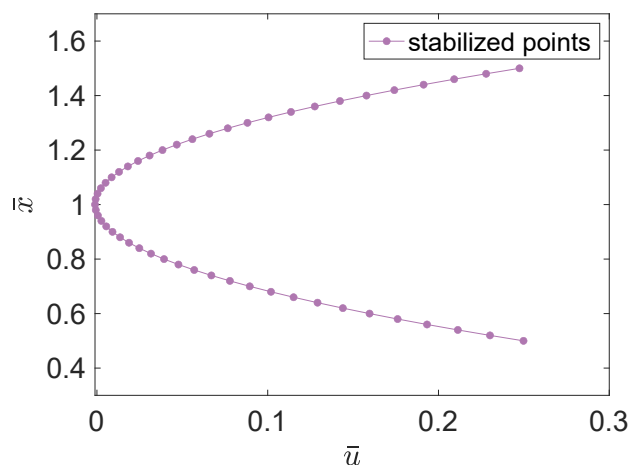


Figure 5. Bifurcation diagram with parameter $\bar{u}$: A set of stabilized points on the stable and unstable branches of the parameter-dependent Poincaré map corresponding to the system (3.7). Here, the controller $u_n = Kh(x_n, v_n)$ is used, where K is obtained from the solution of the convex program (2.14). The function h is defined as $h(x, v) = [x \ v \ x^2 \ xv \ v^2 \ x^3 \ x^2v \ xv^2 \ v^3]^T$. In order to drive the system to different points on the branches, the reference r is varied.

Algorithm 3.1 Trajectory-based stabilization of periodic orbits.

- 1: Construct a section to obtain a parameter-dependent Poincaré map.
- 2: Choose a function h , which contains the basis functions.
- 3: Generate data of the form (2.12) by simulating the extended system (3.6) for $r = 0$ and a chosen set of control parameter values.
- 4: Solve the convex program (2.14).
- 5: Compute the gain matrix K by using (2.15).
- 6: Use the obtained controller $u_n = Kh(x_n, v_n)$ in the extended system (3.6) for different nonzero reference values r in order to drive the system to different points on the branches.

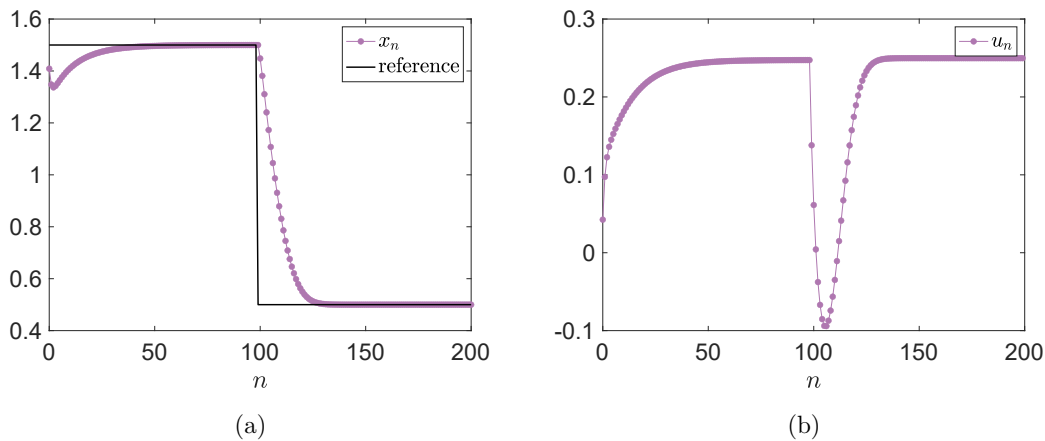


Figure 6. Simulated time series of the Poincaré map corresponding to system (3.7). The control signal is set to $u_n = Kh(x_n, v_n)$, where K is obtained from the solution of the convex program (2.14). For $0 \leq n \leq 98$, the reference is set to $r = 1.5$, while for $99 \leq n \leq 200$ we choose $r = 0.5$. This induces a switching from the fixed point 1.5 on the stable branch to the fixed point 0.5 on the unstable branch. (a) Time series of the state together with the reference signal. (b) Time series of the feedback control signal.

To demonstrate the global stabilization property of the obtained controller, we again solve a switching problem from one point of the stable branch to one point of the unstable branch. Figure 6(a) shows the time series of the state as well as the reference signal, and Figure 6(b) depicts the corresponding control signal. Observe that after changing the reference, the controller drives the state to the selected point on the unstable branch.

4. Conclusions. Trajectory-based control methods allow for an automatic construction of a stabilizing controller based on one a priori recorded input-state trajectory. Section 2 shows that by utilizing feedback linearization, the linear setting can be extended to a class of nonlinear systems. Based on this extension, discrete-time control systems that are induced by parameter-dependent Poincaré maps are amenable to this method. As a result, unstable periodic orbits can be stabilized by the constructed controller, which is obtained by solving a convex program. This circumvents the problem of finding adequate control gains.

Furthermore, in the case of a full nonlinearity cancellation, the controller globally stabilizes the unstable fixed point. Consequently, there is no need for a nearby initialization. The embedding of the trajectory-based scheme into an integral control framework in section 3 allows for stabilization of nonzero fixed points, which makes it possible to use the method for control-based continuation problems. But, in contrast to continuation schemes that are based on linear feedback control, switching between a point on the stable branch to one on the unstable branch is possible without requiring the assumption that the points lie in a small neighborhood. Also, the robustness against external disturbances is increased by the global stabilization property, which is useful in an experimental setup.

There are several possible further research directions in this area. First, the proposed integral control scheme in section 3 assumes that the full state is available for feedback, which typically is not the case in experimental setups. Thus, either a state estimation scheme is needed or one would have to resort to output feedback, where the output includes all the measurement variables [26]. State estimation is possible by using delay coordinates [38, 28]. The influence of such modifications on the stabilization properties of the proposed method will be studied in the future. Furthermore, the presented method is not restricted to single-input systems. Thus, also Poincaré maps that depend on a multidimensional parameter can be handled. This could be used to discover more complex bifurcation diagrams. Also, periodically driven oscillators are amenable to the stabilization scheme. In fact, for this class of systems, it is possible to construct a globally defined Poincaré map; see, e.g., [19]. Finally, a validation of the integral control scheme in a laboratory experiment would be useful and may lead to further research questions. For example, the influence of introducing delay into the integral scheme on the stability could be studied.

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Chapter 5

Data-driven model predictive control of a nonlinear ball-on-a-wheel system

Data-Driven Model Predictive Control of a Nonlinear Ball-on-a-Wheel System

Niklas Kruse, Alexander Wache, Harald Aschemann and Jens Starke

Abstract—The topic of this paper is the design of a data-driven model predictive controller for a nonlinear ball-on-a-wheel system. A Koopman predictor is identified using simulation data of a ball-on-a-wheel system model. The challenge lies in using data, which is characterized by the instability of the given system. This limits the time horizon during which meaningful time series can be extracted. Within this scenario, the predictor is included in a model predictive control scheme. This controller successfully tracks a specified non-flat output trajectory subject to input and output constraints. Finally, the prediction quality of the Koopman predictor is compared with a linear predictor based on dynamic mode decomposition with control.

I. INTRODUCTION

Data-driven design methods form an attractive alternative for the active control of systems, where an accurate model is not given. These methods can be classified into two different approaches.

Given measurements of a system, a control law can be computed directly that, for example, forces the system on a given reference trajectory [1], [2]. Here, an explicit model identification is omitted. On the other hand, one may first identify a model and, afterwards, apply model-based design methods [3], [4]. A strategy which corresponds to the latter approach is the combination of a model identification and a model predictive control scheme that allows for reference tracking with given constraints on the input and output of a system. An identification becomes possible by using the Koopman operator and extended dynamic mode decomposition (EDMD) [5].

This paper successfully transfers ideas of [5] to a data-driven predictive controller applied in a neighborhood of an unstable state in a tracking scenario of a nonlinear ball-on-wheel (BoW) system. Model-based control strategies for this BoW system, where the model equations are derived from first principles, already exist [6], [7]. However, this work is characterized by the following points:

- The successfully obtained control scheme is purely data-driven in the sense that we only use previously generated input-output time series of a BoW system simulation.

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- Simulation data is collected in open-loop operation and in the neighborhood of an unstable equilibrium point. This generates bounds on the time horizon during which meaningful time series can be extracted.
- The natural system output is not a flat output and, therefore, the considered tracking problem is not directly amenable to flatness-based control strategies [8]–[10].
- In terms of the identified predictor, we provide a comparison concerning the prediction quality of a Koopman predictor and a linear predictor based on dynamic mode decomposition with control (DMDc) for different measurement noise levels.

The remainder of this paper is structured as follows. Sect. II contains a description of a BoW system, the used simulation model and a linear stability analysis. In Sect. III, the data-driven identification of a predictor based on the Koopman operator and EDMD is described. Subsequently, this predictor is combined with a model predictive control scheme in Sect. IV. Finally, in Sect. V, the proposed control scheme is tested via simulation of the BoW system. The prediction quality of the identified Koopman predictor is compared with a prediction based on DMDc [3]. This work is concluded in Sect. VI.

II. MODEL DERIVATION AND STABILITY ANALYSIS

A BoW system represents a multibody system consisting of a driven wheel and a ball, which rolls without slip on the wheel as depicted in Fig. 1.

The ball has the mass m_B , radius r and moment of inertia I_B . The wheel with radius R and mass moment of inertia I_W is driven by an electric motor with an underlying velocity

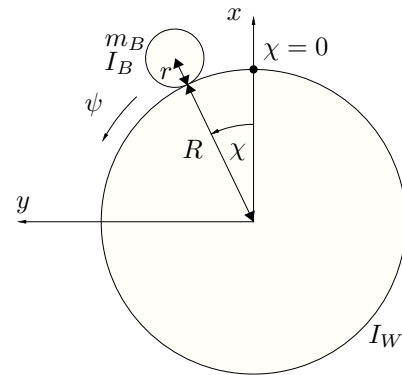


Fig. 1: A Ball-on-a-wheel system. A ball with mass m_B and mass moment of inertia I_B is balanced on a driven wheel with χ as angular deviation from the upper equilibrium point.

control. The state of the system is uniquely determined by the generalized coordinates $\mathbf{q}$ and velocities $\dot{\mathbf{q}}$. We choose $\mathbf{q} = (\psi, \chi)$, whereby ψ measures the absolute angle of the wheel and χ describes the angular deviation w.r.t. the current position of the ball and the upper equilibrium, where the ball's center coincides with the x -axis. The input u of the system is defined to be the desired angular velocity $\dot{\psi}_d$ of the wheel. The dynamics of the angular velocity $\dot{\psi}$ is governed by a first-order lag system for the underlying control of the electric motor

$$T_1 \ddot{\psi} + \dot{\psi} = \dot{\psi}_d, \quad (1)$$

with time constant T_1 .

By using the Lagrange formalism and combining the resulting equations of motion with the motor dynamics (1), one obtains the nonlinear control system

$$\dot{\mathbf{x}}_f = \mathbf{f}_f(\mathbf{x}_f, u) = \begin{bmatrix} \dot{\psi} \\ \dot{\chi} \\ a\dot{\psi} \\ b\dot{\psi} + c \sin \chi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -a \\ -b \end{bmatrix} u, \quad (2)$$

$$y_f = \chi, \quad (3)$$

with the full state $\mathbf{x}_f = (\mathbf{q}, \dot{\mathbf{q}})$, the output $y_f = \chi$ and constants

$$a = -\frac{1}{T_1}, \quad (4)$$

$$b = -\frac{I_B R^2 + I_B R r}{T_1 (R + r)^2 (m_B r^2 + I_B)}, \quad (5)$$

$$c = \frac{g m_B r^2}{(R + r)^2 (m_B r^2 + I_B)}, \quad (6)$$

where g denotes the gravitational acceleration. The model parameters correspond to a BoW laboratory test rig. In future work, it is planned to implement and validate the data-driven control at this experimental setup. The right-hand side of the system (2) is independent of the state variable ψ and thus, in the remainder of this paper, we consider the reduced-order form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) = \begin{bmatrix} x_3 \\ a x_2 \\ b x_2 + c \sin x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ -a \\ -b \end{bmatrix} u, \quad (7)$$

$$y = x_1, \quad (8)$$

with the reduced state $\mathbf{x} = (x_1, x_2, x_3) = (\chi, \dot{\psi}, \dot{\chi})$ and output $y = x_1 = \chi$.

Obviously, the upper equilibrium point $\mathbf{x} = 0$ represents an unstable stationary point of the uncontrolled system $\mathbf{f}(\mathbf{x}, 0)$. Indeed, if $\mathbf{A} = (\partial f_i / \partial x_j)$ denotes the Jacobian of $\mathbf{f}(\mathbf{x}, 0)$ at $\mathbf{x} = 0$, then we obtain the linear system

$$\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & a & 0 \\ c & b & 0 \end{bmatrix} \mathbf{x}, \quad (9)$$

with eigenvalues $(\lambda_1, \lambda_2, \lambda_3) = (a, -\sqrt{c}, \sqrt{c})$. In the case of our experimental setup we have $(\lambda_1, \lambda_2, \lambda_3) \approx (-667, -11, 11)$ and thus the point $\mathbf{x} = 0$ is unstable.

Nevertheless, we want to force the system output χ on a specified reference trajectory in the neighborhood of the unstable equilibrium $\mathbf{x} = 0$. Furthermore, we want to obtain the necessary control law without the model (7), using only a-priori measured input-output time series from model simulation. This is done by a data-driven model predictive control scheme. For this scheme a predictor is identified using the Koopman operator and EDMD. The following section gives a brief introduction into the method behind this identification.

III. DATA-BASED IDENTIFICATION OF A KOOPMAN PREDICTOR

Assume, according to [5], that a discrete-time dynamical system of the form

$$\mathbf{x}_{i+1} = \mathbf{F}(\mathbf{x}_i, u_i), \quad (10)$$

$$y_i = h(\mathbf{x}_i), \quad (11)$$

with unknown functions $\mathbf{F} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is given. In our case, this dynamical system is a discretization of (7)-(8). The only information we want to use in the remainder of this section are input-output time series $\{(u_k, y_k)\}_{k=1}^m = \{(u_1, y_1), (u_2, y_2), \dots, (u_m, y_m)\}$, where u_k and y_k denote the k -th measurement of the input and output and m is the number of measured data points. We aim at a linear system of the form

$$\mathbf{z}_{i+1} = \mathbf{A} \mathbf{z}_i + \mathbf{B} u_i, \quad (12)$$

$$\hat{y}_i = \mathbf{C} \mathbf{z}_i, \quad (13)$$

where $\mathbf{z}_i \in \mathbb{R}^N$ is called the lifted state, $\hat{y}_i$ denotes the prediction of the output y_i of the discrete-time dynamical system (10)-(11), and N is the dimension of the lifted state space $\mathbb{R}^N$, which typically exceeds the state space dimension n . This linear predictor is also called Koopman predictor. The Koopman operator framework allows for finding a global linear representation of a nonlinear system at the expense of infinite dimensions. However, considering a finite projection (12)-(13) is sufficient in the treated case [5]. In the following, we describe the identification of a linear predictor of the form (12)-(13) using only the input-output time series $\{(u_k, y_k)\}_{k=1}^m$.

A. Partitioning of the Data

The first step is to arrange the data $\{(u_k, y_k)\}_{k=1}^m$ in matrices $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{U}$. These matrices are of the form

$$\mathbf{X} = [\zeta_1 \quad \dots \quad \zeta_M], \quad \mathbf{Y} = [\zeta_1^+ \quad \dots \quad \zeta_M^+], \quad (14)$$

$$\mathbf{U} = [\bar{u}_1 \quad \dots \quad \bar{u}_M], \quad (15)$$

where

$$\zeta_i = [y_{i+2} \quad u_{i+1} \quad y_{i+1} \quad u_i \quad y_i]^T, \quad (16)$$

$$\zeta_i^+ = [y_{i+3} \quad u_{i+2} \quad y_{i+2} \quad u_{i+1} \quad y_{i+1}]^T, \quad (17)$$

$$\bar{u}_i = u_{i+2}. \quad (18)$$

Here, ζ_i denotes the i -th delay coordinate. For example, if we set $m = 4$, the time series $\{(u_k, y_k)\}_{k=1}^4$ is partitioned into

$$\zeta_1 = [y_3 \quad u_2 \quad y_2 \quad u_1 \quad y_1]^T, \quad (19)$$

$$\zeta_1^+ = [y_4 \quad u_3 \quad y_3 \quad u_2 \quad y_2]^T, \quad (20)$$

$$\bar{u}_1 = u_3, \quad (21)$$

and the data matrices are given as

$$\mathbf{X} = \zeta_1, \quad \mathbf{Y} = \zeta_1^+ \quad U = \bar{u}_1, \quad (22)$$

with $M = 1$. Note that in this case, it is not possible to construct ζ_2^+ since y_5 is not included in the time series $\{(u_k, y_k)\}_{k=1}^4$. Consequently, the number of columns M of the data matrices $\mathbf{X}$, $\mathbf{Y}$ and U depends on m .

B. Identification of a Koopman predictor

A Koopman predictor is identified by solving an optimization problem in order to find the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of (12)-(13), which are considered in two subsequent optimization problems. These optimization problems are based on the EDMD, described in the following. The first optimization

$$\min_{\mathbf{A}, \mathbf{B}} \|\mathbf{Y}_{\text{lift}} - \mathbf{A}\mathbf{X}_{\text{lift}} - \mathbf{B}U\|_F \quad (23)$$

involves all possible matrices $\mathbf{A}$ and $\mathbf{B}$. Here, $\|\cdot\|_F$ denotes the Frobenius norm and the matrices $\mathbf{X}_{\text{lift}}$ and $\mathbf{Y}_{\text{lift}}$ are given as

$$\mathbf{X}_{\text{lift}} = [\psi(\zeta_1) \quad \dots \quad \psi(\zeta_M)], \quad (24)$$

$$\mathbf{Y}_{\text{lift}} = [\psi(\zeta_1^+) \quad \dots \quad \psi(\zeta_M^+)], \quad (25)$$

where

$$\psi(\zeta) = [\psi_1(\zeta) \quad \dots \quad \psi_N(\zeta)]^T \quad (26)$$

is a vector that contains real or complex-valued nonlinear basis functions. Thus, one has to choose adequate basis functions, which span the possible nonlinearities of the given system (10)-(11). The solution of the optimization problem (23) is given by

$$[\mathbf{A} \quad \mathbf{B}] = \mathbf{Y}_{\text{lift}}(\mathbf{X}_{\text{lift}} U)^\dagger, \quad (27)$$

where † denotes the pseudoinverse. The matrix $\mathbf{C}$ of the Koopman predictor (12)-(13) is computed as

$$\mathbf{C} = \mathbf{X}\mathbf{X}_{\text{lift}}^\dagger, \quad (28)$$

which is the optimal solution of the second optimization problem

$$\min_{\mathbf{C}} \|\mathbf{X} - \mathbf{C}\mathbf{X}_{\text{lift}}\|_F. \quad (29)$$

The direct computation of the pseudoinverse in (27) and (28) is replaced with the associated normal equations, which can be solved more efficiently as explained in [5]. Within these two steps, hence, we have identified a linear predictor of the form (12)-(13) using only the input-output measurement $\{(u_k, y_k)\}_{k=1}^m$ of the given dynamical system (10)-(11). Subsequently, we want to explain how to use the identified Koopman predictor in a model predictive control scheme to track a specified reference trajectory.

IV. MODEL PREDICTIVE CONTROL FRAMEWORK

In Sect. III, we showed how to obtain a Koopman predictor from a given input-output time series $\{(u_k, y_k)\}_{k=1}^m$. This allows to predict the output y of a dynamical system, where the right-hand side is unknown. In this work, the dynamical system is given as a discretization of (7)-(8). Starting from the computed Koopman predictor, we obtain a control law that tracks a specified reference trajectory for the output y . The control law is computed via a model predictive control scheme, which allows for imposing constraints on the input and output of the system [11]. Traditionally a model predictive control scheme uses model equations derived from first principles to predict the behavior over finite time horizons. In the present work, these model equations are replaced by a Koopman predictor, which makes the control scheme applicable even to systems where model equations are unknown or cannot be derived.

A model predictive control scheme solves an optimization problem at fixed time t_k that runs over a finite time horizon and is initialized with the measurement y_k . Only the first input value of the optimal input is applied to the system at time t_k . At times t_{k+1} , t_{k+2} , $\dots$ this procedure is repeated in a shifted time horizon and, hence, the control scheme accounts for measurements of the system.

For a given sample time Δt , T_p denotes the length of the prediction window, which specifies the optimization horizon $N_p = T_p/\Delta t$. At a time t_k during the closed-loop operation, the given measurements (u_k, y_k) are used to initialize an optimization in the following way. Recall that the data (u_k, y_k) together with previous measurements specify ζ_k . But ζ_k determines an initial value $z_0 = \psi(\zeta_k)$ of the identified Koopman predictor (12)-(13). Now, the optimization problem

$$\min_{\mathbf{u}, \mathbf{z}} J(\mathbf{u}, \mathbf{z}), \quad (30)$$

is solved, where the feasible set is the polyhedron given by the constraints

$$\mathbf{z}_{i+1} = \mathbf{A}\mathbf{z}_i + \mathbf{B}u_i, \quad (31)$$

$$\mathbf{D}[\mathbf{u} \quad \mathbf{z}]^T \leq 0. \quad (32)$$

Here, $\mathbf{u}$ and $\mathbf{z}$ denote the vectors $(u_0, u_1, \dots, u_{N_p-1})$ and $(z_0, z_1, \dots, z_{N_p})$. The matrix $\mathbf{D}$ is defined by the constraints on the inputs u_i and the outputs y_i of the system. The deviation from a given reference trajectory is penalized by a quadratic cost function of the form

$$J(\mathbf{u}, \mathbf{z}) = [\mathbf{u} \quad \mathbf{z}]^T \mathbf{P} [\mathbf{u} \quad \mathbf{z}]^T + \mathbf{q}^T [\mathbf{u} \quad \mathbf{z}]^T + r, \quad (33)$$

where the matrix $\mathbf{P}$, the vector $\mathbf{q}$ and the scalar r are chosen appropriately as is explained in [5]. For the quadratic optimization problem (30)-(32) efficient solvers exist [12]. Finally, the first component u_0^* of the optimal vector $\mathbf{u}^*$ is applied to the given dynamical system at the fixed time instant t_k . The described procedure is now repeated at the subsequent times t_{k+1} , t_{k+2} , $\dots$ resulting in a feedback law $\mathbf{z} \mapsto \mathbf{u}(\mathbf{z})$, which tracks a desired reference trajectory. In

the following section, this control strategy is applied in a simulation scenario of the nonlinear BoW model (7)-(8).

V. NUMERICAL RESULTS AND COMPARISON

In this section, the described identification and model predictive control scheme are combined in order to track a specified reference trajectory of a BoW system. After the identification of a Koopman predictor of the form (12)-(13), the model predictive control scheme described in Sect. IV is tested via simulation of the nonlinear BoW model (7)-(8).

The remainder of this section is structured as follows: In Subsect. V-A, we describe the identification step using simulation data of a BoW model. Subsequently, in Subsect. V-B, the identified predictor is included in a model predictive control scheme, and a desired reference trajectory is tracked in a simulation of the nonlinear model (7)-(8).

A. Identification of a Koopman Predictor

The only information we want to consider for the identification of a Koopman predictor of the form (12)-(13) are input-output time series $\{(u_k, y_k)\}_{k=1}^m$.

First, it is described how we generated the data out of a system simulation. For this, the BoW model (7)-(8) is discretized by a fourth-order Runge-Kutta method to obtain a discrete-time dynamical system of the form (10)-(11), where the discretization time step is set to $\Delta t = 0.01$ s. We run 10 simulations, where each simulation generates an input-output trajectory $\{(u_k, y_k)\}_{k=1}^{50}$. The input is determined by uniformly distributed random numbers between 1 and -1 and the initial conditions for the state are uniformly distributed random numbers between 0.1 and -0.1 . The generated output trajectories are shown in Fig. 2.

Recall that the measured output $y = \chi$ is the angular deviation from the upper equilibrium position of the ball, which is unstable. Therefore, after randomly initializing the system in a neighborhood of the equilibrium position, the difference angle χ rapidly increases in the positive or negative direction depending on the initial value. Thus, for the identification we could only use trajectories that diverge from the equilibrium position.

For the partitioning of the collected data in the matrices (14)-(15), we choose three Gaussian basis functions [5]. Combined with the delayed state ζ in (16)-(17) this results in an eight-dimensional lifted state z . The dynamics of the lifted state z is governed by the Koopman predictor (12)-(13), which is computed by solving the optimization problems (23) and (29).

B. Tracking of a desired reference signal

The tracking of a specified reference trajectory is tested in a simulation of the nonlinear BoW model (7)-(8). After the identification of a Koopman predictor based on simulation data, the proposed model predictive control scheme is applied in a simulation.

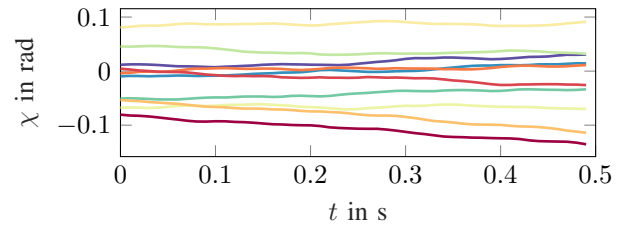


Fig. 2: Simulated output trajectories for 10 different initial conditions. The initial conditions are uniformly distributed random numbers between 0.1 and -0.1 . This data is used for the identification of a Koopman predictor.

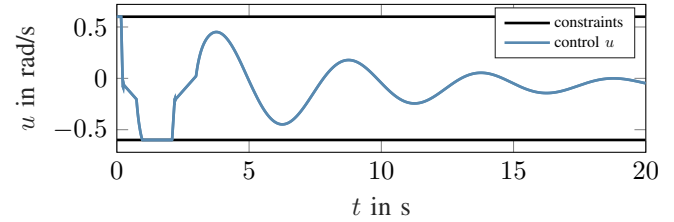


Fig. 3: Resulting control u , which tracks the desired reference trajectory. The constraints on the input are set to 0.6 and -0.6 . The used Koopman predictor is obtained from simulation data.

The control u , the simulated output $y = \chi$ in closed-loop operation and the tracking error are shown in Fig. 3, 4 and 5. The closed loop is simulated for $T = 20$ s. Applying the control shown in Fig. 3 to the BoW system results in the output trajectory χ , which is depicted in Fig. 4. The marked input and output constraints are satisfied, and the desired reference trajectory is tracked with high accuracy.

This can be understood by investigating the prediction quality of the identified Koopman predictor. In Fig. 6, the prediction quality is compared with the model output and a linear prediction model, which is identified with DMDc [3].

The Koopman and DMDc predictors are identified based on simulation data. The trajectories are generated in open-

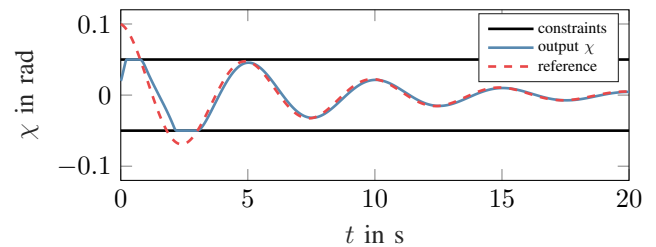


Fig. 4: The simulated output χ (blue) together with the desired reference trajectory (red) and the chosen output constraints (black). The constraints on the output are set to 0.05 and -0.05 and the simulation of $T = 20$ s with a prediction horizon of $T_p = 0.8$ s is initialized with $x_0 = (\chi_0, \psi_0, \dot{\chi}_0) = (0.02, 0, 0)$. The used Koopman predictor is obtained from simulation data.

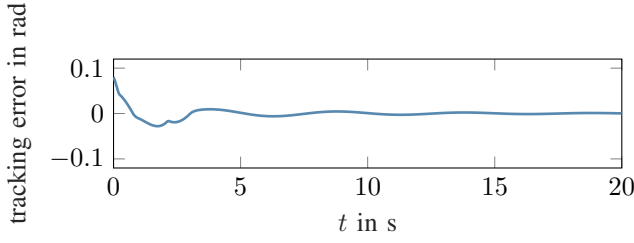


Fig. 5: Resulting tracking error between the simulated output χ and the desired reference trajectory.

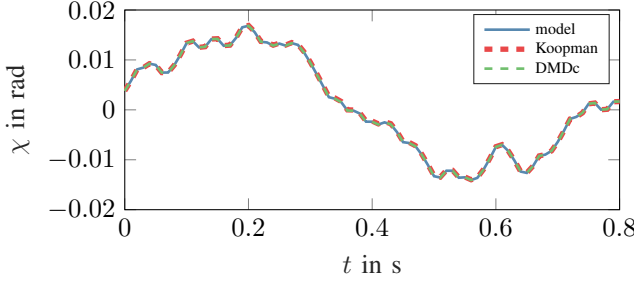


Fig. 6: Comparison of the prediction quality of a Koopman predictor (red) and a predictor based on dynamic mode decomposition with control (DMDc) (green). Both predictors are identified with simulation data. The trajectories including the model output (blue) are generated by a random input signal and are shown within the prediction horizon $T_p = 0.8$ s.

loop operation with a random input signal. The prediction of the Koopman and DMDc predictor coincides and shows a high accuracy in the chosen time window.

By additionally considering a perturbed output

$$\bar{y}_k = y_k + \varepsilon \rho_k, \quad k = 1, 2, \dots, m, \quad (34)$$

where ρ_k are uniformly distributed random numbers in the interval $(-1, 1)$ and ε denotes the amplitude of the perturbation, the robustness of the Koopman predictor is investigated. For comparison, we also show the prediction quality of a DMDc predictor, which is trained with a perturbed state time series with the same noise amplitude ε . The resulting predictions for different values of ε are shown in Fig. 7 and 8. For an increasing noise amplitude ε , the prediction quality of the DMDc predictor decreases. The prediction quality of the Koopman predictor, however, remains unchanged. The resulting tracking performance together with the corresponding tracking errors, where the identification is based on noisy output time series, is shown in Fig. 9, 10, 11 and 12. Despite an increasing noise amplitude, a high tracking accuracy is observed. As is shown in [5], the Koopman approach can be extended to perturbed systems. Given a stochastic disturbance with known distribution and a sequence of control inputs, this extension makes it possible to approximately predict the distribution of the state variable.

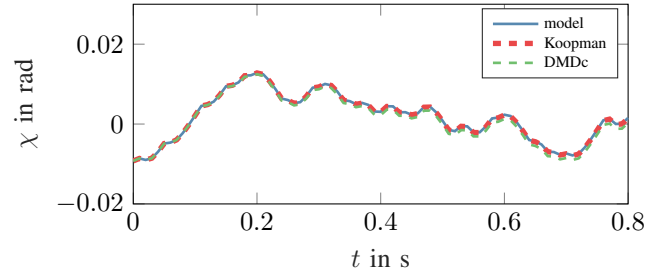


Fig. 7: Comparison of the prediction quality of a Koopman predictor (red) and a predictor based on dynamic mode decomposition with control (DMDc) (green). Both predictors are identified with simulation data, where the output/state time series is perturbed with a noise amplitude $\varepsilon = 10^{-6}$. The trajectories including the model output (blue) are generated by a random input signal and are shown within the prediction horizon $T_p = 0.8$ s.

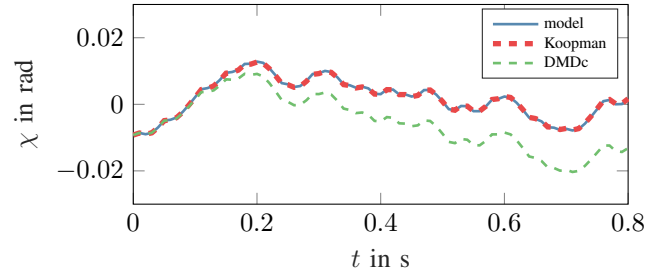


Fig. 8: Comparison of the prediction quality of a Koopman predictor (red) and a predictor based on dynamic mode decomposition with control (DMDc) (green). Both predictors are identified with simulation data, where the output/state time series is perturbed with a noise amplitude $\varepsilon = 10^{-5}$. The trajectories including the model output (blue) are generated by a random input signal and are shown within the prediction horizon $T_p = 0.8$ s.

VI. CONCLUSIONS

In this work, a data-driven model predictive control scheme is proposed that allows for tracking of a specified reference trajectory and could be successfully applied to a BoW system. The challenge relies on the fact that data is only available in a neighborhood of an unstable equilibrium of the system, which limits the time horizon during which meaningful time series can be collected. Furthermore, the considered output is not flat, i.e. the tracking problem is not directly amenable to flatness-based control strategies. When using previously obtained simulation data in the identification of the used Koopman predictor, a high tracking accuracy is achieved. This is based on the high prediction quality, which is observed for the Koopman predictor. In future work, the proposed data-driven model predictive controller will be implemented and validated on a BoW test rig.

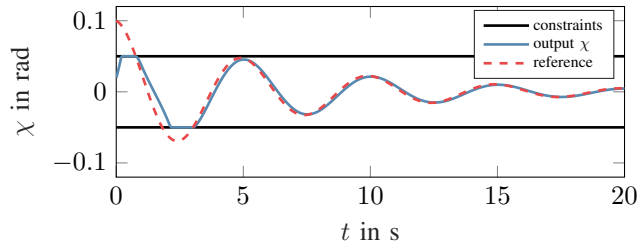


Fig. 9: The simulated output χ (blue) together with the desired reference trajectory (red) and the chosen output constraints (black). The constraints on the output are set to 0.05 and -0.05 and the simulation of $T = 20$ s with a prediction horizon of $T_p = 0.8$ s is initialized with $x_0 = (\chi_0, \psi_0, \dot{\chi}_0) = (0.02, 0, 0)$. The used Koopman predictor is obtained from simulation data, where the output time series is perturbed with a noise amplitude $\varepsilon = 10^{-6}$.

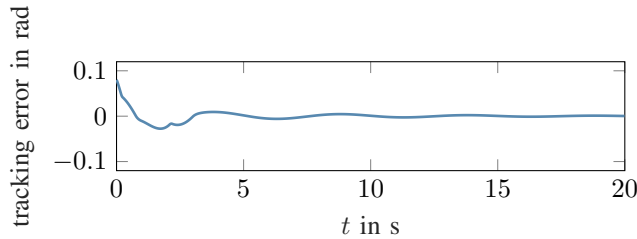


Fig. 10: Resulting tracking error between the simulated output χ and the desired reference trajectory for the case $\varepsilon = 10^{-6}$.

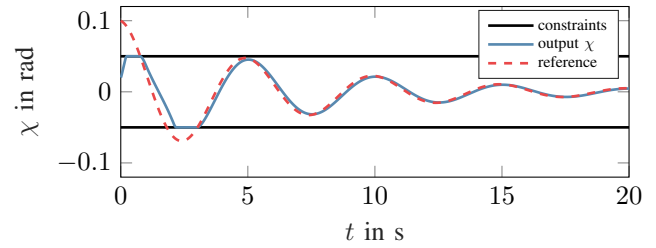


Fig. 11: The simulated output χ (blue) together with the desired reference trajectory (red) and the chosen output constraints (black). The constraints on the output are set to 0.05 and -0.05 and the simulation of $T = 20$ s with a prediction horizon of $T_p = 0.8$ s is initialized with $x_0 = (\chi_0, \psi_0, \dot{\chi}_0) = (0.02, 0, 0)$. The used Koopman predictor is obtained from simulation data, where the output time series is perturbed with a noise amplitude $\varepsilon = 10^{-5}$.

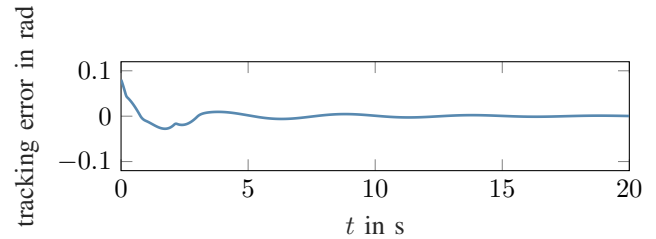


Fig. 12: Resulting tracking error between the simulated output χ and the desired reference trajectory for the case $\varepsilon = 10^{-5}$.

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Chapter 6

Exposing hidden periodic orbits in scanning force microscopy

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Exposing hidden periodic orbits in scanning force microscopy



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The nonlinear interaction between the tip of a scanning probe microscope (SPM) and a sample is manifested in the emergence of bifurcations and unstable branches in the frequency response of a driven cantilever. While extensively investigated theoretically, exploring the unstable branch in an actual SPM experiment is lacking so far, reflecting the broader challenge in studying mechanical nanojunction oscillators under strongly varying external forces. Here we demonstrate experimental tracking of unstable periodic orbits between two saddle-node bifurcation points in the attractive regime, revealing the full set of stationary oscillatory states. This is achieved by a minimally invasive control scheme based on fast adaptive phase extraction and Fourier discretisation of the tip dynamics. Stabilization of unstable branches of oscillating AFM cantilevers opens avenues for novel experimental modes, potentially enabling ultrasensitive surface detection at considerably large amplitudes with minimal tip-surface interaction, new insights in tip-surface interaction mechanisms, as well as new AFM modes enabling arbitrary setpoint choice while inherently avoiding discontinuities.

Control of nanomechanical dynamics is of extreme technological importance. Applications range from motion and position sensors¹, detection of chemical and biological species^{2–4}, transducers and microfluidic transport⁵, to more fundamental demands such as gravitational wave detection and quantum state analysis^{6–8}. Typically, the resulting dynamics are treated linearly, i.e., in terms of a driven, damped harmonic oscillator, where a stationary state corresponds to a periodic orbit. As for any oscillator, however, increasingly nonlinear external forces result in drastic changes in the frequency response, up to the emergence of bifurcation points or even chaotic behavior^{9–20}. For micro-cantilevers in the vicinity of a surface, the nonlinearity can originate from repulsive (i.e., hardening), attractive (i.e., softening), or combined interaction^{9,21,22}. As a result, multiple branches of stable states emerge^{23,24}, and the system may change quasi-erratically between them, triggered by external disturbances such as varying ambient conditions or noise.

These stable states are separated by unstable states, which, however, are experimentally not directly accessible because of their unstable character: infinitesimal deviations from the ideal stationary point in Fourier space rapidly drive the oscillator toward one of the stable solutions. Experimental access to unstable states has been realized via control-based continuation methods for macroscopic mechanical systems^{25–29}, and via phase control for miniaturized oscillators³⁰, offering a variety of advantages over the uncontrolled system, such as an extended parameter space for reliable operation,

or improved noise performance and hence resolution³¹. The first attempts to apply control-based continuation methods to atomic force microscopy can be found in ref. 32. However, the proposed method was only tested in a simulation of a model.

In this work, we explore nonlinear surface-engaged nanomechanical oscillators as they are employed in dynamic scanning force microscopy (SFM), particularly the concept of stabilizing an unstable branch of periodic orbits of an SFM cantilever in the vicinity of a solid sample surface. Scanning force microscopy, also known as atomic force microscope (AFM), is a versatile and popular method for revealing surface properties on the nanoscale. These include morphology, atomic and molecular structure, stiffness, surface potential, magnetic field, and many more^{33,34}. Material parameters can be extracted, e.g., employing phase imaging, force-distance curves³⁴, multimodal SFM³⁵, or by analyzing transient deflection data³⁶, e.g., by using sparse identification of nonlinear dynamics (SINDy)³⁷. Application scenarios of SFM in the past decades range from atomic and molecular systems to life-sciences^{34,38–40}. In short, in this method, a sharp tip attached to a microcantilever beam is approached to the sample surface. The resulting beam deflection due to tip-sample interaction is measured and processed further, usually in a closed-loop manner. Most SFM modes rely on the periodic mechanical actuation of the cantilever on one of its eigenmodes, where continuous changes of the periodic orbits are indicative of varying sample properties, e.g., as a function of the lateral position of the tip.

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While usually treated as a harmonic oscillator, tip-sample interaction forces are highly nonlinear, hence the SFM represents a prototype nanomechanical system. However, transferring control concepts to the SFM is not straightforward, for both technical and physical reasons: when interacting with a surface, the attractive van-der-Waals interaction dominates all other forces⁴¹, if not the even larger capillary forces come into play, e.g., due to condensation of a water meniscus. Therefore, the relevant external forces acting on the oscillator are fundamentally different compared to macroscopic or intrinsically nonlinear systems, and the same holds for the shape of frequency-response curves³⁴. Since the tip-sample distance critically affects the oscillator's properties on the nano- or even atomic scale, stable and precise distance control is crucial. In this work, we are aiming at stabilizing unstable states of a conventional SFM with nanometer amplitudes, while leaving the system's intrinsic properties unchanged via non-invasive control-based continuation⁴². As a useful side effect, unwanted sudden jumps between temporarily stable periodic orbits can be avoided, leading to further robustness of SFM measurements, which otherwise has to be actively enforced using dedicated approaches⁴³.

Results and discussion

Bifurcation diagram of a nonlinear oscillator

Figure 1 shows a typical bifurcation diagram of a nonlinear oscillator with an attractive (i.e., softening) external force. A natural set of independent parameters for an SFM consists of the driving frequency f and the (mean) tip-sample separation s , which both serve as bifurcation parameters in our setup. The latter adjusts the strength of the external nonlinear force. The state can be characterized by the cantilever's amplitude $A(f)$ (Fig. 1a and phase $\phi(f)$ of the deflection with respect to the driving frequency (Fig. 1b), although other descriptors may be appropriate as well, such as a complex Fourier vector²⁸. Starting far away from the sample (i.e. large s), the

frequency-response $A(f)$ resembles that of a typical driven, damped harmonic oscillator with its maximum at the resonance frequency f_0 . As the nonlinearity increases (i.e. s decreases), the resonance initially shifts, then deforms, until it finally enters the bistable regime where A is not uniquely determined by f : for a given frequency f , there are two stable periodic orbits separated by an unstable one. In a pseudo-3D representation (Fig. 1), the set of amplitudes or phases of stationary states in Fourier space describes a folded surface, respectively.

In experimental frequency sweeps, the bistability manifests as sudden amplitude jumps, also called catastrophes, accompanied by a pronounced hysteresis behavior (Fig. 2a). The resulting shape of $A(f)$ is occasionally referred to as *shark fin*, where the orientation of the shark fin depends on the sign of the nonlinear force. The branches of unstable states extend within the region between the two bifurcation points (red curve indicated in Fig. 1a and cover a considerable part of the amplitude range. At the smallest separation ($s \approx 61$ nm in Fig. 2a), the shark fin appears to point to both directions, left and right, for the frequency down-sweep, as well as for the up-sweep.

This is a consequence of the coexistence of attractive and repulsive forces, resulting in a complex double-bent frequency-response curve³⁴. Some of the bifurcation points are hardly visible, because the sudden amplitude change is small, e.g., for the forward sweep (blue curves). However, the same bifurcation points are also formed in the phase response (Fig. 2b), where they are clearly visible. The unstable part of the fold in Fig. 1a, b is inaccessible by simple frequency sweeps. In the following we present a method enabling measurement of such hidden folds of the response surface in SFM experiments.

Tracking unstable states

Our mathematical and experimental approach is summarized graphically in Fig. 3 and described in detail in the Methods section. In short, the tip of an

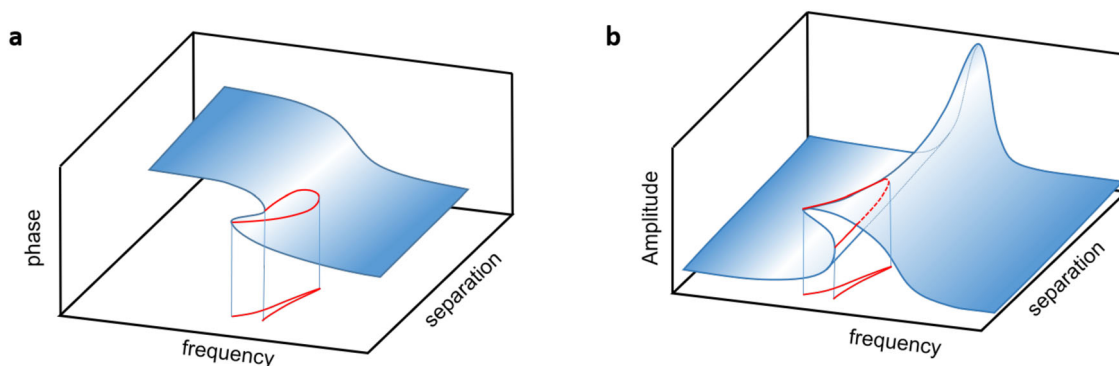


Fig. 1 | Schematic rendering of the complete response surfaces of Amplitude and Phase. **a** amplitude A and **b** phase ϕ of a surface-engaged nano-oscillator under attractive interaction. The red lines indicate positions of bifurcation points and their projection onto the separation-frequency plane.

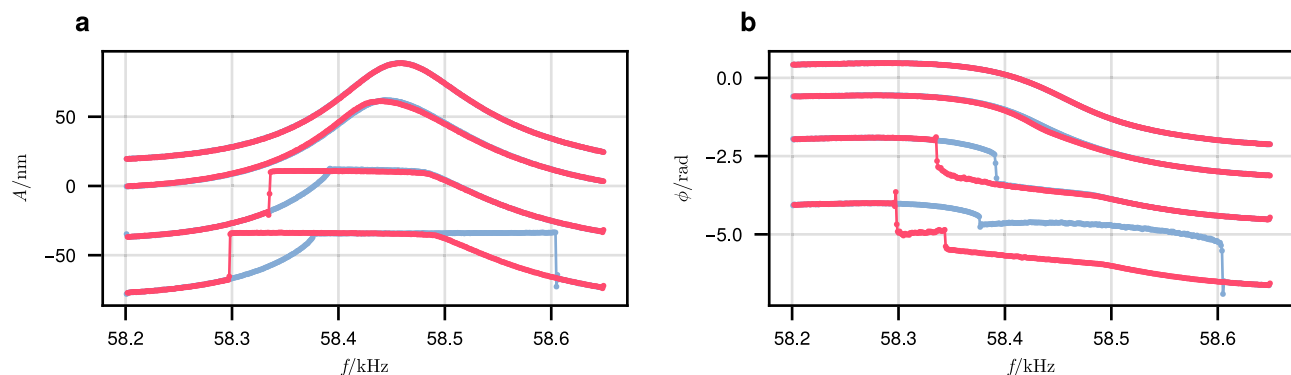
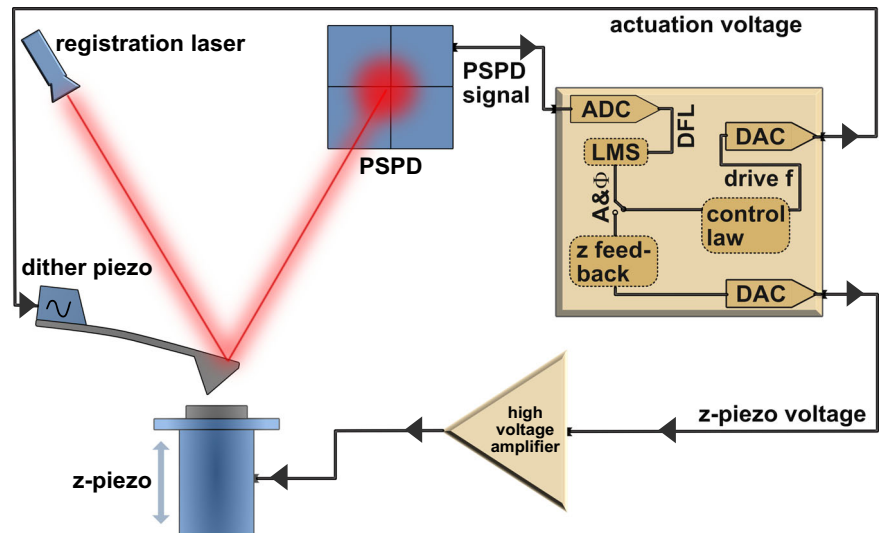


Fig. 2 | Frequency sweeps at different separations. **a** Amplitude A and **b** phase ϕ on a silicon sample for various separations s , ranging from a free oscillation (largest separation, top), to slightly attractive, attractive, and coexisting attractive and

repulsive (lowest separation, bottom) forces. Red curves indicate sweeps from high to low frequencies and blue curves from low to high. The curves are offset vertically by an arbitrary amount for clarity.

Fig. 3 | Simplified scheme of the experiment. Blue-colored parts symbolize parts of the commercial SFM setup, beige components were added later on and integrated via a breakout-box. As the z-piezo voltage is kept constant during control attempts, the tip-sample distance control is omitted for simplicity. PSPD position-sensitive photodiode, ADC analog digital converter, DFL deflection, LMS least-mean square algorithm, DAC digital analog converter.



SFM is approached to a silicon sample, and a periodic force is applied to the cantilever. The phase of the first Fourier component of the resulting oscillation state is extracted and used as an input parameter for a non-invasive control scheme, implemented in a real-time signal processing unit (RTSPU). The control acts on the frequency $f = \frac{1}{2\pi} \frac{d}{dt} \psi$, which results in a time-dependent forcing signal of the form $\Gamma \sin(\psi(t))$, where the accumulated phase $\psi(t)$ (often referred to as *instantaneous phase*) is obtained by integrating the proportional plus integral controller

$$\frac{d}{dt} \psi = \omega_s + K_p(\phi - \phi^*) + \lambda \quad (1)$$

$$\frac{d}{dt} \lambda = K_i(\phi - \phi^*) \quad (2)$$

with initial conditions $\psi(0) = 0$ and $\lambda(0) = 0$, a starting angular frequency ω_s , the target phase ϕ^* , the integral part λ , and control gains K_p , K_i . Note that this control law is non-invasive by design⁴⁴, in that it does not alter the system's dynamics once the control target is reached. Instead of the commonly utilized lock-in amplifier for phase extraction, we implemented a successive phase detector based on the normalized least-mean-square (nLMS) method⁴⁵, which turned out to be an essential ingredient due to its superior rate of convergence⁴⁶. Once a stationary point in Fourier space is reached with sufficient accuracy, a new control step is started where the target phase is decremented by a constant value, such that the entire relevant phase range is sampled in both, stable and unstable regions. Then, the tip-sample separation is changed such that the full set of periodic orbits is tracked while still residing in the predominantly attractive regime of the tip-surface interaction. Hence, double-bent bifurcation diagrams, as described in the previous section, are largely avoided for simplicity.

Unveiling the hidden folded surface

Figure 4a shows an experimental example of a controlled sweep through the phase range from $\phi \approx 1.00$ rad to $\phi \approx -0.48$ rad, including stable and unstable branches of the bifurcation diagram. The blue and red lines are conventional frequency sweeps (i.e., at disabled control loop), showing the bifurcation points where sudden jumps occur, depending on the sweep direction (cf. Fig. 2b). These curves serve as a reference, and facilitate the identification of the area in the (f, ϕ) -plane where the unstable branch is expected. When the control scheme is enabled and the phase is varied from large values in decreasing directions. Initially, the upper stable branch is followed (black-filled circles). At the bifurcation point the frequency direction reverses thanks to control-based continuation, thus tracking the unstable branch without occurrence of any catastrophes. Finally, the

frequency reverses again, and the lower stable branch is followed. Therefore, the full set of stationary oscillatory states is revealed for this particular tip-sample separation s . Repeating this scheme for various tip-sample separations s unveils the entire fold, as shown in Fig. 5. When reducing the tip-sample distance, the influence of nonlinear interaction increases and so does the range along both phase and frequency of the unstable region, resulting in higher path lengths of the unstable branch. A general observation is, that the data scatter more pronounced within the unstable branches compared to the stable ones which points to a high sensitivity to external influences, see discussion below. From the data in Fig. 5, the location of the cusp, i.e., the point in the (f, s) -plane where the two branches of saddle-node bifurcations start to occur, can be extracted, which is located around $f = 58.43$ kHz and $s = 103$ nm.

Transient data for phase and frequency is shown in Fig. 4b, illustrating the dynamic behavior of the control scheme, particularly for the unstable branch. A magnified view of the transient phase evolution for one of the control steps is displayed in Fig. 4c. The target phase is approached quickly after each control step, while the remaining deviation is reduced on a slower timescale (see Fig. 4c), reflecting the typical behavior of the utilized proportional-integral (PI) control scheme, see equations (1) and (2). Finally, the phase clearly converges towards the target.

All control steps, including those on the unstable branch, reach the control target within the given period of time, where some minor oscillations are sufficiently damped by the control (see Fig. 4c). Occasional oscillations in the s -dependent curves visible in the frequency transients (Fig. 5, transients not shown) still result in much smaller uncertainties compared to the interpoint distance between two adjacent control steps.

Numerical simulations

In order to ensure the feasibility of our control approach prior to actual experiments, we employed a simple driven point mass model in common tip-sample interaction potentials, such as the Lennard-Jones or Derjaguin-Muller-Toporov model. However, quantitative a-priori modeling of unstable branches is essentially obstructed because the nature of the tip-sample system strongly depends on tip and sample materials, ambient conditions (particularly humidity), and time scales involved. Commonly accepted ingredients for a mathematical SFM model are a weak, long-range attractive interaction dominated by van-der-Waals forces (range of the order of 100 nm) and a short-range repulsive force caused by the Pauli principle (range of the order of 1 nm) and/or contact mechanical forces, e.g. treated within the Hertz model. Other attractive, repulsive, and dissipative contributions to the interaction force have been considered^{19,47,48}, but crucial parameters (e.g., the dissipation coefficient) are essentially unknown. Hence, we did not try to find a physically accurate model but rather adhered

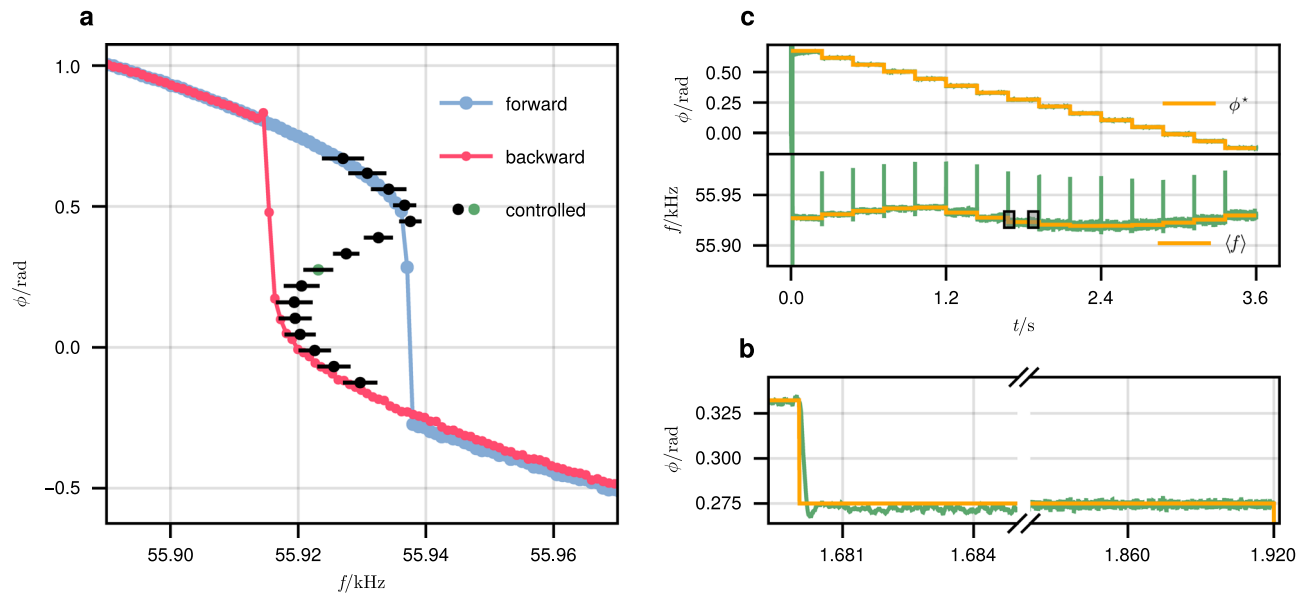


Fig. 4 | Controlled frequency sweep accessing unstable states. a Phase response curve of a frequency sweep with disabled control in forward (blue dotted line) and backward (red dotted line) direction. With enabled control, the unstable branch is tracked (black-filled circles). Control parameters are: $K_p = 12.6 \text{ kHz rad}^{-1}$ and $K_1 = 113 \text{ kHz rad}^{-1} \text{ s}^{-1}$. Error bars represent the mean standard deviation (see

Methods section). **b** Transient behavior (green) of phase ϕ (top) and frequency f (bottom). The target phase ϕ^* (top) and the mean value of the last 2000 frequency values (bottom) are shown for comparison (orange). **c** Is a magnified view of the eighth control step indicated by rectangles in (b) and by a green-filled circle in (a).

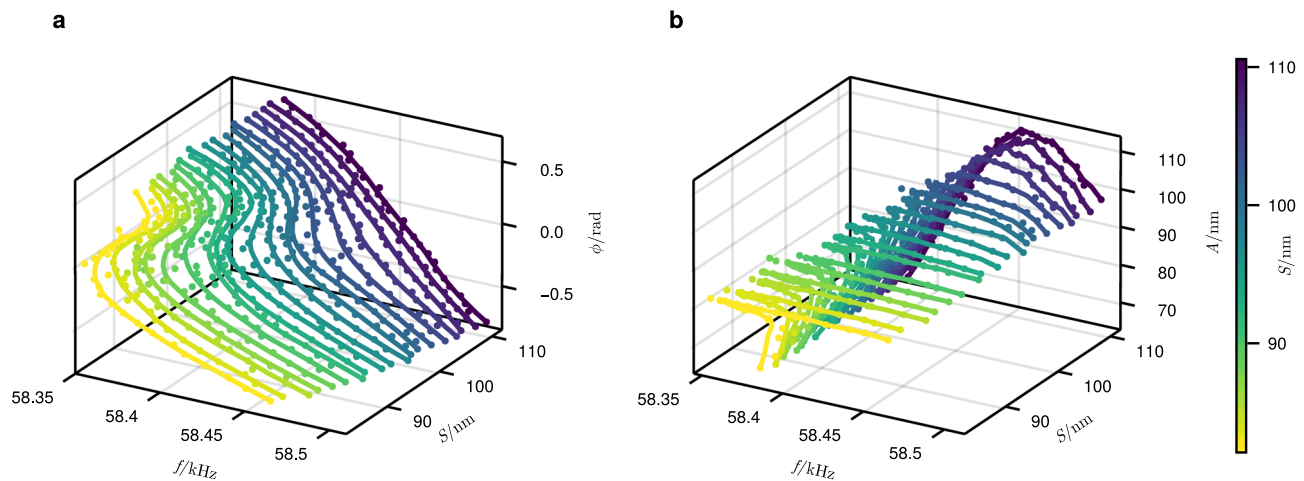


Fig. 5 | Complete measured response surfaces. a Phase $\phi(f, s)$ and **b** amplitude $A(f, s)$ of an oscillating cantilever in attractive interaction with the silicon surface. The indicated mean tip-sample separation s corresponds to the amplitude setpoint

(hard-surface approximation), which is varied in steps of 1.85 nm. An interactive version of these plots can be found in the [Figshare](#) repository.

to the simple Lennard-Jones model and chose parameters that qualitatively reproduce the experimental curves (Fig. 4).

Figure 6 shows simulation results for a Lennard-Jones potential with viscous dissipation as described in ref. 49. The control law (1) and (2), as well as the Least-Mean-Square (LMS) algorithm to obtain the Fourier coefficients *on the fly*, are realized with a similar time discretisation as in the experiment, while the equations of motion of the point mass oscillator are numerically integrated with a Runge-Kutta solver using adaptive step sizes. Despite the model's simplicity, interaction forces and amplitudes are comparable to the experiment (see “Methods” section). Indeed the unstable branch is successfully tracked with suitable control parameters ($K_p = 0.4 \cdot f_0$, $K_1 = 0.004 \cdot \frac{f_0}{s}$, $f_0 = 55 \text{ kHz}$), which are kept constant. The transient behavior in Fig. 6b shows that after some initial oscillations, the target phase and the instantaneous frequency are stable after a few ms, and thus confirms fast

convergence to the stationary points in Fourier space. This proves that the feedback control scheme is indeed capable of stabilizing unstable states in the desired region of the bifurcation diagram.

Figure 6c shows the 3D two-parameter bifurcation diagram of the numerical model, i.e., frequency responses with enabled control for various tip-sample separations using the same control parameters for the entire simulation. All essential features, such as the initial shift of the frequency-response curve for large s , the occurrence of the unstable branch, its increasing length in the (ϕ, f) parameter space, and the emergence of a two-dimensional hidden fold are reproduced. This confirms the validity of experimental results, particularly the shape of the unstable branches. The remaining quantitative deviations are most likely due to the oversimplified model (interaction forces, point approximation).

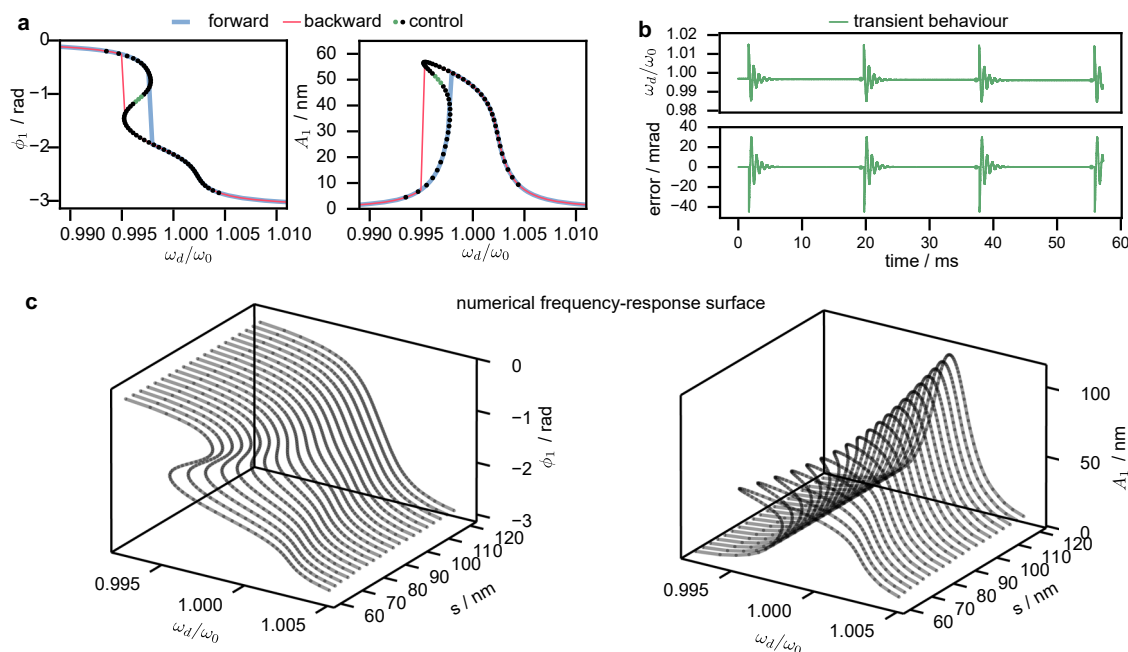


Fig. 6 | Simulated controlled sweeps. **a** Simulated frequency-response curves of a driven point mass in a Lennard-Jones potential with viscous damping at a separation of 60 nm. Black dots represent responses with enabled control, whereas red and blue lines correspond to backward and forward frequency sweeps with disabled control accordingly, indicating jump behavior at saddle-node bifurcation points.

b Transient behavior of the driving frequency and phase-lag error towards formerly unstable periodic orbits, highlighted as green dots in (a). **c** Numerical frequency-response surfaces of phase ϕ (left) and amplitude A (right) where we choose the separation as the second bifurcation parameter.

Control behavior

While our results imply that tracking of unstable branches is successful, some uncertainties remain, resulting in potentially imprecise determination of the native states in the (f, s, ϕ) -phase space. Possible sources of errors are imperfect achievement of the control goal due to small oscillations around it, incomplete convergence, electronic noise resulting in fluctuations of the retrieved oscillation state (i.e., phase and amplitude), and variations of external forces, dynamically changing the characteristics of the oscillator. The main compromise we have to achieve is the total period of time available for the control sweep because thermal drift results in a time-dependent variation of the tip-surface distance and, hence, of the interaction force. Even sub-nanometer changes within a few minutes result in altered frequency-response curves, which impedes unambiguous characterization of the reference frequency sweeps (blue and red curves in Fig. 4a). In addition, a true stationary point in Fourier space does not exist, because the orbits lose their strict periodicity, and the phase to be targeted becomes a function of time. To avoid these complications, we limit the period of time per control step and rather occasionally accept a certain amount of remaining deviation from the control target. This may introduce a systematic error which, in principle, can be avoided by a more stable setup (e.g. employing active temperature control), by faster control schemes (e.g., beyond a simple PI controller), and by a more sophisticated termination criterion for each control step, hence saving some time, particularly in the less critical stable regions of the bifurcation diagram. Results shown in Figs. 4 and 5 are not affected by long-term drift, as evidenced by the coinciding (uncontrolled) frequency sweeps before and after the control sweep, respectively.

In our simple control approach, we selected static control parameters K_p and K_i , which apparently are not optimal for the entire range of the unstable branch. Although we obviously stay within the basin of attraction, we expect an even more robust behavior and faster convergence by a refined choice of parameters, e.g., in an adaptive manner⁵⁰. The employed control not only enables stabilization of unstable branches, but also offers accelerated convergence to the stable states, resulting in much faster frequency sweeps. The effect is similar to that of decreasing the Q-factor via feedback,

also known as Q-control^{22,51}, which can drastically reduce the system's response time to a changing excitation frequency.

Envisioned applications

Once successful stabilization on an unstable branch is achieved, the question of potential benefits emerges. We propose, that imaging on an unstable state—in the following referred to as unstable branch scanning force microscopy (UB-SFM)—potentially results in highly sensitive, yet robust imaging conditions, as argued in the following: consider the conditions for “soft” SFM, where the interaction between tip and sample is to be minimized, sometimes referred to as non-contact mode. A widely used strategy for achieving this goal is to increase the spring constant and decrease the oscillation amplitude, which has been particularly successful for the case of tuning-fork-based SFM operated in vacuum⁵², but is also pursued for cantilever-based microscopes in air.

A limiting factor, especially at ambient or liquid conditions, is the signal-to-noise ratio which ultimately sets a lower bound to the employed amplitude for practical purposes: using the amplitude modulation (AM)-detection scheme and typical cantilevers with normal spring constants k around 1 Nm^{-1} the noise level in the vertical z coordinate corresponds to about 0.1 nm ^{34,38,53}. Furthermore, amplitudes smaller than the sample corrugation result in a high probability of tip or sample modification, because of the predominant sideways forces exerted in such situations. Hence, for robust, long-term, and large-scale imaging, larger amplitudes are generally beneficial. This, however, is not easily possible in the attractive region of the tip-sample interaction because of the small range of amplitude setpoints required for this scenario, and due to the emergence of bifurcations (see Fig. 1), frequently encountered as erratic jumps of phase and amplitude in practical SFM experiments. Choosing the repulsive regime (often referred to as tapping mode), in turn, impedes the requirement of minimized tip-sample interaction. Forcing the system to remain on the upper stable branch in the attractive regime⁴³ implies a weak amplitude dependence as a function of frequency due to the flat shape of the *shark fin*, hence, the sensitivity on tip-surface interaction forces is limited. Only on the unstable branch, large

amplitudes can be combined with a high sensitivity (i.e., large $|\frac{\partial A}{\partial f}|$) while entirely remaining in the attractive regime. Particularly when combined with the increasingly popular light-driven excitation scheme (e.g., via photothermal actuation)⁵⁴, we expect UB-SFM to enable ultra-soft imaging conditions, being beneficial for soft samples, such as those encountered in biological systems.

Another application is touchless sensing of the effective tip-sample potential and dissipation at nanometric resolution. Further refinements of the interaction model, such as those collected in ref. 48 may result in quantitative data obtained simultaneously with imaging. As a side effect, erratic jumps between different stable orbits are avoided with enabled control, although other, dedicated approaches for this purpose have been proposed⁴³. In a wider scope, our methodology is applicable to any micro- or nanomechanical system comprising a mechanical nanojunction, i.e., nearby interfaces with controlled (semi-) contact. Avoidance of stiction in controlled micro- or nanomechanical structures thus appears feasible, with numerous applications from sensors to miniaturized mechanical transmissions employing articulating surfaces.

An interesting topic for future investigation is also the question of how noise-reduction capabilities of a suitable control scheme, as shown for example in ref. 31, can lead to enhanced performance in an actual SFM experiment.

All these applications require implementing continuous scanning while stabilizing the unstable state. While this may be a technical challenge, based on our work, there are no fundamental obstacles.

Methods

Experimental realization

The experiment is based on a commercial SFM setup (NT-MDT NTEGRA Spectra Solar) with an external signal access module (see Fig. 3 for a simplified scheme). For versatile implementation of arbitrary control algorithms, signal generation and processing are accomplished by a real-time signal processing unit (RTSPU, *ADwin Pro II*, Jäger Messtechnik GmbH, Germany) based on a hybrid hardware combining a fast central processing unit (CPU), field-programmable gate arrays (FPGA), and fast 16-bit digital to analog and analog to digital converters (DAC, ADC). Time-discrete versions of equations (1) and (2) are used using a simple forward Euler method with a step-size of 1.2 μ s, implemented within the ADwinC software package (Jäger Messtechnik GmbH, Germany). One of the RTSPU outputs is used for the actuation voltage applied to the dither piezo of the SFM, and the resulting cantilever deflection is measured by the integrated position-sensitive photodiode (PSPD) and fed to an RTSPU input for further processing. A second RTSPU output is connected to a high-voltage amplifier (Nanonis HVA4, SPECS Surface Nano Analysis GmbH, Germany), providing the z-piezo voltage for controlling the tip-sample separation.

Cantilevers with a low resonance frequency of nominally $f_0 = 45$ kHz and high Q-factor ($Q \approx 400$) are chosen (*Extra Tall PointProbePlus SFM Tips*, NanoWorld AG, Switzerland) for moderate controller speed requirements. Typical free oscillation amplitudes are ≈ 70 nm, and tip-sample distances are chosen such that the predominant forces remain attractive (with the exception of the respective lowest curve in Fig. 2a, b, where the repulsive force sets in as evident from the additional bifurcation point for each sweep direction).

All measurements are conducted on a Si(111) sample with a native oxide layer. Ambient humidity ranges from 17% to 36%. To minimize thermal drift and electromagnetic noise, the SFM setup is enclosed by a grounded protective box. The tip-sample separation is estimated from the oscillation amplitude, calibrated by force-distance curves on the Si sample. The phase is used as the control target because compared to the amplitude, it is more distinct between the two stable states (see Fig. 2), and it behaves monotonically within the region of the unstable branch, simplifying the sampling algorithm. Each experimental control run consists of frequency sweeps (forward/backward) with disabled control, followed by the controlled phase sweep from high to low values, and another set of frequency sweeps (forward/backward). The uncontrolled frequency sweeps serve as

a reference and enable identification of the approximate location of the unstable branch in the (f, ϕ) plane. We used control gains in a range from $K_I = 113 \text{ kHz rad}^{-1} \text{ s}^{-1}$ to $202 \text{ kHz rad}^{-1} \text{ s}^{-1}$ and $K_P = 11.4 \text{ kHz rad}^{-1}$ to $12.6 \text{ kHz rad}^{-1}$. Typical durations are $\approx 1.2 \mu$ s per individual control cycle, resulting in about 240 ms for one control run (i.e., for each value of the target phase ϕ^*). This reflects a compromise between minimal drift and good data quality. The last $\approx 40\%$ of data points of each control step are used for deducing statistical figures (i.e., mean and its standard deviation), represented by data points and error bars plotted in Fig. 4a (where the phase error is much lower than the size of the graph markers, hence hardly visible). A full version of the ADwinC code, including nLMS implementation, signal generation, and control algorithm, can be found in the Figshare repository.

For the response surfaces $A(f, s)$ and $\phi(f, s)$ as shown in Fig. 5 the tip-sample separation is varied in steps of 1.85 nm between each control run. This is done by setting the respective new z-piezo voltage and employing a PI feedback loop to stabilize the z position (i.e., keeping the amplitude constant). Then, the feedback is disabled again, and the next control run is started. Due to technical restrictions on the communication between the RTSPU system and the personal computer, only part of all control runs is available as a full data set for the response surface. For control steps without full datasets, only the final values of phase and frequency are taken as a result, hence these data points show increased scatter.

Phase extraction

For both, experiment and simulation, fast phase extraction is crucial. Our approach is based on a least-mean square algorithm. In dynamic mode SFM, a mono-harmonic excitation force of the form $\Gamma \sin(\psi(t))$ is applied. We observe the deflection response $x(t)$ using a truncated Fourier series

$$x(t) = \frac{1}{2} X_0(T) + \sum_{i=1}^N X_{i,s}(T) \sin(\omega_i t) + X_{i,c}(T) \cos(\omega_i t) \quad (3)$$

We will refer to

$$X(T) = (X_0(T), X_{1,s}(T), X_{1,c}(T), \dots, X_{N,s}(T), X_{N,c}(T))^T \in \mathbb{R}^{2N+1} \quad (4)$$

as the harmonic state, to the Fourier coefficients $X_0, X_{i,s}, X_{i,c}$ as the harmonic variables, which change at a slow timescale $T \gg 2\pi / \min \omega_i$, and towards N as the truncation order. To obtain the harmonic variables, which coincide with the Fourier coefficients, we use a Least-Mean-Square (LMS) algorithm in contrast to the standard Lock-In-Amplifier setup. For general decomposition practices the LMS algorithm was introduced for noise canceling applications in⁴⁵, in atomic force microscopy this method was investigated in⁴⁶, for continuation purposes the LMS algorithm was first used in⁵⁵, a comprehensive overview of general adaptive filters can be found in⁵⁶. Time-explicit update equations for the harmonic state X can be found in the “Methods” section.

In the experiment as well as in the simulation, we get information on the current tip displacement x with a sampling time Δt . Using the index as the time discretisation $x(n\Delta t) = x|_n$, the LMS Algorithm updates the harmonic state X time explicitly by

$$\begin{aligned} X|_{n+1} &= X|_n + \mu c(\psi|_n) e|_n \\ e|_n &= x|_n - X^T|_n c(\psi|_n) \\ \psi|_{n+1} &= \psi|_n + \Delta t \omega_d, \end{aligned}$$

where μ denotes the step-size. We exploit the linear parametric form of the truncated Fourier series and

$$c(\psi|_n) = \left(\frac{1}{2}, \sin(\psi|_n), \cos(\psi|_n), \dots, \sin(N\psi|_n), \cos(N\psi|_n) \right)^T \in \mathbb{R}^{2N+1}$$

describes the current basis vector with the instantaneous phase $\psi|_n$ of the actuator $\Gamma \cos \psi|_n$ as an argument to ensure synchronized clocks between measurement and forcing. The error $e|_n$ is the difference between current measurement $x|_n$ and approximation of the displacement $\tilde{x}|_n = X^\top|_n c(\psi|_n)$. In quadrature interpretation we obtain amplitudes A_i and phaselags ϕ_i from harmonic state X via

$$A_i(T) = \sqrt{(X_{i,s}(T))^2 + (X_{i,c}(T))^2} \quad \phi_i(T) = \arctan_2(X_{i,c}(T), X_{i,s}(T)).$$

Whereas in experimental settings we only use the fundamental $i = 1$ to manipulate the driving frequency, in principle we can take into account higher harmonics. Experimental data of the deflection signal (see Supplementary Fig. 1) illustrates that mono-harmonic approximation of the measured signal is reasonable.

Simulations

For simulations, we use a normalized time, which can be seen as the number of cycles at the natural frequency ω_0 of the cantilever. In the following, we will refer to t as the normalized time. The driven point mass in the Lennard-Jones potential with viscous damping can then be written as a coupled ordinary differential equation (ODE)

$$\begin{aligned} \frac{d}{dt}x_1 &= x_2 \\ \frac{d}{dt}x_2 &= -x_1 - \frac{1}{Q}x_2 + \frac{f}{Q}\sin x_3 - \frac{\omega_0\gamma}{(s-x_1)^3}x_2 \\ &\quad - 12\frac{V_0}{k\sqrt{(x_1-s)^2}}\left(\left(\frac{\sigma^2}{\sqrt{(x_1-s)^2}}\right)^6 - \left(\frac{\sigma^2}{\sqrt{(x_1-s)^2}}\right)^3\right) \\ \frac{d}{dt}x_3 &= \frac{\omega_d}{\omega_0} + u \end{aligned}$$

and is integrated numerically in Julia by an adaptive Runge-Kutta solver TSIT5() of DifferentialEquations.jl⁵⁷. We thereby choose the parameters, $k = 0.7 \text{ Nm}^{-1}$, $Q = 400$, $s = 60 \text{ nm}$, $\sigma = 2.8 \text{ nm}$, $f = 130 \text{ nm}$, $\omega_0 = 2\pi \cdot 55000 \text{ rad s}^{-1}$, $V_0 = 4.2 \cdot 10^{-18} \text{ J}$, $\gamma = 0.6 \cdot 10^{-31}$. In the open loop and also closed-loop simulation we vary the normalized driving frequency $\frac{\omega_d}{\omega_0}$ after the transients die out by a discrete callback. In order to check convergence towards a steady state we use an adapted Welford algorithm in combination with a circular buffer. The variance Σ , and mean value $\langle y \rangle$ of the circular buffer with head H can be obtained time explicitly by

$$\begin{aligned} \langle y \rangle|_{n+1} &= \langle y \rangle|_n + \frac{y|_n - H}{L} \\ \Sigma|_{n+1} &= \Sigma|_n + (y|_n + H - \langle y \rangle|_n - \langle y \rangle|_{n+1})(y|_n - H). \end{aligned}$$

In our case, we store transient phase-lag values ϕ_1 in a circular buffer of length $L = 40$ with a sampling frequency of 15.3 cycles at the resonance frequency. We identify steady-state behavior and do a continuation step in the phase-lag target ϕ_1^* once the conditions

$$\langle \phi_1 \rangle|_{n+1} \in [\phi_1^* - \Delta\phi_1^*, \phi_1^* + \Delta\phi_1^*] \quad \text{and} \quad \sqrt{\frac{\Sigma|_{n+1}}{L}} \leq \sigma_{\text{tol}}$$

are fulfilled. Hereby we choose $\Delta\phi_1^* = 0.0005 \text{ rad}$ and $\sigma_{\text{tol}} = 0.00001 \text{ rad}$. For the LMS algorithm, we truncate the Fourier series at the 4th harmonic, use the step-size $\mu = 0.01$, and update the harmonic state as well as the instantaneous phase ψ , 10 times every cycle at the natural frequency ω_0 , which corresponds to $\Delta t = 0.01$. For further details and exact implementations we would like to refer to the code, which is available in the public [GitHub](#) repository.

Data availability

Datasets are available via [Figshare](#).

Code availability

The code used for the experimental stabilization of unstable orbits is also available via [Figshare](#). Simulations shown in this paper are available at [GitHub](#).

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Author contributions

J.S., S.S., and I.B. conceptualized the project. H.W. performed the simulations, realized the experimental nLMS implementation, and wrote the corresponding sections of this paper. L.B. realized experimental implementation, carried out the experiments, and evaluated experimental Data. I.B. and H.W. participated in the experiments. I.B., S.S., and L.B. wrote the paper with contributions and comments from all authors. I.B., S.S., J.S., N.K., W.J., H.W., and L.B. discussed the results.

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Chapter 7

**Detecting stability and bifurcation
points in control-based continuation
for a physical experiment of the
Zeeman catastrophe machine**

Detecting Stability and Bifurcation Points in Control-Based Continuation for a Physical Experiment of the Zeeman Catastrophe Machine*

Anna Dittus[†], Niklas Kruse[‡], Ingo Barke[‡], Sylvia Speller[‡], and Jens Starke[†]

Abstract. Feedback-based control of dynamical systems enables tracking of unstable branches that cannot be reached by the free evolution of a simulated or physical system. As the stationary points are stabilized with these control methods, the stability information on the original system gets lost in the data from the transient behavior reaching these equilibria. We introduce, by means of a physical experiment of the Zeeman catastrophe machine, a method to construct the whole bifurcation diagram including stability information about the underlying uncontrolled equilibria and the location of bifurcation points.

Key words. bifurcation analysis, control-based continuation, stability determination, noninvasive control, feedback control, stabilization of unstable equilibria, laboratory experiment

MSC codes. 34D20, 34H05, 37G10, 37M10, 70-05, 70K50, 93D15

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1. Introduction. Unstable states of dynamical systems contain relevant system information since they play an important role for the separatrix between different regions of attraction. The traditional approach in science starts with experimental observations which serve as a basis for the development of a model, enabling a simplified mathematical description. This model can then be used for prediction to be verified and might lead to further refinements. Naturally, the information about unstable states is by definition inaccessible in an experiment. One approach to gaining information about unstable states is to use feedback control to stabilize them [29, 34, 35]. But then, stability information gets lost in the gained data. To overcome this problem, one could switch off the control for a short time interval and investigate the uncontrolled behavior [7]. In extreme cases, systems in uncontrolled states can destroy the whole physical setup such that alternatives are required. Well-known examples of such events are bridge oscillations, the parameter-dependent climate change, or vibration motion in gears taking destructive amplitudes.

In this paper, we present a method that allows us to control all equilibrium points and still gain stability information from the data. The proposed method offers a substantial increase

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in information from experiments compared to traditional time series analysis which focuses on transients and ω -limit sets like stable stationary points or stable periodic orbits.

This is illustrated by the example of the Zeeman catastrophe machine. The machine was suggested by Zeeman [39] as a didactic model to illustrate the effects of small changes in parameters on nonlinear systems. It is a simple, clear, mechanical experiment for which mathematical models are available. Therefore, it is a good prototype experiment for comparison and to verify concepts.

The paper is organized as follows. In section 2 we introduce a method to stabilize stationary points while determining the stability of the original uncontrolled system. After a short outline of already existing control-based methods to stabilize unstable equilibria or periodic orbits of dynamical systems and approaches to gain stability information about the underlying original system in subsection 2.1, we shortly introduce our control-based approach used for the presented method in subsection 2.2. In subsections 2.3 and 2.4, conditions for controllability and stability for dynamical systems to use the method of subsection 2.2 are given. The approach to investigating stability information on the underlying uncontrolled system out of the gained data is presented in subsection 2.5. The Zeeman catastrophe machine is presented in section 3 and taken as a prototype experiment to verify the concept presented in section 2. A full bifurcation diagram is achieved by control-based continuation and the stability of the gained equilibrium points can be computed. In general, we assume no a priori knowledge about a model describing the system's dynamics. To verify the method introduced in section 2, the results of section 3 are compared with an analytical model in section 4. We conclude with a summary and an outlook of the results in section 5.

2. Control-based continuation for laboratory experiments. Standard methods of numerical bifurcation analysis are often difficult or impossible to be used in laboratory experiments as they require initialization at any point in state space. In such situations, it turned out that a control-based approach allowed the performance of similar tasks of bifurcation analysis with reduced accessibility.

2.1. Methods based on feedback control. Several noninvasive feedback control methods are known and used to stabilize unstable equilibria or unstable periodic orbits. In the following, we will define a feedback control as *noninvasive* if the control vanishes in an equilibrium point or periodic orbit of the original uncontrolled system at any time. All of these methods require prior knowledge either about the equilibria's or periodic orbits' time profile or about the explicit function of the underlying differential equation. This circumstance makes them highly attractive for stabilizing physical systems.

The class of washout filter-aided feedback control methods introduced by Abed et al. in [1, 13, 16] can be understood as a proportional-integral control to stabilize unstable stationary points of $\dot{x} = f(x)$. The closed-loop system with state variable x , scalar control input u , and washout filter w can be written as

$$(2.1a) \quad \dot{x} = Mx + Nu + h(x, u),$$

$$(2.1b) \quad \dot{w} = Q(x - w),$$

$$(2.1c) \quad u = K(x - w),$$

where M is the Jacobian of the function f evaluated at a given point x_0 , N is the derivative of f with respect to u , Q is a nonsingular matrix, K is the feedback gain matrix, and $h(\cdot, \cdot)$ are higher-order terms with respect to x . A comparison of (2.1) with the proportional control combined with a state observer (2.2) reveals the similarity of the structures of the methods. Linear stability theory shows that it is possible to stabilize equilibria or periodic orbits of the closed-loop system (2.1) in the linear sense; see [13].

A similar approach is followed by the OGY method introduced by Ott, Grebogi, and Yorke in [21]. This method aims to stabilize periodic orbits of dynamical systems by kicking them in the desired direction. The necessary information is gained by consideration of a Poincaré section. So, only very little knowledge is needed for the stabilization. On the other hand, one has to identify the Poincaré section and calculate precisely the necessary perturbations. A successful application of an OGY method on a model of an atomic force microscope was done in [18]. Stabilization of unstable periodic orbits with a feedback control using a time-delayed signal has been introduced by Pyragas [28] and applied, e.g., on oscillatory electrodisolution of nickel in [15].

Another noninvasive method to stabilize equilibria is an approach with a state observer introduced by Pyragas, et al. [29, 30]. They propose a proportional feedback control while automatically finding the control target by an additional equation and choosing the parameters to stabilize the whole controlled system.

For many applications, stabilization is needed for an autonomous system of the form $\dot{x} = f(x, \mu)$ with state vector $x \in \mathbb{R}^n$ and bifurcation parameter $\mu \in \mathbb{R}$. A method based on a proportional control combined with a state observer [29, 30] has the following form,

$$(2.2a) \quad \dot{x} = f(x, \mu) + k_1(y - x),$$

$$(2.2b) \quad \dot{y} = k_2(y - x)$$

with control parameters $k_1, k_2 \in \mathbb{R}$, the fixed bifurcation parameter μ , and dynamic variable y which represents the role of the control target. As already mentioned, this is similar to the scheme (2.1) of the washout filter-aided feedback control [38]. This has been used in [22, 29, 35] to stabilize and track stationary points either on the stable or on the unstable branch. With this method, it is only possible to track the curve of stationary points around fold bifurcation points by change of the sign for parameter k_2 .

With pseudoarclength control, stationary points are found by adding an orthogonality condition, and hence one more dimension to the set of equations which are solved by a Newton method. By searching the next stationary point on a line perpendicular to the line connecting the two previously found equilibria, the problem of singularity around a fold can be overcome. In [35], the authors combine pseudoarclength continuation with a feedback control law that drives a microscopically defined system automatically to a steady state on a coarse scale. This allows for producing a bifurcation diagram of a macroscopic dynamical system while knowing only equations on a microscopic scale.

Inspired by the ideas in [28], Sieber and Krauskopf [34] proposed a method based on time-delayed feedback control to stabilize periodic orbits (see also [33]). It can be embedded into a pseudoarclength continuation for direct tracking of periodic orbits and also around fold bifurcations of experiments without further knowledge about the dynamics. The lack

of an explicit model leaves the question about a suitable feedback control, as this is problem dependent. An experimental application of the control-based continuation is presented in [33] on the continuation of periodic orbits through a saddle-node bifurcation point for a pendulum.

Another approach to exploring bifurcation diagrams for laboratory experiments has been proposed by Barton and Sieber [3]. The method uses first a secant prediction in the (x, μ) -plane to estimate the position of the next equilibrium. Starting with the newly estimated parameter value for the equilibrium, an adequate feedback control is constructed having an ω -limit point on the branch of the bifurcation diagram. When the transients have settled, the system is assumed to have reached a (stabilized) equilibrium.

All of the works cited above focus on the noninvasive stabilization of stationary points or periodic orbits. If the considered dynamical system fulfills the controllability and observability conditions, it is possible to determine bifurcation diagrams with these control schemes. None of the therein presented methods provide stability information on the original system. In the following, examples from the literature are provided with approaches on how to obtain this information.

A continuation scheme based on local proportional-derivative (PD) control for experiments with forced nonlinear impact oscillators is given in [7, 8, 32]. It is possible to predict the next equilibrium by pseudoarclength continuation and to control the system's state by the difference between the trajectories of the uncontrolled state and the predicted state. The necessary Newton step for pseudoarclength control is done in Fourier space where periodic orbits appear as equilibrium points as in [33]. In [7], the authors determined the stability of the uncontrolled system by switching off the control for a short time step. Then the trajectories diverged from the reference trajectories, and the unstable branches served as the separatrix. This allowed for estimating Lyapunov exponents in an experiment.

Another method to estimate the original Floquet exponents of already stabilized periodic orbits has been introduced by Barton in [2]. The author proposes a least-squares fit of the measured data to an autoregressive model with exogenous inputs by perturbing the orbit slightly in its neighborhood in state space. This determines the Floquet exponents and hence the desired stability information. This approach has been already applied successfully on a nonlinear tuned mass-damper system and allowed the author to obtain a full bifurcation diagram including stability information.

The works [2, 7] address the gain of stability information by different approaches. Nevertheless, the methods require either invasion into the measurement in case of perturbation or some robustness of the system in the case of switching the control off.

In the following subsection 2.2, we will present a scheme that enables one to produce a complete bifurcation diagram including stability information gained only from the data obtained during the control.

2.2. Control scheme on the parameter. For simplicity with respect to presentation, application, data analysis, and the solution, we limit ourselves in the following to a 2-dimensional system. This limitation is sufficient for the Zeeman catastrophe machine since its phase space consists of a 1-dimensional state ϕ in physical space and its derivative $\dot{\phi}$. A generalization to the n -dimensional case is possible.

We consider an autonomous dynamical system of the form

$$(2.3) \quad \dot{x} = f(x, \mu),$$

with state variable $x = (x_1, x_2) \in \mathbb{R}^2$ and bifurcation parameter $\mu \in \mathbb{R}$. We refer to (2.3) as the original system since no control is applied to it.

The system (2.3) describes the dynamics of the state x depending on x and μ . Usually, the curve of stationary points would be regarded as a function x depending on μ . In the neighborhood of a fold, this approach is not even locally possible (for illustration, see Figure 5 in section 4). In numerical bifurcation analysis, this problem is solved by pseudoarclength continuation. This overcomes the singularity problem by extending the system (2.3) by an additional dimension. In the following, a different approach to a similar problem is presented. The proportional control scheme (2.2) does not allow for tracking stationary points in the neighborhood of fold bifurcation points. To overcome this problem, we change to a control for the state x by adapting the 1-dimensional parameter μ in the original system (2.3) since the curve of stationary points is a function $\mu(x)$ in x . We use the fact that higher-dimensional systems are controllable by lower-dimensional control [37]. Starting from a previous stationary point in the neighborhood, a target x^* is defined for the system. To drive x to the target x^* , the difference $(x - x^*)$ has to be considered. To reach the new target through the proposed additive feedback control, we extend the original system (2.3) by an additional differential equation for the parameter μ . The aim is to adapt μ proportionally to the difference $(x - x^*)$. This leads to the following scheme for local dynamic control,

$$(2.4a) \quad \dot{x} = f(x, \mu),$$

$$(2.4b) \quad \dot{\mu} = -K(x - x^*) = -K_P(x_1 - x_1^*) - K_D(x_2 - x_2^*),$$

where $K = (K_P, K_D) \in \mathbb{R}^2$ defines the feedback gain. In the literature, this control scheme is often called *PD-control*. This means we search for an unknown equilibrium $(\bar{x}, \bar{\mu})$ for which we set in advance a target value x^* for the equilibrium of the original system (2.3) as x -components of the equilibrium of the controlled system (2.4), i.e., a target for the dynamics (2.4a). The distance between the actual state and the target defines the change of the bifurcation parameter μ and hence indirectly the change of the state x . For an illustration of the interaction under control (2.4) between the first component of the state, x_1 and the bifurcation parameter μ , see Figure 1.

The x -components of the equilibria of the controlled system (2.4) are identical to the equilibrium points of the original system (2.3): If $(\bar{x}, \bar{\mu})$ is a stationary point of (2.3) corresponding to $\bar{x}$, i.e., $\dot{x} = 0$, and if furthermore, the state is identical to the control target, i.e., $(\bar{x} - x^*) = 0$. Then $\dot{\mu} = 0$, and $(\bar{x}, \bar{\mu})$ must be a stationary point of the system (2.4). If, to the contrary, $(\bar{x}, \bar{\mu})$ is a stationary point of (2.4), we get $\dot{x} = 0$, and $(\bar{x}, \bar{\mu})$ must be a stationary point of (2.3). This means that the proposed method is noninvasive, i.e., the control disappears at an equilibrium point.

2.3. Controllability for dynamical systems. Controllability describes a system's property to be moved by external input from any initial state to any final state in finite time [37]. In the following, we state the conditions for a system to be controlled by (2.4b). We assume that

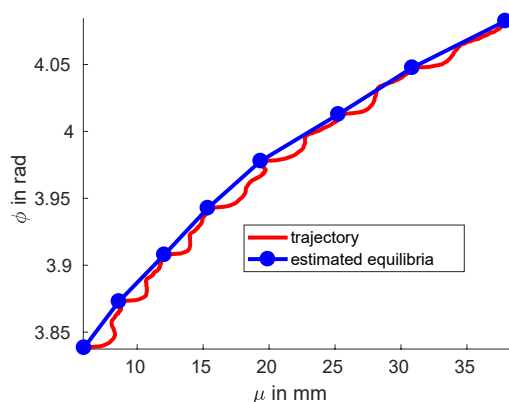


Figure 1. Magnified detail for the continuation of the curve of stationary points by control (2.4) for the resulting experimental system (3.1). Estimated equilibria are marked by a blue dot connected by blue line segments which approximate the position of equilibrium points in-between. The trajectory in (ϕ, μ) -space is represented in red.

the controlled system (2.4) can be approximated by its linearization in a neighborhood around each stationary point. As a consequence, we can use results from control theory for continuous so-called linear time-invariant (LTI) systems [37] to give conditions for the controllability of the controlled system (2.4). The systems are called LTI because the linearization matrix is constant. For reasons of clarity and to gain a linear representation, the following notation distinguishes sharply between system and control. We use the 1-dimensional control vector $u \in \mathbb{R}$ for the system

$$(2.5) \quad \dot{v} = Cv(t) + Du(t),$$

with the state vector

$$v := \begin{pmatrix} x \\ \mu \end{pmatrix} \in \mathbb{R}^3,$$

the input matrix

$$D := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3,$$

and the state matrix

$$C := \left[\begin{array}{c|c} J & f_\mu \\ \hline 0 & 0 \end{array} \right] \in \mathbb{R}^{3 \times 3},$$

where $J \in \mathbb{R}^{2 \times 2}$ is the Jacobian of f with respect to x , and $f_\mu \in \mathbb{R}^{2 \times 1}$ is the gradient of f with respect to μ . The resulting controllability matrix [37] is

$$(2.6) \quad R = \left[\begin{array}{c|c|c} D & CD & C^2D \end{array} \right] = \left[\begin{array}{c|c|c} 0 & 0 & 1 \\ 0 & CD & C^2D \end{array} \right] = \left[\begin{array}{c|c|c} 0 & f_\mu & J \cdot f_\mu \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right] \in \mathbb{R}^{3 \times 3}.$$

Following the results of LTI systems, we know that the original system (2.3) is controllable by (2.4) if R has full rank. This is fulfilled if $[f_\mu | J \cdot f_\mu]$ has full rank. So, controllability depends only on J and f_μ , hence on the original dynamical system (2.3). The condition “ $\text{rank}(R) = n$ ” is equivalent to the condition “ $\text{rank}([sI - C \ D]) = n$ for all s ” which is also known as the Popov–Belevitch–Hautus test [4, 14, 25]. Often, the system (2.5) is considered in a more general manner when the output is a function of the state v .

By definition, the linearization in a bifurcation point is singular. This requires considering these cases separately. In the specific case of the control (2.4b), we perform the control by change of the parameter μ and not of x , as μ is a function of x and not the other way around. Hence, no singularities of the controlled system (2.4) appear at fold bifurcation points.

Usually, controllability is scrutinized together with observability, i.e., the system’s property to estimate the current state only from the output information. In the presented case of system (2.3), we assume that the first component x_1 of the state is directly measurable and the second component $x_2 = \dot{x}_1$ can be approximated by the difference quotient or similar methods from the measurements of x_1 . Hence, the full state trajectory is constructed only from input and output measurements. Therefore, it is not required to investigate observability. For a generalization of the following results, further investigations are necessary to guarantee observability.

2.4. Stability of controlled dynamical systems. In order to find sufficient conditions for the control parameter K to guarantee the linear stability of the controlled system (2.4), it can be considered in the matrix-vector notation

$$(2.7) \quad \frac{d}{dt} \begin{pmatrix} x \\ \mu \end{pmatrix} = \begin{pmatrix} f(x, \mu) \\ -K(x - x^*) \end{pmatrix} \in \mathbb{R}^3$$

of a 3-dimensional state $(x, \mu)^T$. Similarly to subsection 2.3, we assume that the system (2.7) can be approximated in the neighborhood of an equilibrium $(\bar{x}, \bar{\mu})^T$ by the linearization

$$\frac{d}{dt} \begin{pmatrix} x \\ \mu \end{pmatrix} = \tilde{J}_{|(\bar{x}, \bar{\mu})^T} \cdot \begin{pmatrix} x \\ \mu \end{pmatrix}$$

of (2.7) with $\tilde{J}_{|(\bar{x}, \bar{\mu})^T}$ as the Jacobian

$$(2.8) \quad \tilde{J} = \left[\begin{array}{c|c} J & f_\mu \\ \hline -K_P & -K_D \end{array} \middle| \begin{array}{c} f_\mu \\ 0 \end{array} \right] \in \mathbb{R}^{3 \times 3}$$

evaluated at $(\bar{x}, \bar{\mu})^T$. This uses the Jacobian J of the original system (2.3) and the gradient $f_\mu \in \mathbb{R}^{2 \times 1}$ of f with respect to μ . If K_P and K_D are chosen such that all eigenvalues of the Jacobian $\tilde{J}$ have negative real parts, the original system (2.3) is stabilized by (2.4) or, respectively, by (2.7). For systems where no explicit model is given, information about the Jacobian J could be obtained by experimental measurements.

2.5. Detecting stability and bifurcation points. Methods like those in [7] that switch off the control to gain stability information by the original system’s behavior entail disadvantages. There are many cases where switching off the control could harm the system. In the following, we present a way to analyze the stability while keeping the control (2.4b) permanently

activated during the experiment. This means we gain stability information about the original dynamics from the closed-loop measurements without further interventions during the experiment. In general, this approach is not limited to the control (2.4). It can be adapted to any other control where a similar separation of the system state's and the bifurcation parameter's behavior is possible. In many cases, no exact mathematical description of the considered dynamical system is available. So, we cannot directly use a detailed analytical detection neither of the stability information nor the bifurcation points. At first, we find an approximate description of the linearization for the original system (2.3) close to each stabilized equilibrium point in order to obtain stability information from experiments. Computing the linearization matrix's eigenvalues allows us to determine the original stability. Even though we assumed in subsections 2.2 to 2.4 the system to be 2-dimensional, we now consider an n -dimensional system $\dot{x} = f(x, \mu)$ with $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ controlled by the 1-dimensional control (2.4b) with parameter μ . The state of the controlled system is denoted as $z := (x, \mu)^T \in \mathbb{R}^{n+1}$. Since the stationary point $\bar{z}$ is not the origin but a shifted center for the linearization, we approximate by Taylor expansion up to the linear term including the constant shift for the affine linear approximation. So, we need to find a least-squares fit G , with $(G)_{ij} = g_{ij}$, of the measured data $\hat{z}$ to the general solution z of

$$(2.9) \quad \dot{z} = Gz - G\bar{z} = Gz - b.$$

The constant term $G\bar{z}$ of the Taylor expansion is due to the shift of the equilibrium point $\bar{z}$. In order to find the best-fitting solution for (2.9) to the measured data $\hat{z} \in \mathbb{R}^{n+1}$, we apply a standard least-squares approach [5, 27]. The collection of data points is referred to as

$$\hat{Z} := [\hat{z}(t_1) \ \hat{z}(t_2) \ \dots \ \hat{z}(t_p)] \quad \text{and} \quad \dot{\hat{Z}} := [\dot{\hat{z}}(t_1) \ \dot{\hat{z}}(t_2) \ \dots \ \dot{\hat{z}}(t_p)],$$

and we make an affine linear ansatz with

$$\Gamma := \begin{bmatrix} 1 & 1 & \dots & 1 \\ \hat{Z}_{1,1} & \hat{Z}_{1,2} & \dots & \hat{Z}_{1,p} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{Z}_{(n+1),1} & \hat{Z}_{(n+1),2} & \dots & \hat{Z}_{(n+1),p} \end{bmatrix} \in \mathbb{R}^{(n+2) \times p}$$

for the least-squares solution Ψ of

$$(2.10) \quad \dot{\hat{Z}} = G\hat{Z} - b \approx \Psi\Gamma.$$

So, we search for the solution Ψ in the least-squares sense of the minimization problem

$$\min_{\Gamma} \left\{ \left\| \Psi\Gamma - \dot{\hat{Z}} \right\|_F \right\},$$

where $\|H\|_F = \sqrt{\sum_{i=1}^q \sum_{j=1}^r |h_{ij}|^2}$ for $H \in \mathbb{R}^{q \times r}$ is the Frobenius norm. We know from classical results [24] that the matrix

$$(2.11) \quad \Psi = \dot{\hat{Z}}\Gamma^+ \in \mathbb{R}^{(n+1) \times (n+2)}$$

with $(\Psi)_{ij} = \psi_{ij}$, $i = 1, \dots, n+1$, $j = 1, \dots, n+2$, and $(\Gamma)_{jk} = \gamma_{jk}$, $j = 1, \dots, n+2$, $k = 1, \dots, p$, minimizes uniquely the quadratic error $\|\Psi\Gamma - \dot{Z}\|_F$, where Γ^+ is the Moore–Penrose pseudoinverse of Γ . So, we search for the least-squares approximation of (2.10) with

$$(2.12a) \quad (\dot{Z})_{ik} = \dot{z}_{ik} = \left(\sum_{j=1}^{n+2} g_{ij} \hat{z}_{jk} \right) - b_i \quad \text{for } i = 1, \dots, n+1, k = 1, \dots, p$$

$$(2.12b) \quad = \psi_{i1} \gamma_{1k} + \sum_{j=2}^{n+2} \psi_{ij} \gamma_{jk} \quad \text{with } \psi_{i1} = -b_i \text{ and } \gamma_{1k} = 1$$

$$(2.12c) \quad = \sum_{j=1}^{n+2} \psi_{ij} \gamma_{jk} = (\Psi\Gamma)_{ik}.$$

In (2.12b), we define the first row of Ψ to contain the information about the center of the linearization, i.e., the stationary point. We used in (2.12b) that the first column of G represents the dynamics of the equilibrium point. Hence, we get for the Jacobian $\tilde{J}$ of the controlled system (2.4) the approximated linearization matrix

$$(2.13) \quad G = (\Psi)_{1, \dots, n+1; 2, \dots, n+2} \in \mathbb{R}^{(n+1) \times (n+1)}.$$

We know from (2.8) that

$$G \approx \left[\begin{array}{c|c} J & f_\mu \\ \hline -K_P & -K_D \\ \hline & 0 \end{array} \right],$$

and hence that the upper left submatrix consisting of the first n rows and columns of G approximate the Jacobian J of the original system (2.3). So, we can estimate the entries of the linearization matrix of the underlying dynamical system in every stationary point. By computing the eigenvalues of J , we can detect whether the equilibrium is stable (all real parts are negative), unstable (there exists a positive real part), or a bifurcation point (at least one real part is zero as a necessary condition). Since the stability information is not required for the control-based continuation, the computations can be done a posteriori after running the whole experiment.

A similar approach replacing the affine linear ansatz of the right-hand side in (2.9) with a more general basis of nonlinear functions in Γ was chosen by Brunton, Proctor, and Kutz in [6]. Their approach for sparse identification of nonlinear dynamics enables approximating the right-hand side of (2.9) even by functions of higher order. As we are only interested in linear stability information, we do not need terms of higher order and restrict ourselves to the affine linear ansatz.

The naive application of identification techniques to measurements of a controlled system is known to result in a bias in the case of relevant noise (see, e.g., [36, Chapter 10]). For the presented example of the Zeeman catastrophe machine, we could see some small noise that influenced the control and hence the controlled system (2.4) only minimally; see Figure 1. Due to its minor relevance to the identification results, we do not consider correction methods

and restrict ourselves to the case of small or absent noise. For noisy systems, more elaborate methods [36] than least-squares schemes might be considered to avoid bias.

3. Application on the Zeeman catastrophe machine. On the basis of the example of the Zeeman catastrophe machine, we demonstrate the proposed method. The catastrophe machine was introduced by Zeeman [39] in 1972 as an example to illustrate bistability and hysteresis [12, 26, 39, 40, 41]. The bifurcation diagram consists of 2 stable branches which are separated by an unstable branch. Each stable branch is connected to one end of the unstable branch over a saddle-node bifurcation point. Due to its clear and simple construction and its well-described dynamics, it has been used many times as a didactic tool to analyze dynamics in various aspects [19, 20, 40, 41]. In addition, it is used as a toy model to test and verify methods to analyze and stabilize dynamical systems [9, 17, 23, 31].

Even though we do not use an explicit mathematical description for the right-hand side $f(x, \mu)$ of the original system (2.3), we can control and stabilize its unstable stationary points by (2.4) using a dynamic controller for the parameter μ . Furthermore, we can additionally detect the stability of the stationary points and the bifurcation points from the data of the controlled experiment by the least-squares approach described in subsection 2.5.

3.1. Setup of the Zeeman catastrophe machine. The Zeeman catastrophe machine consists of two tension springs with tensionless length L_0 and spring constant k , and a mass-carrying rotatable disk fixed at the origin $O = (0, 0)$ which is the rotational axis of the disk and coincides with the center of mass. Figure 2(a) shows a sketch of the machine. A photo of the physical realization used for the experiments analyzed in the present paper is shown in Figure 2(b). The technical details for the experimental setup can be found in Appendix A. One spring with dynamical length L_1 is attached to a fixed end $A = (-a, 0)$ and the point P on the radius of the disk. The second spring with dynamical length L_2 is attached to the point P on the disk and the movable end $B = (\lambda, \mu)$. The angle ϕ is measured between the x -axis and the line connecting the origin with P . The angle ϕ and its velocity $\dot{\phi}$ define together the state vector

$$x(t) = \begin{pmatrix} \phi(t) \\ \dot{\phi}(t) \end{pmatrix}.$$

The ordinate of point B is defined as the parameter μ . A change of μ is realized by the stepper motor moving point B in the y -direction; see Figure 2. The dynamic change of the bifurcation parameter μ can be understood in the sense of PD control as described in subsection 2.2. This allows us to do continuation over the fold points without the problem of a singularity. The desired velocity of μ is calculated in (2.4b) by the difference of the actual state to the target $x^* = (\phi^*, \dot{\phi}^*)^T$. We set the target angle ϕ^* for the next stationary point, and require its target angular velocity to be $\dot{\phi}^* = 0$ because we aim to reach an equilibrium of $\dot{x} = f(x, \mu)$. Following the procedure, as described in subsection 2.2, we get the controlled system

$$(3.1) \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\mu} \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} \phi \\ \dot{\phi} \\ \mu \end{pmatrix} = \begin{pmatrix} f_1(x, \mu) \\ f_2(x, \mu) \\ -K_P(\phi - \phi^*) - K_D(\dot{\phi} - \dot{\phi}^*) \end{pmatrix} = \begin{pmatrix} f_1(\phi, \dot{\phi}, \mu) \\ f_2(\phi, \dot{\phi}, \mu) \\ -K_P(\phi - \phi^*) - K_D\dot{\phi} \end{pmatrix},$$

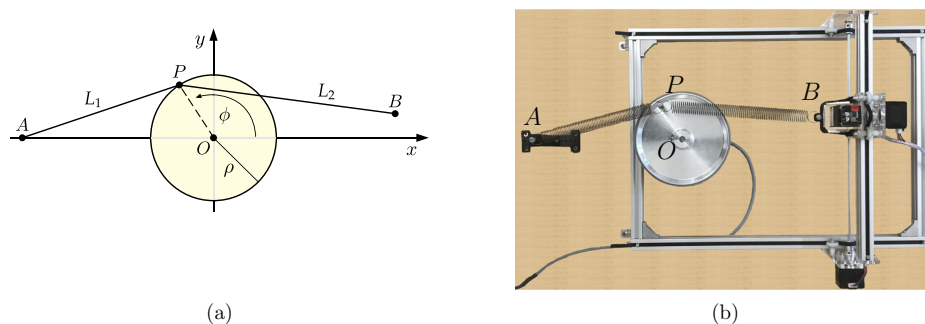


Figure 2. The Zeeman catastrophe machine illustrated as (a) sketch and (b) photo of the experiment. One spring is attached to a fixed end $A = (-a, 0)$ and a point P on the disk with distance ρ to the origin. The other spring is attached to P and a movable end $B = (\lambda, \mu)$ whose y -coordinate μ is the chosen bifurcation parameter. The position B can be moved vertically to the line through A and O by a stepper motor, and the angle ϕ of the disk is measured optically by a rotary encoder; see Appendix A for the technical details.

where $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with $f = (f_1, f_2)^T$ describes the dynamics of ϕ and $\dot{\phi}$. The function f is assumed to not be explicitly known.

Tables 1 and 2 in Appendix C with all introduced variables and quantities support the overview throughout the paper.

3.2. Controllability of the Zeeman catastrophe machine. As outlined in subsection 2.3, the linearization of the controlled system (3.1) in the matrix-vector form is

$$(3.2) \quad \frac{d}{dt} \begin{pmatrix} \phi \\ \dot{\phi} \\ \mu \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial f_2}{\partial \phi} & \frac{\partial f_2}{\partial \dot{\phi}} & \frac{\partial f_2}{\partial \mu} \\ -K_P & -K_D & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \dot{\phi} \\ \mu \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial f_2}{\partial \phi} & \frac{\partial f_2}{\partial \dot{\phi}} & \frac{\partial f_2}{\partial \mu} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \dot{\phi} \\ \mu \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \dot{\mu}.$$

Even though we do not assume any knowledge about the function $f = (f_1, f_2)^T$, we know that in the case of the Zeeman catastrophe machine it holds that $\dot{x}_1 = x_2$ and hence $f_1(x, \mu) = \dot{\phi}$. Following control theory [37], the linearization (3.2) can be seen as the time-invariant control system (2.5) with the 1-dimensional control vector u , the state vector

$$(3.3) \quad v := \begin{pmatrix} \phi \\ \dot{\phi} \\ \mu \end{pmatrix},$$

the state matrix

$$C := \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial f_2}{\partial \phi} & \frac{\partial f_2}{\partial \dot{\phi}} & \frac{\partial f_2}{\partial \mu} \\ 0 & 0 & 0 \end{pmatrix},$$

and the input matrix

$$(3.4) \quad D := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This results in the controllability matrix (cf. (2.6))

$$R = [D \ C \ D \ C^2 D] = \begin{pmatrix} 0 & c_{1,3} & c_{1,1}c_{1,3} + c_{1,2}c_{2,3} \\ 0 & c_{2,3} & c_{1,3}c_{2,1} + c_{2,2}c_{2,3} \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\partial f_2}{\partial \mu} \\ 0 & \frac{\partial f_2}{\partial \mu} & \frac{\partial f_2}{\partial \phi} \cdot \frac{\partial f_2}{\partial \mu} \\ 1 & 0 & 0 \end{pmatrix}.$$

In general, explicit equations for the right-hand side of the original system (2.3) are not available. Hence, one cannot give a strict mathematical proof for a physical experiment that the controllability conditions found in subsection 2.3 are fulfilled. In particular, it is not possible to show directly that the controllability matrix (2.6) has full rank. But we can give some evidence that the system of the physical experiment of the Zeeman catastrophe machine is controllable by estimating the entries of the linearization matrices for the least-squares method from subsection 2.5. Assuming that the autonomous system (2.4a) is continuous in x and μ , and locally monotone, we first calculate the entries of the linearization matrix by a simple difference quotient using the measured data. With the moving average over 10 neighboring stationary points, we robustly obtain $\frac{\partial f_2}{\partial \mu} < 0$. In addition, the same is possible for the other entries of the linearization matrix. We conclude that $\frac{\partial f_2}{\partial \phi} < 0$ and that $\frac{\partial f_2}{\partial \phi}$ exists. Therefore, the controllability matrix R must have full rank. Hence, the system of our physical experiment is controllable by the feedback control (2.4) in a neighborhood of the stationary points including the unstable ones.

3.3. Stability analysis of the controlled Zeeman catastrophe machine. In subsection 2.4, the general conditions for stabilizing the stationary points are discussed. In the following, an estimation for the eigenvalues of the linearization matrix is given to verify that stationary points can be stabilized by the control (2.4b). For the controlled Zeeman catastrophe machine, we have a 3-dimensional system with a linearization matrix given in (3.2). For the stability analysis, we compute the determinant of the linearization matrix and subsequently apply the Routh–Hurwitz criterion. The controlled system (2.4) is stable if the negative characteristic polynomial $p(\nu)$, i.e., the normed polynomial with leading coefficient 1,

$$(3.5) \quad -p(\nu) = \nu^3 - \frac{\partial f_2}{\partial \phi} \nu^2 + \left(K_D \frac{\partial f_2}{\partial \mu} - \frac{\partial f_2}{\partial \phi} \right) \nu + K_P \frac{\partial f_2}{\partial \mu},$$

of the linearization matrix in (3.2) is a Hurwitz polynomial. A necessary condition for all real parts of the eigenvalues ν being negative is that all coefficients of (3.5) are positive. This requires

$$\begin{aligned} K_D \frac{\partial f_2}{\partial \mu} > \frac{\partial f_2}{\partial \phi} &\implies K_D \begin{cases} > \left(\frac{\frac{\partial f_2}{\partial \phi}}{\frac{\partial f_2}{\partial \mu}} \right) & \text{if } \frac{\partial f_2}{\partial \mu} > 0, \\ < \left(\frac{\frac{\partial f_2}{\partial \phi}}{\frac{\partial f_2}{\partial \mu}} \right) & \text{if } \frac{\partial f_2}{\partial \mu} < 0, \end{cases} \\ K_P \frac{\partial f_2}{\partial \mu} > 0 &\implies \text{sign}(K_P) = \text{sign} \left(\frac{\partial f_2}{\partial \mu} \right). \end{aligned}$$

The second necessary condition for the negativity of the real parts of the eigenvalues is that the Hurwitz determinant and all its principal minors are positive. We get the Hurwitz determinant

$$(3.6) \quad H_3 = \begin{vmatrix} \frac{\partial f_2}{\partial \phi} & K_P \frac{\partial f_2}{\partial \mu} & 0 \\ 1 & \frac{\partial f_2}{\partial \phi} - K_D \frac{\partial f_2}{\partial \mu} & 0 \\ 0 & \frac{\partial f_2}{\partial \phi} & K_P \frac{\partial f_2}{\partial \mu} \end{vmatrix} = K_P \frac{\partial f_2}{\partial \mu} \cdot \left(\frac{\partial f_2}{\partial \phi} \left(\frac{\partial f_2}{\partial \phi} - K_D \frac{\partial f_2}{\partial \mu} \right) - K_P \frac{\partial f_2}{\partial \mu} \right)$$

with its principal minors

$$(3.7) \quad H_1 = \left| \frac{\partial f_2}{\partial \phi} \right|$$

and

$$(3.8) \quad H_2 = \begin{vmatrix} \frac{\partial f_2}{\partial \phi} & K_P \frac{\partial f_2}{\partial \mu} \\ 1 & \frac{\partial f_2}{\partial \phi} - K_D \frac{\partial f_2}{\partial \mu} \end{vmatrix} = \frac{\partial f_2}{\partial \phi} \left(\frac{\partial f_2}{\partial \phi} - K_D \frac{\partial f_2}{\partial \mu} \right) - K_P \frac{\partial f_2}{\partial \mu}.$$

From $H_1 > 0$, we get the condition

$$(3.9) \quad \left(\frac{\partial f_2}{\partial \phi} \right) \neq 0.$$

Additionally, we get from $H_2 > 0$ the condition

$$(3.10) \quad \begin{aligned} & \frac{\partial f_2}{\partial \phi} \left(\frac{\partial f_2}{\partial \phi} - K_D \frac{\partial f_2}{\partial \mu} \right) - K_P \frac{\partial f_2}{\partial \mu} > 0 \\ \implies & \frac{\partial f_2}{\partial \phi} \left(\frac{\partial f_2}{\partial \phi} - K_D \frac{\partial f_2}{\partial \mu} \right) > K_P \frac{\partial f_2}{\partial \mu} > 0 \\ \implies & K_D \begin{cases} < \left(\frac{\frac{\partial f_2}{\partial \phi}}{\frac{\partial f_2}{\partial \mu}} \right) - \frac{K_P}{\frac{\partial f_2}{\partial \phi}} & \text{if } \text{sign} \left(\frac{\partial f_2}{\partial \mu} \right) = \text{sign} \left(\frac{\partial f_2}{\partial \phi} \right), \\ > \left(\frac{\frac{\partial f_2}{\partial \phi}}{\frac{\partial f_2}{\partial \mu}} \right) - \frac{K_P}{\frac{\partial f_2}{\partial \phi}} & \text{if } \text{sign} \left(\frac{\partial f_2}{\partial \mu} \right) \neq \text{sign} \left(\frac{\partial f_2}{\partial \phi} \right). \end{cases} \end{aligned}$$

From $H_3 > 0$, we get the additional condition

$$(3.11) \quad \begin{aligned} & K_P \frac{\partial f_2}{\partial \mu} > 0 \\ \implies & \text{sign}(K_P) = \text{sign} \left(\frac{\partial f_2}{\partial \mu} \right). \end{aligned}$$

It remains to study the signs of the partial derivatives of f_2 . We can take again the averaged linearization matrices like in subsection 3.2, and consider the entries of interest. We find that $\frac{\partial f_2}{\partial \phi} < 0$, which is a condition for stability. Additionally, we already know from the entries of the linearization in subsection 3.2 that $\frac{\partial f_2}{\partial \mu} < 0$ which provides the condition that K_P must be negative. Then, we can again compute $\frac{\partial f_2}{\partial \phi}$ by the difference quotient out of the measured

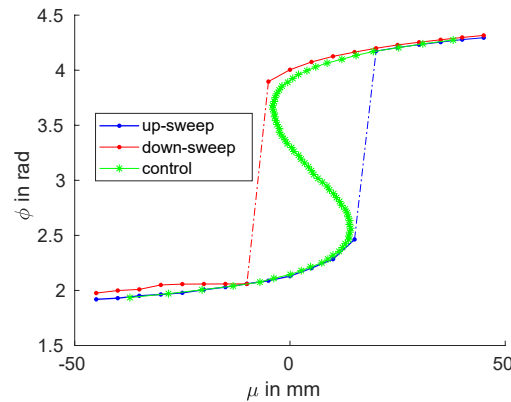


Figure 3. Parameter sweep for the Zeeman catastrophe machine with stable states compared to controlled stationary states. The blue curve is achieved by an upsweep and the red curve by a downsweep. Dashed lines indicate that there are no measurements for the jumps. The stationary points achieved by control (3.1) with control parameters $(K_P, K_D) = (-2, -3)$ are marked by a curve with green asterisks.

data and by taking the moving average over 10 neighboring stationary points. Finally, we can establish that $\frac{\partial f_2}{\partial \phi} / \frac{\partial f_2}{\partial \mu} > -1.8$ holds for all stationary points in the region of interest. Hence, the controlled system (3.1) is stable for $K_P < 0$ and $K_D < -1.8 + K_P / \frac{\partial f_2}{\partial \phi}$. Aiming for a safety margin for inexact assumptions concerning the model and noise in the experiment, led to the choice $K_P = -2$ and $K_D = -3$. The numerical evaluation of the analytical model (4.1) in section 4 confirms that this results in the desired stability information. The successful stabilization of unstable states in the experiment, i.e., the convergence towards the equilibrium points (see Figure 3), verifies additionally the stability of the controlled system (3.1).

3.4. Detecting stability and bifurcation points in the experiment. The asymptotically stable points of the uncontrolled experiment can be found as ω -limit sets without any control carrying out a quasi-stationary sweep. To find the stable branches of the system, we start with B at one of the edges of the rail for the motor. For each position $\bar{\mu}$, we let the system evolve until we cannot detect any movement, i.e., until the measured absolute value of the velocity of ϕ and hence of $\dot{\phi}$ is below a tolerance of $0.00122 \frac{\text{rad}}{\text{s}}$ ($= \frac{0.001^\circ}{0.0143 \text{s}}$). We take this position for the stationary point $(\bar{\phi}, 0, \bar{\mu})$. Then, we increase $\bar{\mu}$ by 5 mm in an upsweep and repeat the procedure until we reach the other edge of the rail. This slow change of parameters results in a quasi-stationary behavior. The second stable branch can be found in a downsweep by starting at the second edge and reducing $\bar{\mu}$ by 5 mm before letting the system run to the next stable point for the new value of $\bar{\mu}$. In Figure 3, the bistability, in particular, the hysteresis, of the system is visible. The two stable branches are separated by an unstable one.

The feedback control (2.4b) allows for finding the unstable stationary branch and the bifurcation points supplementing the sweep information. To localize the unstable branch, we start again with B on one edge of the rail, i.e., on the stable branch outside the bistable regime. Choosing $K_P = -2$ and $K_D = -3$ so that the controlled experimental system (3.1) is stable, the curve of the bifurcation diagram can be obtained by letting the system (3.1) evolve until a stationary point is reached. We aim to approach the curve of equilibria by controlling

the parameter μ by a control law given as the differential equation (2.4b). The equation for μ is an extension of the original system. The movement by changing μ brings the system to the branch of stationary points. By construction, the set of stationary points is identical to the one of the original system. So, we move B by the motor according to the discretized version of the scheme for the feedback control (2.4b):

$$(3.12) \quad \mu_{k+1,l} = \mu_{k,l} + \Delta t \left(-K_P(\phi_{k,l} - \phi_l^*) - K_D \left(\frac{\phi_{k+1,l} - \phi_{k,l}}{\Delta t} \right) \right)$$

with k indicating the number of updates for stabilizing the l th found stationary point $\tilde{\phi}_l$. As soon as the system has reached the stationary point, we increase the target ϕ^* by 0.0349 rad ($= 2^\circ$), and let the controlled system (3.1) evolve again and let it settle at the stationary point. The result is a curve containing two stable branches which are connected with a branch of unstable stationary points. For comparison, Figure 3 also shows the stable branches obtained by quasi-stationary up-and-down sweeps. There is a slight mismatch most likely due to the imprecision induced by the timing belt mechanism or missed transitions from the rotary encoder.

Since the stationary points of Figure 3 are reached by direct evolution of the controlled system (3.1), their asymptomatic stability is confirmed. The method proposed in subsection 2.5 can be used to detect the stability information and in particular the bifurcation points of the uncontrolled system. The result is presented in Figure 4. The eigenvalue information shown in Figure 4 turns out to be rather robust, even though the individual entries of the linearization matrix (2.13) are difficult to determine from experimental data. In the area of the assumed bifurcation point, the colors represent the largest real part of the eigenvalues with blue as negative, red as positive, and green as zero values. Additionally, the insets in Figure 4 show the maxima of the real parts of the eigenvalues depending on the angle ϕ . To extract stability information, sufficiently different states have to be contained in the time series. Therefore, the initial point should not be too close to the ω -limit set. If the tension of the timing belt is not sufficiently large, the resulting data contain further outliers compared to Figure 4 due to measurement noise and slackness in the experimental data.

For producing Figure 4(b), we took for the stability information the moving average of 2 neighboring stationary points, (ϕ_{l-1}, μ_{l-1}) and (ϕ_{l+1}, μ_{l+1}) , of Figure 4(a) to get a better estimate. The positions are the same as in Figure 4(b). To avoid giving too much weight to the noisy behavior close to the asymptotically stable point, we took only the first 40 time steps with $\Delta t = 1/70 \text{ s}$ (or less, if the system settled earlier) in the neighborhood of each equilibrium point.

During the course of the experiments, we noticed that some of the experimental limitations unaddressed by the control algorithm can become critical. The rotary encoder implies two restrictions: finite resolution due to the discretization of angles and limited speed of signal transitions or their detection. While there were no indications of any resolution-related effects, the rate of detection of signal transitions is particularly critical. Once such a transition is missed, the angle variable ϕ is offset, hence, the originally desired control target ϕ^* cannot be reached anymore. For faster systems (e.g., lower mass or stronger springs), a more performant signal processing hardware to read out the angular position is required. On the other hand, the digital angle readout essentially results in the absence of measurement noise. Since we

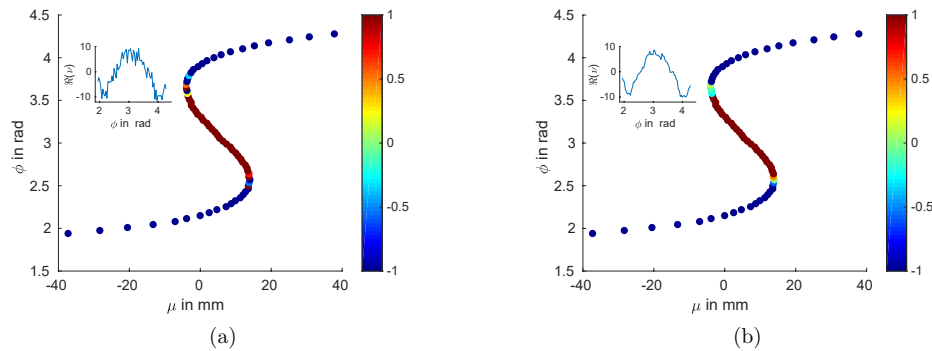


Figure 4. Stability evaluated in the controlled physical experiment by approximating G for $\dot{Z} = G\hat{Z} - b$ with the least-squares approach from subsection 2.5. Stability is visualized by color code for the largest real part of the eigenvalues of G in (2.13). The scale of color is limited to the interval $[-1, 1]$ to emphasize the behavior in the neighborhood of the bifurcation points. Insets show maxima of real parts of the eigenvalues against ϕ . (a) Stability information evaluated in each stationary point taking data points in the neighborhood of each stationary point. These points require a distance large enough to the stationary point such that they cover information on the dynamic behavior. (b) Stability information evaluated as a moving average including the 2 neighboring points of the equilibrium points in (a). The 2 neighboring points for the mean value in (b) define the interval $[\phi^* - \Delta\phi, \phi^* + \Delta\phi]$ with $\Delta\phi = 0.0349 \text{ rad} (= 2^\circ)$.

discovered very little noise in the case of a sufficiently tightened belt, its impact is not studied within this paper. Hence, when applying our approach to other systems or sensor devices, the control and the system identification method may have to be adapted to cope with noise.

A further experimental complication is the static friction, leading to zero angular velocity prior to reaching the target state. Particularly when frequently reaching static states (e.g., for the sweep with stable points with disabled control, see Figure 3) this leads to an imprecise determination of the angle for stationary points. In that case, we employed a simple scheme to induce additional motion for overcoming the static friction; see Appendix A.

Finally, the control parameters themselves are critical in practice, because the final state is only reached after several oscillatory periods around the stationary point. Thus, tracing the entire bifurcation diagram requires a long time, i.e., 15 minutes compared to a simple up-or-downsweep of 3 minutes. In section 2 we gave lower bounds for the absolute values of the control parameters K_P and K_D . Increasing the absolute values of $K = (K_P, K_D)$ leads to faster convergence to the ω -limit set. For physical experiments, there are upper bounds due to possible technical realizations (e.g., motors and belts). Larger values for the control parameters K_P and K_D speed up the process but at some point, the motion system can no longer follow the control output due to physical restrictions. Mechanically more sensitive, “real-world” systems may not allow for spending that time, e.g., due to changes in environmental parameters such as the temperature, continuously altering the characteristics of the system under investigation. This problem could be overcome by more sensitive measurement hardware. However, for a simple system such as the Zeeman machine, this is less of a problem.

4. Mathematical model for verification. The main target of this paper is to determine the stability of equilibrium points of a feedback-controlled system. Since in general no explicit

mathematical description is assumed to be available, all usable information must be taken from data. Nevertheless, in the case of the Zeeman catastrophe machine, there is a good model available for comparison between analytical and data-based results. This mathematical model serves for verification of the analysis and results from physical experiments and the obtained data in section 3.

The model can be derived from classical mechanics, e.g., in [19], using the Lagrange formalism. Modification of the geometry and the mass distribution of the disk change the trajectories towards the stationary point, whereas the positions of the stationary points remain unaffected by these alterations. Therefore, we assume for simplicity reasons a pointlike mass m at the point P which approximates the behavior of the disk. The friction of the joints at points A , B , and P as well as the friction in the rotational axis are summarized in the term $-d\dot{\phi}$. With mass m , this model is given as

$$(4.1a) \quad \frac{d\phi}{dt} = \dot{\phi},$$

$$(4.1b) \quad \frac{d\dot{\phi}}{dt} = \frac{k}{m\rho} \left(\frac{L_1(\phi) - L_0}{L_1(\phi)} \cdot a \cdot \sin \phi + \frac{L_2(\phi) - L_0}{L_2(\phi)} \cdot (\mu \cdot \cos \phi - \lambda \cdot \sin \phi) \right) - d \cdot \dot{\phi}$$

with the spring lengths

$$\begin{aligned} L_1(\phi) &= \sqrt{(\rho \cdot \cos \phi + a)^2 + (\rho \cdot \sin \phi)^2} \quad \text{and} \\ L_2(\phi) &= \sqrt{(\lambda - \rho \cdot \cos \phi)^2 + (\mu - \rho \cdot \sin \phi)^2}. \end{aligned}$$

Like in section 3, we have a 1-dimensional control with the equation

$$(4.2) \quad u = \dot{\mu} = -K_P(\phi - \phi^*) - K_D\dot{\phi}.$$

Details of controllability and stability analysis for this model can be found in Appendix B.

The following values are chosen for the model simulation in order to obtain a good comparison to the experiment described in section 3: spring constant $k = 5 \text{ N/m}$, mass $m = 0.3 \text{ kg}$, distance $\rho = 0.065 \text{ m}$ from the origin to the point P on the disk, distance $a = 0.244 \text{ m}$ from the origin to the fixed end A , distance $\lambda = 0.23 \text{ m}$ from the origin O to the movable end B , and length $L_0 = 0.055 \text{ m}$ of the springs. We assume $d = 1 \text{ s}^{-1}$ with damping constant $\delta = d \cdot m$. The control parameters $K_P = -2$ and $K_D = -3$ are chosen like for the experiment. This choice is derived and justified for unknown values in Appendices B.1 and B.2, and guarantees controllability and stability for the controlled model system (4.1) and (4.2).

To detect stability information on the underlying uncontrolled system (4.1), we search for the optimal approximation for the linearization in the least-squares sense with respect to experimental data proposed in subsection 2.5. The stability estimations obtained by the least-squares method are compared with the analytically gained information from the Jacobian of model (4.1). The results are shown in Figure 5. The largest real parts of the eigenvalues of the analytically derived linearization matrix are shown in Figure 5(a), whereas the results in Figure 5(b) are obtained from the least-squares solution (2.13) approximating the Jacobian for the numerically generated data of model (4.1) and (4.2). A comparison of Figure 5 with Figure 4

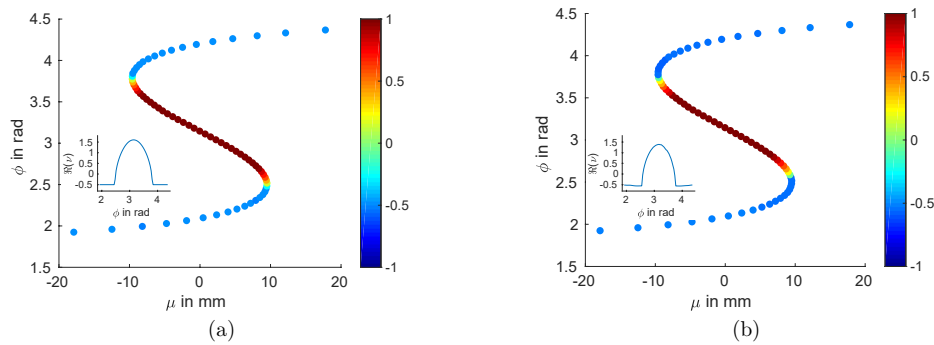


Figure 5. Stability evaluation for the original simulation model (4.1) with data from the system controlled by (4.2). Stability is visualized by color code for the largest real part of the eigenvalues of G . The scale of color is limited to the interval $[-1, 1]$ to emphasize the behavior in the neighborhood of the bifurcation points. Insets show maxima of real parts of the eigenvalues against ϕ . (a) Stability computed by the largest real part of eigenvalues of the linearization matrix for the analytical model (4.1). (b) Stability computed by eigenvalues from G for the least-squares approach $\hat{Z} = G\hat{Z} - b$ in subsection 2.5 with data points in the neighborhood of each stationary point. These points require a distance large enough to the stationary point such that they cover information on the dynamic behavior.

shows that despite its simplification, model (4.1) and (4.2) describes well the experimental behavior summarized in the bifurcation diagram Figure 4.

Because of the absence of noise in the numerical model, it is even possible to obtain the already known control parameter $K = (K_P, K_D) = (-2, -3)$ directly from the estimated linearization matrix (2.13) with a relative error of less than 2%.

5. Conclusion. This paper aims to find a method to first perform control-based continuation along stable and unstable branches of a bifurcation diagram, and second to determine the stability along these branches. While conventional feedback-controller methods are typically invasive and aim primarily for robustness or general applicability, the presented control scheme (2.4) is noninvasive and allows for tracking stationary points even around folds without initializing the full state, as is necessary e.g., for the Newton method.

It has been demonstrated for a laboratory experiment of the Zeeman catastrophe machine that the unstable branch of stationary points can be stabilized by the proposed method. Furthermore, it is even possible to estimate a full bifurcation diagram including stability information from experimental data through the proposed least-squares method. Bifurcation points are identified by a sign change of the real parts of the eigenvalues of the linearization. It is unnecessary to switch off the control or run the experiment at the borders of stable behavior to obtain stability information on the original unstabilized system. Especially for cases where the physical setup might be damaged when the system converges to an undesired ω -limit set, the presented approach enables obtaining stability information and locating bifurcation points.

In section 3, the approach to determine stability information has been applied successfully by a time series analysis of the controlled system in the experiment of the Zeeman catastrophe machine. The continuation begins on a stable branch which becomes unstable at a certain

point. The change of stability corresponds to a change of sign in the largest eigenvalue of the linearization matrix. This locates a bifurcation point. In the experiment of the Zeeman catastrophe machine, two saddle-node bifurcation points were located. As the method does not use any specific knowledge or properties of the dynamics, it could be applied to many other dynamical laboratory experiments. Hence, information on these systems is extended beyond traditional experimental access. In future works, systems with other types of bifurcations and other laboratory experiments could be considered under the discussed assumptions of subsection 2.3 (e.g., controllability and observability). For complex systems with unknown macroscopic description, the relevant macroscopic variables could be identified by e.g., proper orthogonal decomposition or diffusion maps.

Appendix A. Technical details of the experiment. The Zeeman machine is experimentally implemented using a commercial low-cost laser engraver kit (ETE ETMATE). It contains all necessary mechanical parts for 2-dimensional motion, including belt-driven linear stages, stepper motors, and a controller board. The latter provides stepper drivers and a microcontroller that takes care of motion path generation based on the open-source firmware Grbl [11]. The Zeeman control algorithm is implemented in Python and runs on a Raspberry Pi (RPi) single-board computer (Raspberry Pi 4 model B, 4 GB RAM) which is connected to the Grbl controller via a universal serial bus interface operating as a virtual serial port. This enables simple motion control via G-Code commands consisting of text strings and coding motion commands. An incremental rotary encoder (Sendix 5000, Fritz Kübler GmbH, 5000 steps per revolution [10]) is used as an optical angle sensor. It provides a two-channel digital signal (5 V, TTL compatible), resulting in two quadrature square waves. Each voltage level transition on either channel corresponds to a change of the angular state whose direction can be unambiguously determined from the quadrature phase, i.e., from the (unchanged) state of the other channel, respectively. Upon each such state change, the stored angle in the control program is updated correspondingly. The RPi input pins can only cope with low-voltage TTL (LVTTTL, 3.3 V) signals, hence a voltage divider is inserted between the rotary encoder and the RPi, acting as a unidirectional level shifter. Since no absolute angle readout is provided by the sensor, the mechanics of the Zeeman catastrophe machine have to be homed, i.e., driven to a defined starting position where the angle is set to a known value. The rotary encoder is mounted on a 3-dimensional-printed plastic holder made from polylactic acid (PLA). A measuring wheel with a radius of 79.6 mm (type 8.0000, Fritz Kübler GmbH) is attached directly to the rotary encoder, providing the distance $\rho = 65$ mm, because the pin for the point P on the disk cannot be installed directly on the edge of the disk (see Figure 2). In addition, the wheel provides enough inertia, thus it sufficiently slows down the mechanical response for real-time processing and communication. Two helical tension springs with tensionless length $L_0 = 55$ mm and spring constant $k = 5$ N/m are attached to the measuring wheel, one spring with length L_1 connected to a pin (point A) on a fixed 3-dimensional-printed PLA holder, and the other one with length L_2 to a 3-dimensional-printed PLA bracket mounted to the engraver head (point B). Point A and the origin, i.e., the center of the disk, have a distance of 24.4 mm, and point B has a distance λ in the x -direction to the origin of 23 mm. To reduce friction, all spring connections are lubricated with oil for bicycle chains.

Particular care has to be taken to enable reliable angle readout and sufficiently fast control of the Zeeman machine. To this end, threaded callbacks and event detection are used in the Python script for monitoring the rotary encoder state. This technique is related to interrupt routines frequently used in microcontroller-based sensor applications, and combines fast response with a low computational load. Furthermore, the predefined limits for acceleration and velocity have been pushed to the mechanical limits of the stepper motor, using the corresponding Grbl commands. In experiments with active control, the output is updated 70 times per second. The angle velocity $\dot{\phi}$ is estimated numerically by a simple difference quotient with time step $\Delta t = 0.0143$ s. When the distance $|\phi^* - \phi|$ between the measured angle and the target is below a tolerance of 0.00175 rad ($= 0.1^\circ$), the absolute value of the angular velocity is below a tolerance of $\frac{0.001^\circ}{0.0143\text{s}} \approx 0.00122 \frac{\text{rad}}{\text{s}}$, and the angle acceleration $\ddot{\phi}$ has an absolute value lower than $\frac{0.001^\circ \cdot \text{s}^{-1}}{0.0143\text{s}} \approx 0.00122 \frac{\text{rad}}{\text{s}^2}$, then the state is considered to be stationary, and the next control target $(\phi^*)_{k+1} = (\phi^*)_k + 0.0349$ rad ($= (\phi^*)_k + 2^\circ$) is defined.

Appendix B. Details of the analytical model.

B.1. Controllability of the model. Using $\frac{\partial f_2}{\partial \phi} = -d$ in (3.2), the linearization of the controlled model system (4.1) and (4.2) can be written in the matrix-vector form

$$(B.1) \quad \frac{d}{dt} \begin{pmatrix} \phi \\ \dot{\phi} \\ \mu \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial f_2}{\partial \phi} & -d & \frac{\partial f_2}{\partial \mu} \\ -K_P & -K_D & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \dot{\phi} \\ \mu \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial f_2}{\partial \phi} & -d & \frac{\partial f_2}{\partial \mu} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \dot{\phi} \\ \mu \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \dot{\mu}$$

with $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $(\phi, \dot{\phi})^T \mapsto (\dot{\phi}, \ddot{\phi}, \mu)^T$ describing the dynamics. The linearization (B.1) can be seen as the system (2.5) with the control vector $u := \dot{\mu}$, the state vector (3.3), the input matrix (3.4), and the state matrix

$$C = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial f_2}{\partial \phi} & -d & \frac{\partial f_2}{\partial \mu} \\ 0 & 0 & 0 \end{pmatrix}.$$

This results in the controllability matrix

$$R = [D \ C \ C D \ C^2 D] = \begin{pmatrix} 0 & c_{1,3} & c_{1,1}c_{1,3} + c_{1,2}c_{2,3} \\ 0 & c_{2,3} & c_{1,3}c_{2,1} + c_{2,2}c_{2,3} \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\partial f_2}{\partial \mu} \\ 0 & \frac{\partial f_2}{\partial \mu} & -d \cdot \frac{\partial f_2}{\partial \mu} \\ 1 & 0 & 0 \end{pmatrix}.$$

A numerical evaluation of the partial derivative (B.3) with realistic values $\lambda = 0.23$ m, $a = 0.244$ m, $\rho = 0.065$ m, $d = 1$ kg/s, $m = 0.3$ kg, $k = 5$ N/m, and $L_0 = 0.055$ m shows that $\frac{\partial f_2}{\partial \mu} < -15 < 0$ for $(\mu, \phi) \in [-0.15, 0.15] \times [1.7, 4.5]$. Therefore, $\text{rank}(R) = 3$, which is a sufficient condition for the controllability of the system at stationary points, as discussed in subsection 2.3. The rank condition is also satisfied for larger parameter intervals for μ and ϕ .

B.2. Analytical stability analysis of the mathematical model. For the Zeeman catastrophe machine, we have a model consisting of the 2-dimensional model system (4.1). We restrict the control to one dimension described by (4.2). Hence, the linearization matrix $\tilde{J}$ of the 2-dimensional system (4.1) with control (4.2) is

$$\tilde{J} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial f_2}{\partial \phi} & -d & \frac{\partial f_2}{\partial \mu} \\ -K_P & -K_D & 0 \end{pmatrix}.$$

For the stability analysis, we consider the Routh–Hurwitz criterion for the determinant of $\tilde{J}$. From subsection 3.3 we know the negative of the characteristic polynomial of $\tilde{J}$ is

$$-p(\nu) = \nu^3 + d\nu^2 + \left(K_D \frac{\partial f_2}{\partial \mu} - \frac{\partial f_2}{\partial \phi}\right)\nu + K_P \frac{\partial f_2}{\partial \mu}.$$

The Routh–Hurwitz criterion gives us the conditions (3.9)–(3.11). We know $\frac{\partial f_2}{\partial \phi} = -d < 0$, and hence get the stability conditions

$$(B.2a) \quad d \neq 0$$

$$(B.2b) \quad \Rightarrow K_D \begin{cases} < \left(\frac{\frac{\partial f_2}{\partial \phi}}{\frac{\partial f_2}{\partial \mu}}\right) + \frac{K_P}{d} & \text{if } \frac{\partial f_2}{\partial \mu} < 0, \\ > \left(\frac{\frac{\partial f_2}{\partial \phi}}{\frac{\partial f_2}{\partial \mu}}\right) + \frac{K_P}{d} & \text{if } \frac{\partial f_2}{\partial \mu} > 0, \end{cases} \quad \text{and}$$

$$(B.2c) \quad \text{sign}(K_P) = \text{sign}\left(\frac{\partial f_2}{\partial \mu}\right).$$

Condition (B.2a) is fulfilled because $d > 0$ is the damping factor. It remains to study the sign of $\frac{\partial f_2}{\partial \mu}$ and the possible values of $\frac{\left(\frac{\partial f_2}{\partial \phi}\right)}{\left(\frac{\partial f_2}{\partial \mu}\right)}$. We have

$$(B.3) \quad \begin{aligned} \frac{\partial f_2}{\partial \mu} &= \frac{k}{m\rho} \left(\frac{\partial}{\partial \mu} \left(\frac{L_2(\phi) - L_0}{L_2(\phi)} \right) \cdot (\mu \cdot \cos \phi - \lambda \cdot \sin \phi) + \frac{L_2(\phi) - L_0}{L_2(\phi)} \cdot \frac{\partial}{\partial \mu} (\mu \cdot \cos \phi - \lambda \cdot \sin \phi) \right) \\ &= \frac{k}{m\rho} \left(\frac{L_0}{L_2(\phi)^{5/2}} \cdot (\mu^2 \cdot \cos \phi - \lambda \cdot \mu \cdot \sin \phi - \mu \cdot \rho \cdot \cos \phi \cdot \sin \phi + \lambda \cdot \rho \cdot \sin^2 \phi) \right. \\ &\quad \left. + \frac{L_2(\phi) - L_0}{L_2(\phi)} \cdot \cos \phi \right). \end{aligned}$$

As we already know from subsection B.1, a numerical evaluation of the partial derivative (B.3) with the values from above shows that $\frac{\partial f_2}{\partial \mu} < 0$ for $(\mu, \phi) \in [-0.15, 0.15] \times [1.7, 4.5]$. So, we

already know that $K_P < 0$ must be chosen to guarantee stability. For the right choice of the control parameters to fulfill (B.2), we have to study

$$\frac{\partial f_2}{\partial \phi} = \frac{k}{m\rho} \cdot \left(-\frac{L_0 \cdot \rho \cdot (a \cdot \sin \phi)^2}{(L_1(\phi))^3} + \frac{L_1(\phi) - L_0}{L_1(\phi)} \cdot a \cdot \cos \phi - \frac{L_0 \cdot \rho \cdot (\lambda \cdot \sin \phi - \mu \cdot \cos \phi)^2}{(L_2(\phi))^3} - \frac{L_2(\phi) - L_0}{L_2(\phi)} \cdot (\mu \cdot \sin \phi + \lambda \cdot \cos \phi) \right).$$

A numerical evaluation of this term for the above-mentioned values $(\mu, \phi) \in [-0.15, 0.15] \times [1.7, 4.5]$ shows that $\frac{\partial f_2}{\partial \phi}$ is bounded and that for the control parameter, a value

$$(B.4) \quad K_D < \min_{(\mu, \phi)} \left\{ \frac{\left(\frac{\partial f_2}{\partial \phi} \right)}{\left(\frac{\partial f_2}{\partial \mu} \right)} \right\} + \frac{K_P}{d}$$

can always be found. Thus, each stationary point of the model system (4.1) can be stabilized by (4.2) with the control parameters K_P and K_D satisfying conditions (B.2c) and (B.4).

Appendix C. Description of variable names and quantities. A summary of variable names and quantities used within the paper can be found in Tables 1 and 2.

Table 1
Quantities with descriptions.

Parameter	Description	Value	Used in
Physical quantities			
Δt	time step for update	$1/70$ s	(3.12)
k	spring constant	5 N/m	(4.1) and (B.3)
m	equivalent point mass of the disk	0.3 kg	(4.1) and (B.3)
ρ	distance from P to O	0.065 m	(4.1) and (B.3)
L_0	length of springs without tension	0.055 m	(4.1) and (B.3)
L_1	length of spring to fixed end		(4.1) and (B.3)
L_2	length of spring to movable end		(4.1) and (B.3)
d	factor in damping constant δ	1 s ⁻¹	(4.1), (B.1), (B.2), and (B.4)
Indirect quantities			
x	state of dynamical system		(2.3), (2.4), (2.7), and (3.1)
x^*	control target for x		(2.4) and (2.7)
K	feedback gain	(K_P, K_D)	(2.4) and (2.7)
K_P	proportional control parameter	-2	(2.4), (2.8), (3.1)–(4.2), and (B.1)–(B.4)
K_D	differential control parameter	-3	(2.4), (2.8), (3.1)–(4.2), and (B.1)–(B.4)
O	origin	$(0, 0)$	
P	fixed point on edge of circular disk		
A	fixed end of spring	$(-a, 0)$	
a	negative x -coordinate of A	0.244 m	(4.1)
B	movable end of spring	(λ, μ)	
λ	x -coordinate of B	0.23 m	(4.1) and (B.3)
μ	y -coordinate of B	$[-0.2, 0.2]$ m	(2.3)–(2.7), (3.1)–(4.2), and (B.1)–(B.4)
ϕ	angle	$[\pi/2, 3\pi/2]$	(3.1)–(4.2) and (B.1)–(B.4)
ϕ^*	control target for ϕ	$[\pi/2, 3\pi/2]$	(3.1), (3.12), and (4.2)

Table 2
Auxiliary variables.

Parameter	Description	Used in
u	input (control) vector	(2.5) and (4.2)
v	state vector	(2.5) and (3.3)
C	state matrix	(2.5) and (2.6)
D	input matrix	(2.5), (2.6), and (3.4)
R	controllability matrix	(2.6)
J	linearization matrix of original dynamical system	(2.6) and (2.8)
$\tilde{J}$	linearization matrix of controlled dynamical system	(2.8)
G	least-squares linearization matrix	(2.9), (2.10), (2.12), and (2.13)
z	solution of $\dot{z} = Gz - b$	(2.9)
$\hat{z}$	measured data of controlled system	(2.12)
$\hat{Z}$	collection of data points	(2.10)–(2.12)
n	dimension of dynamical system	(2.12) and (2.13)
p	number of data points	(2.12)
Ψ	least-squares solution of $\dot{Z} = \Psi\Gamma$	(2.10)–(2.13)
Γ	coefficient matrix	(2.10)–(2.12)
H_1, H_2, H_3	Hurwitz determinants	(3.6)–(3.8)
ν	eigenvalue of H_3	(3.5)

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