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Periodic Character of a Difference Equation

ABSTRACT. In this note we prove that every positive solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{p + x_{n-1} + x_n}, \quad n = 0, 1...$$

where $p \in [0, \infty)$ and the initial conditions x_{-1}, x_0 are positive real numbers, converges to a, not necessarily prime, periodic-two solution. This result confirms Conjecture 7.5.2 in [1] (with $q = 1$). Also, we show that the positive solutions of Eq.(1) converge to the corresponding periodic-two solutions geometrically.

KEY WORDS AND PHRASES. Period two solution, difference equation, positive solution, asymptotics

1 Introduction

In this note we consider the periodic character of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{p + x_{n-1} + x_n}, \quad n = 0, 1, \dots \tag{1}$$

where $p \in [0, \infty)$ and the initial conditions x_{-1}, x_0 are positive real numbers. In fact we consider the case $p \in (0, 1)$ since when $p \geq 1$ the zero equilibrium of Eq.(1) is obviously global attractor of all positive solutions of Eq.(1), see [1, Theorem 7.4.1 *(a)*]. The case $p = 0$ was considered, for example, in [1, p. 61, (ii)].

Our motivation here stems from Conjecture 7.5.2 in [1]:

Conjecture 1 *Assume that*

$$p < 1$$

Show that every positive solution of Eq.(1) *converges to a, not necessarily prime, periodic-two solution.*

Note that when $p<1$ all prime period-two solutions of Eq.(1) are given by

$$...\phi, 1-p-\phi, \phi, 1-p-\phi, ...$$

with

$$0 \leq \phi \leq 1-p \quad \text{and} \quad \phi \neq \frac{1-p}{2},$$

see, [1, p. 134].

Recently there has been a great interest in studying the periodic nature of nonlinear difference equations. For some recent results concerning, among other problems, the periodic nature of scalar nonlinear difference equations see for example, [1, 2], [4]-[9] and references therein.

Our aim in this note is to confirm Conjecture 1. Also, we show that the positive solutions of Eq.(1) converge to the corresponding periodic-two solutions geometrically and we look for their asymptotics.

2 Main results

In this section we prove the main results in this note.

Theorem 1 *Consider the difference Eq.(1) where $p\in(0,1)$ and initial conditions x_{-1}, x_0 are positive real numbers. Then every positive solution of Eq.(1) converges to a, not necessarily prime, periodic-two solution (ρ_0, ρ_1), such that $p+\rho_0+\rho_1=1$. If $p+x_0+x_{-1}>1$ the sequences $x_{2n+i}, (i=1,2)$ are decreasing, if $p+x_0+x_{-1}<1$ the sequences $x_{2n+i}, (i=1,2)$ are increasing, and if $p+x_0+x_{-1}=1$ the sequence x_n is a periodic-two solution of Eq.(1).*

Proof: By the change of variables $x_n=\frac{1}{z_n}$, Eq.(1) becomes

$$z_{n+1}=\frac{z_n+z_{n-1}+pz_nz_{n-1}}{z_n}. \tag{2}$$

From (2) we have

$$\begin{aligned} z_{n+1}-z_{n-1} &= \frac{z_n+z_{n-1}+pz_nz_{n-1}-z_nz_{n-1}}{z_n} \\ &= \frac{z_n+z_{n-1}+pz_nz_{n-1}-z_{n-1}-z_{n-2}-pz_{n-1}z_{n-2}}{z_n} \\ &= \frac{(pz_{n-1}+1)(z_n-z_{n-2})}{z_n} \end{aligned}$$

and consequently

$$z_{n+1}-z_{n-1}=(z_1-z_{-1})\prod_{i=1}^{n}\frac{pz_{i-1}+1}{z_i}. \tag{3}$$

From (3) we obtain that the signum of $z_{n+1} - z_{n-1}$ remains invariant for $n \in \mathbf{N}$ and that the sequences (z_{2n+i}), $i = 0, 1$, are nondecreasing or nonincreasing at the same time which implies that the sequences (x_{2n+i}), $i = 0, 1$, are nonincreasing or nondecreasing at the same time. Since

$$z_1 - z_{-1} = \frac{p + x_0 + x_{-1} - 1}{x_{-1}}$$

we see from (3) that if $p + x_0 + x_{-1} < 1$, then the sequences (x_{2n+i}), $i = 0, 1$ are increasing, if $p + x_0 + x_{-1} > 1$, the sequences (x_{2n+i}), $i = 0, 1$ are decreasing and if $p + x_0 + x_{-1} = 1$, then $(x_{-1}, x_0, x_{-1}, x_0, \ldots)$ is a periodic-two solution of Eq.(1).

First suppose that the sequences (x_{2n+i}), $i = 0, 1$ are decreasing, that is $p + x_0 + x_{-1} > 1$. Then there are finite limits

$$\lim_{n\to\infty} x_{2n+i} = \rho_i, \quad i = 0, 1\,.$$

It is clear that (ρ_0, ρ_1) is a two cycle of Eq.(1). Suppose that both of them are equal to zero. Since (x_{2n+i}), $i = 0, 1$ are decreasing from (1) we obtain

$$p + x_{n-1} + x_n > 1, \quad n = 0, 1, \ldots\,. \tag{4}$$

Letting $n \to \infty$ in (4) we obtain $p \geq 1$ which is a contradiction. Hence $(\rho_0, \rho_1) \neq (0, 0)$ and as we mentioned above it is a two cycle of Eq.(1).

Without loss of generality we may assume that $\rho_1 \neq 0$. Then letting $n \to \infty$ in the equation

$$x_{2n+1} = \frac{x_{2n-1}}{p + x_{2n-1} + x_{2n}}$$

we obtain the equality $p + \rho_0 + \rho_1 = 1$.

Now suppose that the sequences (x_{2n+i}), $i = 0, 1$ are increasing, that is $p + x_0 + x_{-1} < 1$. Then there are finite or infinite limits

$$\lim_{n\to\infty} x_{2n+i} = \rho_i, \; i = 0, 1\,.$$

By a result of L. Berg [2, p. 1070] all solutions of Eq.(1) are bounded, hence $\rho_i < \infty$, $i = 0, 1$. On the other hand, since (x_{2n+i}), $i = 0, 1$ are increasing $\rho_0 > x_0 > 0$ and $\rho_1 > x_1 > 0$. Similarly as above we obtain that (ρ_0, ρ_1) is a two cycle of Eq.(1) and $p + \rho_0 + \rho_1 = 1$.

Finally, for the initial conditions $x_{-1} = x_0 = (1 - p)/2$, we have $x_n = (1 - p)/2, n \geq -1$, which shows that there is a solution which converges to a not prime period-two solution.

Remark 1 Note that the condition $p + x_0 + x_{-1} > 1$, (e.g. condition (4) for $n = 0$) implies (4) for all greater n, that is, for $n \geq 1$, moreover the sequence $u_n = p + x_{n-1} + x_n$ is also decreasing.

Also, the condition $p + x_0 + x_{-1} < 1$ and (1) imply that the sequence $u_n = p + x_{n-1} + x_n$ is increasing and

$$p + x_{n-1} + x_n < 1, \quad n = 0, 1, \dots .$$

From this and by Theorem 1 it follows that the distance from the point (x_{n-1}, x_n) to the limit line $p + x + y = 1$, i.e.,

$$d_n = \frac{p + x_n + x_{n-1} - 1}{\sqrt{2}},$$

also converges monotonously to zero (we use here Hesse's normal form).

For the readers who are interested in this area we leave the following open problem.

Open Problem 1 *Let*

$$\dots, \rho_0, 1 - p - \rho_0, \rho_0, 1 - p - \rho_0, \dots$$

be a positive two cycle of Eq.(1). *Find the basin of attraction of this two cycle.*

The following result gives an estimation of the convergence rate of the positive solutions of Eq.(1).

Theorem 2 *Every positive solution of Eq.*(1) *converges to the corresponding periodic-two solution* (ρ_0, ρ_1) *geometrically, that is, there is an* $M > 0$ *and* $q \in (0, 1)$ *such that*

$$|x_{2n} - \rho_0| + |x_{2n+1} - \rho_1| \le M q^{2n}, \quad n \ge 0.$$

Proof: As we have seen in the proof of Theorem 1, using the change $x_n = \frac{1}{z_n}$ we obtain

$$z_{n+1} - z_{n-1} = \frac{(p z_{n-1} + 1)(z_n - z_{n-2})}{z_n}.$$

If we go back to the sequence x_n we have

$$\frac{p + x_n + x_{n-1} - 1}{x_{n-1}} = \frac{p + x_{n-1}}{x_{n-1}} x_n \frac{p + x_{n-1} + x_{n-2} - 1}{x_{n-2}},$$

that is,

$$d_n = (p + x_{n-1}) \frac{x_n}{x_{n-2}} d_{n-1},$$

where $d_n = \frac{p + x_n + x_{n-1} - 1}{\sqrt{2}}$, and consequently

$$d_n = (p + x_{n-1}) \frac{x_n}{x_{n-2}} (p + x_{n-2}) \frac{x_{n-1}}{x_{n-3}} d_{n-2}. \tag{5}$$

Let $\varepsilon \in (0, (1-(1+p)^2/4))$. Since the sequences (x_{2n+i}), $i = 0, 1$, are convergent, from (5) we have that for such chosen ε there is an $n_0 \in \mathbf{N}$ such that

$$|d_n| \le (\,(p+\rho_0)(1-\rho_0)+\varepsilon\,)|d_{n-2}| \le \left(\left(\frac{1+p}{2}\right)^2+\varepsilon\right)|d_{n-2}|, \tag{6}$$

for every $n \ge n_0$.

In view of the choice of ε we see that $r = \left(\frac{1+p}{2}\right)^2 + \varepsilon < 1$. From this and (6), using the following equality

$$|d_{2n+1}| = \frac{|x_{2n+1}-\rho_1+x_{2n}-\rho_0|}{\sqrt{2}} = \frac{|x_{2n+1}-\rho_1|+|x_{2n}-\rho_0|}{\sqrt{2}}$$

we see that for $q = \sqrt{r}$ we can obtain the result easily. Note that in the last equality we have used the fact that the sequences x_{2n+i}, $i = 0, 1$, converge monotonously to ρ_i, $i = 0, 1$.

Corollary 1 *The distance d_n from the point (x_{n-1}, x_n) to the limit line $p+x+y=1$, converges to zero monotonously and geometrically.*

3 The case of nonnegative solutions of Eq.(1)

If $x_{-1} = 0$ or $x_0 = 0$, from (1) we obtain $x_{2n-1} = 0$ or $x_{2n} = 0$, for all $n \ge 0$. Further, if $x_{-1} = 0$ then Eq.(1) becomes

$$x_{2n} = \frac{x_{2n-2}}{p+x_{2n-2}}.$$

This is a Riccati equation (see [1, Section 1.6]) for x_{2n} with the elementary solution

$$x_{2n} = \frac{x_0(1-p)}{x_0+(1-p-x_0)p^n}, \qquad n \ge 0\,. \tag{7}$$

From (7) we see that $\lim_{n\to\infty} x_{2n} = 1-p$, so far as x_0 is different from 0. Similarly we can treat the case $x_0 = 0, x_{-1} \ne 0$. The case $x_0 = x_{-1} = 0$ yields the constant solution $x_n = 0$ for all $n \ge -1$.

We believe that only these solutions satisfy the condition $\rho_0\rho_1 = 0$, where as before ρ_i, $i = 0, 1$ denote the limits $\lim_{n\to\infty} x_{2n+i}$. Hence we leave the following conjecture:

Conjecture 1 ([3]) *For positive initial values x_{-1} and x_0 there are no solutions of Eq.(1) such that $\rho_0\rho_1 = 0$.*

4 Asymptotically two-periodic solutions

Theorem 2 motivated us to study the asymptotics of the solutions of Eq.(1), as well as the corresponding ones for the sequence d_n.

Let $u_n = x_{2n-1}$ and $v_n = x_{2n}$, then (1) can be written as the following system

$$u_{n+1} = \frac{u_n}{p + u_n + v_n}$$

$$v_{n+1} = \frac{v_n}{p + u_{n+1} + v_n}. \tag{8}$$

We expect that the asymptotically two-periodic solutions have the following form (see [2, p.1066])

$$u_n = \rho + \sum_{k=1}^{\infty} a_k c^k t^{nk}, \quad \text{and} \quad v_n = 1 - \rho - p + \sum_{k=1}^{\infty} b_k c^k t^{nk}\,, \tag{9}$$

where $t \in (0,1)$ is unknown and c an arbitrary real number.

Substituting (9) into system (8) and comparing the coefficients we obtain

$$a_1 t^n = a_1 t^{n+1} + \rho(a_1 + b_1)t^n, \quad \text{and} \quad b_1 t^n = b_1 t^{n+1} + (1 - \rho - p)(a_1 t^{n+1} + b_1 t^n),$$

which implies

$$(1 - t - \rho)a_1 = \rho b_1, \quad \text{and} \quad t(1 - \rho - p)a_1 = (\rho + p - t)b_1\,. \tag{10}$$

This system has a nontrivial solution a_1, b_1 if and only if its determinant vanishes, i.e

$$t^2 - (\,1 + p + \rho(1 - \rho - p)\,)t + (1 - \rho)(\rho + p) = 0\,. \tag{11}$$

The only solution of (11) with t contained in $(0,1)$ is $t = (1-\rho)(\rho+p)$ and the corresponding solution of system (10) is $a_1 = \rho$, $b_1 = (1 - \rho)(1 - \rho - p)$ up to a constant factor c which already appears in the series (9). Therefore

$$u_n = \rho + \rho c t^n + \mathcal{O}(t^{2n}), \quad \text{and} \quad v_n = 1 - \rho - p + (1 - \rho)(1 - \rho - p)ct^n + \mathcal{O}(t^{2n})\,. \tag{12}$$

The asymptotic formulas (12) for u_n and v_n remain valid in the limit cases $\rho = 0$ and $\rho = 1 - p$, with $t = p$, where they express the asymptotic behaviour of the explicitly known solutions with one vanishing initial value (see Section 3). Note that the asymptotic formulas (12) can also be obtained in the case $u_n = x_{2n}$ and $v_n = x_{2n+1}$. We leave the following conjecture:

Conjecture 2 *Let $p \in (0,1)$ and (x_n) be a nonnegative solution of Eq.(1) such that $(x_{2n-1}, x_{2n}) \to (\rho, 1 - \rho - p)$, as $n \to \infty$. Then*

(a) $x_{2n-1} = \rho + \rho c t^n + \mathcal{O}(t^{2n})$;

(b) $x_{2n} = 1 - \rho - p + (1-\rho)(1-\rho-p)ct^n + \mathcal{O}(t^{2n})$;

where $t = (1-\rho)(\rho+p)$ *and the constant* c *depends on initial values* x_{-1} *and* x_0.

If this conjecture is true, then it follows:

Corollary 2 *Let* $p \in (0,1)$ *and* (x_n) *be a nonnegative solution of Eq.*(1) *such that* $(x_{2n-1}, x_{2n}) \to (\rho, 1-\rho-p)$, *as* $n \to \infty$. *Then the distance* d_n *from the point* (x_{n-1}, x_n) *to the limit line* $p + x + y = 1$, *has the following asymptotics*

$$d_n = \frac{c}{\sqrt{2}} e_n (1-t)(\sqrt{t})^n + \mathcal{O}(t^n),$$

where $t = (1-\rho)(\rho+p)$, $e_{2n} = 1, e_{2n+1} = 1-\rho$, *for* $n \geq 0$, *and the constant* c *depends on initial values* x_{-1} *and* x_0.

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On the Dynamics of $x_{n+1} = (bx_{n-1}^2)(A + Bx_{n-2})^{-1}$

ABSTRACT. We investigate the boundedness, the global stability, and the periodic nature of the nonnegative solutions of the equation in the title with nonnegative parameters

KEY WORDS AND PHRASES. difference equations, boundedness, global asymptotic stability, semi-cycles

1 Introduction and Preliminaries

In this paper we consider the third order nonlinear rational difference equation

$$x_{n+1} = \frac{bx_{n-1}^2}{A + Bx_{n-2}}, \quad n = 0, 1, ... \tag{1}$$

where the parameters A, B, and b and the initial conditions x_{-2}, x_{-1} and x_0 are arbitrary non-negative real numbers. We investigate the boundedness, the global stability and the periodic nature of the solutions of the Eq. (1).

Recently there has been a great interest in studying rational [1, 2, 5, 7] and nonrational nonlinear difference equations [3, 5, 8, 9, 10, 11, 12], see also the references therein. Some of the results recently obtained in this field can be applied in studying some mathematical biology models, population dynamic etc., see [3, 4, 12].

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}) \tag{2}$$

with $x_{-2}, x_{-1}, x_0 \in I$ (where I is some interval of real numbers).

The *linearized equation* of Eq. (2) about an equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = c_1 y_n + c_2 y_{n-1} + c_3 y_{n-2}, \quad n = 0, 1, ... \tag{3}$$

where

$$c_1 = \frac{\partial f}{\partial x}(\bar{x}, \bar{x}, \bar{x}), \quad c_2 = \frac{\partial f}{\partial y}(\bar{x}, \bar{x}, \bar{x}), \quad c_2 = \frac{\partial f}{\partial z}(\bar{x}, \bar{x}, \bar{x})$$

The characteristic equation of Eq. (3) is

$$\lambda^3 - c_1\lambda^2 - c_2\lambda - c_3 = 0. \tag{4}$$

Theorem A ([6, Theorem 1]) *(Linearized Stability Theorem) The following statements are true.*

a) *If all roots of Eq. (4) have modulus less than one, then the equilibrium $\bar{x}$ of Eq. (2) is locally asymptotically stable.*

b) *If at least one of the roots of Eq. (4) has modulus greater than one, then the equilibrium $\bar{x}$ of Eq. (2) is unstable.*

A necessary and sufficient condition for all roots of Eq. (4) to have modulus less than one is the following:

$$|c_1 + c_3| < 1 - c_2, \quad |c_1 - 3c_3| < 3 + c_2, \quad \text{and} \quad c_3^2 - c_2 - c_1c_3 < 1.$$

In this case, the locally asymptotically stable equilibrium $\bar{x}$ is called a *sink*.

The equilibrium $\bar{x}$ of Eq. (2) is called a *saddle point* if there exits a root of Eq. (4) with absolute value less than one and a root of Eq. (4) with absolute value greater than one. In particular a saddle point equilibrium is unstable.

2 The case $AB = 0$

In this section we shortly discuss the case when one of the parameters in the Eq. (1) is zero, where we have the following two nontrivial cases:

$$x_{n+1} = \frac{bx_{n-1}^2}{Bx_{n-2}}, \quad n = 0, 1, ... \tag{5}$$

$$x_{n+1} = \frac{b}{A}x_{n-1}^2, \quad n = 0, 1, ... \tag{6}$$

In each of the above equations we assume that all parameters in the equations are positive. Equations (5) and (6) are nonlinear third and second order respectively, and the change of variables $x_n = e^{y_n}$ reduce the equations (5) and (6) to a third and second order linear difference equation respectively, which can be solved. The details we leave to the reader.

3 Main Results

In this section we investigate the dynamics of Eq. (1) under the assumption that all parameters in Eq. (1) are positive and the initial conditions are nonnegative.

The change of variables $x_n = \frac{A}{B} y_n$ reduces Eq. (1) to the difference equation

$$y_{n+1} = \frac{r y_{n-1}^2}{1 + y_{n-2}}, \quad n = 0, 1, \dots \tag{7}$$

where $r = \frac{b}{B}$.

It is easy to see that $\bar{y}_1 = 0$ is always an equilibrium point and when $r > 1$ we have also a positive equilibrium point $\bar{y}_2 = \frac{1}{r-1}$.

3.1 An Oscillation Result

Lemma 1 *Assume that $r > 1$ and let $\{y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (7) such that either*

$$y_{-2}, y_0 \geq \bar{y}_2 \quad \textit{and} \quad y_{-1} < \bar{y}_2 \tag{8}$$

or

$$y_{-2}, y_0 < \bar{y}_2 \quad \textit{and} \quad y_{-1} \geq \bar{y}_2 \tag{9}$$

then $\{y_n\}_{n=-2}^{\infty}$ oscillates about $\bar{y}_2$ with semi-cycle of length one.

Proof: We will assume that (8) holds. The case where (9) holds is similar and will be omitted. From (7) we obtain

$$y_1 = \frac{r y_{-1}}{1 + y_{-2}} < \frac{r \bar{y}_2}{1 + \bar{y}_2} = \bar{y}_2 \quad \text{and} \quad y_2 = \frac{r y_0}{1 + y_{-1}} > \frac{r \bar{y}_2}{1 + \bar{y}_2} = \bar{y}_2.$$

Using induction the result follows.

3.2 Existence of Prime Period-Two Solutions

In this subsection, we show that Eq. (7) has prime period-two solutions.

Theorem 2 *Eq. (7) has eventually nonnegative prime period-two solutions if and only if either*

$$y_{-1} = 0 \quad \textit{and} \quad y_0 = \frac{1}{r} \tag{10}$$

or

$$y_0 = 0 \quad \textit{and} \quad \frac{y_{-1}^2}{1 + y_{-2}} = \frac{1}{r^2}, \tag{11}$$

the period-two solution must be in the form

$$\dots, 0, \frac{1}{r}, 0, \frac{1}{r}, \dots \,. \tag{12}$$

Proof: Assume that

$$\dots, \phi, \psi, \phi, \psi, \dots$$

is a nonnegative prime period-two solution of Eq. (7).
Then

$$\phi = \frac{r\phi^2}{1+\psi} \quad \text{and} \quad \psi = \frac{r\psi^2}{1+\phi} \ . \tag{13}$$

Hence $\phi - \psi = r(\phi^2 - \psi^2)$, and consequently

$$\phi + \psi = \frac{1}{r}. \tag{14}$$

From Eqs.(13) and (14) we get the period-two solution in form (12). If $y_{2k+1} = 0$ for some $k \in \mathbf{N}$ then from (7), it follows that $y_{2n-1} = 0$, $n = 0, 1, \dots$, $y_{2n} = 1/r$, $n = 1, 2, \dots$, and y_{-2} is arbitrary. If $y_{2l} = 0$ for some $l \in \mathbf{N}$, then $y_{2n} = 0$, $n = 1, 2, \dots$, $y_{2n-1} = 1/r$, $n = 1, 2, \dots$, and $\frac{ry_{-1}^2}{1+y_{-2}} = y_1 = \frac{1}{r}$, as desired.

3.3 Local and Global Stability

As we have already noted $\bar{y}_1 = 0$ is always an equilibrium solution of Eq. (7). Furthermore when $r > 1$, Eq. (7) also possesses the positive equilibrium $\bar{y}_2 = \frac{1}{r-1}$.

Theorem 3 *Consider Eq. (7). Then the following results hold:*

(i) *The zero equilibrium point is locally asymptotically stable.*

(ii) *Assume that $r > 1$ then the equilibrium point $\bar{y}_2 = \frac{1}{r-1}$ is unstable. In particular $\bar{y}_2$ is a saddle point.*

Proof: The linearized equation associated with Eq. (7) about $\bar{y}_i$, $i = 1, 2$, has the form

$$z_{n+1} - \frac{2r\bar{y}_i}{1+\bar{y}_i} z_{n-1} + \frac{r\bar{y}_i^2}{(1+\bar{y}_i)^2} z_{n-2} = 0 \quad , \qquad n = 0, 1, \dots \ .$$

So the linearized equation of Eq. (7) about $\bar{y}_1 = 0$ is $z_{n+1} = 0$, $n = 0, 1, \dots$, and the characteristic equation about $\bar{y}_1 = 0$ is $\lambda^3 = 0$ so proof of (i) follows immediately from Theorem A.

The linearized equation of Eq. (7) about $\bar{y}_2 = \frac{1}{r-1}$ is $z_{n+1} = 2z_{n-1} - \frac{1}{r} z_{n-2}$, $\quad n = 0, 1, \dots$, and the characteristic equation is

$$\lambda^3 - 2\lambda + \frac{1}{r} = 0, \quad \text{with} \quad r > 1.$$

Set

$$f(\lambda) = \lambda^3 - 2\lambda + \frac{1}{r}. \tag{15}$$

Then $f(1) = -1 + \frac{1}{r} < 0$ and $\lim_{\lambda \to +\infty} f(\lambda) = +\infty$, so $f(\lambda)$ has at least a zero in $(1, \infty)$ and the product of the moduli of the zeros of the function f is $\frac{1}{r} < 1$, hence there exists a root in the unit disk. This completes the proof.

Theorem 4 *The zero equilibrium point of Eq. (7) is globally asymptotically stable relative to the set*

$$S = [0, \infty) \times [0, 1/r]^2 \setminus A. \tag{16}$$

where

$$A = \{(x, y, z) | (y, z) = (0, 1/r) \quad or \quad \big(y^2/(1 + x), z\big) = (1/r^2, 0)\},$$

with $(y_{-2}, y_{-1}, y_0) \in S$.

Proof: By Theorem 3 we know that $\bar{y}_1 = 0$ is locally asymptotically stable equilibrium point of Eq. (7), and so it suffices to show that $\bar{y}_1 = 0$ is a global attractor of Eq. (7) relative to S. So let $\{y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (7), such that $(y_{-2}, y_{-1}, y_0) \in S$. We show that $\lim_{n\to\infty} y_n = 0$. We have

$$y_1 = \frac{ry_{-1}^2}{1 + y_{-2}} \leq ry_{-1}^2 \leq y_{-1} \leq \frac{1}{r}$$

$$y_2 = \frac{ry_0^2}{1 + y_{-1}} \leq ry_0^2 \leq y_0 \leq \frac{1}{r}.$$

By induction we obtain

$$0 \leq y_{n+1} = \frac{ry_{n-1}^2}{1 + y_{n+2}} \leq ry_{n-1}^2 \leq y_{n-1} \leq \frac{1}{r}, \quad n = 0, 1, ...,$$

that is, $0 \leq y_n \leq \frac{1}{r}$, $n = -1, 0, 1, ...,$ and $\{y_{2n}\}_{n=-1}^{\infty}$ and $\{y_{2n-1}\}_{n=0}^{\infty}$ are non-increasing and bounded. Hence, there are finite limits

$$\lim_{n\to\infty} y_{2n} = M \quad \text{and} \quad \lim_{n\to\infty} y_{2n-1} = L,$$

moreover, in view of (16), we have

$$M, L \in [0, 1/r). \tag{17}$$

Letting $n \to \infty$ in (7) we obtain

$$M = \frac{rM^2}{1 + L} \quad \text{and} \quad L = \frac{rL^2}{1 + M}.$$

Now, we want to prove that $M = L = 0$. We consider the following cases:
(i) If $M = 0$ and $L \neq 0$ then $L = \frac{1}{r}$, which is a contradiction to (17).
(ii) If $M \neq 0$ and $L = 0$, then $M = \frac{1}{r}$, a contradiction.
(iii) If $M \neq 0$ and $L \neq 0$, then we have

$$1 + L = rM \quad \text{and} \quad 1 + M = rL$$

which implies $L - M = r(M - L)$. Hence $M = L = 1/(r - 1)$, which is a contradiction. Thus $L = M = 0$, as desired.

3.4 Existence of Unbounded Solutions

In this subsection we show that when $r > 1$ Eq. (7) possesses unbounded solutions.

Theorem 5 *Assume that $r > 1$. Then Eq. (7) possesses unbounded solution. In particular, every solution of Eq. (7) which oscillate about the equilibrium $\bar{y}_2 = \frac{1}{r-1}$ with semi-cycle of length one is unbounded.*

Proof: We prove that every solution $\{y_n\}_{n=-2}^{\infty}$of Eq. (7) which oscillates with semi-cycles of length one is unbounded (see Lemma 1 for the existence of such solutions). Let $r > 1$ and without loss of generality that $\{y_n\}_{n=-2}^{\infty}$ is such that

$$y_{2n-1} < \bar{y}_2 \quad \text{and} \quad y_{2n} > \bar{y}_2 \quad \text{for} \quad n \geq 0.$$

Then

$$y_{2n+2} = \frac{ry_{2n}^2}{1 + y_{2n-1}} > \frac{ry_{2n}^2}{1 + \frac{1}{r-1}} = (r-1)y_{2n}^2 > y_{2n}.$$

and

$$y_{2n+3} = \frac{ry_{2n+1}^2}{1 + y_{2n-2}} < \frac{ry_{2n+1}^2}{1 + \frac{1}{r-1}} = (r-1)y_{2n+1}^2 < y_{2n+1}$$

from which it follows that there are $\lim_{n\to\infty} y_{2n} = M$ and $\lim_{n\to\infty} y_{2n+1} = m \in [0, \bar{y}_2)$. If $M = \infty$, there is nothing to prove. Hence, assume that $M < \infty$. As in the proof of Theorem 3 we can see that $m \neq 0$ and $M < \infty$ is impossible. If $m = 0$ and $M < \infty$ then $M = \frac{1}{r} < \frac{1}{r-1} = \bar{y}_2$, a contradiction. Hence $M = \infty$, from which the result follows.

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On the dual König property of the order-interval hypergraph of a new class of poset

ABSTRACT. Let P be a finite poset. We consider the hypergraph $\mathcal{H}(P)$ whose vertices are the elements of P and whose edges are the maximal intervals of P. It is known that $\mathcal{H}(P)$ has the König and dual König properties for the class of series-parallel posets. Here we introduce a new class which contains series-parallel posets and for which the dual König property is satisfied. For the class of N-free posets, again a generalization of series-parallel posets, we give a counterexample to see that the König property is not satisfied.

1 Introduction

Let P be a finite poset. A subset I of P of the form $I = \{v \in P : p \leq v \leq q\}$ (denoted $[p, q]$) is called an interval. It is maximal if p (resp. q) is a minimal (resp. maximal) element of P. Denote by $\mathcal{I}(P)$ the family of maximal intervals of P. The hypergraph $\mathcal{H}(P) = (P, \mathcal{I}(P))$, briefly denoted $\mathcal{H} = (P, \mathcal{I})$, whose vertices are the elements of P and whose edges are the maximal intervals of P is said to be the *order-interval hypergraph of* P. The *line-graph* $L(\mathcal{H})$ of $\mathcal{H}$ is a graph whose vertices are points $e_1, \dots, e_m$ representing the edges $I_1, \dots, I_m$ of $\mathcal{H}$, the vertices e_i, e_j being adjacent iff $I_i \cap I_j \neq \emptyset$. The dual $\mathcal{H}^*$ of the order-interval hypergraph $\mathcal{H}$ is a hypergraph whose vertices $e_1, \dots, e_m$ correspond to intervals of P and whose edges are $X_i = \{e_j : x_i \in I_j\}$.

Let α, ν, τ and ρ be the independence, matching, edge-covering and vertex-covering number of a hypergraph $\mathcal{H}$, respectively. $\mathcal{H}$ has the König property if $\nu(\mathcal{H}) = \tau(\mathcal{H})$ and it has the dual König property if $\nu(\mathcal{H}^*) = \tau(\mathcal{H}^*)$, i.e $\alpha(\mathcal{H}) = \rho(\mathcal{H})$ since $\alpha(\mathcal{H}) = \nu(\mathcal{H}^*)$ and $\rho(\mathcal{H}) = \tau(\mathcal{H}^*)$. This class of hypergraphs has been studied intensively in the past and one finds interesting results from an algorithmic point of view as well as min-max relations [2]-[6], [9].

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A poset P is said to be a series-parallel poset, if it can be constructed from singletons using only two operations: disjoint sum and linear sum. It may be characterized by the fact that it does not contain the poset N of Figure 3 as an induced subposet [13], [14].

Let P be a finite poset. The graph $G_P = (P, E_P)$, with $xy \in E_P$ if $x < y$ or $y < x$ is the comparability graph of the poset P. $G = (V, E)$ is a comparability graph if there is a poset P such that $G \sim G_P$.

It is known that the cographs, i.e. graphs without an induced path of length 4, are comparability graphs of series-parallel posets [7]. The cographs belong to the class of distance-hereditary graphs, which has been studied in graph theory [7]. A possible definition of a distance-hereditary graph is as follows: G is a distance-hereditary graph iff G has no induced gem, house, hole (cycle of length at least 4) and domino (see Figure 1).

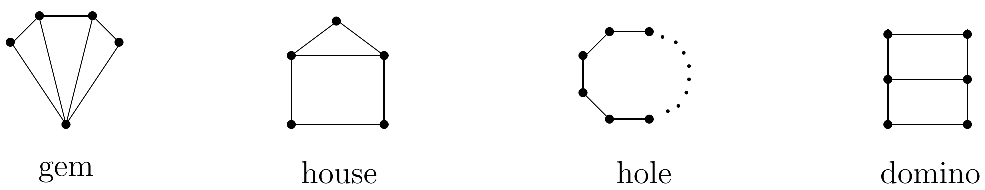

Figure 1

We investigate a class of posets that contains the series-parallel posets and whose members have comparability graphs which are distance-hereditary graphs or generalizations of them.

A poset P is in the class $\mathcal{Q}$ (resp. $\mathcal{Q}'$) if it has no induced subposet isomorphic to P_1, P_2, P_3 (resp. P_1, P_2, P_3, P_4) of Figure 2 and their duals, where P_3 has n vertices, $n \geq 6$. Obviously the class $\mathcal{Q}'$ is included in $\mathcal{Q}$. We prove that if P is in $\mathcal{Q}$, then $\mathcal{H}(P)$ has the dual König property.

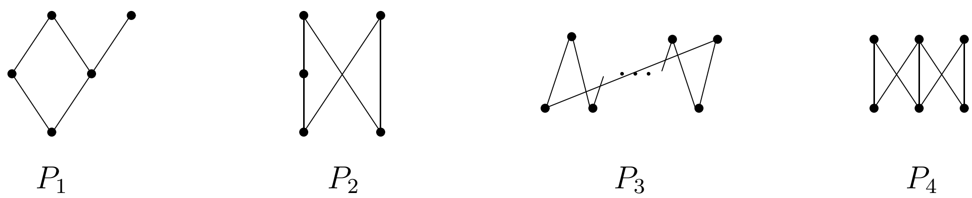

Figure 2

We characterize the comparability graphs of the class of posets in $\mathcal{Q}'$ in terms of four forbidden subgraphs.

Proposition 1 *Let G be the comparability graph of the poset P. Then G contains no induced gem, house, domino and even hole if and only if $P \in \mathcal{Q}'$.*

Proof: We prove this result in four steps.

Step 1. The graph G contains no induced gem if and only if P contains neither P_1 nor P_1^* as an induced subposet. Indeed, assume that P has an induced P_1 (resp. P_1^*) and let x, y, z, t, u be the elements of P_1 such that $x < y < z > t > x$ and $t < u$ (resp. $t > u$). We immediately deduce a gem with edges xy, xz, xt, xu, yz, zt and tu (resp. zy, zx, zt, zu, yx, xt and tu) of G. Conversely, suppose that the graph G has an induced gem whose edges are xy, xz, xt, xu, yz, zt and tu. The subgraph of G induced by $\{x, y, z\}$ (resp. $\{x, z, t\}$) is a triangle, hence x, y, z (resp. x, z, t) form a chain of P. As $yt \notin E$, we obtain only six possibilities: $z < y < x > t > z$ or $x < y < z > t > x$ or $y, t < z < x$ or $t, y < x < z$ or $x < z < y, t$ or $z < x < y, t$. In virtue of the existence of the triangle induced by $\{x, u, t\}$, we infer that the first case gives $z < y < x > t > u, z$, the second $z > y > x < t < z, u$, the third $t < u < x > z > t, y$, the fifth $t > u > x < z < y, t$, without another comparability relation, and the fourth and sixth lead to a contradiction. Hence, we have obtained in each case either P_1 or P_1^*.

Step 2. The graph G contains no induced house if and only if P contains no P_2 as an induced subposet. Indeed, assume that P has an induced P_2 and let x, y, z, t, u be the elements of P_2 such that $x < y < z > t < u > x$. We immediately deduce a house with edges xy, yz, xz, zt, tu and ux of G. Conversely, suppose that G has an induced house whose edges are xy, yz, xz, zt, tu and ux. Since $xy \in E$, the elements x and y are comparable. First, assume that $x < y$. As $yz \in E$, we have $y < z$ or $z < y$. In fact $z < y$ leads to a contradiction. To see this, note that if $z < y$ holds, then $z < x < y$ or $x < z < y$. In the first case, from $ux \in E$, we deduce $u > x$, i.e. $u > z$, or $u < x$, i.e. $u < y$, both impossible since uz and uy are not edges of E. In the second case, $zt \in E$ implies $z < t$, i.e., $x < t$ or $z > t$, i.e. $t < y$, both impossible since xt and ty are not edges of E. Hence $z < y$ is impossible. From $tz \in E$ and $yt \notin E$, we obtain $x < y < z > t$ and these are the only comparability relations. As $tu \in E$ and $uz \notin E$, we deduce $t < u$. Finally, the only possibility for the relation between x and u is $x < u$. Hence, P_2 is obtained as an induced subposet. Adopting the same argument for $y < x$, we obtain P_2 as an induced subposet with the ordering $y < x < z > t < u > y$.

Step 3. It is easy to see that G contains no induced even hole if and only if P does not contain a P_3 as an induced subposet.

Step 4. The graph G contains no induced domino if and only if P contains no P_4 as an induced subposet. Indeed, assume that P has an induced P_4 and let x, y, z, t, u, v be elements of P_4 such that $x < t, u$ and $y < t, u, v$ and $z < u, v$. We immediately deduce a domino with edges xt, ty, yv, vz, zu, ux and uy of G. Conversely, suppose that G has an induced domino whose edges are xt, ty, yv, vz, zu, ux and uy. Hence, $ux, uy, uz \in E$ and $xy, yz, xz \notin E$ lead to $x, y, z < u$ or $u < x, y, z$ with $x \parallel y$, $y \parallel z$ and $x \parallel z$. We consider only the first possibility because the other may be settled by duality. From $yt \in E$ (resp. $yv \in E$) and $ut \notin E$ (resp. $uv \notin E$), we obtain $y < t$ (resp. $y < v$). For the remaining edges xt and zv, we have only the possibilities $x < t$ and $z < v$. Obviously, there are no other comparability relations between these elements. □

By Proposition 1, the comparability graph of a poset in $\mathcal{Q}'$ is a distance-hereditary graph, because the comparability graph of any poset cannot contain an odd hole: Each transitive orientation of an odd hole contains two consecutive arcs xy and yz which imply the chord xz.

In order to prove the dual König property of $\mathcal{H}(P)$ when P is in the class $\mathcal{Q}$, let us introduce two observations. We recall that the vertices of the line-graph $L(\mathcal{H}^*(P))$ are the points of P and two vertices are adjacent iff they belong to the same interval of P.

Observation 1 *Assume that P has no induced subposet isomorphic to P_1 and P_1^*. Let $u, v, w \in P$ with $u \parallel v$. If there exist two intervals I and I' such that $u, v \in I$ and $v, w \in I'$, then $u \in I'$.*

Proof: Let $I = [p, q]$ and $I' = [p', q']$. If $u \notin I'$, then $u \not< q'$ or $p' \not< u$. In the first case, the poset induced by $\{p, u, v, q, q'\}$ and P_1 are isomorphic. In the second case, the poset induced by $\{p, p', u, v, q\}$ and P_1^* are isomorphic, both impossible. □

By Observation 1, one can say that the existence of two edges uv and vw of the line-graph $L(\mathcal{H}^*(P))$ with the above mentioned properties enables us to affirm that uw is an edge, too.

Observation 2 *Assume that P has no induced subposet isomorphic to P_1, P_1^* and P_3. Let the 'zig zag' $u_1 < u_2 > u_3 < \cdots > u_{i-1} < u_i$, be given by i elements of P, linking u_1 to u_i, where i is even, $i \geq 6$. If u_1 and u_i belong to the same interval of P, then there exists at least another comparability relation between $u_1, \ldots, u_i$, different from $u_1 < u_i$ and $u_i < u_1$.*

Proof: If $u_1 > u_i$, then $u_1 > u_{i-1}$. If $u_1 \parallel u_i$, then from Observation 1, $u_i, u_2 \in I_1$, where I_1 is the interval containing u_1 and u_2. If $u_1 < u_i$, then there exists at least another comparability relation between $u_1, \ldots, u_i$, different from $u_1 < u_i$ and $u_i < u_1$, because otherwise the poset induced by $\{u_1, u_2, \ldots, u_i\}$ and P_3 would be isomorphic. □

Theorem 1 *Let $\mathcal{H}(P)$ be the order-interval hypergraph of a poset P of the class $\mathcal{Q}$. Then the line-graph $L(\mathcal{H}^*(P))$ is perfect.*

Proof: It is enough to verify that the line-graph $L(\mathcal{H}^*(P))$ is a Meyniel graph, i.e. each cycle of odd length at least 5 has at least two chords. Meyniel [11] proved the perfectness of Meyniel graphs.

Let $\mathcal{C} = (a_1, \ldots, a_k)$ be a cycle of odd length k, $k \geq 5$. Let us denote by $I_i = [p_i, q_i]$ the interval of P containing both a_i and a_{i+1} and by $I = [p, q]$ the interval of P containing both a_1 and a_k.

Case 1. $a_1 \parallel a_2$. From Observation 1, we have $a_1a_3 \in I_2$ and $a_2a_k \in I$.

Case 2. $a_1 < a_2$. We distinguish three subcases:

Case 2.1. $a_2 \parallel a_3$. From Observation 1, we have $a_1, a_3 \in I_1$ and $a_2, a_4 \in I_3$.

Case 2.2. $a_2 < a_3$. We immediately deduce the existence of the chord a_1a_3 of $\mathcal{C}$. Let us determine another chord.

Case 2.2.1. $a_3 < a_4$ or $a_3 \parallel a_4$. Then a_2a_4 is a chord of $\mathcal{C}$. Indeed, $a_3 < a_4$ implies $a_2 < a_4$ and from Observation 1, $a_3 \parallel a_4$ leads to $a_2, a_4 \in I_2$.

Case 2.2.2. $a_3 > a_4$. Then a_3a_5 is a chord of $\mathcal{C}$ if $a_4 > a_5$ or $a_4 \parallel a_5$. Indeed, $a_4 > a_5$ implies $a_3 > a_5$ and from Observation 1, $a_4 \parallel a_5$ leads to $a_3, a_5 \in I_3$.

Now let $a_4 < a_5$. In the case $k = 5$, we have three possibilities: If $a_1 > a_5$ or $a_1 \parallel a_5$, then a_2a_5 is a chord of $\mathcal{C}$. Indeed, $a_1 > a_5$ implies $a_2 > a_5$ and from Observation 1, $a_1 \parallel a_5$ leads to $a_2, a_5 \in I_1$. If $a_1 < a_5$, then we must have another comparability relation between the elements a_1, a_2, a_3, a_4, a_5, i.e. the existence of a new chord, because otherwise the poset induced by $\{a_1, a_2, a_3, a_4, a_5\}$ and P_2 would be isomorphic. In the case $k > 5$, consider the 'zig zag' $a_1 < a_3 > a_4 < a_5 > \cdots > a_{i-1} < a_i$ linking a_1 to a_i where i is a maximum odd integer, $5 \leq i \leq k$. If $i = k$, i.e a_1, a_i are in the same interval of P, we use Observation 2 to affirm the existence of the second chord. If $i < k$, we have again three possibilities:

If $a_{i+1} > a_i$ or $a_{i+1} \parallel a_i$, then $a_{i-1}a_{i+1}$ is a chord of $\mathcal{C}$. Indeed, $a_i < a_{i+1}$ implies $a_{i+1} > a_{i-1}$ and from Observation 1, $a_i \parallel a_{i+1}$ leads to $a_{i-1}, a_{i+1} \in I_{i-1}$. If $a_{i+1} < a_i$, the cases $a_{i+1} > a_{i+2}$ and $a_{i+1} \parallel a_{i+2}$ give a new chord a_ia_{i+2} since $a_{i+2} < a_{i+1}$ implies $a_{i+2} < a_i$ and from Observation 1, $a_{i+1} \parallel a_{i+2}$ implies $a_i, a_{i+2} \in I_i$.

Case 2.3. $a_2 > a_3$. We distinguish three subcases:

Case 2.3.1. $a_3 \parallel a_4$. From Observation 1, $a_2, a_4 \in I_2$ and $a_3a_5 \in I_4$.

Case 2.3.2. $a_3 > a_4$. Then a_2a_4 is a chord of $\mathcal{C}$ since $a_4 < a_3 < a_2$. Now, if $a_4 > a_5$ or $a_4 \parallel a_5$, we deduce the chord a_3a_5 since $a_4 > a_5$ implies $a_3 > a_5$ and from Observation 1, $a_4 \parallel a_5$ implies $a_3, a_5 \in I_3$. If $a_4 < a_5$, then the corresponding part of this case in Case 2.2.2. remains valid here by considering the 'zig zag' $a_1 < a_2 > a_4 < a_5 > \cdots > a_{i-1} < a_i$.

Case 2.3.3. $a_3 < a_4$. If $a_4 < a_5$, then $a_3 < a_5$, i.e. a_3a_5 is a chord of $\mathcal{C}$. For obtaining the second chord, we continue as in Case 2.2.2 (from the same situation $a_4 < a_5$). Here the 'zig zag' is $a_1 < a_2 > a_3 < a_5 > a_6 < \cdots > a_{i-1} < a_i$. If $a_4 \parallel a_5$, then from Observation 1, we have on the one hand $a_3, a_5 \in I_3$. On the other hand $a_1, a_4 \in I$ if $k = 5$ and $a_4, a_6 \in I_5$

otherwise. If $a_4 > a_5$ and $k = 5$, then either $a_5 < a_1$ (resp. $a_1 < a_5$) or $a_1 \parallel a_5$. If $a_5 < a_1$ (resp. $a_1 < a_5$), not only $a_5 < a_2$ (resp. $a_1 < a_4$), i.e. a_2a_5 (resp. a_1a_4) is a chord of $\mathcal{C}$ but again, it must exist another comparability relation between elements $a_1, \ldots, a_5$ because otherwise, the poset induced by these elements and P_2 would be isomorphic. If $a_1 \parallel a_5$, we have by Observation 1, $a_2, a_5 \in I_1$ and $a_1, a_4 \in I_4$, hence a_2a_5 and a_1a_4 are chords of $\mathcal{C}$.

If $a_4 > a_5$ and $k > 5$, consider the 'zig zag' $a_1 < a_2 > a_3 < \cdots < a_{i-1} > a_i$, where i is a maximum odd integer, $5 \leq i < k$.
If $i = k$, we have either, $a_1 > a_i$ (resp. $a_1 < a_i$) or $a_1 \parallel a_i$. If $a_1 > a_i$ (resp. $a_1 < a_i$), a_2a_i (resp. a_1a_{i-1}) is a chord of $\mathcal{C}$. Moreover there exists another comparability relation between elements $a_2, \ldots, a_i$ (resp. $a_1, \ldots, a_{i-1}$) because otherwise the poset induced by these elements and P_3 would be isomorphic. If $a_1 \parallel a_i$, by Observation 1, $a_1, a_{i-1} \in I_{i-1}$ and $a_2, a_i \in I_1$.
If $i < k$, we have three subcases:

Case 2.3.3.1. $a_{i+1} \parallel a_i$. Then from Observation 1, $a_{i-1}, a_{i+1} \in I_{i-1}$ and $a_i, a_{i+2} \in I_{i+1}$.

Case 2.3.3.2. $a_{i+1} < a_i$. We immediately deduce $a_{i+1} < a_{i-1}$, i.e. the chord $a_{i-1}a_{i+1}$ of $\mathcal{C}$.

If $a_{i+1} > a_{i+2}$, then a_ia_{i+2} is a chord of $\mathcal{C}$. If $a_{i+1} \parallel a_{i+2}$, then from Observation 1, $a_ia_{i+2} \in I_i$. If $a_{i+1} < a_{i+2}$, we continue as in Case 2.2.2. with the zig zag' $a_1 < a_2 > \cdots < a_{i-1} > a_{i+1} < a_{i+2}$.

Case 2.3.3.3. $a_{i+1} > a_i$. If $a_{i+1} < a_{i+2}$, then $a_i < a_{i+2}$ and hence a_ia_{i+2} is a chord of $\mathcal{C}$. In the case $k = i + 2$, we have either $a_1 < a_{i+2}$ which leads to the existence of another comparability relation between the elements $a_1, \ldots, a_i, a_{i+2}$, i.e. a new chord, since otherwise the poset induced by these elements and P_3 would be isomorphic, or $a_1 > a_{i+2}$ or $a_1 \parallel a_{i+2}$. These last possibilities give the chord a_1a_{i+1} of $\mathcal{C}$ because $a_{i+2} < a_1$ implies $a_{i+1} < a_1$ and from Observation 1, $a_1 \parallel a_{i+2}$ implies $a_1, a_{i+1} \in I_{i+1}$.
In the case $i + 2 < k$, we consider the 'zig zag' $a_1 < a_2 > \cdots < a_{i-1} > a_i < a_{i+2}$ and we continue as in Case 2.2.2. by substituting the elements $a_3, \ldots, a_{i-2}, a_{i-1}, a_i$ by $a_2, \ldots, a_{i-1}, a_i, a_{i+2}$, respectively.
If $a_{i+1} \parallel a_{i+2}$, then from Observation 1, $a_i, a_{i+2} \in I_i$ and $a_1, a_{i+1} \in I$ (resp. $a_{i+1}, a_{i+3} \in I_{i+2}$) if $k = i + 2$ (resp. $k > i + 2$).

Case 3. $a_2 < a_1$. By duality, this case is similar to Case 2.

Finally, we have obtained in each case at least two chords of $\mathcal{C}$ and the proof is complete. $\square$

Let $\mathcal{H} = (E_1, \ldots, E_m)$ be a hypergraph. We say that $\mathcal{H}$ has the *Helly property* or is a *Helly hypergraph* if every intersecting family of $\mathcal{H}$ is a star, i.e. for $J \subset \{1, \ldots, m\}$, $E_i \cap E_j \neq \emptyset$, for $i, j \in J$, implies $\cap_{j \in J} E_j \neq \emptyset$. A good characterization of a Helly hypergraph, due to Berge and Duchet [1], is given by the following property:

For any three vertices a_1, a_2, a_3 the family of edges containing at least two of the vertices a_i

has a non-empty intersection.

Theorem 2 *Let $\mathcal{H}(P)$ be the order-interval hypergraph of a poset P which has no induced subposet isomorphic to P_1 and P_1^*. Then $\mathcal{H}^*(P)$ is a Helly hypergraph.*

Proof: In the class of order-interval hypergraphs of posets, $\mathcal{H}^*(P)$ is a Helly hypergraph if and only if $\mathcal{H}(P)$ is a Helly hypergraph [5]. Consequently, we can verify this property for the hypergraph $\mathcal{H}(P)$.

Let $\mathcal{I} = \{I_1, \dots, I_m\}$ be the family of maximal intervals of P. We suppose that there exist three vertices a_1, a_2, a_3 of P such that $\cap_{j \in J} I_j = \emptyset$ where $J = \{j : |I_j \cap \{a_1, a_2, a_3\}| \geq 2\}$. Hence, $|J| \geq 3$ and there exists three edges, say w.l.o.g $I_1 = [p_1, q_1]$, $I_2 = [p_2, q_2]$, $I_3 = [p_3, q_3]$, such that:

$$\begin{aligned} a_2, a_3 \in I_1 \quad &\textit{and} \quad a_1 \notin I_1 \\ a_1, a_3 \in I_2 \quad &\textit{and} \quad a_2 \notin I_2 \\ a_1, a_2 \in I_3 \quad &\textit{and} \quad a_3 \notin I_3 \end{aligned}$$

From Observation 1, we have $a_1 \in I_1$ if $a_1 \parallel a_2$, and $a_2 \in I_2$ if $a_1 < a_2$ and $a_2 \parallel a_3$. If $a_1 < a_2$ and $a_2 < a_3$, we have immediately $a_2 \in I_2$. Again, we obtain $a_3 \in I_3$, if $a_1 < a_2$ and $a_3 < a_2$. Indeed, we must have $a_1 \parallel a_3$ because $a_1 < a_3$ (resp. $a_3 < a_1$) implies $p_3 < a_1 < a_3 < a_2 < q_3$ (resp. $p_1 < a_3 < a_1 < a_2 < q_1$), i.e. $a_3 \in I_3$ (resp. $a_1 \in I_1$). Moreover, $p_2 \neq p_3$, because otherwise $p_3 = p_2 < a_3 < a_2 < q_3$ and hence, $a_3 \in I_3$. Consequently, the poset induced by $\{p_2, p_3, a_1, a_3, a_2\}$ and P_1^* are isomorphic. By duality, the remaining case, namely $a_2 < a_1$, leads to a contradiction as well. □

A hypergraph $\mathcal{H}$ is said to be normal if every partial hypergraph $\mathcal{H}'$ has the coloured edge property, i.e. it is possible to colour the edges of $\mathcal{H}'$ with $\triangle(\mathcal{H}')$ colours, where $\triangle(\mathcal{H}')$ represents the maximum degree of $\mathcal{H}'$. Several sufficient conditions exist for a hypergraph to have the König property [1]. One of them is its normality. A hypergraph $\mathcal{H}$ is normal iff it satisfies the Helly property and the line-graph $L(\mathcal{H})$ is a perfect graph. This characterization enables us to derive the following corollary.

Corollary 3 *Let $\mathcal{H}(P)$ be the order-interval hypergraph of a poset P of the class $\mathcal{Q}$. Then every subhypergraph of $\mathcal{H}(P)$ has the dual König property.*

Proof: By Theorem 1 and Theorem 2, $\mathcal{H}^*(P)$ is normal and consequently every partial hypergraph is again normal. As the dual of a partial hypergraph of $\mathcal{H}^*(P)$ is a subhypergraph of $\mathcal{H}(P)$, we deduce that every subhypergraph of $\mathcal{H}(P)$ has the dual König property. □

2 N-free posets

Another natural and interesting generalization of series-parallel posets is the class of N-free poset. A poset is called N-free iff its Hasse-diagram does not contain the N from

Figure 3 as an induced subgraph [12], i.e. if there do not exist vertices $v_1, \dots, v_4$ such that $v_1 \prec v_3 \succ v_2 \prec v_4$ and $v_1 \parallel v_4$.

There is a characterization of series-parallel posets within the class of N-free posets [8]. It states that a poset P is a series-parallel iff P is N-free and does not contain the poset N' of Figure 3 as an induced subposet.

N N'

Figure 3

Unfortunately, if the poset P is N-free, the König property is not satisfied in general. The poset of Figure 4, gives a counterexample since $\nu(\mathcal{H}(P_1)) = 1$ and $\tau(\mathcal{H}(P_1)) = 2$. Moreover, $\mathcal{H}^*(P)$ is not normal. To see this, consider the poset P_2 of Figure 4. The line-graph $L(\mathcal{H}^*(P))$ contains an induced odd cycle C_5 given by the vertices $\{2, 3, 4, 12, 13\}$ and hence $L(\mathcal{H}^*(P))$ is not perfect.

P_1 P_2

Figure 4

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Strong Convergence and pseudo Stability for Operators of the ϕ-accretive type in uniformly smooth Banach Spaces

ABSTRACT. Let X be a uniformly Banach space and let $T : X \to X$ be a ϕ-strongly quasi-accretive operator. It is proved that, under suitable conditions, the Ishikawa iterative process with errors both converges strongly to the unique zero of T and is pseudo stable. A few related results deal with the convergence and stability of the Ishikawa iterative process with errors to the solutions of the equations $Tx = f$ and $x + Tx = f$, respectively, when $T : X \to X$ is ϕ-strongly accretive. Our results extend, improve, and unify the results due to Chidume [2], [3] and Zhou [18].

KEY WORDS AND PHRASES. Ishikawa iterative process with errors, ϕ-strongly quasi-accretive operator, ϕ-strongly accretive operator, stability, uniformly smooth Banach space.

1 Introduction

Let X be a Banach space with norm $\|\cdot\|$ and the dual space X^*. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \mathrm{Re}\langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is known that if X is uniformly smooth, then J is single valued and is uniformly continuous on any bounded subset of X.

The symbols $D(T), R(T), F(T), N(T)$ stand for the domain, the range, the fixed point set and the kernel of T, respectively, where $N(T) = \{x \in D(T); Tx = 0\}$.

Let $T : D(T) \subseteq X \to X$ be an operator and I denote the identity mapping on X.

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Definition 1.1 (i) *T is called to be strongly accretive if there exists a constant $k \in (0,1)$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying*

$$Re\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2;$$

(ii) *T is said to be ϕ-strongly accretive if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying*

$$Re\langle Tx - Ty, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|;$$

(iii) *T is said to be ϕ-strongly quasi-accretive if $N(T) \neq \emptyset$ and if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x \in D(T)$ and $y \in N(T)$, there exists $j(x-y) \in J(x-y)$ satisfying*

$$Re\langle Tx, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|.$$

The classes of operators appearing Definition 1.1 have been used and studied by several authors (see, e.g., [1]-[4], [8], [10], [12]-[16], [18]). It is known that the classes of strongly accretive operators and ϕ-strongly accretive operators with a nonempty kernel are proper subclasses of the classes of ϕ-strongly accretive operators and ϕ-strongly quasi-accretive operators, respectively.

Let us recall the following iterative schemes due to Mann [11], Ishikawa [9] and Liu [10], respectively.

Definition 1.2 (i) *Let $D(T)$ be a convex subset of X with $D(T) = R(T)$. For any given $x_0 \in D(T)$, the sequence $\{x_n\}_{n=0}^{\infty}$ in $D(T)$ defined by*

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n, \;\; x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, n \geq 0$$

is called the Ishikawa iteration sequence, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ satisfying certain conditions;

(ii) *If $\beta_n = 0$ for all $n \geq 0$ in (i), then the sequence $\{x_n\}_{n=0}^{\infty}$ in $D(T)$ defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \;\; n \geq 0,$$

is called the Mann iterative sequence;

(iii) *For any given $x_0 \in D(T)$, the sequence $\{x_n\}_{n=0}^{\infty}$ in $D(T)$ defined by*

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n + v_n, \;\; x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n + u_n, \;\; n \geq 0,$$

is called the Ishikawa iteration sequence with errors, where $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are sequences in X and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ satisfying suitable conditions;

(iv) *If, $\beta_n = \|v_n\| = 0$ for all $n \geq 0$ in (iii), then the sequence $\{x_n\}_{n=0}^{\infty}$ in $D(T)$ now defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n + u_n,\ n \geq 0,$$

is called the Mann iteration sequence with errors.

It is clear that the Ishikawa and Mann iterative sequences are all special cases of the Ishikawa iterative sequences with errors.

Let $T : X \to X$ be an operator and $\{\alpha_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$. Assume that $x_0 \in X$ and $x_{n+1} = f(T, \alpha_n, x_n)$ defines an iteration scheme which produces a sequence $\{x_n\}_{n=0}^{\infty} \subset X$. Suppose that, furthermore, that $F(T) \neq \emptyset$ and that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q \in F(T)$. Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X and define $\{\varepsilon_n\}_{n=0}^{\infty} \subset [0, \infty)$ by $\varepsilon_n = \|y_{n+1} - f(T, \alpha_n, y_n)\|$.

Definition 1.3 (i) *The iterative scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, \alpha_n, x_n)$ is called T-stable if $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$;*

(ii) *The iterative scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, \alpha_n, x_n)$ is called almost T-stable if $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = q$;*

(iii) *The iterative scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, \alpha_n, x_n)$ is called pseudo T-stable if $\lim_{n\to\infty} \alpha_n = 0$ and $\varepsilon_n = o(\alpha_n)$ implies that $\lim_{n\to\infty} y_n = q$.*

Osilike [16] pointed out that T-stability implies almost T-stability, and the converse does not hold in general. Clearly, an iteration scheme $\{x_n\}_{n=0}^{\infty}$ which is T-stable is pseudo T-stable. In section 2, we shall show that an iteration which is pseudo T-stable may fail to be T-stable.

Several researchers proved that the Mann iterative scheme, Ishikawa iterative scheme, the Mann iterative scheme with errors and Ishikawa iterative scheme with errors can be used to approximate solutions of the equations $Tx = f$ and $x + Tx = f$, where T is continuous strongly accretive or continuous ϕ-strongly accretive operators (see, e.g. [2]-[4], [12], [15], [18]).

Rhoades [17] obtained that the Mann and Ishikawa iterative schemes may exhibit different behaviors for different classes of nonlinear mappings. Several stability results for certain classes of nonlinear mappings have been established by a few researchers (see, e.g. [5]-[7], [13], [14], [16]). Harder and Hicks [7] revealed the importance of investigating the stability of various iteration schemes for various classes of nonlinear mappings. In [13], [14] and [16], Osilike established the stability and almost stability of certain Mann and Ishikawa iteration procedures for the classes of Lipschitz strongly accretive operators and Lipschitz ϕ-strongly accretive operators in real q-uniformly smooth Banach spaces and real Banach spaces, respectively.

For ϕ-strongly quasi-accretive operators without Lipschitz assumption, the possibility of establishing corresponding stability results has not been explored yet within our knowledge.

The aim of this paper is to establish the strong convergence and pseudo stability of the Ishikawa iterative scheme with errors to zeros of ϕ-strongly quasi-accretive operators in uniformly smooth Banach spaces. A few related results deal with the strong convergence and pseudo stability of the Ishikawa iterative scheme with errors to solutions of the equation $Tx = f$ and $x + Tx = f$, respectively, where $T : X \to X$ is ϕ-strongly accretive. The convergence results in this paper are generalizations and improvements of the corresponding results due to Chidume [2], [3] and Zhou [18].

We shall make use of the following result.

Lemma 1.1 ([1]) *Let X be a Banach space. Then for all $x, y \in X$, $j(x+y) \in J(x+y)$*

$$\|x + y\|^2 \leq \|x\|^2 + 2\mathrm{Re}\langle y,\ j(x + y)\rangle.$$

2 Main results

Theorem 2.1 *Let X be a uniformly Banach space and let $T : X \to X$ be a ϕ-strongly quasi-accretive operator. Suppose that the range $(I - T)$ of either or T is bounded and that $S = I - T$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ and $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are sequences in X satisfying*

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \|v_n\| = 0; \tag{2.1}$$

$$\sum_{n=0}^{\infty} \alpha_n = \infty; \tag{2.2}$$

$$\|u_n\| = o(\alpha_n). \tag{2.3}$$

Suppose that $\{x_n\}_{n=0}^{\infty}$ is the sequence generated from arbitrary $x_0 \in X$ by

$$z_n = (1 - \beta_n)x_n + \beta_n S x_n + v_n, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S z_n + u_n, \quad n \geq 0. \tag{2.4}$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique zero q of T and it is pseudo $(I - T)$-stable.

Proof: Since T is ϕ-strongly quasi-accretive, it follows that $N(T)$ is a singleton, say, $\{q\}$. It is easy to see that S has a unique fixed point q, and that

$$\mathrm{Re}\langle Sx - q, j(x - q)\rangle \leq \|x - q\|^2 - \phi(\|x - q\|)\|x - q\|, \quad x \in X. \tag{2.5}$$

Now we show that $R(S)$ is bounded. In fact, if $R(I-T)$ is bounded, so is $R(S)$; if $R(T)$ is bounded, then

$$\|Sx\| \leq \|x-q\| + \|q\| + \|Tx\| \leq \phi^{-1}(\|Tx\|) + \|q\| + \|Tx\|$$

for all $x \in X$. That is, $R(S)$ is bounded. Using (2.1) and (2.3), we conclude that there exists a nonnegative sequence $\{r_n\}_{n=0}^{\infty}$ such that

$$\|u_n\| = r_n\alpha_n, \quad n \geq 0; \tag{2.6}$$

$$\lim_{n\to\infty} r_n = 0. \tag{2.7}$$

Let $A = \text{diam}R(S) + \|x_0 - q\|$ and $B = A + \sup\{\|v_n\| : n \geq 0\} + \sup\{r_n : n \geq 0\}$. Next we show by induction that

$$\|x_n - q\| \leq A + \sup\{r_n : n \geq 0\} \leq B, \quad n \geq 0. \tag{2.8}$$

Obviously, (2.8) is true for $n = 0$. Suppose that (2.8) is true for some $n \geq 0$. It follows from (2.4) and (2.6) that

$$\begin{aligned} \|x_{n+1} - q\| \leq& (1-\alpha_n)\|x_n - q\| + \alpha_n\|Sz_n - q\| + \|u_n\| \\ \leq& (1-\alpha_n)[A + \sup\{r_n : n \geq 0\}] + \alpha_n A + \alpha_n r_n \\ \leq& A + \sup\{r_n : n \geq 0\}. \end{aligned}$$

Hence (2.8) is true for all $n \geq 0$.

In view of (2.4) and (2.8), we infer that

$$\begin{aligned} \|z_n - q\| \leq& (1-\beta_n)\|x_n - q\| + \beta_n\|Sx_n - q\| + \|v_n\| \\ \leq& (1-\beta_n)[A + \sup\{r_n : n \geq 0\}] + \beta_n A + \|v_n\| \\ \leq& B \end{aligned} \tag{2.9}$$

for all $n \geq 0$. It follows from Lemma 1.1, (2.4), (2.8) and (2.9) that

$$\begin{aligned} \|z_n - q\|^2 =& \|(1-\beta_n)(x_n - q) + \beta_n(Sx_n - q) + v_n\|^2 \\ \leq& (1-\beta_n)^2\|x_n - q\|^2 + 2\beta_n \text{Re}\langle Sx_n - q, j(z_n - q)\rangle \\ & + 2\text{Re}\langle v_n, j(z_n - q)\rangle \\ \leq& (1-\beta_n)^2\|x_n - q\|^2 + 2B^2B\beta_n + 2B\|v_n\| \end{aligned} \tag{2.10}$$

for all $n \geq 0$. Using Lemma 1.1, (2.4)-(2.6) and (2.8)-(2.10), we get that

$$\begin{aligned}
\|x_{n+1}-q\|^2 =&\|(1-\alpha_n)(x_n-q)+\alpha_n(Sz_n-q)+u_n\|^2 \\
\leq&(1-\alpha_n)^2\|x_n-q\|^2+2\alpha_n\mathrm{Re}\langle Sz_n-q, j(x_{n+1}-q)\rangle \\
&+2\mathrm{Re}\langle u_n, j(x_{n+1}-q)\rangle \\
\leq&(1-\alpha_n)^2\|x_n-q\|^2+2\alpha_n\mathrm{Re}\langle Sz_n-q, j(z_n-q)\rangle \\
&+2\alpha_n\mathrm{Re}\langle Sz_n-q, j(x_{n+1}-q)-j(z_n-q)\rangle+2B\|u_n\| \\
\leq&(1-\alpha_n)^2\|x_n-q\|^2+2\alpha_n[\|z_n-q\|^2-\phi(\|z_n-q\|)\|z_n-q\|] \\
&+2\alpha_n B\|j(x_{n+1}-q)-j(z_n-q)\|+2B\|u_n\| \\
\leq&[(1-\alpha_n)^2+2\alpha_n(1-\beta_n)^2]\|x_n-q\|^2+4B\alpha_n\beta_n+4B^2B\alpha_n\|v_n\| \\
&-2\alpha_n\phi(\|z_n-q\|)\|z_n-q\| \\
&+2\alpha_n B\|j(x_{n+1}-q)-j(z_n-q)\|+2B\|u_n\| \\
\leq&\|x_n-q\|^2-2\alpha_n\phi(\|z_n-q\|)\|z_n-q\|+\alpha_n t_n
\end{aligned} \tag{2.11}$$

for all $n \geq 0$, where

$$t_n = B^2\beta_n+4B\beta_n+4B\|v_n\|+2B\|j(x_{n+1}-q)-j(z_n-q)\|+2Br_n, \quad n \geq 0.$$

Since j is uniformly continuous on each bounded subset of X and

$$\begin{aligned}
\|x_{n+1}-q-(z_n-q)\| \leq&\alpha_n\|x_n-Sz_n\|+\beta_n\|x_n-Sx_n\|+\|u_n\|+\|v_n\| \\
\leq&2B(\alpha_n+\beta_n)+\|u_n\|+\|v_n\| \to 0
\end{aligned}$$

as $n \to \infty$, it follows that $\lim_{n\to\infty}\|j(x_{n+1}-q)-j(z_n-q)\| = 0$. Thus, by (2.1), (2.6) and (2.7) we have

$$\lim_{n\to\infty} t_n = 0. \tag{2.12}$$

Put $\inf\{\|z_n-q\| : n \geq 0\} = r$. We claim that $r = 0$. Otherwise $r > 0$. Thus (2.12) ensures that there exists a positive integer N such that $t_n \leq \phi(r)r$ for all $n \geq N$. From (2.11) we obtain that for all $n \geq N$,

$$\begin{aligned}
\|x_{n+1}-q\|^2 \leq&\|x_n-q\|^2-2\alpha_n\phi(r)r+\alpha_n\phi(r)r \\
\leq&\|x_n-q\|^2-\alpha_n\phi(r)r,
\end{aligned}$$

which implies that

$$\phi(r)r\sum_{n=N}^{\infty}\alpha_n \leq \sum_{n=N}^{\infty}(\|x_n-q\|^2-\|x_{n+1}-q\|^2) = \|x_N-q\|^2.$$

That is, $\sum_{n=0}^{\infty} \alpha_n < \infty$ contradicting (2.2). Therefore $r = 0$. Thus there exists a subsequence $\{\|z_{n_k} - q\|\}_{k=0}^{\infty}$ of $\{\|z_n - q\|\}_{n=0}^{\infty}$ such that $\lim_{k\to\infty} \|z_{n_k} - q\| = 0$. It follows from (2.1), (2.4), (2.6) and (2.7) that

$$\begin{aligned}\|x_{n_k} - q\| \leq& \|z_{n_k} - q\| + \beta_{n_k}\|x_{n_k} - Sx_{n_k}\| + \|v_{n_k}\| \\ \leq& \|z_{n_k} - q\| + 2B\beta_{n_k} + \|v_{n_k}\| \to 0\end{aligned}$$

as $k \to \infty$. That is,

$$\lim_{k\to\infty} \|x_{n_k} - q\| = 0. \tag{2.13}$$

By virtue of (2.1)-(2.3), (2.12) and (2.13), we conclude that for given $\varepsilon > 0$, there exists positive numbers k_0 and $p = n_{k_0}$ such that

$$\begin{aligned}&\|x_p - q\| \leq \varepsilon, \quad \max\{\alpha_n, \beta_n\} \leq \frac{\varepsilon}{16B}, \\ &\max\{\|u_n\|, \|v_n\|\} \leq \frac{\varepsilon}{8}, \quad t_n \leq \phi(\frac{1}{2}\varepsilon)\varepsilon, \quad n \geq p.\end{aligned} \tag{2.14}$$

By induction we show that

$$\|x_{p+m} - q\| \leq \varepsilon, \quad m \geq 0. \tag{2.15}$$

Note that (2.14) ensures that (2.15) holds for $m = 0$. Suppose that (2.15) holds for some $m \geq 0$. If $\|x_{p+m+1} - q\| > \varepsilon$, then (2.14), (2.8) and (2.4) yield that

$$\begin{aligned}\|x_{p+m} - q\| \geq& \|x_{p+m+1} - q\| - \alpha_{p+m}\|Sz_{p+m} - x_{p+m}\| - \|u_{p+m}\| \\ >& \varepsilon - \frac{\varepsilon}{16B} \cdot 2B - \frac{\varepsilon}{8} = \frac{3}{4}\varepsilon\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\|z_{p+m} - q\| \geq& \|x_{p+m} - q\| - \beta_{p+m}\|Sx_{p+m} - x_{p+m}\| - \|v_{p+m}\| \\ >& \frac{3}{4}\varepsilon - \frac{\varepsilon}{16B} \cdot 2B - \frac{\varepsilon}{8} = \frac{1}{2}\varepsilon.\end{aligned} \tag{2.17}$$

It follows from (2.11), (2.14), (2.16) and (2.17) that

$$\begin{aligned}\varepsilon^2 <& \|x_{p+m+1} - q\|^2 \\ \leq& \|x_{p+m} - q\|^2 - 2\alpha_{p+m}\phi(\|z_{p+m} - q\|)\|z_{p+m} - q\| + \alpha_{p+m}t_{p+m} \\ \leq& \varepsilon^2 - \alpha_{p+m}\phi(\frac{1}{2}\varepsilon)\varepsilon + \alpha_{p+m}\phi(\frac{1}{2}\varepsilon)\varepsilon = \varepsilon^2,\end{aligned}$$

which is impossible. Hence $\|x_{p+m+1} - q\| \leq \varepsilon$. That is, (2.15) holds for all $m \geq 0$. Thus (2.15) yields that $\lim_{n\to\infty} x_n = q$.

Let $\{y_n\}_{n=0}^{\infty}$ be any given sequence in X and define $\{\varepsilon_n\}_{n=0}^{\infty}$ by

$$w_n = (1-\beta_n)y_n + \beta_n S y_n + v_n, \quad n \ge 0; \tag{2.18}$$

$$\varepsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T w_n - u_n\|, \quad n \ge 0.$$

Put $p_n = y_{n+1} - (1-\alpha_n)y_n - \alpha_n T w_n - u_n$. Then

$$y_{n+1} = (1-\alpha_n)y_n + \alpha_n T w_n + u_n + p_n, \quad n \ge 0. \tag{2.19}$$

Suppose that $\varepsilon_n = o(\alpha_n)$. By (2.3), we get that

$$\|u_n + p_n\| \le \|u_n\| + \varepsilon_n = o(\alpha_n),$$

which implies that $\|u_n + p_n\| = o(\alpha_n)$. It follows from the above conclusion that the sequence $\{y_n\}_{n=0}^{\infty}$ defined by (2.18) and (2.19) converges strongly to q. That is, $\{x_n\}_{n=0}^{\infty}$ is pseudo T-stable. This completes the proof.

Theorem 2.2 *Let X, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be as in Theorem 2.1. Let $T : X \to X$ be a ϕ-strongly accretive operator and the range of either $(I-T)$ or T be bounded. Suppose that the equation $Tx = f$ has a solution for a given $f \in X$ and that $Sx = f + x - Tx$ for all $x \in X$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in X$ by (2.4) converges strongly to the unique solution of the equation $Tx = f$ and it is pseudo S-stable.*

Proof: Since T is ϕ-strongly accretive and the equation $Tx = f$ has a solution, it follows that the equation $Tx = f$ has a unique solution. The rest of the proof is identical the proof of Theorem 2.1 and is therefore omitted. This completes the proof.

Theorem 2.3 *Let X, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be as in Theorem 2.1. Let $T : X \to X$ be a ϕ-strongly accretive operator and the range of either $(I+T)$ or T be bounded. Suppose that the equation $x + Tx = f$ has a solution for a given $f \in X$ and that $Sx = f - Tx$ for all $x \in X$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in X$ by (2.4) converges strongly to the unique solution of the equation $x + Tx = f$ and it is pseudo S-stable.*

Proof: Let $A = I + T$. Then A is ϕ-strongly accretive and the range either A or $(I-A)$ is bounded. Clearly, $x + Tx = f$ becomes $Ax = f$ and $Sx = f - Tx = f + x - Ax$ for all $x \in X$. Hence Theorem 2.3 follows from Theorem 2.2. This completes the proof.

Remark 2.1 The boundedness of $R(T)$ or $R(I-T)$ in Theorems 2.1 and 2.2 can be replaced by the boundedness of $\{Tx_n\}_{n=0}^{\infty}$ and $\{Tz_n\}_{n=0}^{\infty}$ or $\{x_n - Tx_n\}_{n=0}^{\infty}$ and $\{z_n - Tz_n\}_{n=0}^{\infty}$.

Remark 2.2 The convergence result in Theorem 2.2 extends, improves and unifies Theorems 1 and 2 of [2], Theorems 7 and 8 of [3] and Theorem 1 of [18] in the following ways:

(a) The Mann iterative schemes in [2, 3] and the Ishikawa iterative schemes in [2, 3, 18] are replaced by the more general Ishikawa iterative scheme with errors.

(b) The strongly accretive operators in [2], [3] and [18] are replaced by the more general ϕ-strongly accretive operators;

(c) That T is Lipschitz in [2] is omitted;

(d) The assumptions of $\alpha_n \leq \beta_n$ in [2], [3], [18], $\sum_{n=0}^{\infty} c_n b(c_n) < \infty$ in [2], [3], $\sum_{n=0}^{\infty} \alpha_n b(\alpha_n) < \infty$ in [2], [3] are superfluous;

(e) The boundedness hypotheses of $R(I-T)$ in [2], [18] and $R(T)$ in [3] are replaced by the boundedness of either $R(I-T)$ or $R(T)$;

The following example reveals that the convergence result in Theorem 2.2 extends properly the corresponding results in [2], [3] and [18].

Example 2.1 Let $X = (-\infty, \infty)$ with the usual norm. Then for any $q > 1$, X is real q-uniformly smooth Banach space. Define $T : X \to X$ by

$$Tx = \begin{cases} x - 1, & \text{if } x < -1 \\ x - \sqrt{-x}, & \text{if } x \in [-1, 0) \\ x, & \text{if } x \in [0, \infty). \end{cases}$$

Clearly $R(T) = X$, $R(I-T)$ is bounded and T is continuous. Note that

$$\lim_{x \to 0^-} \frac{Tx - T0}{x - 0} = \lim_{x \to 0^-} \left(1 + \frac{1}{\sqrt{-x}} \right) = \infty.$$

Hence T is not Lipschitz. Take $\phi(t) = \frac{1}{2}t$ for all $t \geq 0$. In order to prove that T is ϕ-strongly accretive, that is,

$$\langle Tx - Ty, j(x-y) \rangle \geq \phi(\|x - y|)\|x - y\|, \quad x, y \in X. \tag{2.20}$$

We have to consider the following cases.

Case 1. Let $x, y \in (-\infty, -1)$ or $x, y \in [0, \infty)$. Then

$$\langle Tx - Ty, j(x-y) \rangle = (x-y)^2;$$

Case 2. Let $x, y \in [-1, 0)$. Then

$$\langle Tx - Ty, j(x-y)\rangle = [x - y - (\sqrt{-x} - \sqrt{-y})](x-y) = \left(1 + \frac{1}{\sqrt{-x} + \sqrt{-y}}\right)(x-y)^2;$$

Case 3. Let $x \in (-\infty, -1)$, $y \in [-1, 0)$. Then

$$\langle Tx - Ty, j(x-y)\rangle = [x - 1 - (y - \sqrt{-y})(x-y) = (x-y)^2 + (1 - \sqrt{-y})(y-x);$$

Case 4. Let $x \in (-\infty, -1)$, $y \in [0, \infty)$. Then

$$\langle Tx - Ty, j(x-y)\rangle = (x - 1 - y)(x-y) = (x-y)^2 + (y-x);$$

Case 5. Let $x \in [-1, 0)$, $y \in [0, \infty)$. Then

$$\langle Tx - Ty, j(x-y)\rangle = (x - \sqrt{-x} - y)(x-y) = (x-y)^2 + \sqrt{-x}(y-x).$$

Therefore (2.20) holds. Since $R(T) = X$, it follows that the equation $Tx = f$ has a solution for any $f \in X$. Set

$$\alpha_n = (1+n)^{-\frac{1}{2}}, \quad \beta_n = (2+2n)^{-\frac{1}{2}}, \quad u_n = (1+n)^{-1}, \quad v_n = (1+n)^{-\frac{1}{3}}, \quad n \ge 0.$$

Then all the assumptions of Theorem 2.2 are fulfilled. But Theorems 1 and 2 in [2], Theorems 7 and 8 in [3], and Theorem 1 in [18] are not applicable since $R(T)$ is unbounded, T is not Lischitz, and $\alpha_n > \beta_n$ for each $n > 0$.

Remark 2.3 Theorems 11 and 12 in [3] are special cases of our Theorem 2.3.

Remark 2.4 For $T : X \to X$ a ϕ-strongly quasi-accretive operator, Theorem 2.1 proves that the Ishikawa iterative scheme with errors considered in Theorem 2.1 is pseudo $(I-T)$-stable. The following example reveals that the iterative scheme is not $(I-T)$-stable.

Example 2.2 Let $X = (-\infty, \infty)$ with the usual norm, $T = I$ and $u_n = v_n = 0$ for all $n \ge 0$. Clearly,

$$\mathrm{Re}\langle Tx - Ty, j(x-y)\rangle = \|x-y\|^2 \ge \phi(\|x-y\|)\|x-y\|, \quad x \in X, \ y \in N(T),$$

where $\phi(t) = \frac{1}{2}t$ for all $t \ge 0$. It follows from Theorem 2.1 that the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in X$ by

$$z_n = (1-\beta_n)x_n + \beta_n S x_n + v_n, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n S z_n + u_n, \quad n \ge 0,$$

converges strongly to the unique zero 0 of T and is pseudo $(I-T)$-stable. Next we prove that it is not pseudo $(I-T)$-stable. Let $y_n = \frac{n}{1+n}$ for all $n \ge 0$. Then

$$\varepsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n S w_n - u_n\| \le \|y_{n+1} - y_n\| + \alpha_n\|y_n\| \to 0$$

as $n \to \infty$. That is, $\lim_{n\to\infty} \varepsilon_n = 0$. But, $\lim_{n\to\infty} y_n = 1 \notin N(T) = \{0\}$.

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A Note on Global Asymptotic Stability of a Family of Rational Equations

ABSTRACT. In this note we prove that all positive solutions of the difference equations

$$x_{n+1} = \frac{1 + x_n \sum_{i=1}^{k} x_{n-i}}{x_n + x_{n-1} + x_n \sum_{i=2}^{k} x_{n-i}}, \quad n = 0, 1, ...,$$

where $k \in \mathbf{N}$, converge to the positive equilibrium $\bar{x} = 1$. The result generalizes the main theorem in the paper: Li Xianyi and Zhu Deming, Global asymptotic stability in a rational equation, *J. Differ. Equations Appl.* **9** (9), (2003), 833-839. We present a very short proof of the theorem. Also, we find the asymptotics of some of the positive solutions.

KEY WORDS AND PHRASES. rational difference equation, global asymptotic stability, equilibrium point, positive solution, asymptotics

1 Introduction

In [11], Xianyi and Deming prove that the positive equilibrium of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1} + 1}{x_n + x_{n-1}}, \quad n = 0, 1, 2, ... \tag{1}$$

with positive initial values x_{-1}, x_0, is globally asymptotically stable.

In [1], Kruse and Nesemann, among other things, proved the following theorem:

Theorem A *Consider the difference equation*

$$x_{n+r} = f(x_{n+r-1}, ..., x_n), \quad n = 0, 1, ... \tag{2}$$

where $r \in \mathbf{N}$, $f : (0, \infty)^r \to (0, \infty)$ is a continuous function with some unique positive equilibrium $\bar{x}$. Suppose that there is an $m \in \mathbf{N}$ such that for all solutions (x_n) of Eq. (2)

$$(x_n - x_{n+m}) \left(\frac{\bar{x}^2}{x_n} - x_{n+m} \right) \leq 0$$

with equality if and only if $x_n = \bar{x}$. Then $\bar{x}$ is globally asymptotically stable.

In this note we consider a family of difference equations of the form

$$x_{n+1} = \frac{1 + x_n \sum_{i=1}^{k} x_{n-i}}{x_n + x_{n-1} + x_n \sum_{i=2}^{k} x_{n-i}}, \quad n = 0, 1, ..., \tag{3}$$

where $k \in \mathbf{N}$ and the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are positive numbers. From the equation

$$\bar{x} = \frac{k\bar{x}^2 + 1}{(k-1)\bar{x}^2 + 2\bar{x}} \tag{4}$$

we see that $\bar{x} = 1$ is a unique positive equilibrium of Eq. (3).

We show that the positive solutions of Eq. (3) have some similar properties with the positive solutions of Eq. (1) and give a very short proof of the following result:

Theorem 1 *The positive equilibrium point $\bar{x}$ of Eq. (3) is globally asymptotically stable.*

This theorem generalizes the main result in [11], since for $k = 1$ Eq. (3) becomes Eq. (1).

For some other globally convergence results and their applications, see, for example, [5, 6, 7, 8, 9, 10].

In the last section we find the asymptotics of some solutions of Eq. (1).

2 Some properties of the positive solutions of Eq. (3)

In this section we prove several results concerning the positive solutions of Eq. (3).

Lemma 1 *A positive solution $(x_n)_{n=-k}^{\infty}$ of Eq. (3) is eventually equal to 1 if and only if*

$$(x_{-1} - 1)(x_0 - 1) = 0. \tag{5}$$

Proof: Assume that Eq. (5) holds. Then by Eq. (3), it is easy to see that the following conclusion is true: if $x_{-1} = 1$ or $x_0 = 1$, then $x_n = 1$ for $n \geq 1$.

Conversely, assume that $(x_{-1} - 1)(x_0 - 1) \neq 0$. We show

$$x_n \neq 1 \text{ for any } n \geq 1 \tag{6}$$

Let $x_N = 1$ with minimally chosen $N \geq 1$.

Clearly

$$1 = x_N = \frac{1 + x_{N-1} \sum_{i=1}^{k} x_{N-1-i}}{x_{N-1} + x_{N-2} + x_{N-1} \sum_{i=2}^{k} x_{N-1-i}}$$

which implies $(1 - x_{N-1})(1 - x_{N-2}) = 0$ and consequently $x_{N-1} = 1$ or $x_{N-2} = 1$, a contradiction with the choice of N and the condition $(x_{-1} - 1)(x_0 - 1) \neq 0$.

Lemma 2 *Let $(x_n)_{n=-k}^{\infty}$ be a positive solution of Eq. (3) which is not eventually equal to* 1. *Then the following statements are true:*

(i) $(x_{n+1} - x_n)(x_n - 1) < 0$ *for* $n \geq 0$,

(ii) $(x_{n+1} - 1)(x_n - 1)(x_{n-1} - 1) > 0$ *for* $n \geq 0$.

Proof: From Eq. (3), we obtain

$$x_{n+1} - x_n = \frac{(1 - x_n)(1 + x_n + x_n \sum_{i=2}^{k} x_{n-i})}{x_n + x_{n-1} + x_n \sum_{i=2}^{k} x_{n-i}}, \quad n = 0, 1, 2, ... \tag{7}$$

and

$$x_{n+1} - 1 = \frac{(x_n - 1)(x_{n-1} - 1)}{x_n + x_{n-1} + x_n \sum_{i=2}^{k} x_{n-i}}, \quad n = 0, 1, 2, ... \tag{8}$$

From (7) and (8), inequalities (i) and (ii) follow according to Lemma 1.

Remark 1 From Lemma 2 we see that the signs of $x_n - 1$, $n \geq 1$ of a positive solution (x_n) of Eq. (3) are determined by x_{-1} and x_0. Hence in the investigation of the semicycle analysis of positive solutions of Eq. (3) we will consider only the terms with the indices greater than or equal to -1.

A *positive semicycle* of a solution (x_n) of Eq.(3) consists of a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all greater than or equal to $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} < \bar{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \bar{x}.$$

A *negative semicycle* of a solution (x_n) of Eq. (3) consists of a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all less than to $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} \geq \bar{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \bar{x}.$$

The first semicycle of a solution starts with the term x_{-1} and is positive if $x_{-1} \geq \bar{x}$ and negative if $x_{-1} < \bar{x}$.

Lemma 3 *For Eq. (3), the following statements are true:*

(i) *There exists a positive solution with a semicycle of Eq. (3) which has an infinite number of terms and monotonically tends to the positive equilibrium point $\bar{x}$;*

(ii) *Every negative semicycle of a solution of Eq. (3), except perhaps for the first, has exactly two terms.*

(iii) *Every positive semicycle of an oscillatory solution of Eq. (3) has exactly one term.*

Proof:

(i) If $x_{-1} > 1$ and $x_0 > 1$, then by Lemma 2 (ii), it follows that $x_n > 1$, $n \geq -1$, i.e. this positive semicycle has infinite number of terms. By Lemma 2 (i), we see that x_n is strictly decreasing for $n \geq 0$. Hence, there is finite $\lim_{n\to\infty} x_n = l > 0$. From this and (4) it follows that $l = \bar{x} = 1$.

(ii) If x_s $(s \geq 0)$ is the first term of a negative semicycle, then from Lemma 2 (ii) we have

$$(x_{s+1} - 1)(x_s - 1)(x_{s-1} - 1) > 0$$

and consequently $x_{s+1} < 1$.

From this and since

$$(x_{s+2} - 1)(x_{s+1} - 1)(x_s - 1) > 0$$

it follows that $x_{s+2} > 1$, from which the result follows.

(iii) If x_p $(p \geq 0)$ is the first term of a positive semicycle of an oscillatory solution of Eq. (3), then from the inequality in Lemma 2 (ii) we have

$$(x_{p+1} - 1)(x_p - 1)(x_{p-1} - 1) > 0.$$

Since $x_{p-1} < 1$ it follows that $x_{p+1} < 1$, as desired.

From Lemmas 1, 2 and 3 it follows the following corollary.

Corollary 1 *Consider Eq. (3). Then a positive solution of Eq. (3) is either eventually equal to 1, or greater than 1 and monotonically tends to 1, or an oscillatory solution of Eq. (3), such that the positive semicycles of the solution have always one term, and the negative semicycles, disregarding the first one, two terms.*

3 Proof of Theorem 1

In this section we prove Theorem 1.

Proof: From (3) we have

$$\begin{aligned} \frac{1}{x_n} - x_{n+1} &= \frac{1}{x_n} - \frac{1 + x_n \sum_{i=1}^{k} x_{n-i}}{x_n + x_{n-1} + x_n \sum_{i=2}^{k} x_{n-i}} \\ &= \frac{(1 - x_n)(x_{n-1}(1 + x_n) + x_n \sum_{i=2}^{k} x_{n-i})}{x_n(x_n + x_{n-1} + x_n \sum_{i=2}^{k} x_{n-i})}. \end{aligned} \tag{9}$$

From (7) and (9) we have

$$(x_n - x_{n+1})\left(\frac{1}{x_n} - x_{n+1}\right) \le 0, \quad n = 0, 1, \ldots .$$

with equality if and only if $x_n = 1$. From this and by Theorem A, we obtain that the positive equilibrium $\bar{x} = 1$ is globally asymptotically stable, as desired.

4 Asymptotics of solutions of Eq. (3)

In this section we find the asymptotics of some solutions of Eq. (3). We use the method described in [3], see also, [2] and [4].

4.1 Asymptotics of nonoscillatory solutions of Eq. (3)

According to Lemma 3 these solutions monotonically tend to 1 as $n \to \infty$. In order to find the asymptotics we make the ansatz $x_n = 1 + y_n$ with $y_n = o(1)$. Equation (3) implies

$$y_{n+1} = \frac{y_n y_{n-1}}{k + 1 + k y_n + \sum_{i=1}^{k} y_{n-i} + y_n \sum_{i=2}^{k} y_{n-i}}. \tag{10}$$

Note that Eq. (10) can be approximated by the equation

$$y_{n+1} = \frac{y_n y_{n-1}}{k+1}, \tag{11}$$

where first we look for positive solutions y_n which correspond to the condition $x_n > 1$ for $n \ge 0$. Taking the logarithm of (11) and making the change $z_n = \ln y_n$, we obtain

$$z_{n+1} - z_n - z_{n-1} = -\ln(k+1). \tag{12}$$

By standard methods it can be shown that the general solution of Eq. (12) has the form.

$$z_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n + \ln(k+1).$$

Hence the general solution of Eq. (11) reads

$$y_n = (k+1) e^{c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n}. \tag{13}$$

For real constants c_j this solution is positive, and it satisfies $y_n = o(1)$ if $c_1 < 0$. Without loss of generality we may assume that $c_1 = -1$, which is shown by a suitable shift of n.

This motivated us to make the ansatz

$$y_n = (k+1)\left(e^{-l^n} + b\psi_n\right), \tag{14}$$

with $\psi_n = \exp(-al^n)$, $a > 1$, where $l = (1+\sqrt{5})/2$.

Setting (14) into (10) and comparing the coeficients we obtain that $a = 1 + l^{1-k}$ and $b = 1$. Now after a shift of n to $n+k$ in (10) we apply Theorem 2.1 in [3]. Let

$$\varphi_n = (k+1)\left(e^{-l^n} + e^{-al^n}\right) \quad \text{and} \quad \psi_n = e^{-al^n}, \tag{15}$$

where a and l are as above and let

$$F(w_0, w_1, ..., w_{k+1}) = (k + 1 + kw_k + w_{k-1} + (w_k + 1)\sum_{i=0}^{k-2} w_i)w_{k+1} - w_k w_{k-1}.$$

The partial derivatives of the function F are the following

$$F_{w_0} = F_{w_1} = \cdots = F_{w_{k-2}} = w_{k+1}(w_k + 1),$$

$$F_{w_{k-1}} = w_{k+1} - w_k, \quad F_{w_k} = w_{k+1}(k + \sum_{i=0}^{k-2} w_i) - w_{k-1},$$

$$F_{w_{k+1}} = k + 1 + kw_k + w_{k-1} + (w_k + 1)\sum_{i=0}^{k-2} w_i.$$

Hence

$$\psi_{n+i}F_{w_i}(\varphi_n, ..., \varphi_{n+k+1}) \sim \psi_{n+i}\varphi_{n+k+1} \sim (k+1)e^{-l^n(al^i + l^{k+1})}$$

for $i = 0, 1, ..., k-2$,

$$\psi_{n+k-1}F_{w_{k-1}}(\varphi_n, ..., \varphi_{n+k+1}) \sim -\psi_{n+k-1}\varphi_{n+k} \sim -(k+1)e^{-l^n(al^{k-1} + l^k)},$$

$$\psi_{n+k}F_{w_k}(\varphi_n, ..., \varphi_{n+k+1}) \sim -\psi_{n+k}\varphi_{n+k-1} \sim -(k+1)e^{-l^n(al^k + l^{k-1})}$$

and

$$\psi_{n+k+1}F_{w_{k+1}}(\varphi_n, ..., \varphi_{n+k+1}) \sim (k+1)\psi_{n+k+1} = (k+1)e^{-l^n(al^{k+1})}.$$

Since $a = 1 + l^{1-k}$ it is easy to see that

$$l^{k+1} + 1 = al^{k-1} + l^k = \min\{al^i + l^{k+1}, (i = 0, 1, ..., k-2), al^{k-1} + l^k, al^k + l^{k-1}, al^{k+1}\},$$

where the minimum is attained at the last but two position.

Thus, for $f_n = e^{-l^n(l^{k+1}+1)}$ we obtain

$$\psi_{n+i}F_{w_i}(\varphi_n, ..., \varphi_{n+k+1}) \sim A_i f_n$$

where $A_i = 0$, $i = 0, 1, 2, ..., k-2, k, k+1$, and $A_{k-1} = -(k+1)$.

Now we prove that

$$F(\varphi_n, ..., \varphi_{n+k+1}) \sim (k+1)^2 e^{-(l^{k+1}+1+l^{1-k})l^n} = o(f_n). \tag{16}$$

For $w_i = \varphi_{n+i} = (k+1)s_i$, $i = 0, 1, ..., k+1$, with $s_i = e^{-l^{n+i}} + e^{-al^{n+i}}$ let $F = (k+1)^2 G$ with

$$G(s_0, s_1, ..., s_{k+1}) = s_{k+1}(1 + ks_k + s_{k-1} + (1 + (k+1)s_k)\sum_{i=0}^{k-2} s_i) - s_k s_{k-1}.$$

It follows

$$G(s_0, s_1, ..., s_{k+1}) = s_{k+1}(1 + s_0 + s_1) - s_k s_{k-1} + o\left(e^{(L+a)l^n}\right)$$

with $L = l^{k+1}$, since the terms $s_{k+1}s_k s_i$ with $i \geq 0$, the terms $s_{k+1}s_i$ for $i \geq 2$, and the terms $s_{k+1}s_k$ for $k \geq 2$ are contained in the remainder term. In the exponents of the terms of the product $s_k s_{k-1}$ there appear the factors of $-l^n$

$$l^k + l^{k-1} = L, \tag{17}$$

$$l^k + al^{k-1} = L + 1, \tag{18}$$

$$al^k + l^{k-1} = L + l \tag{19}$$

and

$$al^k + al^{k-1} = aL. \tag{20}$$

The corresponding factors concerning the product $s_{k+1}(1 + s_0 + s_1)$ are

$$(17),\ (20),\ (18),\ L + a,\ aL + 1,\ a(L+1),\ (19),\ L + al,\ aL + l,\ a(L+l).$$

The terms with a number cancel. The smallest term of the remaining ones is $L + a$. Hence (16) is proved.

From all above mentioned the conditions of Theorem 2.1 in [3] are satisfied for $m = k+1$, hence for every $\varepsilon > 0$, Eq. (3) has a solution y_n in the stripe $\varphi_n - \varepsilon\psi_n \leq y_n \leq \varphi_n + \varepsilon\psi_n$ for sufficiently large $n_0 = n_0(\varepsilon)$, with φ_n and ψ_n defined in (15).

4.2 Asymptotics of oscillatory solutions of Eq. (3)

The signs of the terms of a solution of Eq. (11) depend on the initial conditions y_0 and y_1. It can easily be seen that the general nontrivial solution of Eq. (11) can be written as $v_n y_n$ where y_n is the positive solution (13) and v_n for $n \geq 0$ one of the four 3-periodic sequences in Table 1.

$v_n^{(i)}$	$v_0^{(i)}$	$v_1^{(i)}$	$v_2^{(i)}$	$v_3^{(i)}$	$v_4^{(i)}$	$v_5^{(i)}$	$v_6^{(i)}$	$v_7^{(i)}$	$v_8^{(i)}$	.	.	.
$v_n^{(1)}$	1	1	1	1	1	1	1	1	1	.	.	.
$v_n^{(2)}$	1	-1	-1	1	-1	-1	1	-1	-1	.	.	.
$v_n^{(3)}$	-1	1	-1	-1	1	-1	-1	1	-1	.	.	.
$v_n^{(4)}$	-1	-1	1	-1	-1	1	-1	-1	1	.	.	.

TABLE 1. Values of the sequences $v_n^{(i)}$, $i = 1, 2, 3, 4$.

These periodic sequences can be represented as $v_n = e^{i\pi t_n}$ where t_n is one of the solutions with integer values mod 2 of Fibonacci's equation in Table 2.

$t_n^{(i)}$	$t_0^{(i)}$	$t_1^{(i)}$	$t_2^{(i)}$	$t_3^{(i)}$	$t_4^{(i)}$	$t_5^{(i)}$	$t_6^{(i)}$	$t_7^{(i)}$	$t_8^{(i)}$	.	.	.
$t_n^{(1)}$	0	0	0	0	0	0	0	0	0	.	.	.
$t_n^{(2)}$	0	1	1	0	1	1	0	1	1	.	.	.
$t_n^{(3)}$	1	0	1	1	0	1	1	0	1	.	.	.
$t_n^{(4)}$	1	1	0	1	1	0	1	1	0	.	.	.

TABLE 2. $t_{n+1}^{(i)} = t_n^{(i)} + t_{n-1}^{(i)} \pmod 2,\ i = 1, 2, 3, 4.$

With some more effort it can be shown analogously as before that Eq. (10) has also solutions which behave asymptotically like the solutions $v_n y_n$ of (11). This result matches with Lemma 2 (ii), which is equivalent to $v_{n+1}v_n v_{n-1} > 0$ for $n \geq 0$, and it also matches with Lemma 3.

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Anti-Maximum Principle for a Schrödinger Equation in $\mathbb{R}^N$, with a non radial potential

ABSTRACT. Anti-maximum for the Schrödinger equation $-\Delta u + q(x)u - \lambda u = f(x)$ in $L^2(\mathbb{R}^N)$ is extended to potentials q non necessarily radial. The anti-maximum is proved in the following form: Let φ_1 denote the positive eigenfunction associated with the principal eigenvalue λ_1 of the Schrödinger operator $\mathcal{A} = -\Delta + q(x)\bullet$ in $L^2(\mathbb{R}^N)$. Assume the potential $q(x)$ grows fast enough near infinity, and the function f satisfy $f \not\equiv 0$ and $0 \le f/\varphi_1 \le C \equiv$ *const* a.e. in $\mathbb{R}^N$. Then there exists a positive number δ (depending upon f) such that, for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$, the inequality $u \le -c\varphi_1$ holds a.e. in $\mathbb{R}^N$, where c is a positive constant depending upon f and λ.

KEY WORDS. Positive or negative solutions; pointwise bounds; principal eigenvalue; positive eigenfunction; strong maximum and anti-maximum principles

1 Introduction

The anti-maximum for the Dirichlet Laplacian defined in a regular bounded domain $\Omega \subset \mathbb{R}^N$ is an important result established first by Ph. Clément and L. A. Peletier [4] and extended to several types of elliptic operators or systems defined in a bounded domain, see e.g. G. Sweers [13], P. Takáč [14],G. Fleckinger et al.[5, 6]. The case of the Schrödinger operator on $\Omega = \mathbb{R}^N$ is more difficult. Indeed, for maximum and anti-maximum in unbounded domain the works of B. Alziary and P. Takáč [3], B. Alziary, G. Fleckinger and P. Takáč [1, 2] and Y. Pinchover [8, 9] show that, on must always take into account the growth of the solution near the infinity.

We investigate here, anti-maximum for a linear partial differential equation with the Schrödinger operator,

$$-\Delta u + q(x)u - \lambda u = f(x) \quad \text{in} \ \ \mathbb{R}^N, \tag{1}$$

Here, f is a given function satisfying $0 \le f \not\equiv 0$ in $\mathbb{R}^N$ $(N \ge 1)$, and λ stands for the spectral parameter. As usual, the Schrödinger operator takes the form $\mathcal{A} = -\Delta + q(x)\bullet$ in

$L^2(\mathbb{R}^N)$ where Δ and $q(x)\bullet$, respectively, denote the selfadjoint Laplace operator and the pointwise multiplication operator by the potential q in $L^2(\mathbb{R}^N)$. Let φ_1 denote the positive eigenfunction of $\mathcal{A}$ associated with the principal eigenvalue λ_1. We recall the definition of φ_1-positivity and φ_1-negativity.

Definition 1.1 *A function $u \in L^2(\mathbb{R}^N)$ is called* φ_1-positive *if there exists a constant $c > 0$ such that*

$$u \geq c\varphi_1 \quad \textit{almost everywhere in } \ \mathbb{R}^N. \tag{2}$$

Analogously, $u \in L^2(\mathbb{R}^N)$ is called φ_1-negative *if there exists a constant $c > 0$ such that*

$$u \leq -c\varphi_1 \quad \textit{almost everywhere in } \ \mathbb{R}^N. \tag{3}$$

To obtain anti-maximun for the Schrödinger operator on $\Omega = \mathbb{R}^N$, we need to assume f in the strongly ordered Banach space X introduced in Alziary and Takáč [3]:

$$X = \{u \in L^2(\mathbb{R}^N) \colon u/\varphi_1 \in L^\infty(\mathbb{R}^N)\} \tag{4}$$

endowed with the ordered norm

$$\|u\|_X = \inf\{C \in \mathbb{R} \colon |u| \leq C\varphi_1 \text{ almost everywhere in } \mathbb{R}^N\}. \tag{5}$$

The ordering "$\leq$" on X is the natural pointwise ordering of functions. This means that X is an ordered Banach space whose positive cone X_+ has nonempty interior $\overset{\circ}{X}_+$. Taking $N \geq 2$, The necessity of such a restriction for the Schrödinger operator in $L^2(\mathbb{R}^N)$ has been justified in [2, Example 4.1 p. 377]. B. Alziary, G. Fleckinger and P. Takáč construct a counterexample to the anti-maximum principle (3) for a positive, radially symmetric function $f \in L^2(\mathbb{R}^N) \setminus X$.

The validity of (2) for a "sufficiently smooth" solution u to Equation (1) is established in Alziary and Takáč [3, Theorem 2.1, p. 284] for a nonnegative function $f \not\equiv 0$ in $L^2(\mathbb{R}^N)$. The inequality (3) is shown in Alziary, Fleckinger and Takáč [1, 2] under considerably more restrictive hypotheses on q and f, since they consider only radially symmetric potentials and they establish the anti-maximum only for f from a Banach space $X^{\alpha,2}$ that contains "sufficiently smooth" perturbations of radially symmetric functions of X.

In the present work we are able to extend this results to some non radial potential and for $f \in X_+ \setminus \{0\}$ and $f \in C^{0,\alpha}(\mathbb{R}^N)$. For either (2) or (3) to be valid, it is necessary and sufficient that the potential $q(x)$, which is assumed to be strictly positive and locally bounded, have a superquadratic growth as $|x| \to \infty$. In particular, $q(x)$ must grow faster than $|x|^2$ as $|x| \to \infty$; the growth like $|x|^{2+\varepsilon}$ with any constant $\varepsilon > 0$ is sufficient. Thus, both (2) and (3) are in general false for the harmonic oscillator, i.e., for $q(x) = |x|^2$ in $\mathbb{R}^N$; see [1], Examples 4.1 and 4.2. As it seems to be inevitable in the theory of Schrödinger operators, we assume that $q(x)$ is a "relatively small" perturbation of a radially symmetric function, $q(x) = q_1(|x|) + q_2(x)$ for $x \in \mathbb{R}^N$.

Y. Pinchover in [8, 9] prove the inequalities (2) and (3) for any solution $f \in X_+ \setminus \{0\}$,but imposing certain growth conditions on the first derivatives of $q(x)$, and assuming the solution u is already in X. Our method combine a comparison result from B. Alziary and P. Takáč [3, Theorem 2.2, p. 285] in the exterior domain $\Omega_R = \{\, x \in \mathbb{R}^N \colon |x| > R \,\}$, for $0 < R < \infty$ and the approach of Y. Pinchover in the proof of [8, theorem 5.3, p.23]. We study the behavior of the principle eigencurve of a certain two parameter eigenvalue problem and prove the anti-maximum principle using a fixed point argument.

This article is organized as follows. In Section 2 we state our main result, Theorem 2.1. There, the inequality (3) for $\lambda_1 < \lambda < \lambda_1 + \delta$ is stated for the solution u of (1). In Section 3 we first recall the comparison result we will used and then give the proof of the main result.

2 The Main Result

Notation. We denote by $\mathbb{R}^N$ the N-dimensional Euclidean space ($N \geq 2$) endowed with the inner product $x \cdot y$ and the norm $|x| = (x \cdot x)^{1/2}$, for $x, y \in \mathbb{R}^N$. We write $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^N_+ = (\mathbb{R}_+)^N \subset \mathbb{R}^N$. For a set $M \subset \mathbb{R}^N$, we denote by ∂M ($\overline{M}$, and $\mathring{M}$, respectively) the boundary (closure, and interior) of the set M in $\mathbb{R}^N$. We use analogous notation for sets in all Banach spaces.

Given a set $\Omega \subset \mathbb{R}^N$ and $1 \leq p \leq \infty$, we use the following standard Banach spaces of functions $f \colon \Omega \to \mathbb{R}$ (or $\mathbb{C}$), see e.g. Gilbarg and Trudinger [7, Chapt. 7]:

$L^p(\Omega)$, where Ω is Lebesgue measurable, is the Lebesgue space of all (equivalence classes of) Lebesgue measurable functions $f \colon \Omega \to \mathbb{R}$ with the norm

$$\|f\|_p \equiv \|f\|_{L^p(\Omega)} \stackrel{\text{def}}{=} \begin{cases} \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p} < \infty & \text{if } 1 \leq p < \infty; \\ \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| < \infty & \text{if } p = \infty. \end{cases}$$

The space $W^{k,p}(\Omega)$, where $k \geq 1$ is an integer and Ω open in $\mathbb{R}^N$, is the Sobolev space of all functions $f \in L^p(\Omega)$ whose all partial derivatives of order $\leq k$ also belong to $L^p(\Omega)$. The norm $\|f\|_{k,p} \equiv \|f\|_{W^{k,p}(\Omega)}$ in $W^{k,p}(\Omega)$ is defined in a natural way.

The local Lebesgue and Sobolev spaces $L^p_{\text{loc}}(\Omega)$ and $W^{k,p}_{\text{loc}}(\Omega)$ are defined analogously.

The holder spaces $\mathcal{C}^{k,\alpha}(\mathbb{R}^N)$ are defined as the subspaces of $\mathcal{C}^k(\mathbb{R}^N)$ consisting of functions whose K-th order partial derivatives are locally Hölder continuous with exponent α.

Finally, for Ω open in $\mathbb{R}^N$, $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ is the space of all infinitely many times differentiable functions $f\colon \Omega \to \mathbb{R}$ with compact support. It is well-known that $\mathcal{D}(\mathbb{R}^N)$ is a dense linear subspace of both $L^p(\mathbb{R}^N)$ and $W^{k,p}(\mathbb{R}^N)$ for $1 \le p < \infty$.

In order to formulate our hypothesis on the potential $q(x)$, $x \in \mathbb{R}^N$, we first introduce the following class of auxiliary functions $Q(r)$ of $r \equiv |x|$, $R_0 \le r < \infty$, for some $R_0 > 0$:

$$\begin{cases} Q(r) > 0, \quad Q \text{ is locally absolutely continuous,} \\ Q'(r) \ge 0, \qquad \text{and there exists a constant } \beta \text{ with} \\ 0 < \beta < \frac{1}{2} \text{ and } \int_{R_0}^\infty Q(r)^{-\beta}\, dr < \infty. \end{cases} \tag{6}$$

We assume that the potential q takes the form

$$q(x) = q_1(|x|) + q_2(x), \quad x \in \mathbb{R}^N,$$

where $q_1(r)$ and q_2 are Lebesgue measurable functions satisfying the following hypothesis, with some auxiliary function $Q(r)$ which obeys (6):

Hypothesis (**H1**) The potential $q\colon \mathbb{R}_+ \to \mathbb{R}$ is locally essentially bounded, $q(r) \ge \text{const} > 0$ for $r \ge 0$, and there exists a constant $c_1 > 0$ such that

$$c_1 Q(r) \le q(r) + \frac{(N-1)(N-3)}{4r^2} \quad \text{for } R_0 \le r < \infty. \tag{7}$$

(**H2**) The potential $q_2\colon \mathbb{R}_+ \to \mathbb{R}$ is locally essentially bounded, $q(x) = q_1(|x|) + q_2(x) \ge \text{const} > 0$ for $r \ge 0$, and there exists a constant $c_2 > 0$ such that

$$|q_2(x)| \le c_2 Q(|x|)^{\frac{1}{2}-\beta} \quad \text{for } x \in \mathbb{R}^N. \tag{8}$$

Notice that the fraction $(N-1)(N-3)/4r^2$ in the inequality (7) is not essential and has been added for convenience in later applications; it can be left out.

Next we introduce the quadratic form

$$(v,w)_q \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} (\nabla v \cdot \nabla w + q(x) v w)\, dx \tag{9}$$

defined for every pair

$$v, w \in V_q \stackrel{\text{def}}{=} \{ f \in L^2(\mathbb{R}^N) \colon (f,f)_q < \infty \}. \tag{10}$$

Notice that V_q is a Hilbert space with the inner product $(v,w)_q$ and the norm $\|v\|_{V_q} = ((v,v)_q)^{1/2}$. The set $\mathcal{D}(\mathbb{R}^N)$ is a dense linear subspace of V_q. By the Lax-Milgram theorem, the Schrödinger operator

$$\mathcal{A} = -\Delta + q(x) \bullet \quad \text{in } L^2(\mathbb{R}^N) \tag{11}$$

is defined to be the selfadjoint operator in $L^2(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} (\mathcal{A}v)w\,dx = (v,w)_q \quad \text{for all} \;\; v,w \in \mathcal{D}(\mathbb{R}^N). \tag{12}$$

We denote by $\mathcal{D}(\mathcal{A})$ its domain. The Banach space $\mathcal{D}(\mathcal{A})$ endowed with the graph norm is compactly embedded into $L^2(\mathbb{R}^N)$, by Rellich's theorem combined with $q(x) \to \infty$ as $|x| \to \infty$.

It is well-known that $\mathcal{A}$ possesses an infinite sequence of positive eigenvalues, $\lambda_1 < \lambda_2 < \cdots \lambda_n \cdots$, and the first one, denote by λ_1, is given by

$$\lambda_1 = \inf\left\{(f,f)_q \colon f \in V_q \;\; \text{with} \;\; \|f\|_{L^2(\mathbb{R}^N)} = 1\right\}, \quad \lambda_1 > 0.$$

The eigenvalue λ_1 is simple with the eigenspace spanned by an eigenfunction $\varphi_1 \in \mathcal{D}(\mathcal{A})$ satisfying $\varphi_1 > 0$ throughout $\mathbb{R}^N$. We normalize φ_1 by the condition $\|\varphi_1\|_{L^2(\mathbb{R}^N)} = 1$. Since $q(x) \equiv q(|x|)$ for $x \in \mathbb{R}^N$, we must have also $\varphi_1(x) \equiv \varphi_1(|x|)$ for $x \in \mathbb{R}^N$. Furthermore, if $u \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}u = f \in L^2(\mathbb{R}^N)$ with $f \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for some p with $2 \le p < \infty$, then the local L^p-regularity theory yields $u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$, see Gilbarg and Trudinger [7, Theorem 9.15, p. 241]. In particular, if $p > N$ then $u \in C^1(\mathbb{R}^N)$, by the Sobolev imbedding theorem [7, Theorem 7.10, p. 155]. It follows that also $\varphi_1 \in C^1(\mathbb{R}^N)$.

The following theorem about φ_1-negativity of u is our main result:

Theorem 2.1 *Let the hypotheses* (**H1**) *and* (**H2**) *be satisfied and q be locally Hölder continuous. Assume that $u \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}u - \lambda u = f \in L^2(\mathbb{R}^N)$, $\lambda \in \mathbb{R}$. Let $f \in X \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$ be a nonnegative function with $f > 0$ in some set of positive Lebesgue measure. Then there exists a positive number δ (depending upon f) such that, for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$, the inequality*

$$u \le -c\varphi_1 \quad \text{in} \;\; \mathbb{R}^N \tag{13}$$

is valid with a constant $c > 0$ (depending upon f and λ).

If we choose $\delta < \lambda_2 - \lambda_1$, for any $\lambda_1 < \lambda < \lambda_1 + \delta$, the solution of the equation, $\mathcal{A}u - \lambda u = f \in L^2(\mathbb{R}^N)$, always exists and is unique. So it suffices to show the existence of a φ_1-negative solution for $\lambda_1 < \lambda < \lambda_1 + \delta$ as in Y. Pinchover [8, 9]. Y. Pinchover proved that for any $f \in X_+ \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$, $f \not\equiv 0$, there exists a positive number δ such that, for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$, any solution u of X is φ_1-negative. Here we prove that $u \in X$. Moreover, his hypothesis on q_1 is much stronger than ours in that he requires that $\log q_1$ be uniformly Lipschitz in $\mathbb{R}^N$ and q_1 itself satisfy $q_1'(r) \ge 0$ and $\int^\infty q_1(r)^{-1/2} dr < \infty$.

3 Proof of the Main Result

We first recall some comparison result and then prove our theorem.

3.1 Preliminary result

The following theorem, proved by B. Alziary and P. Takáč in [3, Theorem 2.2 p. 285], establish a comparison result for positive solution $u(x)$ and $u_1(x)$ of the Schrödinger equation in the exterior domain Ω_R with the potentials $q(x)$ and $q_1(x)$, respectively, and $f \equiv 0$ in Ω_R:

Theorem 3.1 *Let the hypotheses* (**H1**) *and* (**H2**) *be satisfied. Furthermore, fix any constant $R \geq R_0$ such that $Q(R)^{\frac{1}{2}+\beta} \geq 2c_2/c_1$. Assume that u and u_1 are two functions of $x \in \mathbb{R}^N$ such that $u,\ u_1 \in \mathcal{D}(\mathcal{A})$, both u and u_1 are positive and continuous throughout $\overline{\Omega}_R$, for some $R > 0$, and the following equations hold in the sense of distributions over Ω_R,*

$$
\begin{aligned}
-\Delta u + q(x)u &= 0 \quad \text{in } \Omega_R, && (14)\\
-\Delta u_1 + q_1(|x|)u_1 &= 0 \quad \text{in } \Omega_R. && (15)
\end{aligned}
$$

Then there exists a positive constant γ (depending only upon the potential q) such that:

$$
\gamma^{-1}\frac{m_u}{u_1(R)}u_1(|x|) \leq u(x) \leq \gamma\frac{M_u}{u_1(R)}u_1(|x|) \text{ for a.e. } x \in \overline{\Omega}_R, \tag{16}
$$

with

$$
m_u = \min_{|x|=R} u(x) \quad \text{and} \quad M_u = \max_{|x|=R} u(x).
$$

3.2 Proof of the Theorem

Since $\mathcal{A}$ has a discrete spectrum, there exists δ_0 such that $(\lambda_1, \lambda_1+\delta_0)\cap\sigma(\mathcal{A}) = \emptyset$. Therefore, it is enough to show that there exists $\delta \leq \delta_0$ such that, for every $\lambda \in (\lambda_1, \lambda_1+\delta)$ the equation $\mathcal{A}u - \lambda u = f$ admits a negative solution u_λ, satifying $-u_\lambda \geq c\varphi_1$ with a positive constant c.

Set $w_\lambda = -u_\lambda$, the equation becomes

$$
(\mathcal{A} + f(x)/w_\lambda - \lambda)w_\lambda = (-\Delta + q(x) + f(x)/w_\lambda - \lambda)w_\lambda = 0 \text{ in } \mathbb{R}^N. \tag{17}
$$

Now, we need to prove that the equation (17) has a positive solution w_λ, satisfying $w_\lambda \geq c\varphi_1$ with a positive constant c.

First, for $\lambda_1 < \lambda \leq \lambda_1 + 1$, we introduce the following set of functions:

$$
\begin{aligned}
Y_\lambda = \{\, u \in \mathcal{D}(\mathcal{A}), \quad & u > 0,\ u(0) = \varphi_1(0),\ \text{and} \\
& \exists V \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N), \quad 0 \leq V \leq 1,\ \text{s.t.} \quad (\mathcal{A} - \lambda + V)u = 0\,\}
\end{aligned} \tag{18}
$$

First we prove that Y_λ *is a nonempty convex compact set.*

- **(i)** Y_λ *is nonempty:* Indeed, for $V_\lambda = \lambda - \lambda_1$, we have $0 \leq V_\lambda \leq 1$ and the eigenfunction φ_1 is solution of the equation $(\mathcal{A} - \lambda + V_\lambda)\varphi_1 = 0$. Therefore $\varphi_1 \in Y_\lambda$.

(ii) Y_λ *is convex:* Let u_1 and u_2 be two functions of Y_λ. This functions u_1 and u_2 satisfy respectively the equations $(\mathcal{A} - \lambda + V_1)u_1 = 0$ and $(\mathcal{A} - \lambda + V_2)u_2 = 0$, with $0 \leq V_1, V_2 \leq 1$. Let $0 < t < 1$ and denote $u_t = tu_1 + (1-t)u_2$. We check easily that u_t is solution of $(\mathcal{A} - \lambda + V_t)u_t = 0$, with $0 \leq t\frac{u_1}{u_t}V_1 + (1 - t\frac{u_1}{u_t})V_2 \leq 1$. So $u_t \in Y_\lambda$.

(iii) Let us prove now that *there exists* $C > 0$ *such that*

$$C^{-1}\varphi_1(x) \leq u(x) \leq C\varphi_1(x) \textit{ for all } x \in \mathbb{R}^n$$

for every $u \in Y_\lambda$ *and* $\lambda_1 \leq \lambda \leq \lambda_1 + 1$.

We introduce now ψ_1 the radial eigenfunction corresponding to the eigenvalue Λ_1 of the Schrödinger operator $-\Delta + q_1(|x|)\bullet$.

Notice that, since $\lambda_1 < \lambda \leq \lambda_1 + 1$ and $0 \leq V \leq 1$, we have

$$q - \lambda_1 - 1 \leq q + V - \lambda \leq q - \lambda_1 + 1.$$

The potential q goes to $+\infty$ as $|x|$ goes to ∞, so there exists R_1 such that $0 < \mathit{const} < q(x) - \lambda_1 - 1 \leq q(x) + V(x) - \lambda \leq q(x) - \lambda_1 + 1$ for all $|x| \geq R_1$. Thus principal eigenvalues corresponding to those potentials on Ω_{R_1} are all positive. We choose R_1 large enough, so that we could apply theorem 3.1 with the potentials $q(x) - \lambda_1 + 1$ and $q_1(|x|) - \Lambda_1$, $q(x) - \lambda_1$ and $q_1(|x|) - \Lambda_1$, or $q(x) - \lambda_1 - 1$ and $q_1(|x|) - \Lambda_1$.

Let us take any $u \in Y_\lambda$. Now we split our proof of **(iii)** into the cases $x \in \overline{B}_{R_1}$ and $x \in \Omega_{R_1}$.

Case $x \in \Omega_{R_1}$ Denote by $\underline{u}$ and $\overline{u}$ the solutions of the following equations:

$$\begin{cases} -\Delta u + (q + V - \lambda)u = 0 & \text{in } \Omega_{R_1} \\ -\Delta \underline{u} + (q - \lambda_1 + 1)\underline{u} = 0 & \text{in } \Omega_{R_1} \\ -\Delta \overline{u} + (q - \lambda_1 - 1)\overline{u} = 0 & \text{in } \Omega_{R_1} \\ \underline{u}(x) = \overline{u}(x) = u(x) & \text{on } \partial\Omega_{R_1} \end{cases} \tag{19}$$

Since $q - \lambda_1 - 1 \leq Q + V - \lambda \leq q - \lambda_1 + 1$, by the weak maximum principle on Ω_{R_1}, we have:

$$\underline{u} \leq u \leq \overline{u} \text{ in } \overline{\Omega}_{R_1} \tag{20}$$

For the eigenfunctions φ_1 and ψ_1, the following equations hold for all $R > 0$,

$$\begin{cases} -\Delta\varphi + (q(x) - \lambda_1)\varphi_1 = 0 & \text{in } \Omega_R, \\ -\Delta\psi_1 + (q_1(|x|) - \Lambda_1)\psi_1 = 0 & \text{in } \Omega_R. \end{cases} \tag{21}$$

So applying the theorem 3.1 on Ω_{R_1} for φ_1 and ψ_1, there exists a positive constant γ, (depending only upon the potential q) such that :

$$\gamma^{-1}\frac{m_{\varphi_1}}{\psi_1(R_1)}\psi_1(|x|) \leq \varphi(x) \leq \gamma\frac{M_{\varphi_1}}{\psi_1(R_1)}\psi_1(|x|) \text{ for a.e. } x \in \overline{\Omega}_{R_1}, \tag{22}$$

with

$$m_{\varphi_1} = \min_{|x|=R_1} \varphi_1(x) \quad \text{and} \quad M_{\varphi_1} = \max_{|x|=R_1} \varphi_1(x).$$

More clearly, there exists a constant $C_1 > 0$ (depending only on q) such that

$$C_1^{-1}\psi_1(|x|) \leq \varphi_1(x) \leq C_1\psi_1(|x|) \text{ for a.e. } x \in \overline{\Omega}_{R_1}. \tag{23}$$

We apply now the theorem 3.1 on Ω_{R_1} for $\overline{u}$ and ψ_1 and for $\underline{u}$ and ψ_1. So there exist two constants $\overline{\gamma}$ and $\underline{\gamma}$ (depending only on q) such that

$$\overline{\gamma}^{-1}\frac{m_{\overline{u}}}{\psi_1(R_1)}\psi_1(|x|) \leq \overline{u}(x) \leq \overline{\gamma}\frac{M_{\overline{u}}}{\psi_1(R_1)}\psi_1(|x|) \text{ for a.e. } x \in \overline{\Omega}_{R_1}, \tag{24}$$

with

$$m_{\overline{u}} = \min_{|x|=R_1} \overline{u}(x) = \min_{|x|=R_1} u(x) \quad \text{and} \quad M_{\overline{u}} = \max_{|x|=R_1} \overline{u}(x) = \max_{|x|=R_1} u(x),$$

and

$$\underline{\gamma}^{-1}\frac{m_{\underline{u}}}{\psi_1(R_1)}\psi_1(|x|) \leq \underline{u}(x) \leq \underline{\gamma}\frac{M_{\underline{u}}}{\psi_1(R_1)}\psi_1(|x|) \text{ for a.e. } x \in \overline{\Omega}_{R_1}, \tag{25}$$

with

$$m_{\underline{u}} = \min_{|x|=R_1} \underline{u}(x) = \min_{|x|=R_1} u(x) \quad \text{and} \quad M_{\underline{u}} = \max_{|x|=R_1} \underline{u}(x) = \max_{|x|=R_1} u(x).$$

Combining (20) ,(23), (24) and (25), we arrive for a.e. $x \in \overline{\Omega}_{R_1}$ at

$$\frac{\underline{\gamma}\, C_1^{-}1}{\psi(R_1)} m_u\, \varphi_1(x) \leq u(x) \leq \frac{\overline{\gamma}\, C_1}{\psi(R_1)} M_u\, \varphi_1(x), \tag{26}$$

with

$$m_u = \min_{|x|=R_1} u(x) \quad \text{and} \quad M_u = \max_{|x|=R_1} u(x).$$

Case $x \in \overline{B}_{R_1}$. By the Harnack inequality on B_{2R_1} (see Gilbarg and Trudinger [7, Corollary 9.25, p.250]), we gate

$$\sup_{B_R} u(x) \leq C_2 \inf_{B_R} u(x) \text{ for all } R < 2R_1.$$

with a constant C_2 depending only on q and R. Then using the condition $u(0) = \varphi_1(0)$ for $u \in Y_\lambda$, we obtain for $R_1 < R < 2R_1$

$$\begin{aligned} M_u &\leq \sup_{B_R} u(x) \leq C_2 \inf_{B_R} u(x) \leq C_2\varphi_1(0), \\ \varphi_1(0) &\leq \sup_{B_R} u(x) \leq C_2 \inf_{B_R} u(x) \leq C_2 m_u. \end{aligned} \tag{27}$$

Then for a.e. $x \in B_{R_1}$,

$$\frac{C_2^{-1}\varphi_1(0)}{max_{B_{2R_1}}\varphi_1(x)}\varphi_1(x) \le \inf_{B_R} u \le u(x) \le \sup_{B_R} u \le \frac{C_2\varphi_1(0)}{min_{B_{2R_1}}\varphi_1(x)}\varphi_1(x). \tag{28}$$

Finally, by (28), (27) and (26), we deduce **(iii)**.

(iv) Y_λ *is compact in* $\mathcal{C}^0(\mathbb{R}^N)$*:* Let $(u_n)_{n\in\mathbb{N}} \in Y_\lambda$ be a sequence. By **(iii)**, we know that the functions $(u_n)_{n\in\mathbb{N}}$ are bounded in $L^\infty(\mathbb{R}^N)$ and by the regularity theory, we know that they are continuous.

For $R > 0$, we denote by $u_n^{(R)}$ the restriction of u_n to $\overline{B}_R(0)$. This restriction satisfy

$$(-\Delta + q + V_n - \lambda)u_n^{(R)} = 0 \text{ in } B_R(0) \tag{29}$$

Using the Schauder estimate it follows that $u_n^{(R)} \in \mathcal{C}^{2,\alpha}(B_R(0))$ and that

$$\|u_n^{(R)}\|_{2,\alpha} \le C\|u_n^{(R)}\|_\infty \tag{30}$$

where $C = C(N, R, q)$ (see Gilbarg and Trudinger [7, Theorem 6.13, p.106 and Theorem 6.2 p.90]). By (30) and **(iii)**, we deduce that $(u_n^{(R)})_{n\in\mathbb{N}}$ and $(\nabla u_n^{(R)})_{n\in\mathbb{N}}$ are bounded in $\mathcal{C}^0(B_R(0))$. So, using theorem of Ascoli, one can extract a subsequence $(u_{n_k}^{(R)})$ such that:

$$\begin{cases} u_{n_k}^{(R)} & \to & u^{(R)} & \text{strongly in } \mathcal{C}^0(B_R(0)), \\ \nabla u_{n_k}^{(R)} & \to & \nabla u^{(R)} & \text{strongly in } \mathcal{C}^0(B_R(0)), \\ \Delta u_{n_k}^{(R)} & \to & \Delta u^{(R)} & \text{strongly in } \mathcal{C}^{0,\alpha'}(B_R(0)) \text{ for some } 0 < \alpha' < \alpha. \end{cases} \tag{31}$$

Then, taking the diagonal subsequence $(u_{n_n}^{(n)})_{n\in\mathbb{N}}$, we construct a subsequence of $(u_n)_{n\in\mathbb{N}}$ wich converge, strongly in $\mathcal{C}^{2,\alpha}(B_R(0))$ for all $R > 0$, to a continuous function u satisfying

$$C^{-1}\varphi_1(x) \le u(x) \le C\varphi_1(x) \text{ for all } x \in \mathbb{R}^n.$$

Thus the subsequence $(u_{n_n}^{(n)})_{n\in\mathbb{N}}$ converge to u strongly in $\mathcal{C}^0(\mathbb{R}^N)$. Indeed, by **(iii)**,

$$\forall\varepsilon > 0, \quad \exists n_0 > 0 \quad \text{such that } \forall x \in \overline{\Omega}_{n_0} \quad \forall n \ge n_0 \quad |u_{n_n}^{(n)}(x) - u(x)| \le \varepsilon,$$

and by the strong convergence of $u_{n_n}^n$ to u in $\mathcal{C}^0(B_{n_0}(0))$,

$$\exists n_1 > 0 \quad \text{such that } \forall n \ge n_1, \quad \forall x \in B_{n_0}(0), \quad |u_{n_n}^{(n)}(x) - u(x)| \le \varepsilon.$$

To finish the proof of the compactness of Y_λ, we have to check that u belongs to Y_λ. Since $V_{n_n} = \frac{\Delta u_{n_n}^{(n)}}{u_{n_n}^{(n)}} - q + \lambda$, it follows that $V_{n_n} \to V$ locally in $\mathcal{C}^{0,\alpha}(\mathbb{R}^N)$, where $0 \le V \le 1$. Hence u satisfy the equation

$$(\mathcal{A} - \lambda + V)u = 0 \quad \text{in } \mathbb{R}^N,$$

and $u \in Y_\lambda$.

Now, for every nonzero, nonnegative, bounded function V and any $t > 0$, we define the operator $\mathcal{A}_t$,

$$\mathcal{A}_t := -\Delta + q + tV.$$

The potential $q_t = q + tV$ has the same properties as q, so the operator $\mathcal{A}_t$ has the same properties than $\mathcal{A}$. This operator $\mathcal{A}_t$ possesses an infinite sequence of positive eigenvalues, and the first one, denote by $\lambda_V(t)$, is given by

$$\lambda_V(t) = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2 + q_t(x)|u|^2\,dx \colon u \in V_q \;\text{ with }\; \|u\|_{L^2(\mathbb{R}^N)} = 1\right\}. \tag{32}$$

The eigenvalue $\lambda_V(t) > 0$ is simple with the eigenspace spanned by an eigenfunction $\varphi_{V,t} \in \mathcal{D}(\mathcal{A}_t)$ satisfying $\varphi_{V,t} > 0$ throughout $\mathbb{R}^N$ and $\|\varphi_{V,t}\|_{L^2(\mathbb{R}^N)} = 1$. The following properties of the curve $\{(t, \lambda_V(t)) \,|\, t > 0\}$ are easy to check with the characterization (32). The function $\lambda(t)$ is a continuous increasing concave function of t such that $\lambda_V(t) \to \lambda_1$ as $t \to 0$. Furthermore, if $V_1 \le V \le V_2$, then

$$\lambda_{V_1}(t) \le \lambda_V(t) \le \lambda_{V_2}(t). \tag{33}$$

Fix $f \in X \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$, $f \ge 0$, by **(iii)**,

$$V_1 := C^{-1}\frac{f}{\varphi_1} \le \frac{f}{u} \le V_2 := C\frac{f}{\varphi_1}, \tag{34}$$

for every $u \in Y_\lambda$ and $\lambda_1 < \lambda \le \lambda_1 + 1$.

It follows, from the properties of the function $\lambda_V(t)$, that there exists δ_0, such that for every $u \in Y_\lambda$ with $\lambda_1 < \lambda \le \lambda_1 + \delta_0$, there exist a unique t_λ and a unique eigenfunction φ of the equation

$$\mathcal{A}_{t_\lambda}\varphi - \lambda\varphi = (-\Delta + q + t_\lambda\frac{f}{u} - \lambda)\varphi = 0,$$

wich satisfy $\varphi(0) = \varphi_1(0)$. We define then the mapping T_λ by $T_\lambda(u) = \varphi$.

We prove now that there exists $\delta > 0$ (depending only on f) such that for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$ we have $T_\lambda \colon Y_\lambda \to Y_\lambda$. By (34) we know that there exists some $\varepsilon > 0$ such that

$$|t| \le \varepsilon \quad \Rightarrow \quad t\frac{f}{u} \le tV_2 \le 1.$$

Since the function $\lambda_{V_1}(t)$ is invertible, with a continuous inverse, there exists $\delta > 0$ such that

$$0 < \lambda_{V_1}(t) - \lambda_1 < \delta \quad \Rightarrow 0 < t < \varepsilon.$$

Using (33), $\lambda_{V_1}(t_\lambda) \le \lambda_V(t_\lambda) = \lambda$, so if $0 < \lambda - \lambda_1 < \delta$ then $t_\lambda \le \varepsilon$. Thus $T_\lambda(u) = \varphi \in Y_\lambda$.

The mapping T_λ is continuous. If a sequence $(u_n)_{n\in\mathbb{N}} \in Y_\lambda$ converge to $u \in Y_\lambda$ in $\mathcal{C}^0(\mathbb{R}^N)$, the corresponding sequence $(v_n = T_\lambda(u_n))_{n\in\mathbb{N}}$ converge to $v = T_\lambda(u)$ in $\mathcal{C}^0(\mathbb{R}^N)$. Indeed, the sequence $(v_n)_{n\in\mathbb{N}}$ is in the compact set Y_λ and any convergent subsequence clearly converges to $v = T_\lambda(u)$.

Applying the Schauder-Tychonoff fixed point theorem to the operator T_λ, we conclude that there exist $t_\lambda > 0$ and $u_\lambda \in Y_\lambda$ such that u_λ is a positive solution of the equation

$$(\mathcal{A} - \lambda + t_\lambda \frac{f}{u_\lambda})u_\lambda = 0 \text{ in } \mathbb{R}^N.$$

So the function $u = -\frac{u_\lambda}{t_\lambda}$ is the negative solution of the equation

$$-\Delta u + q(x)u - \lambda u = f \text{ in } \mathbb{R}^n$$

and this function satisfy the φ_1-negativity,

$$u \leq -\frac{C^{-1}}{t_\lambda}\varphi_1.$$

■

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On the Zeros of an Infinitely Often Differentiable Function and their Derivatives

ABSTRACT. In this paper, we investigate the structure of an infinitely often differentiable real function f defined on the interval $[0,1]$. We show that for such a function the set $\{t : \exists n \in \mathbb{N}_0 : f^{(n)}(t) = 0,\ f^{(n+1)}(t) \neq 0\}$ is at most countable, and if f is not a polynomial then the set $\{t : f^{(n)}(t) \neq 0,\ \forall n \in \mathbb{N}_0\}$ has the power $\mathbf{c}$.

KEY WORDS. C^∞-functions, derivatives of higher order, Cantor sets, Theorem of Cantor-Bendixsohn, sets of first category.

In this paper we investigate real functions f on $[0,1]$ which are infinitely often differentiable, where in the endpoints we consider the one-side derivatives. For such a given f we define the sets

$$E = \{t : \exists n \in \mathbb{N}_0 : f^{(n)}(t) = 0\} \tag{1}$$

and their complement

$$D = \{t : f^{(n)}(t) \neq 0,\ \forall n \in \mathbb{N}_0\}, \tag{2}$$

i.e. $E \cup D = [0,1]$. Obviously, if f is a polynomial then $E = [0,1]$. But it holds also the conversion:

Theorem 1 ([3], [5]) *Let f be an infinitely often differentiable real function over $[0,1]$. If $E = [0,1]$ then f is a polynomial.*

Obviously, for a polynomial f the set D from (2) is empty, so that $D = \emptyset$ if and only if f is a polynomial according to Theorem 1. In this paper we investigate the case $D \neq \emptyset$ and prove a general assertion concerning the structure of an infinitely often differentiable real function (Proposition 3). Theorem 1 is an immediately consequence of Proposition 3. The main results of this note are Theorem 6 and 7 which are proved by means of Proposition 3. In order to prove Proposition 3 we need some preparations.

Lemma 2 *Every closed set $F \subseteq [0,1]$ has a unique representation as union of three disjoint sets*

$$F = A_0 \cup B_0 \cup C_0 \tag{3}$$

where A_0 is an open set, B_0 is a nowhere dense perfect set and C_0 is at most countable, where A_0, B_0 and C_0 can be empty.

Proof: We assume that the closed set F is not countable. Then, owing to the Theorem of Cantor-Bendixsohn, cf. [6], p. 55, it is representable in the form

$$F = P_0 \cup Q_0$$

where P_0 is a nonempty perfect set and where Q_0 is at most countable. If P_0 is nowhere dense then it follows (3) with $A_0 = \emptyset$, $B_0 = P_0$ and $C_0 = Q_0$. Assume that P_0 is dense in the intervals $[a_n, b_n]$ $(n \in \mathbb{N}_0)$ where these intervals are maximal then we put

$$A_0 = \bigcup_n (a_n, b_n) \tag{4}$$

which is an open set with $A_0 \subseteq P_0$ since P_0 is closed. Consequently, the set $F_1 = P_0 \setminus A_0$ is nowhere dense and closed, and it holds $A_0 \cap F_1 = \emptyset$. If F_1 is countable then (3) is valid with A_0 from (4), $B_0 = \emptyset$ and $C_0 = F_1 \cup Q_0$. If the closed set F_1 is not countable then, again by the Theorem of Cantor-Bendixsohn, it is representable as

$$F_1 = P_1 \cup Q_1$$

where P_1 is a nonempty perfect set and where Q_1 is at most countable. In this case (3) is valid with A_0 from (4), $B_0 = P_1$ and $C_0 = Q_0 \cup Q_1$.

Assume that besides of (3) for F there exist a further representation

$$F = A_1 \cup B_1 \cup C_1. \tag{5}$$

If $A_0 \neq A_1$ then we can assume that there exist a point $x_0 \in A_0 \setminus A_1$. This means that there exist an interval $(\alpha, \beta) \subset A_0 \setminus A_1$. Since $F \setminus A_1 = B_1 \cup C_1$ is a set of first category and (α, β) is a set of second category by a Theorem of Baire, cf. e.g. [4], the relation $(\alpha, \beta) \subseteq F \setminus A_1$ is impossible. This implies that the case $A_0 \neq A_1$ cannot be. In the case $A_0 = A_1$ the set $P = F \setminus A_0 = F \setminus A_1$ is closed. Therefore it holds $B_0 = B_1$ since this set is exactly equal to the set of all points of condensation of P, cf. [6]. Finally, it follows $C_0 = C_1$, too ■

On the structure of an infinitely often differentiable function we have the

Proposition 3 *Let f be an infinitely often differentiable real function over $[0,1]$. Then the set E of all points t for which there exists an integer $n \in \mathbb{N}_0$ such that $f^{(n)}(t) = 0$ has a unique representation as union of three disjoint sets*

$$E = A \cup B \cup C \tag{6}$$

which have the following form: A is an open set, i.e.

$$A = \bigcup_j (\alpha_j, \beta_j), \tag{7}$$

B is the union of at most countably many nowhere dense perfect sets B_n with $B_n \subseteq B_{n+1}$, and C is at most countable, where A, B and C can be empty. In the case $A \neq \emptyset$ the function f is a polynomial on each interval $[\alpha_j, \beta_j]$.

Proof: Obviously, E is the union of the sets $E_n = \{t : f^{(n)}(t) = 0\}$ $(n \in \mathbb{N}_0)$, which are closed owing to the continuity of $f^{(n)}$. Hence, according to Lemma 2 for each $n \in \mathbb{N}_0$ the set E_n is representable as union of three disjoint sets

$$E_n = A_n \cup B_n \cup C_n \tag{8}$$

where A_n is an open set, B_n is a nowhere dense perfect set and C_n is at most countable, where A_n, B_n and C_n can be empty. Hence, for the union E of all E_n is representable as (6) where A and B are the union of all A_n, B_n, respectively, and

$$C = \bigcup_n C_n \setminus (A \cup B)$$

is at most countable. Thus A is an open set which has the form (7) where the components (α_i, β_i) are pairwise disjoint, and $A \cap C = B \cap C = \emptyset$.

For $t \in A_n$ and $t \in B_n$ we have $f^{(n+1)}(t) = 0$ so that $A_n \subseteq A_{n+1}$ and $B_n \subseteq B_{n+1}$, respectively. Hence, $A_n \cap B_n = \emptyset$ for all n implies that $A \cap B = \emptyset$, too.

The sets A_n, B_n and C_n are unique determined according to Lemma 2. This implies the uniqueness of A, B and C in (6).

Finally let be $A \neq \emptyset$. We remember that $A_m \subseteq A_n$ for $n > m$. Assume that $I_n = (a_n, b_n)$ and $I_m = (a_m, b_m)$ are components of A_n and A_m, respectively, then either $I_n = I_m$ or $\overline{I}_n \cap \overline{I}_m = \emptyset$. This follows from the fact that $f^{(n-1)}(t) = c \neq 0$ for $t \in \overline{I}_n$ and $f^{(n-1)}(t) = 0$ for $t \in \overline{I}_m$. Consequently, f is a polynomial on each interval $[\alpha_j, \beta_j]$. ■

Remarks 4 **1.** In case $E = [0,1]$ we have $A = (0,1)$, $C = \{0,1\}$, and f is a polynomial on $[0,1]$ so that Theorem 1 is a consequence of Proposition 3.

2. In case $A \neq \emptyset$ the endpoints of each component (α_i, β_i) belong to E. Between two intervals (α_i, β_i), (α_j, β_j) of A there exists at least one point $t_0 \notin E$. If namely $(\alpha_i, \beta_j) \subseteq E$ where $\alpha_i < \alpha_j$ then, owing to Theorem 1, the function f is equal to a polynomial of degree n. Hence, $(\alpha_i, \beta_j) \subseteq A_n$ which is impossible in view of the unique representation of A_n according to Proposition 3.

Let us consider some examples for the different possibilities of the sets E, A, B, C in Proposition 3. Obviously, if f is a polynomial then $E = [0,1]$, $A = (0,1)$, $B = \emptyset$ and $C = \{0,1\}$, but also the case $E = \emptyset$ is possible, e.g. for $f(t) = e^t$. For further possibilities let us consider the homogeneous integral-functional equation

$$\phi(t) = b \int_{at-a+1}^{at} \phi(\tau)d\tau \qquad \left(b = \frac{a}{a-1}\right) \tag{9}$$

with the real variable t and a parameter $a > 1$, cf. [1], [2]. The solutions of (9) were studied for $a = 3$ in Wirsching [9], for $a = 2$ in Schnabl [7] and Volk [8], and for $a > \frac{3}{2}$ in Wirsching [10]. In [1] it was shown that for $a > 1$ equation (9) has a C^∞-solution with the support $[0,1]$ which is uniquely determined by the normalization

$$\int_0^1 \phi(t)dt = 1. \tag{10}$$

In case $a = 2$ the solution ϕ has the property $\phi^{(n)}(t) = 0$ if and only if $t = \frac{k}{2^n}$ with $k \in 0, 1, \ldots, 2^n$, cf. [2], formula (4.8), so that in this case we have $A = B = \emptyset$ and C is the countable set of all dyadic rational numbers in $[0,1]$. In case $a > 2$ the solution ϕ is a polynomial on each component of an open Cantor set G with Lebesgue measure $|G| = 1$, and the set of all $t \notin G$ with $\phi^{(n)}(t) = 0$ with a certain $n \in \mathbb{N}$ is countable, cf. formula (4.7) in [2]. Hence, in this case we have $A = G$, i.e. $\overline{A} = [0,1]$, $B = \emptyset$ and C is the set of all endpoints of the components of G.

The following example shows that also the case $B \neq \emptyset$ is possible.

Example 5 Let f_0 be any infinitely often differentiable function over $[0,1]$ with $f_0(t) > 0$ for $0 < t < 1$ and $f_0^{(k)}(0) = f_0^{(k)}(1) = 0$ for all $k \in \mathbb{N}_0$, e.g.

$$f_0(t) = e^{\frac{1}{t(1-t)}}. \tag{11}$$

For a given nowhere dense perfect set $B_0 \subseteq [0,1]$ with $0, 1 \in B_0$ the open complement $G = [0,1] \setminus B_0$ is representable as union of pairwise disjoint intervals (a_j, b_j) $(j \in \mathbb{N})$. We define a function f by $f(t) = 0$ for $t \in B_0$ and by

$$f(t) = c_j f_0\left(\frac{t-a_j}{b_j-a_j}\right)$$

for $a_j < t < b_j$, and

$$c_j = \frac{1}{j\, M_j} \tag{12}$$

where

$$M_j = \max_{k\in\{0,\dots,j\}} \max_{a_j<t<b_j} \frac{1}{(b_j-a_j)^k \, \min\,(t-a_j, b_j-t)} \left| f_0^{(k)}\left(\frac{t-a_j}{b_j-a_j}\right)\right| . \tag{13}$$

The number M_j exists in view of the continuity of $f_0^{(k)}$ and $f_0^{(k+1)}(0) = f_0^{(k+1)}(1) = 0$ so that $c_j > 0$ for all j. Consequently, it holds $E_0 = B_0$. Obviously, for $a_j < t < b_j$ and $k \in \mathbb{N}_0$ it holds

$$f^{(k)}(t) = \frac{c_j}{(b_j-a_j)^k} f_0^{(k)}\left(\frac{t-a_j}{b_j-a_j}\right). \tag{14}$$

We show by induction with respect to k that $f^{(k)}(t) = 0$ for $t \in B_0$. This is true for $k = 0$ according to the definition of f. Assume that this is true for a fixed k. Let $t_0 \in B_0$ and $t_n \neq t_0$ a sequence which converges to t_0. If $t_n \in B_0$ then

$$\frac{f^{(k)}(t_n) - f^{(k)}(t_0)}{t_n - t_0} = 0.$$

Hence, it suffices to consider the case that $t_n \in [0,1] \setminus B_0$ for all $n \in \mathbb{N}$, i.e. $t_n \in (a_{j_n}, b_{j_n})$. Obviously, we need to investigate only two cases: **1.** the sequence j_n is bounded and **2.** $j_n \to \infty$ as $n \to \infty$. The first case is only possible if for $n \geq n_0$ all t_n belong to the same interval (a_j, b_j) and t_0 is an endpoint of (a_j, b_j). Then we have

$$\lim_{n\to\infty} \frac{f^{(k)}(t_n) - f^{(k)}(t_0)}{t_n - t_0} = 0$$

in view of $f_0^{(k+1)}(0) = f_0^{(k+1)}(1) = 0$. In the second case we can choose an integer n_0 such that $j_n \geq k$ for $n \geq n_0$. From (14) and $f^{(k)}(t_0) = 0$ we obtain

$$\left| \frac{f^{(k)}(t_n) - f^{(k)}(t_0)}{t_n - t_0} \right| = \frac{c_{j_n}}{(b_{j_n} - a_{j_n})^k \, |t_n - t_0|} \left| f_0^{(k)}\left(\frac{t_n - a_{j_n}}{b_{j_n} - a_{j_n}}\right)\right| .$$

Since $|t_0 - t_n| \geq \min(t_n - a_{j_n}, b_{j_n} - t_n)$ we get for $n \geq n_0$ in view of (12), (13) and $k \leq j_n$ that

$$\left| \frac{f^{(k)}(t_n) - f^{(k)}(t_0)}{t_n - t_0} \right| \leq \frac{1}{j_n} \to 0$$

for $n \to \infty$. Altogether we obtain $f^{(k+1)}(t_0) = 0$. According to Proposition 3 it holds $B_0 \subseteq B$ so that here we have an example for an infinitely often differentiable function f with $B \neq \emptyset$.

Theorem 6 *Let f be an infinitely often differentiable real function over $[0,1]$. Then the set $M = \{t : \exists n \in \mathbb{N}_0 : f^{(n)}(t) = 0, f^{(n+1)}(t) \neq 0\}$ is at most countable.*

Proof: For $n \in \mathbb{N}_0$ let M_n the set of all points $t \in [0,1]$ with $f^{(n)}(t) = 0$ and $f^{(n+1)}(t) \neq 0$. Hence, $M_n \subseteq E_n$ with the notations of Proposition 3, cf. (8), where A_n is an open set. Let (α, β) be a component of A_n then f is a polynomial of degree m. Hence, for $n < m$ the number of points $t \in (\alpha, \beta)$ with $f^{(n)}(t) = 0$ is finite and for $n \geq m$ there is no point with $f^{(n+1)}(t) \neq 0$. It follows that $M_n \cap A_n$ is at most countable. For $t \in B_n$ we have $f^{(n+1)}(t) = 0$ so that $M_n \cap B_n = \emptyset$, i.e. $M_n \subseteq A_n \cap C_n$. It follows that M is at most countable ■

Obviously, for a polynomial f the set $D = \{t : f^{(n)}(t) \neq 0,\ \forall n \in \mathbb{N}_0\}$ is empty.

Theorem 7 *Let f be an infinitely often differentiable real function over $[0,1]$. If f is not a polynomial then the set $D = \{t : f^{(n)}(t) \neq 0,\ \forall n \in \mathbb{N}_0\}$ has the power $\mathbf{c}$.*

Proof: Let D be a nonempty set. We apply Proposition 3 with the introduced notations. Obviously, the set D is the complement of E so that $E \subset [0,1]$ since $D \neq \emptyset$. We consider two cases:

1. Assume that there exists an interval $I = (a,b)$ without points of A. Then according to Proposition 3 it holds the disjoint decomposition

$$I = (I \cap B) \cup (I \cap C) \cup (I \cap D)$$

where the first and the second set on the right-hand side are sets of first category. Consequently, $I \cap D$ is a set of second category and so D has the power $\mathbf{c}$, cf. [4], 10.12.

2. Assume that $[0,1] \setminus A$ is nowhere dense in $[0,1]$, i.e. $\overline{A} = [0,1]$ where because of $D \neq \emptyset$ the case $A = (0,1)$ is excluded in view of Remark 4.1. It follows from Proposition 3 that A is the union of countably many open intervals (α_i, β_i) which are pairwise disjoint, cf. (7). Hence, the set $[0,1] \setminus A$ is a nowhere dense perfect set. Then there exists a continuous increasing function g with $g(0) = 0$, $g(1) = 1$ and $g(t) = g_i$ for $t \in (\alpha_i, \beta_i)$ with $g_i \neq g_j$ for $i \neq j$ where the countable set $g(A)$ of all g_i is dense in $[0,1]$, cf. Cantor's stair function. For the set

$$A^* = \bigcup_i [\alpha_i, \beta_i]$$

we have $g(A^*) = g(A) = \{g_i\}$ and the restriction of g to $[0,1] \setminus A^*$ is even strictly increasing and has the following property:

(i) The map $g : ([0,1] \setminus A^*) \mapsto [0,1] \setminus g(A^*)$ is bijektive.

According to Remark 4 the set D from (2) is a subset of $[0,1] \setminus A^*$. Next we show that for all n the sets $g(B_n)$ are nowhere dense. Assume that there exists an n such that $g(B_n)$ is dense in an interval (g_i, g_j) with $i \neq j$ then $[g_i, g_j] \subseteq g(B_n)$ since $g(B_n)$ is closed in view of the continuity of g. This implies owing to **(i)** that all points of the set $(\alpha_i, \beta_j) \setminus A$ belong to $B_n \subseteq E$ which is impossible, cf. Remark 4. Consequently, $g(B_n)$ is nowhere dense so that $g(B)$ is a set of first category. This is true also for the union $g(A) \cup g(B) \cup g(C)$ since $g(A)$ and $g(C)$ are at most countable sets. This implies that $g(D)$ is a set of second category so that it has the power $\mathbf{c}$, cf. [4]. Since $D \subseteq [0,1] \setminus A^*$ it follows from **(i)** that also the set D has the power $\mathbf{c}$ $\blacksquare$

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A Note on Iteratively Extendable Strings*

ABSTRACT. This scientific note introduces the notion of an iteratively extendable string within a language. It demonstrates that every language that has such an iteratively extendable string z contains infinitely many strings whose length is divisible by the length of z. Some consequences and applications of this result are given.

KEY WORDS. Formal languages, Pumping lemmas, Primes

1 Introduction

Consider a language, L. A string of the form $u_0v_1u_1v_2...u_{n-1}v_nu_n$ in L, where $v_1v_2...v_n$ is non-empty, is iteratively extendable within L if $u_0v_1^mu_1v_2^m...u_{n-1}v_n^mu_n$ is also in L, for every $m \geq 0$. In this scientific note, we prove that if there exists an iteratively extendable, z, within L, then L contains infinitely many strings whose length is divisible by the length of z. As a consequence of this, L contains infinitely many strings whose length differs from any prime. Thus, if there is a pumping lemma for a language family, such as the pumping lemma for the family of ETOL languages of finite index, then every infinite language in this family contains infinitely many strings whose length differs from any prime.

2 Definitions

This paper assumes that the reader is familiar with the theory of formal languages (see [1, 2, 3, 5]). For an alphabet, V, V^* represents the free monoid generated by V under the operation of concatenation. The identity of V^* is denoted by ε. Set $V^+ = V^* - \{\varepsilon\}$; algebraically, V^+ is thus the free semigroup generated by V under the operation of concatenation. For $w \in V^*$, $|w|$ denotes the length of w.

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Now, we introduce the notion of an iteratively extendable string within a language. Let $L \subseteq V^*$. A string $w \in L$ is *iteratively extendable within* L if $w = u_0v_1u_1v_2...u_{n-1}v_nu_n$ for some $n \geq 1$, where $u_i, v_j \in V^*$, $0 \leq i \leq n$, $1 \leq j \leq n$, $|v_1v_2...v_n| \geq 1$ and $u_0v_1^mu_1v_2^m...u_{n-1}v_n^mu_n \in L$ for all $m \geq 0$.

3 Results

Theorem 3.1 *Let L be a language over an alphabet V. For every iteratively extendable string $z \in L$, there exists an infinite language $L_z \subseteq L$ such that for each $x \in L_z$, $|x|$ is divisible by $|z|$.*

Proof: Let L be a language over an alphabet V. Let z be an iteratively extendable string in L. That is, $z = u_0v_1u_1v_2...u_{n-1}v_nu_n$ for some $n \geq 1$, where $u_i, v_j \in V^*$, $0 \leq i \leq n$, $1 \leq j \leq n$, $|v_1v_2...v_n| \geq 1$ and $u_0v_1^mu_1v_2^m...u_{n-1}v_n^mu_n \in L$ for all $m \geq 0$. Set $L_z = \{u_0v_1^ju_1v_2^j...u_{n-1}v_n^ju_n : j = i.|u_0v_1u_1v_2...u_{n-1}v_nu_n| + 1 \text{ for } i \geq 0\}$. Clearly, L_z is infinite and $L_z \subseteq L$. Consider any string $u_0v_1^ju_1v_2^j...u_{n-1}v_n^ju_n \in L_z$ with $j = i.|u_0v_1u_1v_2...u_{n-1}v_nu_n| + 1$ for some $i \geq 0$. Observe that $|u_0v_1^ju_1v_2^j...u_{n-1}v_n^ju_n| = |u_0v_1u_1v_2...u_{n-1}v_nu_n| + |v_1^{j-1}| + |v_2^{j-1}| + ... + |v_n^{j-1}| = |u_0v_1u_1v_2...u_{n-1}v_nu_n| + (j-1).|v_1| + (j-1).|v_2| + ... + (j-1).|v_n| = |u_0v_1u_1v_2...u_{n-1}v_nu_n| + (j-1).|v_1v_2...v_n| = |u_0v_1u_1v_2...u_{n-1}v_nu_n| + i.|u_0v_1u_1v_2...u_{n-1}v_nu_n|.|v_1v_2...v_n| = |u_0v_1u_1v_2...u_{n-1}v_nu_n|.(1 + i.|v_1v_2...v_n|) = |z|.(1 + i.|v_1v_2...v_n|)$. Thus, Theorem 3.1 holds. ■

Corollary 3.2 *Let L be a language and $z \in L$ be iteratively extendable string; then, L contains infinitely many strings whose length is divisible by $|z|$.*

To demonstrate some applications of the previous corollary, recall that almost every textbook about formal languages proves that $\{a^n : n \text{ is a prime}\}$ is not regular in a rather complex way (c.f. Example 3.2 in [1], Example 8.8 in [2], Example 4.1.3 in [3], and Example 7.3.2 in [5]). Notice, however, that Corollary 3.2 immediately implies this result because every infinite regular language contains infinitely many iteratively extendable strings as follows from the regular pumping lemma (see Section 4.1 in [3]). From a broader perspective, if there is a pumping lemma for a language family, then this family contains no infinite language in which the length of every string equals a prime, such as $\{a^n : n \text{ is a prime}\}$. To illustrate, consider the pumping lemma for the family of ETOL languages of finite index (see Theorem 3.13 in [4]). That is, let $G = (V, P, S, \Sigma)$ be an ETOL system of index k (for some $k \geq 1$) and let $L(G)$ be infinite. Then, there exist positive integers e and $\overline{e}$ such that, for every string w in $L(G)$ that is longer than e, there exists a positive integer $n \leq 2k$ such that w can be written in the form $w = u_0v_1u_1v_2...u_{n-1}v_nu_n$ with $|v_i| < \overline{e}$ for $1 \leq i \leq n$, $|v_1v_2...v_n| \geq 1$ and for every positive integer m, the string $u_0v_1^mu_1v_2^m...u_{n-1}v_n^mu_n \in L$. By Corollary 3.2 above, this family does not contain any infinite language in which the length of each string equals a prime.

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Finite and conditional completeness properties of generalized ordered sets

ABSTRACT. In particular, we show that if X is a set equipped with a transitive relation $\leq$, then the following completeness properties are equivalent:

(1) $\operatorname{lb}(\{x, y\}) \neq \emptyset$ for all $x, y \in X$, and $\inf(A) \neq \emptyset$ for all $A \subset X$ with $A \neq \emptyset$ and $\operatorname{lb}(A) \neq \emptyset$;

(2) $\inf(\{x, y\}) \neq \emptyset$ for all $x, y \in X$, and $\inf(A) \neq \emptyset$ for all $A \subset X$ with $A \neq \emptyset$, $\operatorname{lb}(A) \neq \emptyset$ and $\operatorname{ub}(A) \neq \emptyset$.

Thus, we obtain a substantial generalization of a basic theorem of Garrett Birkhoff which says only that in a conditionally complete lattice every nonempty subset which has a lower bound has a greatest lower bound.

KEY WORDS AND PHRASES. Generalized ordered sets, lower bound and infimum completenesses.

Introduction

Throughout this paper, X will denote an arbitrary set equipped with an arbitrary binary relation $\leq$. Thus, X may be considered as a generalized ordered set or an ordered set without axioms.

The set X will be called reflexive, transitive, antisymmetric and total if the relation $\leq$ has the corresponding property. If X is total, then for any $x, y \in X$ we have either $x \leq y$ or $y \leq x$. Thus, in particular, X is reflexive.

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For any $A \subset X$, the members of the families

$$\operatorname{lb}(A) = \big\{ x \in X : \quad \forall\ a \in A : \quad x \leq a \big\}$$

and

$$\operatorname{ub}(A) = \big\{ x \in X : \quad \forall\ a \in A : \quad a \leq x \big\}$$

are called the lower and upper bounds of A in X, respectively. And the members of the families

$$\min(A) = A \cap \operatorname{lb}(A), \qquad \max(A) = A \cap \operatorname{ub}(A),$$
$$\inf(A) = \max\big(\operatorname{lb}(A)\big), \qquad \sup(A) = \min\big(\operatorname{ub}(A)\big)$$

are called the minima, maxima, infima and suprema of A in X, respectively.

First, we show that the following extension of [2, Lemma 2.23, p. 46] is true.

Lemma *If X is transitive, and moreover $A_i \subset X$ and $\inf(A_i) \neq \emptyset$ for all $i \in I$, then*

$$\operatorname{lb}\Big(\bigcup_{i\in I} A_i\Big) = \operatorname{lb}\Big(\bigcup_{i\in I} \inf(A_i)\Big) \quad \textit{and} \quad \inf\Big(\bigcup_{i\in I} A_i\Big) = \inf\Big(\bigcup_{i\in I} \inf(A_i)\Big).$$

Then, by using this lemma, we show that the following generalization of [1, Theorem 9, p. 115] is also true.

Theorem *If X is transitive, then the following completeness properties are equivalent:*

(1) $\operatorname{lb}\big(\{x, y\}\big) \neq \emptyset$ *for all* $x, y \in X$, *and* $\inf(A) \neq \emptyset$ *for all* $A \subset X$ *with* $A \neq \emptyset$ *and* $\operatorname{lb}(A) \neq \emptyset$;

(2) $\inf\big(\{x, y\}\big) \neq \emptyset$ *for all* $x, y \in X$, *and* $\inf(A) \neq \emptyset$ *for all* $A \subset X$ *with* $A \neq \emptyset$, $\operatorname{lb}(A) \neq \emptyset$ *and* $\operatorname{ub}(A) \neq \emptyset$.

Remark If in particular X is partially ordered, then by using the above lemma we also show that the following completeness properties are equivalent:

(1) $\inf(\{x, y\}) \neq \emptyset$ for all $x, y \in X$;

(2) $\inf(A) \neq \emptyset$ for every finite, nonvoid subset A of X.

In this respect, it is noteworthy that to prove a counterpart of the above equivalence for lb instead of inf, the transitivity of the relation $\leq$ is again sufficient.

1 Lower and upper bounds

Concerning lower and upper bounds, we shall only quote here the following simple theorems of [5] .

Theorem 1.1 *If* $A_i \subset X$ *for all* $i \in I$, *then*

$$\mathrm{lb}\Big(\bigcup_{i \in I} A_i\Big) = \bigcap_{i \in I} \mathrm{lb}\,(A_i)\,.$$

Corollary 1.2 *If* $A \subset B \subset X$, *then* $\mathrm{lb}\,(B) \subset \mathrm{lb}\,(A)$.

Proof: Note that $\mathrm{lb}\,(B) = \mathrm{lb}\,(A \cup B) = \mathrm{lb}\,(A) \cap \mathrm{lb}\,(B) \subset \mathrm{lb}\,(A)$.

Corollary 1.3 *If* $A \subset X$, *then* $\mathrm{lb}\,(A) = \bigcap_{a \in A} \mathrm{lb}\,(a)$, *where* $\mathrm{lb}\,(a) = \mathrm{lb}\,\big(\{a\}\big)$.

Theorem 1.4 *If* $A\,, B \subset X$, *then*

$$A \subset \mathrm{lb}\,(B) \quad\Longleftrightarrow\quad B \subset \mathrm{ub}\,(A)\,.$$

Corollary 1.5 *If* $A \subset X$, *then* $A \subset \mathrm{ub}\,\big(\mathrm{lb}\,(A)\big)$.

Proof: Clearly, $\mathrm{lb}\,(A) \subset \mathrm{lb}\,(A)$. Hence, by Theorem 1.4, the required inclusion already follows.

Theorem 1.6 *If* $A \subset X$, *then*

$$\min\,(A) = A \cap \inf\,(A) \qquad \textit{and} \qquad \inf\,(A) = \mathrm{lb}\,(A) \cap \mathrm{ub}\,\big(\mathrm{lb}\,(A)\big)\,.$$

Corollary 1.7 *If* $A \subset X$, *then* $\min\,(A) \subset \inf\,(A) \subset \mathrm{lb}\,(A) \subset \mathrm{lb}\,\big(\inf\,(A)\big)$.

Proof: By Theorem 1.6, we have not only $\min\,(A) \subset \inf\,(A) \subset \mathrm{lb}\,(A)$, but also $\inf\,(A) \subset \mathrm{ub}\,\big(\mathrm{lb}\,(A)$. Hence, by Theorem 1.4, the required inclusion already follows.

The importance of reflexivity, totality and antisymmetry will only be illuminated here by the following basic theorems of [6].

Theorem 1.8 *If* $\Phi = \mathrm{lb}\,, \min$ *or* $\inf$, *then the following assertions are equivalent :*

(1) X *is reflexive ;*

(2) $x \in \Phi\,(x)$ *for all* $x \in X$.

Theorem 1.9 *The following assertions are equivalent :*

(1) X *is reflexive ;*

(2) $\min(x) \neq \emptyset$ *for all* $x \in X$ *;*

(3) $\min(x) = \{x\}$ *for all* $x \in X$*.*

Theorem 1.10 *The following assertions are equivalent :*

(1) X *is total ;*

(2) $\min\big(\{x, y\}\big) \neq \emptyset$ *for all* $x, y \in X$*.*

Theorem 1.11 *If X is reflexive and* $\Phi = \min$ *or* $\inf$*, then the following assertions are equivalent :*

(1) X *is antisymmetric ;*

(2) $\operatorname{card}\big(\Phi(A)\big) \leq 1$ *for all* $A \subset X$*.*

Corollary 1.12 *If X is reflexive and antisymmetric, then* $\inf(x) = \{x\}$ *for all* $x \in X$ *.*

2 The importance of transitivity

Concerning the importance of transitivity, we shall only quote here the following basic theorems of [6]. Hints for the proofs are included only for the reader's convenience.

Theorem 2.1 *The following assertions are equivalent :*

(1) X *is transitive ;*

(2) $y \in \operatorname{lb}(x)$ *and* $z \in \operatorname{lb}(y)$ *imply* $z \in \operatorname{lb}(x)$ *for all* $x, y, z \in X$*;*

(3) $B \subset \operatorname{lb}(A)$ *and* $C \subset \bigcup_{b\in B} \operatorname{lb}(b)$ *imply* $C \subset \operatorname{lb}(A)$ *for all* $A, B \subset X$*.*

Proof: To prove the less obvious implication $(2) \Longrightarrow (3)$, suppose that (2) and the conditions of (3) hold. If $c \in C$, then since $C \subset \bigcup_{b\in B} \operatorname{ub}(b)$ there exists $b \in B$ such that $c \in \operatorname{lb}(b)$. Moreover, if $a \in A$, then since $b \in B \subset \operatorname{lb}(A)$ we have $b \in \operatorname{lb}(a)$. Hence, by using (2), we can already infer that $c \in \operatorname{lb}(a)$. Now, since $a \in A$ and $c \in C$ were arbitrary, it is clear that $c \in \operatorname{lb}(A)$, and thus $C \subset \operatorname{lb}(A)$. Therefore, (3) also holds.

From the above theorem, it is clear that in particular we also have

Corollary 2.2 *If* X *is transitive, then* $x \in \operatorname{lb}(A)$ *and* $y \in \operatorname{lb}(x)$ *imply* $y \in \operatorname{lb}(A)$ *for all* $A \subset X$ *and* $x, y \in X$.

Theorem 2.3 *If* X *is transitive, then*

$$\operatorname{lb}(x) = \operatorname{lb}(A) \qquad \textit{for all} \qquad A \subset X \quad \textit{and} \quad x \in \inf(A).$$

Proof: If $x \in \inf(A)$, then by Corollaries 1.7 and 1.2 we have $\operatorname{lb}(A) \subset \operatorname{lb}\big(\inf(A)\big) \subset \operatorname{lb}(x)$ even if X is not transitive.

Moreover, if $x \in \inf(A)$, then Corollary 1.7 we also have $x \in \operatorname{lb}(A)$. Hence, by Corollary 2.2, it is clear $y \in \operatorname{lb}(x)$ implies $y \in \operatorname{lb}(A)$. Therefore, $\operatorname{lb}(x) \subset \operatorname{lb}(A)$ is also true.

Corollary 2.4 *If* X *is transitive, then*

$$\operatorname{lb}(A) = \operatorname{lb}\big(\inf(A)\big) \qquad \textit{for all} \qquad A \subset X \quad \textit{with} \quad \inf(A) \neq \emptyset.$$

Proof: By Theorems 1.1 and 2.3, it is clear that

$$\operatorname{lb}\big(\inf(A)\big) = \bigcap_{x \in \inf(A)} \operatorname{lb}(x) = \bigcap_{x \in \inf(A)} \operatorname{lb}(A) = \operatorname{lb}(A).$$

Now, in addition to the results of [6], we can also easily prove the following

Theorem 2.5 *If* X *is transitive and* $A_i \subset X$ *for all* $i \in I$, *then*

(1) $\operatorname{lb}\Big(\bigcup_{i \in I} A_i\Big) = \operatorname{lb}\bigg(\Big(\bigcup_{i \in J} A_i\Big) \cup \Big(\bigcup_{i \in I \setminus J} \inf(A_i)\Big)\bigg)$,

(2) $\inf\Big(\bigcup_{i \in I} A_i\Big) = \inf\bigg(\Big(\bigcup_{i \in J} A_i\Big) \cup \Big(\bigcup_{i \in I \setminus J} \inf(A_i)\Big)\bigg)$,

where $J = \big\{ i \in I : \ \inf(A_i) = \emptyset \big\}$.

Proof: By Theorem 1.1 and Corollary 2.4, we have

$$\operatorname{lb}\Big(\bigcup_{i \in I} A_i\Big) = \bigcap_{i \in I} \operatorname{lb}(A_i) =$$

$$\Big(\bigcap_{i \in J} \operatorname{lb}(A_i)\Big) \cap \Big(\bigcap_{i \in I \setminus J} \operatorname{lb}(A_i)\Big) = \Big(\bigcap_{i \in J} \operatorname{lb}(A_i)\Big) \cap \Big(\bigcap_{i \in I \setminus J} \operatorname{lb}\big(\inf(A_i)\big)\Big) =$$

$$\operatorname{lb}\Big(\bigcup_{i \in J} A_i\Big) \cap \operatorname{lb}\Big(\bigcup_{i \in J} \inf(A_i)\Big) = \operatorname{lb}\bigg(\Big(\bigcup_{i \in J} A_i\Big) \cup \Big(\bigcup_{i \in I \setminus J} \inf(A_i)\Big)\bigg).$$

Hence, by the definition of inf, it is clear that (2) is also true.

From Theorem 2.5, we can at once get the following generalization of the second part of [2, Lemma 2.23, p. 46].

Corollary 2.6 *If X is transitive, and moreover $A_i \subset X$ and $\inf(A_i) \neq \emptyset$ for all $i \in I$, then*

(1) $\operatorname{lb}\Big(\bigcup_{i\in I} A_i\Big) = \operatorname{lb}\Big(\bigcup_{i\in I} \inf(A_i)\Big)$;

(2) $\inf\Big(\bigcup_{i\in I} A_i\Big) = \inf\Big(\bigcup_{i\in I} \inf(A_i)\Big)$.

3 Finite lower bound completenesses

Definition 3.1 *We say that*

(1) X *is two-lb-complete if* $\operatorname{lb}\big(\{x, y\}\big) \neq \emptyset$ *for all* $x, y \in X$;

(2) X *is two-inf-complete if* $\inf\big(\{x, y\}\big) \neq \emptyset$ *for all* $x, y \in X$;

(3) X *is finitely quasi-lb-complete if* $\operatorname{lb}(A) \neq \emptyset$ *for all finite, nonvoid subset* A *of* X;

(4) X *is finitely quasi-inf-complete if* $\inf(A) \neq \emptyset$ *for all finite, nonvoid subset* A *of* X.

Remark 3.2 By Corollary 1.7, it is clear that 'two-inf-completeness' implies 'two-lb-completeness', and 'finite quasi-inf-completeness' implies 'finite quasi-lb-completeness'.

Moreover, by using the well-orderedness of the set $\mathbb{N}$ of all natural numbers, we can prove the following

Theorem 3.3 *If X is transitive, then the following assertions are equivalent:*

(1) X *is two-lb-complete;*

(2) X *is finitely quasi-lb-complete.*

Proof: By Definition 3.1, it is clear that $(2) \Longrightarrow (1)$ even if X is not partially ordered.

To prove the converse implication, suppose on the contrary that (1) holds, but (2) does not hold. That is, $\operatorname{lb}\big(\{x, y\}\big) \neq \emptyset$ for all $x, y \in X$, and $\operatorname{lb}(A) = \emptyset$ for some finite, nonvoid subset A of X.

Denote by $\mathcal{A}$ the family of all finite, nonvoid subsets A of X such that $\operatorname{lb}(A) = \emptyset$. Then, by the above assumptions, it is clear that $\mathcal{A} \neq \emptyset$ and $\operatorname{card}(A) > 2$ for all $A \in \mathcal{A}$. Define

$$M = \big\{ \operatorname{card}(A) : \quad A \in \mathcal{A} \big\}.$$

Then, we evidently have $\emptyset \neq M \subset \mathbb{N}$ such that $1 \notin M$ and $2 \notin M$.

Hence, since $\mathbb{N}$ is well-ordered, we can infer that $\min(M) \neq \emptyset$. Therefore, there exists $n \in \min(M)$. This implies that $n \in M$ and $n \in \mathrm{lb}(M)$. Hence, it is clear that $2 < n \in \mathbb{N}$ such that $n \leq m$ for all $m \in M$. Moreover, we can also state that there exists $A \in \mathcal{A}$ such that $n = \mathrm{card}(A)$.

Thus, we can choose $a \in A$, and define $B = A \setminus \{a\}$. Then, it is clear that B is a finite nonvoid subset of X such that $k = \mathrm{card}(B) < \mathrm{card}(A) = n$. Therefore, $\mathrm{lb}(B) \neq \emptyset$ also holds. Namely, $\mathrm{lb}(B) = \emptyset$ would imply that $B \in \mathcal{A}$. Hence, we could infer that $k = \mathrm{card}(B) \in M$, and thus $n \leq k$, which would be a contradiction.

Now, we can choose $\beta \in \mathrm{lb}(B)$ and $\gamma \in \mathrm{lb}\big(\{a, \beta\}\big)$. Then, by Theorem 1.1, it is clear that $\gamma \in \mathrm{lb}(a)$ and $\gamma \in \mathrm{lb}(\beta)$. Hence, by using Corollary 2.2, we can infer that $\gamma \in \mathrm{lb}(B)$. Therefore, by Theorem 1.1, we also have $\gamma \in \mathrm{lb}(a) \cap \mathrm{lb}(B) = \mathrm{lb}\big(\{a\} \cup B\big) = \mathrm{lb}(A)$. This contradiction proves that $(1) \Longrightarrow (2)$.

A particular case of the following theorem is usually considered to be quite obvious in the advanced theory of lattices. The proofs given here and in [4, p. 40] show that this attitude cannot be completely justified.

Theorem 3.4 *If X is partially ordered, then the following assertions are equivalent:*

(1) *X is two-inf-complete;*

(2) *X is finitely quasi-inf-complete.*

Proof: By Definition 3.1, it is clear that $(2) \Longrightarrow (1)$ even if X is not partially ordered.

To prove the converse implication, suppose on the contrary that (1) holds, but (2) does not hold. Denote by $\mathcal{A}$ the family of all finite, nonvoid subsets A of X such that $\inf(A) = \emptyset$. Then, by using a similar argument as in the proof of Theorem 3.3, we can see that there exists $A \in \mathcal{A}$ such that by choosing $a \in A$ and defining $B = A \setminus \{a\}$, we already have $\inf(B) \neq \emptyset$.

Now, by Theorem 1.11, it is clear that there exists $x \in X$ such that $\inf(B) = \{x\}$. Moreover, by Corollary 1.12, we also have $\inf\big(\{a\}\big) = \{a\}$. Hence, by using Corollary 2.6, we can infer that

$$\inf(A) = \inf\big(\{a\} \cup B\big) = \inf\big(\inf\big(\{a\}\big) \cup \inf(B)\big) = \inf\big(\{a\} \cup \{x\}\big) = \inf\big(\{a, x\}\big).$$

However, this is already a contradiction. Namely, by $A \in \mathcal{A}$, we have $\inf(A) = \emptyset$. While, by (1), we have $\inf\big(\{a, x\}\big) \neq \emptyset$. Therefore, the implication $(1) \Longrightarrow (2)$ is also true.

4 Conditional infimum completenesses

Definition 4.1 *We say that*

(1) X *is pseudo-inf-complete if* $\inf(A) \neq \emptyset$ *for all* $A \subset X$ *with* $\operatorname{lb}(A) \neq \emptyset$ *;*

(2) X *is semi-inf-complete if* $\inf(A) \neq \emptyset$ *for all* $A \subset X$ *with* $A \neq \emptyset$ *and* $\operatorname{lb}(A) \neq \emptyset$ *;*

(3) X *is almost pseudo-inf-complete if* $\inf(A) \neq \emptyset$ *for all* $A \subset X$ *with* $\operatorname{lb}(A) \neq \emptyset$ *and* $\operatorname{ub}(A) \neq \emptyset$ *;*

(4) X *is almost semi-inf-complete if* $\inf(A) \neq \emptyset$ *for all* $A \subset X$ *with* $A \neq \emptyset$ *,* $\operatorname{lb}(A) \neq \emptyset$ *and* $\operatorname{ub}(A) \neq \emptyset$ *.*

Remark 4.2 Thus, 'pseudo-inf-complete' implies both 'semi-inf-complete' and 'almost pseudo-inf-complete', and 'almost pseudo-inf-complete' implies 'almost-semi-inf-complete'.

Moreover, by using Corollary 2.6, we can also prove the following

Theorem 4.3 *If* X *is transitive and* $\operatorname{ub}(X) \neq \emptyset$*, then the following assertions are equivalent :*

(1) X *is two-lb-complete and pseudo-inf-complete ;*

(2) X *is two-inf-complete and almost pseudo-inf-complete .*

Proof: By the corresponding definitions, it is clear that (1) $\Longrightarrow$ (2) even if X is not transitive or $\operatorname{ub}(X) = \emptyset$. Moreover, from Remark 3.2 we know that the first part (2) always implies that of (1). Therefore, to prove the converse implication (2) $\Longrightarrow$ (1), we need only show that (2) implies the second part of (1).

For this, assume that (2) holds, and moreover $A \subset X$ such that $\operatorname{lb}(A) \neq \emptyset$. If $A = \emptyset$, then by the corresponding definitions it is clear that

$$\inf(A) = \inf(\emptyset) = \max\big(\operatorname{lb}(\emptyset)\big) = \max(X) = \operatorname{ub}(X),$$

and thus $\inf(A) \neq \emptyset$. Therefore, we may assume that $A \neq \emptyset$, i.e., there exists $a \in A$. Define

$$B = \bigcup_{x \in A} \inf\big(\{a, x\}\big).$$

Then, by Corollary 2.6, it is clear that

$$\operatorname{lb}(B) = \operatorname{lb}\bigg(\bigcup_{x \in A} \inf\big(\{a, x\}\big)\bigg) = \operatorname{lb}\bigg(\bigcup_{x \in A} \{a, x\}\bigg) = \operatorname{lb}(A).$$

Moreover, by using the duals of Theorems 1.1 and Corollary 1.2, and Corollaries 1.7 and 1.5, we can see that

$$\operatorname{ub}(B) = \operatorname{ub}\Big(\bigcup_{x \in A} \inf(\{a, x\}) \Big) = \bigcap_{x \in A} \operatorname{ub}(\inf(\{a, x\})) \supset \bigcap_{x \in A} \operatorname{ub}(\operatorname{lb}(\{a, x\})) \supset \bigcap_{x \in A} \{a, x\} \supset \{a\}.$$

Therefore, $\operatorname{lb}(B) \neq \emptyset$ and $\operatorname{ub}(B) \neq \emptyset$ also hold. Thus, by the almost pseudo-inf-completeness of X, we also have $\inf(B) \neq \emptyset$.

Now, it remains to note that by Corollary 2.6 we also have

$$\inf(A) = \inf\Big(\bigcup_{x \in A} \{a, x\} \Big) = \inf\Big(\bigcup_{x \in A} \inf(\{a, x\}\Big) = \inf(B).$$

Therefore, $\inf(A) \neq \emptyset$ also holds, and thus X is pseudo-inf-complete.

The following theorem is a generalization of the first part of [1, Theorem 9, p. 115]. Our subsequent sketch of the proof shows that the two and a half line proof given there may only be considered as a hint.

Theorem 4.4 *If X is transitive, then the following assertions are equivalent:*

(1) *X is two-lb-complete and semi-inf-complete;*

(2) *X is two-inf-complete and almost semi-inf-complete.*

Proof: Again, it is clear that (1) $\Longrightarrow$ (2) even if X is not transitive. Moreover, the first part (2) always implies that of (1). Therefore, to prove the converse implication (2) $\Longrightarrow$ (1), we need only show that (2) implies the second part of (1).

For this, assume that (2) holds, and moreover $A \subset X$ such that $A \neq \emptyset$ and $\operatorname{lb}(A) \neq \emptyset$. Choose $a \in A$, and define

$$B = \bigcup_{x \in A} \inf\big(\{a, x\}\big).$$

Then, it is clear that $\emptyset \neq B \subset X$. Namely, by the two-inf-completeness of X and the definition of B, we evidently have $\emptyset \neq \inf\big(\{a, a\}\big) \subset B$.

Moreover, from the proof of Theorem 4.3, we can see that $\operatorname{lb}(B) \neq \emptyset$ and $\operatorname{ub}(B) \neq \emptyset$ also hold. Thus, by the almost semi-inf-completeness of X, we also have $\inf(B) \neq \emptyset$. Now, it remains to note that by the proof of Theorem 4.3, we also have $\inf(A) = \inf(B)$. Therefore, $\inf(A) \neq \emptyset$ also holds, and thus X is semi-inf-complete.

5 Two illustrating examples

Example 5.1 If $X = \{ a, b, c \}$ such that we only have

$$a \leq b, \qquad b \leq c, \qquad c \leq a \qquad \text{and} \qquad x \leq x \qquad \text{for all} \qquad x \in X,$$

then X is total and antisymmetric. Moreover, X is two-inf-complete, but not finitely quasi-lb-complete. Thus, by Remark 3.2, X is also two-lb-complete, but not finitely quasi-inf-complete.

To check that X is not finitely quasi-lb-complete, note that

$$\text{lb}(a) = \{ a, c \}, \qquad \text{lb}(b) = \{ a, b \}, \qquad \text{lb}(b) = \{ b, c \}.$$

Therefore, by Corollary 1.3, we have

$$\text{lb}(X) = \text{lb}(a) \cap \text{lb}(b) \cap \text{lb}(c) = \emptyset,$$

and thus X is not finitely quasi-lb-complete.

Moreover, we can quite similarly see that

$$\text{lb}\big(\{ a, b \}\big) = \{a\}, \qquad \text{lb}\big(\{ a, c \}\big) = \{c\}, \qquad \text{lb}\big(\{ b, c \}\big) = \{b\}.$$

Hence, since by the dual of Theorem 1.8 we have $x \in \max(x)$ for all $x \in X$, it is already clear that

$$\inf\big(\{ x, y \}\big) = \max\big(\text{lb}\big(\{ x, y \}\big)\big) \neq \emptyset$$

for all $x, y \in X$ with $x \neq y$. Moreover, by Theorem 1.8, we also have $x \in \inf(x)$, and hence $\inf(x) \neq \emptyset$ for all $x \in X$. Therefore, X is two-inf-complete.

Remark 5.2 In addition to Example 5.1 and Corollary 2.4, it is worth noticing that if X is reflexive, antisymmetric and

$$\text{lb}(A) = \text{lb}\big(\inf(A)\big)$$

for all $A \subset X$ with $\text{card}(A) = 2$ and $\inf(A) \neq \emptyset$, then X is necessary transitive. Thus, by Theorem 3.4, X is finitely quasi-inf-complete if and only if it is two-inf-complete.

To check the transitivity of X, by Theorem 2.1 it is enough to show only that if $x \in X$,

$$y \in \text{lb}(x), \qquad z \in \text{lb}(y) \qquad \text{and} \qquad x \neq y,$$

then $z \in \text{lb}(x)$. For this, note if $A = \{ x, y \}$, then by Theorem 1.8 and Corollaries 1.3 and 1.5 we have

$$y \in \text{lb}(x) \cap \text{lb}(y) = \text{lb}(A) \qquad \text{and} \qquad y \in A \subset \text{ub}\big(\text{lb}(A)\big).$$

Hence, by Theorems 1.6 and 1.11, it is clear that

$$y \in \mathrm{lb}\,(A) \cap \mathrm{ub}\,\big(\mathrm{lb}\,(A)\big) = \inf\,(A)\,, \qquad \text{and thus} \qquad \{y\} = \inf\,(A)\,.$$

Now, by using our former assumptions and observations, we can already easily see that

$$z \in \mathrm{lb}\,(y) = \mathrm{lb}\,\big(\inf\,(A)\big) = \mathrm{lb}\,(A) = \mathrm{lb}\,(x) \cap \mathrm{lb}\,(y) \subset \mathrm{lb}\,(x)\,.$$

Example 5.3 If $X = \{\, a,\, b,\, c,\, d \,\}$ such that we only have

$$a \le a\,, \qquad a \le c\,, \qquad a \le d\,, \qquad b \le d\,, \qquad c \le d\,,$$

then X is transitive and antisymmetric. Moreover, X is almost semi-inf-complete, but not semi-inf-complete.

To check this, note that

$$\begin{array}{llll} \mathrm{lb}\,(a) = \{a\}\,, & \mathrm{lb}\,(b) = \emptyset\,, & \mathrm{lb}\,(c) = \{a\}\,, & \mathrm{lb}\,(d) = \{\, a,\, b,\, c \,\}\,; \\ \mathrm{ub}\,(a) = \{\, a,\, c,\, d \,\}\,, & \mathrm{ub}\,(b) = \{d\}\,, & \mathrm{ub}\,(c) = \{d\}\,, & \mathrm{ub}\,(d) = \emptyset\,. \end{array}$$

Hence, by Theorem 1.6 and the dual of Corollary 1.3, it is clear that

$$\inf\,(d) = \mathrm{lb}\,(d) \cap \mathrm{ub}\,\big(\mathrm{lb}\,(d)\big) = \mathrm{lb}\,(d) \cap \mathrm{ub}\,(a) \cap \mathrm{ub}\,(b) \cap \mathrm{ub}\,(c) = \emptyset\,.$$

Therefore, X is not semi-inf-complete.

Moreover, by Corollary 1.3, it is clear that, for any $A \subset X$,

$$\mathrm{lb}\,(A) \ne \emptyset \implies A \subset \{\, a,\, c,\, d \,\} \qquad \text{and} \qquad \mathrm{ub}\,(A) \ne \emptyset \implies A \subset \{\, a,\, b,\, c \,\}\,.$$

Therefore, if $A \ne \emptyset$, $\mathrm{lb}\,(A) \ne \emptyset$ and $\mathrm{ub}\,(A) \ne \emptyset$, then we necessarily have

$$A = \{a\} \qquad \text{or} \qquad A = \{c\} \qquad \text{or} \qquad A = \{\, a,\, c \,\}\,.$$

Hence, by Corollary 1.3, it is clear that $\mathrm{lb}\,(A) = \{a\}$. Moreover, by Theorem 1.6, it is clear that

$$\inf\,(A) = \mathrm{lb}\,(A) \cap \mathrm{ub}\,\big(\,\mathrm{lb}\,(A)\big) = \mathrm{lb}\,(a) \cap \mathrm{ub}\,(a) = \{a\}\,.$$

Therefore, X is almost semi-inf-complete.

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Iterative Processes with Random Errors for Fixed Point of Φ-Pseudocontractive Operator*

ABSTRACT. The purpose of this paper is to introduce Φ-pseudo-contractive operators—a class of operators which is much more general than the important class of strongly pseudo-contractive operators and ϕ-strongly pseudocontractive operators, and to study problems of approximating fixed points by Ishikawa and Mann iterative processes with random errors for Φ-pseudocontractive operators. As applications, the iterative approximative methods for the solution of equation with Φ-accretive operator are obtained. The results presented in this paper improve, generalize and unify the corresponding results of Chang [3]-[4], Chidume [5]-[10], Deng [12], Ding [13]-[14], Liu [16], Osilike [18], Xu [19], Zhou [20].

KEY WORDS AND PHRASES. Duality mapping, Mann iteration sequence, Ishikawa iteration sequence, Φ -pseudocontractive operator.

1 Introduction and Preliminaries

Throughout this paper, we assume that X is a real Banach space with dual X^*, $(\cdot,\ \cdot)$ denotes the generalized duality pairing. The mapping $J : X \to 2^{X^*}$ defined by

$$Jx = \{j \in X^* : (x, j) = \|x\|\|j\|, \|j\| = \|x\|\}^{[1]} \quad \forall x \in X \tag{1.1}$$

is called the normalized duality mapping.

We recall the following two iterative processes due to Ishikawa [15] and Mann [17], respectively.

(a) Let K be a nonempty convex subset of X, and $T : K \longrightarrow K$ be a mapping. For any given $x_0 \in K$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \qquad y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad (n \geq 0)$$

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is called Ishikawa iteration sequence, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying some conditions.

(b) In particular, if $\beta_n = 0$ for all $n \geq 0$ in (a), then $\{x_n\}$ defined by

$$x_0 \in K, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad (n \geq 0)$$

is called the Mann iteration sequence.

The consideration of error terms is an important part of any theory of iteration methods. For this reason, Xu [19] introduced the following definitions.

(A) Let K be a nonempty convex subset of X and $T : K \longrightarrow K$ a mapping. For any given $x_0 \in K$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad y_n = \hat{\alpha}_n x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n \quad (n \geq 0) \tag{1.2}$$

is called Ishikawa iteration sequence with random errors. Here $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K; $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\}$ and $\{\hat{\gamma}_n\}$ are six sequences in $[0,1]$ satisfying

$$\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1, \quad for \ \ all \ \ n \geq 0.$$

(B) In particular, if $\hat{\beta}_n = \hat{\gamma}_n = 0$ for all $n \geq 0$ in (A), the $\{x_n\}$ defined by

$$x_0 \in K, \quad x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n, \quad (n \geq 0) \tag{1.3}$$

is called Mann iteration sequence with random errors .

Note that the Ishikawa and Mann iterative processes are all special cases of the Ishikawa and Mann iterative processes with random errors.

Now, we introduce Φ-pseudocontractive operators as follows.

Definition 1.1 *Let K be nonempty subset of X. An operator $T : K \to X$ is said to be Φ-pseudocontractive, if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ and $j(x-y) \in J(x-y)$ such that*

$$(Tx - Ty, \ j(x-y)) \leq \|x-y\|^2 - \Phi(\|x-y\|) \quad \forall x, y \in K. \tag{1.4}$$

An operator $A : K \to X$ is said to be Φ-accretive, if

$$(Ax - Ay, \ j(x-y)) \geq \Phi(\|x-y\|) \ \ \forall x, y \in K. \tag{1.5}$$

Remark 1.1 Obvious, if a Φ-pseudocontractive operator has a fixed point then it is unique.

The pseudocontractive operator is intimately connected with accretive operator [11]. It is easy to verify that the operator T is Φ-pseudoaccretive if and only if $I - T$ is Φ-accretive where I is a identity mapping on X. Hence, the mapping theory for accretive operators is intimately connected with the fixed point theory for pseudocontraction operators.

We like to point out: every ϕ-strongly pseudocontractive operator must be the Φ-pseudocontractive operator with Φ:$[0, \infty) \to [0, \infty)$ defined by $\Phi(s) = \phi(s)s$, and every strongly pseudocontractive operator is ϕ-strongly pseudocontractive with $\phi : [0, \infty) \to [0, \infty)$ defined by $\phi(s) = ks$ where $k \in (0,\ 1)$.

In 1994, Chidume proved a related result that deals with the Ishikawa iterative approximation of the fixed point for the class of Lipschitz strictly pseudocontractive mappings in uniformly smooth Banach space. At the same time, he put forth an open problem: It is not known whether or not the Ishikawa iteration method converges for a continuous strongly pseudocontractive mapping. Recently, this open problem has been studied extensively by researchers (see, for example[3-4, 6-10, 12-14, 18-20]) in the case of T is strongly pseudocontractive or ϕ-strongly pseudocontractive operators respectively.

The objective of this paper is to introduce the Φ-pseudocontractive operators — a class of operators which is much more general than the important class of strongly pseudocontractive operators and ϕ-strongly pseudocontractive operators, and to study problems of approximating fixed point by Ishikawa and Mann iterative processes with random errors for Φ-pseudocontractive operators. We will prove that the answer of Chidume's open problem is affirmative if X is an arbitrary Banach space and $T : K \to K \subset X$ is uniformly continuous Φ-quasicontractive. furthermore, if X is an uniformly smooth Banach space and T may be not continuous, the answer of Chidume's open problem also is affirmative. As applications, the iterative approximation methods for the solution of equation with Φ-accretive operator are obtained. The results presented in this paper improve, generalize and unify results of Chang [3]-[4], Chidume [5]-[10], Deng [12], Ding [13]-[14], Liu [16], Osilike [18], Xu [19], Zhou [20].

The following two Lemmas play crucial roles in the proofs of our main results.

Lemma 1.1 ([4]) *If X be a real Banach space then there exists $j(x + y) \in J(x + y)$ such that*

$$\|x + y\|^2 \leq \|x\|^2 + 2(y,\ j(x + y)) \quad \forall x,\ y \in X. \tag{1.6}$$

Lemma 1.2 ([2](Browder)) *X is uniformly smooth(equivalently X^* is uniformly convex) Banach space if and only if J is single-valued and uniformly continuous on any bounded subset of X.*

2 The Convergence Theorems in Arbitrary Banach Space

If X is an arbitrary real Banach space with dual X^*, we can prove following theorems.

Theorem 2.1 *Let X be an arbitrary real Banach space with dual X^* and $K \subset X$ a nonempty bounded convex subset. Let $T : K \to K$ be an uniformly continuous Φ-pseudocontractive mapping. Suppose the Ishikawa iteration sequence $\{x_n\}$ with random errors be defined by* (1.2) *with parameters*

(i) $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \hat{\beta}_n = \lim_{n\to\infty} \hat{\gamma}_n = 0$ *and* $\sum_{n=0}^{+\infty} \beta_n = +\infty$;

(ii) $\gamma_n = o(\beta_n)$.

If $F(T) \neq \emptyset$ then for arbitrary $x_0 \in K$, $\{x_n\}$ converges strongly to unique fixed point of T.

Proof: From Remark 1.1, we have that $F(T) = \{q\}$. Putting $M = \sup\{\|x\| : x \in K\} + \|q\|$. Since $\|y_n - x_{n+1}\| = \|(\hat{\alpha}_n - \alpha_n)x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n - \beta_n T y_n - \gamma_n u_n)\| \to 0(as\ n \to \infty)$, therefore,

$$e_n := \|Ty_n - Tx_{n+1}\| \to 0(as\ n \to \infty)$$

by the uniformly continuity of T.

Let $2\sigma = \inf\{\|x_{n+1} - q\| : \ n \geq 0\}$. If $\sigma > 0$, then $\Phi(\|x_{n+1} - q\|) > \Phi(\sigma)$ for all $n \geq 0$. From the conditions (i) and (ii) there exists an integer $N_0 > 0$ such that

$$0 \leq \gamma_n, \beta_n \leq \frac{1}{6} \quad and \quad o(\beta_n) \leq \beta_n \Phi(\sigma) \quad \forall n \geq N_0. \tag{2.7}$$

By (1.4), (1.6) and (2.7) we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\alpha_n(x_n - q) + \beta_n(Ty_n - q) + \gamma_n(u_n - q)\|^2 \\ &\leq \alpha_n^2\|x_n - q\|^2 + 2\beta_n(Ty_n - q,\ j(x_{n+1} - q)) \\ &\quad +2\gamma_n(u_n - q,\ j(x_{n+1} - q)) \\ &\leq \alpha_n^2\|x_n - q\|^2 + 2\beta_n(Ty_n - Tx_{n+1},\ j(x_{n+1} - q)) \\ &\quad +2\beta_n(Tx_{n+1} - q,\ j(x_{n+1} - q)) + 2M^2\gamma_n \\ &\leq (1 - \beta_n - \gamma_n)^2\|x_n - q\|^2 + 2\beta_n\|x_{n+1} - q\|^2 \\ &\quad -2\beta_n\Phi(\|x_{n+1} - q\|) + 2M\beta_n e_n + 2M^2\gamma_n \\ &\leq \|x_n - q\|^2 + \tfrac{3}{2}M^2\beta_n^2 + 3M\beta_n e_n + 3M^2\gamma_n \\ &\quad -2\Phi(\|x_{n+1} - q\|)\beta_n \\ &= \|x_n - q\|^2 + o(\beta_n) - 2\Phi(\|x_{n+1} - q\|)\beta_n \end{aligned} \tag{2.8}$$

for all $n \geq N_0$. It follows from (2.8) that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + o(\beta_n) - 2\Phi(\sigma)\beta_n \leq \|x_n - q\|^2 + -\Phi(\sigma)\beta_n$$

for all $n \geq N_0$. By induction, we obtain

$$\Phi(\sigma) \sum_{j=N}^{+\infty} \beta_j \leq \|x_N - q\|^2 \leq M^2. \tag{2.9}$$

(2.9) is in contradiction with $\sum_{j=0}^{+\infty} \beta_j = +\infty$. From this contradiction, we get $\sigma = 0$. Therefore, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to q$ as $j \to \infty$. For any given $\varepsilon > 0$ there exists an integer $j_0 \geq N_0$ such that $\|x_{n_j} - q\| < \varepsilon$ for all $j \geq j_0$. If j_0 is fixed, we will prove that $\|x_{n_{j_0}+k} - q\| < \varepsilon$ for all integers $k \geq 1$.

The proof is by induction. For $k = 1$, suppose $\|x_{n_{j_0}+1} - q\| \geq \varepsilon$. It follows from (2.8) and $\Phi(\|x_{n_{j_0}+1} - q\|) \geq \Phi(\varepsilon)$ that

$$\varepsilon^2 \leq \|x_{n_{j_0}+1} - q\|^2 \leq \|x_{n_{j_0}} - q\|^2 + o(\beta_{n_{j_0}}) - 2\beta_{n_{j_0}}\Phi(\varepsilon) \leq \|x_{n_{j_0}} - q\|^2 < \varepsilon^2.$$

It is a contradiction. Hence, $\|x_{n_{j_0}+1} - q\| < \varepsilon$ holds for $k = 1$. Assume now that $\|x_{n_{j_0}+p} - q\| < \varepsilon$ for some integer $p > 1$. We prove $\|x_{n_{j_0}+p+1} - q\| < \varepsilon$. Again, assuming the contrary, Using (2.8), $\Phi(\|x_{n_{j_0}+p+1} - q\|) > \Phi(\varepsilon)$ and (2.7), as above, it leads to a contradiction as follows

$$\varepsilon^2 \leq \|x_{n_{j_0}+p+1} - q\|^2 \leq \|x_{n_{j_0}+p} - q\|^2 + o(\beta_{n_{j_0}+p}) - 2\beta_{n_{j_0}+p}\Phi(\varepsilon) \leq \|x_{n_{j_0}+p} - q\|^2 < \varepsilon^2$$

Where $n_{j_0} + p \geq n_{j_0} \geq j_0 \geq N_0$. Therefore, $\|x_{n_{j_0}+k} - q\| < \varepsilon$ holds for all integers $k \geq 1$, so that $x_{n_{j_0}+k} \to q$ as $k \to \infty$.
The Proof is completed.□

Remark 2.1 Theorem 2.1 improves a number of results (for example, Theorem 3.4 of [7] and Theorem 4 of [11]). A prototype for $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\}$ and $\{\hat{\gamma}_n\}$ in Theorem 2.1 is

$$\alpha_n = \frac{n^2 + 3n + 1}{(n+2)^2}, \quad \beta_n = \frac{1}{n+2}, \quad \gamma_n = \frac{1}{(n+2)^2}, \quad \hat{\alpha}_n = \frac{n+1}{n+3}$$

and

$$\hat{\beta}_n = \hat{\gamma}_n = \frac{1}{n+3} \quad \forall n \geq 0.$$

Theorem 2.2 *Let X, K and T be as in Theorem 2.1. If q is a fixed point of T in K and the Mann iteration sequence $\{x_n\}$ is defined by (1.3) with parameters*

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{+\infty} \beta_n = +\infty$;

(ii) $\gamma_n = o(\beta_n)$ *then $\{x_n\}$ converges strongly to unique fixed point of T.*

Theorem 2.3 *Suppose that $K \subset X$ is a nonempty bounded convex subset with $K + K \subseteq K$ and $A : K \to K$ is an uniformly continuous Φ-accretive operator. For any given $f \in K$ the equation $Ax = f$ has unique solution in K.*

Proof: We define $S : K \to K$ by $Sx = f + x - Ax$ for all $x \in K$. It is easy to see that S is uniformly continuous Φ-pseudocontractive. Clearly, q is a fixed point of S in K if and only if that q is a solution of the equation $Ax = f$. It follows from Theorem 2.1 or Theorem 2.2 above that the equation $Ax = f$ has unique solution in K.
The proof is completed. □

3 The Convergence Theorems in Uniformly Smooth Banach Space

Let X be a real uniformly smooth Banach space. Now we prove the following theorems.

Theorem 3.1 *Suppose that $K \subset X$ is a nonempty bounded convex subset and $T : K \to K$ is a Φ-pseudocontractive operator. If T has a fixed point and the Ishikawa iteration sequence $\{x_n\}$ is defined by (1.2) with parameters*

(i) $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \hat{\beta}_n = 0$ *and* $\sum_{n=0}^{+\infty} \beta_n = +\infty$;

(ii) $\hat{\gamma}_n = o(\hat{\beta}_n)$ *and* $\gamma_n = o(\beta_n)$,

then iteration sequence $\{x_n\}$ converges strongly to unique fixed point of T.

Proof: From Definition 1.1, we know that $F(T)$ is singleton. Setting $F(T) = \{q\}$ and $M = \sup\{\|x\| : x \in K\} + \|q\|$. Since $\|(y_n - q) - (x_{n+1} - q)\| = \|(\hat{\alpha}_n - \alpha_n)x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n - \beta_n T y_n - \gamma_n u_n\| \to 0$ $(as\ n \to \infty)$ and $\|(y_n - q) - (x_n - q)\| = \|(\hat{\alpha}_n - 1)x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n\| \to 0$ $(as\ n \to \infty)$, thus the uniformly continuity of j ensures that

$$e_n := \|j(y_n - q) - j(x_{n+1} - q)\| \to 0\ (as\ n \to \infty)$$

and

$$s_n := \|j(y_n - q) - j(x_n - q)\| \to 0\ (as\ n \to \infty).$$

Using (1.4) and (1.6), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n(x_n - q) + \beta_n(Ty_n - q) + \gamma_n(u_n - q)\|^2 \\
&\le \|\alpha_n(x_n - q)\|^2 + 2\beta_n(Ty_n - q,\ j(x_{n+1} - q)) \\
&\quad +2\gamma_n(u_n - q,\ j(x_{n+1} - q)) \\
&\le \|\alpha_n(x_n - q)\|^2 + 2\beta_n(Ty_n - q,\ j(y_n - q)) \\
&\quad +2\beta_n(Ty_n - q,\ j(x_{n+1} - q) - j(y_n - q)) \\
&\quad +2M^2\gamma_n \\
&\le \alpha_n^2\|x_n - q\|^2 + 2\beta_n\|y_n - q\|^2 - 2\beta_n\Phi(\|y_n - q\|) \\
&\quad +2M\beta_n e_n + 2M^2\gamma_n \\
&\le (1 - \beta_n)^2\|x_n - q\|^2 + 2\beta_n\|y_n - q\|^2 \\
&\quad -2\beta_n\Phi(\|y_n - q\|) + o(\beta_n)
\end{aligned} \tag{3.10}$$

for all $n \geq 0$. Similarly,

$$\begin{aligned}
\|y_n - q\|^2 &= \|\hat{\alpha}_n(x_n - q) + \hat{\beta}_n(Tx_n - q) + \hat{\gamma}_n(v_n - q)\|^2 \\
&\leq \|\hat{\alpha}_n(x_n - q)\|^2 + 2\hat{\beta}_n(Tx_n - q,\ j(y_n - q)) \\
&\quad +2\hat{\gamma}_n(u_n - q,\ j(y_n - q)) \\
&\leq \hat{\alpha}_n^2\|x_n - q\|^2 + 2\hat{\beta}_n(Tx_n - q,\ j(x_n - q)) \\
&\quad +2\hat{\beta}_n(Tx_n - q,\ j(y_n - q) - j(x_n - q)) + 2M^2\hat{\gamma}_n \\
&\leq \hat{\alpha}_n^2\|x_n - q\|^2 + 2\hat{\beta}_n\|x_n - q\|^2 - 2\hat{\beta}_n\Phi(\|x_n - q\|) \\
&\quad +2M\hat{\beta}_n s_n + 2M^2\hat{\gamma}_n \\
&\leq \|x_n - q\|^2 + M^2\hat{\beta}_n^2 + 2M\hat{\beta}_n s_n + 2M^2\hat{\gamma}_n \\
&\leq \|x_n - q\|^2 + o(\hat{\beta}_n)
\end{aligned} \tag{3.11}$$

for all $n \geq 0$. Substituting (3.11) into (3.10) and simplifying, we obtain

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + o(\beta_n) - 2\beta_n\Phi(\|y_n - q\|) \quad \forall n \geq 0 \tag{3.12}$$

where $o(\beta_n) \geq 0$. Let $2\sigma = \inf\{\|y_n - q\| :\ n \geq 0\}$. If $\sigma > 0$, then $\Phi(\|y_n - q\|) > \Phi(\sigma) > 0$ for all $n \geq 0$, and so, there exists an integer $N > 0$ such that $o(\beta_n) < \beta_n\Phi(\sigma)$ for all $n \geq N$. It follows from (3.12) that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \beta_n\Phi(\sigma) \quad \forall n \geq N.$$

By induction, we obtain

$$\Phi(\sigma)\sum_{j=N}^{+\infty}\beta_j \leq \|x_N - q\|^2 \leq M^2. \tag{3.13}$$

(3.13) is in contradiction with $\sum_{j=0}^{+\infty}\beta_j = +\infty$. It follows from the contradiction that $\sigma = 0$. Therefore, there exists a subsequence $\{y_{n_j}\} \subset \{y_n\}$ such that $y_{n_j} \to q$ as $j \to \infty$. Since $\lim_{j\to\infty}\|x_{n_j} - q\| = \lim_{j\to\infty}\hat{\alpha}_{n_j}\|x_{n_j} - q\| \leq \lim_{j\to\infty}\|y_{n_j} - q\| + M\lim_{j\to\infty}(\hat{\beta}_{n_j} + \hat{\gamma}_{n_j}) = 0$, the subsequence $\{x_{n_j}\}$ converges strongly to q. So, we know that maybe $\{x_n\}$ converges to q and we cannot assure $\{x_n\}$ is not convergent. But, there are other conditions of $\{x_n\}$, such that $\{x_n\}$ converges to q. Since $x_{n_j} \to q$ as $j \to \infty$, for any given $\varepsilon > 0$ there exists an integer $j_0 > 0$ such that $\|x_{n_j} - q\| < \varepsilon$ for all $j \geq j_0$, and $2M(|\alpha_n - \hat{\alpha_n}| + \beta_n + \hat{\beta_n} + \gamma_n + \hat{\gamma_n}) < \varepsilon$ and $o(\beta_n) \leq \beta_n\Phi(\varepsilon/2)$ for all $n \geq n_{j_0}$. If j_0 is fixed, we will prove that $\|x_{n_{j_0}+k} - q\| < \varepsilon$ for all integers $k \geq 1$.

The proof is by induction. For $k = 1$, suppose $\|x_{n_{j_0}+1} - q\| \geq \varepsilon$. Then, (1.2) implies that $\|y_{n_{j_0}} - q\| > \varepsilon/2$. In fact, we have

$$\varepsilon \leq \|x_{n_{j_0}+1} - q\| \leq \|y_{n_{j_0}} - q\| + M(|\alpha_{n_{j_0}} - \hat{\alpha_{n_{j_0}}}| + \beta_{n_{j_0}} + \hat{\beta_{n_{j_0}}} + \gamma_{n_{j_0}} + \hat{\gamma_{n_{j_0}}}) < \|y_{n_{j_0}} - q\| + \varepsilon/2.$$

From $\Phi(\|y_{n_{j_0}} - q\|) > \Phi(\varepsilon/2)$ and using (3.12), we obtain

$$\varepsilon^2 \leq \|x_{n_{j_0}+1} - q\|^2 \leq \|x_{n_{j_0}} - q\|^2 + o(\beta_{n_{j_0}}) - 2\beta_{n_{j_0}}\Phi(\varepsilon/2) \leq \|x_{n_{j_0}} - q\|^2 < \varepsilon^2.$$

It is a contradiction. So, $\|x_{n_{j_0}+1}-q\|<\varepsilon$ holds for $k=1$. Assume now that $\|x_{n_{j_0}+p}-q\|<\varepsilon$ for some integer $p>1$. We prove $\|x_{n_{j_0}+p+1}-q\|<\varepsilon$. Again, assuming the contrary, as above, it leads to a contradiction. Hence, $\|x_{n_{j_0}+k}-q\|<\varepsilon$ holds for all integers $k\geq 1$, so that $x_n\to q$ as $n\to\infty$, i.e., $\lim_{k\to\infty} x_{n_{j_0}+k}=q$.
The Proof is completed. □

Remark 3.1 Theorem 3.1 gives an affirmative answer to Chidume's open problem when T is Φ-quasicontractive. The corresponding results (see, for example, Theorem 3.3 of [4], Theorem 2 of [5], Theorem 3.1 of [13], Theorem 2 of [16] and Theorem 3.3 of [19]) are all special cases of Theorem 3.1 in the following senses:

1) T may not be continuous, therefore, T may not be Lipschitz, also;

2) T may not be strongly pseudocontractive or ϕ-strongly pseudocontractive;

3) the random errors of iterative processes have been considered appropriately;

4) the condition (iii) of Chidume's Theorem in [5] is dropped.

We like to point out: the iteration parameters $\{\alpha_n\}$, $\{\beta_n\},\{\gamma_n\},\{\hat{\alpha}_n\},\{\hat{\beta}_n\}$ and $\{\hat{\gamma}_n\}$ in Theorem 3.1 do not depend on any geometric structure of the Banach space X and on any property of the operator T, but, the selection of the parameters is deal with the convergence rate of the iteration. A prototype for $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\hat{\alpha}_n\},\{\hat{\beta}_n\}$ and $\{\hat{\gamma}_n\}$ in our theorem is

$$\alpha_n=\hat{\alpha}_n=\frac{n^2+3n+1}{(n+2)^2},\quad \beta_n=\hat{\beta}_n=\frac{1}{n+2}$$

and

$$\gamma_n=\hat{\gamma}_n=\frac{1}{(n+2)^2}\quad \forall n\geq 0.$$

In the Theorem 3.1, if $\hat{\beta}_n=\hat{\gamma}_n=0$ for all $n\geq 0$, then we obtain a result that deals with the Mann iterative process with random errors as follows.

Theorem 3.2 *Let K be a nonempty bounded convex subset of X and $T: K\to K\subset X$ a Φ-pseudocontractive operator. If T has a fixed point and the Mann iteration sequence $\{x_n\}$ is defined by (1.3) with parameters*

(i) $\lim_{n\to\infty}\beta_n=0$ *and* $\sum_{n=0}^{+\infty}\beta_n=+\infty$;

(ii) $\gamma_n=o(\beta_n)$.

Then $\{x_n\}$ converges strongly to unique fixed point of T.

Theorem 3.3 *Suppose that $K \subset X$ is a nonempty bounded convex subset with $K + K \subseteq K$ and $A : K \to K$ is a Φ-accretive operator. For any given $f \in K$ the equation $Ax = f$ has unique solution in K.*

Proof: We define $S : K \to K$ by $Sx = f + x - Ax$ for all $x \in K$. It is easy to see that S is Φ-pseudocontractive. Clearly, q is a fixed point of S if and only if that q is a solution of the equation $Ax = f$. It follows from Theorem 2.1 or Theorem 2.2 above that the equation $Ax = f$ has an unique solution in K.
The proof is completed. □

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