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BEZUGSMÖGLICHKEITEN: Universität Rostock
Universitätsbibliothek, Schriftentausch
18051 Rostock
Tel.: +49-381-498 22 81
Fax: +49-381-498 22 68
e-mail: maria.schumacher@ub.uni-rostock.de

Universität Rostock
Institut für Mathematik
18051 Rostock
Tel.: +49-381-498 6551
Fax: +49-381-498 6553
e-mail: romako@uni-rostock.de

DRUCK: Universitätsdruckerei Rostock

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Lothar Berg

Iterative functional equations

ABSTRACT. Functional equations for iterates are solved or approximated by means of associated difference equations. Some examples and three Open Problems are pointed out.

KEY WORDS. Functional equations, difference equations, iterates, approximations.

1 Introduction

Let y be a strictly monotonous continuous function $\mathbb{R} \mapsto I \subset \mathbb{R}$ and $y^{[-1]}$ its inverse $I \mapsto \mathbb{R}$, then the functions

$$f^{[n]}(x) = y\left(n + y^{[-1]}(x)\right) \tag{1.1}$$

with integer n are the n-th iterates of the strictly increasing function $f = f^{[1]} : I \mapsto I$. Classical books concerning iterates are Aczél [1] and Kuczma [4], and recent surveys are Baron and Jarczyk [3], and Targonski [8]. Let us mention that the functions (1.1) are solutions of the well known translation equation, which also can be considered in the multidimensional case, cf. [2].

The connection (1.1) between y and the iterates of f enables us to solve the functional equation

$$f^{[k]} = F\left(x, f, \dots, f^{[k-1]}\right) \tag{1.2}$$

with $k \in \mathbb{N}$ and $F : I^k \mapsto I$ by means of the associated difference equation

$$y(t+k) = F(y(t), y(t+1), \dots, y(t+k-1)) \tag{1.3}$$

with the same F. Namely, any strictly monotonous continuous solution $y : \mathbb{R} \mapsto I$ of (1.3) yields a strictly increasing solution $f : I \mapsto I$ by means of (1.1).

In the case that F is a homogeneous function of degree 1, equation (1.2) has the so called characteristic solutions $f = rx$, so far as r is a real solution of the characteristic equation

$$r^k = F(1, r, \dots, r^{k-1}),$$

cf. Matkowski and Zhang [5] for linear homogeneous F.

In the following we mostly deal with the case $k = 2$. In the next section we give some examples for the method in the Introduction, and in the last two sections we deal with further methods for solving (1.2), at least approximatively. Three Open Problems are offered to the reader.

2 Linear associated equations

As a special case of (1.2) we consider the functional equation with constant coefficients

$$f^{[2]} = af + bx, \tag{2.1}$$

which was studied in detail by Matkowski and Zhang [5], see also the references therein. The associated difference equation (1.3) reads

$$y(t+2) = ay(t+1) + by(t) \tag{2.2}$$

and has the general solution

$$y(t) = \begin{cases} cp^t + dq^t & \text{for } p \neq q, \\ (ct+d)p^t & \text{for } p = q, \end{cases} \tag{2.3}$$

where c and d are arbitrary 1-periodic functions, and p, q the solutions of the corresponding characteristic equation

$$r^2 = ar + b. \tag{2.4}$$

In the following we restrict ourselves to the case that p, q are positive, and c, d some real constants.

The case of arbitrary real p, q different from zero can be reduced to the foregoing one by determining the function $g = f^{[2]}$ out of the equation

$$g^{[2]} + (2b - a^2)g + b^2x = 0$$

with the squares of p, q as zeros of its characteristic equation. Afterwards, f is to determine as fractional iterate $f = g^{[1/2]}$, cf. [4, Theorems 15.7 and 15.9]. However, not all fractional iterates are solutions of (2.1), especially, in the case $pq < 0$ there exist at most the characteristic solutions of (2.1), disregarding exceptional cases, cf. [5, Theorem 6]. Let us mention that, if f is a solution of (2.1), then $g = -f(-x)$ is also a solution of it (what is trivial for odd f).

Applying formula (2.3) in the case $p \neq q$, we find that the iterates (1.1) of the solutions f of (2.1) can be written as

$$f^{[n]}(x) = cp^{n+y^{[-1]}(x)} + dq^{n+y^{[-1]}(x)} . \tag{2.5}$$

We introduce a real constant $s \neq 1$ such that

$$q = p^s , \tag{2.6}$$

and moreover the function

$$u(x) = cp^{y^{[-1]}(x)} .$$

Hence, (2.5) can be written as

$$f^{[n]}(x) = p^n u + Ap^{sn}u^s$$

with $A = dc^{-s}$. Since $f^{[0]}(x) = x$, we have to determine u by inversion of

$$x = u + Au^s , \tag{2.7}$$

and the foregoing functions turn into

$$f^{[n]}(x) = p^{sn}x + (p^n - p^{sn})u . \tag{2.8}$$

Since u must be strictly monotonous in x, we have to choose $A > 0$ for $s > 0$, and $A < 0$ for $s < 0$, whereas A remains arbitrary for $s = 0$.

Let us consider three examples for solutions of equation (2.1).

Example 2.1 In the case $s = -1$ we find from (2.7)

$$u = \frac{1}{2}\left(x + \sqrt{x^2 - 4A}\right) ,$$

where also the negative sign of the root would be possible, and (2.8) with $k = 1$ yields the solutions

$$f(x) = \frac{1}{2}\left(p + \frac{1}{p}\right) x + \frac{1}{2}\left(p - \frac{1}{p}\right) \sqrt{x^2 - 4A} . \tag{2.9}$$

Example 2.2 In the case $s = 2$ we find from (2.7)

$$u = -B + \sqrt{B^2 + 2Bx}$$

with $B = 1/(2A)$ and $x \geq 0$, and therefore the solutions

$$f(x) = p^2 x + (p - p^2)\left(\sqrt{B^2 + 2Bx} - B\right) \tag{2.10}$$

with $B = 1/(2A)$ and $x \geq 0$.

Example 2.3 In the case $s = 3$ we get analogously by means of Cardano's formula the solutions

$$f(x) = p^3 x + \frac{3}{2}C(p - p^3)\left(\sqrt[3]{\sqrt{x^2 + C^3} + x} - \sqrt[3]{\sqrt{x^2 + C^3} - x}\right) \tag{2.11}$$

with $C^3 = 4/(27A)$.

Further examples are possible, e.g. in the cases $s = -2, 0$ or $\frac{3}{2}$.

3 Approximate solutions

In order to find an approximate solution of (1.2) with continuous F, we consider (1.3) for integer $t = n$ and write this equation as

$$y_{n+k}(z) = F(y_n(z), y_{n+1}(z), \ldots, y_{n+k-1}(z)), \tag{3.1}$$

where z is a certain parameter. Let the solution $y_n(z)$ be continuous and strongly increasing in z, so that $x = y_n(z)$ can be inverted by $z = y_n^{[-1]}(x)$, and put $f_n(x) = y_{n+1}\left(y_n^{[-1]}(x)\right)$. Replacing z in (3.1) by $y_n^{[-1]}(x)$ and considering that $y_{n+2}\left(y_n^{[-1]}(x)\right) = f_{n+1}(f_n(x))$, $y_{n+3}\left(y_n^{[-1]}(x)\right) = f_{n+2}(f_{n+1}(f_n(x)))$ etc., we see that the resulting equation converges to (1.3) in case that f_n converges to f.

We try this method for the example

$$f^{[2]} = x(1 + f) \tag{3.2}$$

with $x \geq 0$ and the nonlinear associated difference equation

$$y_{n+2} = y_n(1 + y_{n+1}) \tag{3.3}$$

from Stević [6], [7].

Proposition *Let y_n be the solution of (3.3) subject to the initial conditions $y_0 = y_1 = z$, then the corresponding functions $f_n = y_{n+1}\left(y_n^{[-1]}\right)$ satisfy*

$$f_{2n} < f_{2n+2} < f_{2n+3} < f_{2n+1} \tag{3.4}$$

for all $x > 0$ and all integers $n \geq 0$.

Proof: Obviously, all y_n and all f_n are continuous and strictly increasing and hence invertible. From the associated difference equation (3.3) it follows

$$f_{n+1}(f_n) = x(1 + f_n)\,, \tag{3.5}$$

and this equation implies

$$f_{n+1} = (1 + x) f_n^{[-1]}\,. \tag{3.6}$$

From $f_0 = x$, $f_1 = x + x^2$, $f_2(f_1) = x + x^2 + x^3$ and $f_3(f_2) = x + x^2 + x^3 + x^4$ it easily follows $f_0 < f_1$, $f_0(f_1) < f_2(f_1)$ and $f_3(f_2) < f_1(f_2)$, so that (3.4) is satisfied for $n = 0$, disregarding the inequality in the middle. From (3.6) it follows

$$\begin{aligned} f_{n+2} - f_{n+1} &= (1 + x)\left(f_{n+1}^{[-1]} - f_n^{[-1]}\right)\,, \\ f_{n+2} - f_n &= (1 + x)\left(f_{n+1}^{[-1]} - f_{n-1}^{[-1]}\right)\,, \end{aligned}$$

and since $f < g$ implies $f^{[-1]} > g^{[-1]}$, the inequalities (3.4) follow for all n by induction. □

Corollary *The functions f_{2n} converge to a function g, and the functions f_{2n+1} to a function h with $g \leq h$ and both*

$$g(h) = x(1 + h)\,, \qquad h(g) = x(1 + g)\,.$$

Open Problem 1 Prove that $g = h$.

By means of DERIVE we find that the solutions of (3.3) with $y_0 = y_1 = z$ are polynomials with the first terms

$$\begin{aligned} y_{2n} &= z + nz^2 + n(n-1)z^3 + n(n-1)^2 z^4 + \frac{n}{6}(n-1)(6n^2 - 13n + 5)z^5\,, \\ y_{2n+1} &= z + nz^2 + n^2 z^3 + \frac{n}{2}(n-1)(2n+1)z^4 + \frac{n}{6}(n-1)(6n^2 - n - 4)z^5\,, \end{aligned}$$

and that the corresponding functions f_n are power series with the first terms

$$\left.\begin{aligned} f_{2n} &= x + nx^3 - \frac{3}{2}n(n+1)x^4 + \frac{n}{2}(4n^2 + 11n + 3)x^5\,, \\ f_{2n+1} &= x + x^2 - nx^3 + \frac{n}{2}(3n+1)x^4 - n^2(2n+1)x^5\,. \end{aligned}\right\} \tag{3.7}$$

Obviously, it is not possible in these formulas to go to the limit $n \to \infty$ term by term. Nevertheless, since

$$nx^3 - \frac{3}{2}n^2x^4 + 2n^3x^5 + \frac{5}{2}n^4x^6 + \dots = \frac{x^2}{2}\left(1 - \frac{1}{(1+nx)^2}\right)$$

for small nx, we expect the approximations

$$f_{2n} = x + \frac{x^2}{2} - \frac{x^2}{2(1+nx)^2}\,, \qquad f_{2n+1} = x + \frac{x^2}{2} + \frac{x^2}{2(1+nx)^2}\,.$$

Open Problem 2 Prove that the functions f_n with the first terms (3.7) converge to a solution f of (3.2) with the expansion

$$f = x + \frac{1}{2}x^2 - \frac{1}{16}x^4 + \frac{1}{16}x^5 - \frac{1}{64}x^6 + \dots \tag{3.8}$$

for small x, and the asymptotic expansion

$$f = x^\lambda + \lambda - \frac{\lambda^2}{x} - \frac{\lambda^3}{2x^\lambda} + \frac{1}{x^{\lambda+1}} + \dots \tag{3.9}$$

as $x \to \infty$, where $\lambda = \frac{1}{2}\left(\sqrt{5}+1\right)$ is the positive root of $\lambda^2 = \lambda + 1$.

The terms in (3.8) and (3.9) can be found by inserting suitable power series with indeterminate coefficients into (3.2) and comparing coefficients.

4 A new method

Let the parameter z in (3.1) be the initial value $x = y_0$. We deal with the question, whether it is possible that a solution $y_n(x)$ of (3.1) yields for $n = 1$ a solution $f(x) = y_1(x)$ of (1.2) in the case $k = 2$. Obviously, the answer is yes, if

$$y_1(y_1(x)) = y_2(x) \tag{4.1}$$

for all $x \in \mathbb{R}$. We discuss this answer in two cases.

4.1 Linear associated equations

As first example we consider once more equation (2.1) with the linear associated equation

$$y_{n+2} = ay_{n+1} + by_n \tag{4.2}$$

for $t = n$. According to $y_0 = x$ we can write the solution (2.3) of (4.2) as

$$y_n(x) = \begin{cases} cp^n + (x-c)q^n & \text{for } p \neq q\,, \\ (cn+x)p^n & \text{for } p = q\,. \end{cases}$$

In the case $p \neq q$ it follows

$$y_1(y_1(x)) = xq^2 + c(p-q)(1+q)\,,$$

and this is equal to $y_2(x)$ for $c(p-1)=0$, i.e. for $c=0$ resp. $p=1$, cf. [5, Theorem 8], where the solution with $p=1$ is excluded.

In the case $p=q$ it follows

$$y_1(y_1(x)) = xp^2 + cp(1+p)\,,$$

and this is equal to $y_2(x)$ also for $c(p-1)=0$, i.e. for $c=0$ resp. $p=1$ cf. [5, Theorem 7]. Hence, we have found very few examples for the equation (4.1), i.e. for $f(x)=y_1(x)$, but we can consider a further one.

4.2 Nonlinear associated equation

Let us return to equation (3.2) with the associated equation (3.3). If we look for a solution of (3.3) being a power series in $x=y_0$, we find by means of DERIVE for the first terms

$$\begin{aligned} y_n(x) = x &+ \frac{n}{2}x^2 + \frac{n}{4}(n-1)x^3 + \frac{n}{16}(2n^2-5n+2)x^4 + \frac{n}{96}(6n^3-26n^2+27n-1)x^5 \\ &+ \frac{n}{384}(12n^4-77n^3+142n^2-49n-34)x^6 + \ldots \end{aligned} \tag{4.3}$$

which for $n=1$ turn into (3.8).

Open Problem 3 Show that $f(x)=y_1(x)$ is indeed a solution of (3.2).

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Author:

Lothar Berg, i.R.
Universität Rostock,
Institut für Mathematik,
18051 Rostock,
Germany

e-mail: lothar.berg@uni-rostock.de

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Sadek Bouroubi, Nesrine Benyahia Tani

Integer partitions into arithmetic progressions

ABSTRACT. Every number not in the form 2^k can be partitioned into two or more consecutive parts. Thomas E. Mason has shown that the number of ways in which a number n may be partitioned into consecutive parts, including the case of a single term, is the number of odd divisors of n. This result is generalized by determining the number of partitions of n into arithmetic progressions with a common difference r, including the case of a single term.

KEY WORDS. Integer partitions, Arithmetic progression, divisors of an integer.

1 Introduction

Let n be an integer. A partition of n is an integral solution of the system:

$$\begin{cases} n = n_1 + \cdots + n_k, \\ 1 \le n_1 \le \cdots \le n_k. \end{cases}$$

The positive integers $n_1, \ldots, n_k$ are called parts, and k is the length of the partition. In partition identities, we are often interested in the number of partitions that satisfy some conditions. For example, the number of partitions into odd parts, the number of partitions into even parts, the number of partitions into even and odd parts [3], the number of partitions into distinct parts, and so on. For more on integer partitions the reader is hereby invited to see for instance [1], [2], [4], [5], [6] and [8]. Thomas E. Mason [7] was interested in the number of ways in which a number n may be partitioned into consecutive parts, including the case of a single term, and he has shown that this number equals the number of odd divisors of n. Let $n = 2^\alpha \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where the $p's$ are distinct odd primes and the $\alpha' s$ are the powers for which they occur. Then the number of ways in which a number n may be partitioned into consecutive parts, including the case of a single term, equals $\prod_{i=1}^{r}(\alpha_i + 1)$, except in the case $n = 2^\alpha$ where the number of ways is 1. In this paper we treat the case of partitions of n into arithmetic progressions.

For $1 \leq r \leq n-2$, if $n_i - n_{i-1} = r$, $i = 2, \ldots, k$, we say that we have a partition into an arithmetic progression with a common difference r.

Our problem can be formulated in terms of partitions: For a given positive integer n, what is the number of partitions of n into an arithmetic progression with a common difference r?

Partitioning n into an arithmetic progression with a common difference r means that there exist two positive integers $l \geq 1$ and $m \geq 0$, such that

$$n = l + (l + r) + (l + 2r) + \cdots + (l + mr). \tag{1}$$

Let us denote such a partition briefly by $\pi(l, m)$.

From (1),

$$n = (m+1)l + \frac{m(m+1)}{2} r.$$

Hence,

$$rm^2 + (2l + r)m + 2l - 2n = 0. \tag{2}$$

The solution of equation (2) in m yields

$$m = \frac{-(2l + r) + \sqrt{(2l - r)^2 + 8rn}}{2r}. \tag{3}$$

Since m is an integer, $(2l - r)^2 + 8rn$ is a perfect square, i.e.,

$$(2l - r)^2 + 8rn = u^2. \tag{4}$$

Then,

$$2rn = \frac{(u - (2l - r))}{2} \times \frac{(u + (2l - r))}{2}.$$

Putting,

$$A = \frac{u - (2l - r)}{2} \quad \text{and} \quad B = \frac{u + (2l - r)}{2},$$

we have

$$A + B = u, \quad \text{and} \tag{5}$$

$$B - A = 2l - r. \tag{6}$$

By (3), (5) and (6) we get

$$l = \frac{r + B - A}{2} \quad \text{and} \quad m = \frac{A - r}{r}. \tag{7}$$

Consequently, r divides A. Hence

$$2n = \frac{A}{r} \times B. \tag{8}$$

Now we discuss the parity of r.

1.1 Arithmetic progressions with an odd common difference r

By (4) u is odd if r is odd. Then in view of (5), A and B must have different parity.

Case 1. If A is even then B is odd and $n = \frac{A}{2r} \times B$. In this case we have

$$l = \frac{r + d - \frac{2rn}{d}}{2} \quad \text{and} \quad m = \frac{2n}{d} - 1,$$

where $d = B$, is an odd divisor of n.

Since $l \geq 1$, d must verify $(2n - d)r \leq d(d-2)$, i.e., $d \geq \frac{2 - r + \sqrt{(r-2)^2 + 8rn}}{2}$.

Case 2. If A is odd then $\frac{A}{r}$ is odd as well and $n = \frac{A}{r} \times \frac{B}{2}$. In this case we have

$$l = \frac{r + \frac{2n}{d} - dr}{2} \quad \text{and} \quad m = d - 1,$$

where $d = \frac{A}{r}$ is an odd divisor of n.

Because $l \geq 1$, d must verify $d(d-1)r \leq 2(n-d)$, i.e., $d \leq \frac{r - 2 + \sqrt{(r-2)^2 + 8rn}}{2r}$.

For every such odd divisor of n there exists a partition into an arithmetic progression with an odd common difference r, and vice versa.

A moment's reflection will show that an odd divisor d of n cannot satisfy the inequalities $(2n - d)r \leq d(d-2)$, $d(d-1)r \leq 2(n-d)$ simultaneously, otherwise $(d-1)r \leq (d-2)$, a contradiction.

1.2 Arithmetic progressions with an even common difference r

By (4) u is even, if r is even. Then in view of (5), A and B must have the same parity.

Since r divides A, A and B are even. Hence from (8)

$$n = \frac{A}{r} \times \frac{B}{2}.$$

From (7) we get

$$l = \frac{r + 2d - \frac{nr}{d}}{2} \quad \text{and} \quad m = \frac{n}{d} - 1,$$

where $d = \frac{B}{2}$ is a divisor of n.

Since $l \geq 1$, d must verify $(n-d)r \leq 2d(d-1)$, i.e., $d \geq \dfrac{2-r+\sqrt{(r-2)^2+8rn}}{4}$.

Now we are able to formulate our results as follows:

Theorem 1 *Let $n \geq 3$ be a positive integer and let $1 \leq r \leq n-2$ be an odd integer. Then the number of partitions of n into an arithmetic progression with an odd common difference r, including the case of a single term, is the number of odd divisors d of n, satisfying $\big(2n-d\big)r \leq d(d-2)$ or $d(d-1)r \leq 2(n-d)$ and for every such odd divisor, the partition $\pi(l,m)$ is given by:*

$$\begin{cases} l = \dfrac{r+\frac{2n}{d}-dr}{2} \ \text{ and } \ m = d-1 & \text{if} \quad d(d-1)r \leq 2(n-d), \\[2ex] l = \dfrac{r+d-\frac{2rn}{d}}{2} \ \text{ and } \ m = \dfrac{2n}{d}-1 & \text{if} \quad (2n-d)r \leq d(d-2). \end{cases}$$

Example 1 An example illustrating the above theorem is the following:

Let $n = 15$ and $r = 3$. The odd divisors of 15 satisfying $(2n-d)r \leq d(d-2)$ or $d(d-1)r \leq 2(n-d)$ are 1, 3 and 15, so 15 admits three partitions into an arithmetic progression of the common difference 3, each one is associated with one of these divisors and this can be shown as follows:

▶ $\boldsymbol{d=1}$ satisfies $d(d-1)r \leq 2(n-d)$. We have $l = \dfrac{r+\frac{2n}{d}-dr}{2} = 15$ and $m = d-1 = 0$. Hence $\pi(l,m) = \mathbf{15}$.

▶ $\boldsymbol{d=3}$ satisfies $d(d-1)r \leq 2(n-d)$. We have $l = \dfrac{r+\frac{2n}{d}-dr}{2} = 2$ and $m = d-1 = 2$. Hence $\pi(l,m) = \mathbf{2+5+8}$.

▶ $\boldsymbol{d=15}$ satisfies $\big(2n-d\big)r \leq d(d-2)$. We have $l = \dfrac{r+d-\frac{2rn}{d}}{2} = 6$ and $m = \dfrac{2n}{d}-1 = 1$. Hence $\pi(l,m) = \mathbf{6+9}$.

Theorem 2 *Let $n \geq 3$ be a positive integer and let $1 \leq r \leq n-2$ be an even integer. Then the number of partitions of n into an arithmetic progression with an even common difference r, including the case of a single term, is the number of divisors d of n, satisfying $(n-d)r \leq 2d(d-1)$ and for every such divisor, the partition $\pi(l,m)$ is given by:*

$$l = \frac{r+2d-\frac{nr}{d}}{2} \quad \text{and} \quad m = \frac{n}{d}-1.$$

Example 2 An example illustrating the above theorem is the following:

Let $n = 30$ and $r = 4$. The divisors of 30 satisfying $(n-d)r \leq 2d(d-1)$ are 10, 15 and 30, so 30 admits three partitions into an arithmetic progression of the common difference 4, each one is associated with one of these divisors and this can be shown as follows:

▶ $\boldsymbol{d = 10} \Rightarrow l = \dfrac{r + 2d - \frac{nr}{d}}{2} = 6$ and $m = \dfrac{n}{d} - 1 = 2$, hence $\pi(l,m) = \mathbf{6{+}10{+}14}$.

▶ $\boldsymbol{d = 15} \Rightarrow l = \dfrac{r + 2d - \frac{nr}{d}}{2} = 13$ and $m = \dfrac{n}{d} - 1 = 1$, hence $\pi(l,m) = \mathbf{13{+}17}$.

▶ $\boldsymbol{d = 30} \Rightarrow l = \dfrac{r + 2d - \frac{nr}{d}}{2} = 30$ and $m = \dfrac{n}{d} - 1 = 0$, hence $\pi(l,m) = \mathbf{30}$.

2 Why does Theorem 1 generalize Mason's theorem ?

Theorem 1 is an extension of Mason's theorem [7]. Indeed, for $r = 1$ and for every odd divisor d of n, one and only one of the two inequalities $d(d-1)r \leq 2(n-d)$, $\big(2n-d\big)r \leq d(d-2)$ necessarily holds. In fact, $r = 1$, $d = 2p+1$ and $n = dk$, imply

$$d(d-1) \leq 2(n-d) \Leftrightarrow p \leq k-1,$$

and

$$2n - d \leq d(d-2) \Leftrightarrow p \geq k.$$

Hence, if $p \leq k-1$, we get

$$l = \frac{1-d+2k}{2} \quad \text{and} \quad m = d-1,$$

otherwise

$$l = \frac{1+d-2k}{2} \quad \text{and} \quad m = 2k-1.$$

For example, $n = 15$ has four odd divisors: 1, 3, 5 and 15:

▶ $\boldsymbol{d = 1} \Rightarrow p = 0$ and $k = 15$, then $l = \dfrac{1-d+2k}{2} = 15$ and $m = d-1 = 0$, hence $\pi(l,m) = \mathbf{15}$.

▶ $\boldsymbol{d = 3} \Rightarrow p = 1$ and $k = 5$, then $l = \dfrac{1-d+2k}{2} = 4$ and $m = d-1 = 2$, hence $\pi(l,m) = \mathbf{4{+}5{+}6}$.

▶ $\boldsymbol{d = 5} \Rightarrow p = 2$ and $k = 3$, then $l = \dfrac{1-d+2k}{2} = 1$ and $m = d-1 = 4$, hence $\pi(l,m) = \mathbf{1{+}2{+}3{+}4{+}5}$.

▶ $\boldsymbol{d = 15} \Rightarrow p = 7$ and $k = 1$, then $l = \dfrac{1+d-2k}{2} = 7$ and $m = 2k-1 = 1$, hence $\pi(l,m) = \mathbf{7{+}8}$.

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Authors:

Sadek Bouroubi
USTHB,
Faculty of Mathematics
Department of Operational Research
Laboratory LAID3
P.O. Box 32
16111 El-Alia
Bab-Ezzouar,
Algeria

e-mail: sbouroubi@usthb.dz
or: bouroubis@yahoo.fr

Nesrine Benyahia Tani
Algiers University,
Faculty of Economics and
Management Sciences,
Department of Commercial Sciences,
Laboratory LAID3
2 Ahmed Waked Street,
Dely Brahim,
Algiers,
Algeria

e-mail: benyahiatani@yahoo.fr

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Théodore K. Boni, Thibaut K. Kouakou

Numerical quenching for a semilinear parabolic equation with Dirichlet-Neumann boundary conditions and a potential

ABSTRACT. This paper concerns the study of the numerical approximation for a semilinear parabolic equation with Dirichlet-Neumann boundary conditions and a potential. Under some conditions, we show that the solution of a semidiscrete form of the above problem quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time, and finally, we give some numerical experiments to illustrate our analysis.

KEY WORDS AND PHRASES. semidiscretization, semilinear parabolic equation, semidiscrete quenching time, convergence.

1 Introduction

Consider the following initial-boundary value problem

$$u_t(x,t) - u_{xx}(x,t) = -b(x)f(u(x,t)), \quad x \in (0,1), \quad t \in (0,T), \tag{1}$$

$$u_x(0,t) = 0, \quad u(1,t) = 1, \quad t \in (0,T), \tag{2}$$

$$u(x,0) = u_0(x) > 0, \quad x \in [0,1], \tag{3}$$

where $f : (0,\infty) \longrightarrow (0,\infty)$ is a C^1 convex, nonincreasing function, $\int_0^\alpha \frac{ds}{f(s)} < \infty$ for any positive real α, $\lim_{s\to 0^+} f(s) = \infty$, $b \in C^1([0,1])$, $b(x) > 0$, $x \in (0,1)$, $b'(0) = 0$, $b'(1) = 0$. The initial datum $u_0 \in C^2([0,1])$, $u_0(x) > 0$, $x \in [0,1]$,

$$u_0''(x) - b(x)f(u_0(x)) < 0, \quad x \in (0,1), \tag{4}$$

$$u_0'(x) > 0, \quad x \in (0,1), \tag{5}$$

$$u_0'(0) = 0, \quad u_0(1) = 1. \tag{6}$$

Here, $(0, T)$ is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is infinite, then we say that the solution u of (1)–(3) exists globally. When T is finite, then the solution u of (1)–(3) develops a singularity in a finite time, namely,

$$\lim_{t \to T} u_{\min}(t) = 0,$$

where $u_{\min}(t) = \min_{0 \le x \le 1} u(x, t)$. In this last case, we say that the solution u of (1)–(3) quenches in a finite time, and the time T is called the quenching time of the solution u. By virtue of the definition of the time T, we have

$$u(x, t) > 0, \quad (x, t) \in [0, 1] \times [0, T).$$

The theoretical study of solutions for semilinear parabolic equations which quench in a finite time, has been the subject of investigations of many authors (see, [2], [4], [7], [8], [10], [18], [21], [22], [29], and the references cited therein). In [7], Boni has proved the local in time existence and uniqueness of a classical solution under the hypotheses given in the introduction. The condition (4) allows the solution u to decrease with respect to the second variable, and the assumption (5) permits the solution to increase in space. Hence, the assumption (5) forces the solution u to attain its minimum at the first node. In the previous studies, with the help of the conditions (4) and (5), it is proved that the solution u of (1)–(3) quenches in a finite time at the first node. In addition, the quenching time is estimated (see, [10]). Let us notice that theoretically, it is not possible to determine the exact value of the quenching time.

In this paper, we are interested in the numerical study of the phenomenon of quenching. More precisely, we want to propose an algorithm which allows us to compute a good approximation of the real quenching time. We start by the construction of a semidiscrete scheme as follows. Let I be a positive integer, and define the grid $x_i = ih$, $0 \le i \le I$, where $h = 1/I$. Let $U_h(t) = (U_0(t), \cdots, U_I(t))^T$, and approximate the solution u of (1)-(3) by the solution $U_h(t)$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) - \beta_i f(U_i(t)), \quad 0 \le i \le I - 1, \quad t \in (0, T_q^h), \tag{7}$$

$$U_I(t) = 1, \quad t \in (0, T_q^h), \tag{8}$$

$$U_i(0) = \varphi_i, \quad 0 \le i \le I, \tag{9}$$

where $\varphi_h > 0$, and

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \tag{10}$$

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \le i \le I-1,$$

β_i and φ_i are approximations of $b(x_i)$ and $u_0(x_i)$, respectively. One may also choose $\beta_i = b(x_i)$ and $\varphi_i = u_0(x_i)$, but sometimes, we are obliged to take approximations, especially when one does not know the exact value of either the potential or the initial datum. On the other hand, there is another motivation which has incited our choice. This motivation is that, we want to know the behavior of the quenching time when one perturbs slightly either the potential or the initial datum. This is very important in certain situations when for instance the exact values do not possess certain properties, which is not the case of the approximated values. One may list some remarks on the motivation of our study. Firstly, let us notice that the semidiscrete solution U_h of (7)–(9) is the solution of a differential system. Secondly, due to the fact that β_i and φ_i are approximations of $b(x_i)$ and $u_0(x_i)$, respectively, the scheme presented in (7)–(9) is not a standard scheme. Although this scheme is not standard, we shall see later that it allows us to obtain good approximations of the continuous quenching time. In addition, we shall observe that the solution of a semilinear parabolic equation and that of a differential system quench in finite times which are practically the same.

Here, $(0, T_q^h)$ is the maximal time interval on which $U_{hmin}(t) > 0$, where $U_{hmin}(t) = \min_{0 \le i \le I} U_i(t)$. When T_q^h is finite, then we say that the solution $U_h(t)$ of (7)–(9) quenches in a finite time, and the time T_q^h is called the semidiscrete quenching time of the solution $U_h(t)$.

In this paper, under some assumptions, we show that the semidiscrete solution quenches in a finite time and estimate its semidiscrete quenching time. We also prove that the semidiscrete quenching time converges to the real one when the mesh size tends to zero. Recently, a similar study has been undertaken by Nabongo and Boni in [25] where they have considered the problem (1)–(3) for the case $b(x) = 1$ and $f(u) = u^{-p}$ with $p > 0$. It is worth noting that the potential and the nonlinearity of the current paper take into account those of Nabongo and Boni in [25]. One may also consult the papers of Nabongo and Boni in [26], [28] where semidiscrete and discrete schemes have been utilized to study the phenomenon of quenching for other parabolic problems. Let us notice that in these papers, the potential is equal one, and thus, the authors have not studied the effect of a perturbation of the potential on the semidiscrete quenching time. One of our source of motivation to undertake our study on numerical quenching comes from the study on numerical blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time) where the authors, for the treatment, have used semidiscrete and discrete schemes (see [1] and [24]). Our paper is organized as follows. In the next section, we give some results about the semidiscrete maximum principle and reveal certain properties of the semidiscrete solution. In the third section, under some conditions, we show that the semidiscrete solution quenches in a finite time and estimate its semidiscrete quenching time. In the fourth section, we also prove the convergence of the semidiscrete quenching time. Finally, in the last section, we give some numerical results to illustrate our analysis.

2 The semidiscrete scheme

In this section, we prove some results about the semidiscrete maximum principle and reveal certain properties concerning the operator δ^2 and the semidiscrete solution.

The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1 *Let $\alpha_h \in C^0([0,T),\mathbb{R}^{I+1})$ and let $V_h \in C^0([0,T),\mathbb{R}^{I+1})$ be such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + \alpha_i(t)V_i(t) \geq 0, \quad 0 \leq i \leq I-1, \quad t \in (0,T), \tag{11}$$

$$V_I(t) \geq 0, \quad t \in (0,T), \tag{12}$$

$$V_i(0) \geq 0, \quad 0 \leq i \leq I. \tag{13}$$

Then, the following estimates hold

$$V_i(t) \geq 0, \quad 0 \leq i \leq I, \quad t \in (0,T).$$

Proof: Let T_0 be any quantity satisfying $T_0 < T$, and introduce the vector $Z_h(t) = e^{\lambda t}V_h(t)$, where λ is such that $\alpha_i(t) - \lambda > 0$ for $t \in [0,T_0]$, $0 \leq i \leq I$. Let

$$m = \min_{t\in[0,T_0]} Z_{hmin}(t).$$

Since the vector $Z_i(t)$ is continuous on the compact $[0,T_0]$, then there exist $i_0 \in \{0,1,\cdots,I\}$ and $t_0 \in [0,T_0]$ such that $m = Z_{i_0}(t_0)$.

If $i_0 = I$, then according to (12), we have $m \geq 0$.

If $i_0 \in \{0,\cdots,I-1\}$, then we observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-k)}{k} \leq 0, \tag{14}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq \quad \text{if} \quad i_0 = 0, \tag{15}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1. \tag{16}$$

Due to (11), a straightforward computation reveals that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (\alpha_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0. \tag{17}$$

It follows from (14)–(16) that $(\alpha_{i_0}(t_0)-\lambda)Z_{i_0}(t_0) \geq 0$, which implies that $Z_{i_0}(t_0) \geq 0$ because $\alpha_{i_0}(t_0) - \lambda > 0$. We deduce that $V_h(t) \geq 0$ for $t \in [0, T_0]$, and the proof is complete. □

The lemma below shows a property of the semidiscrete solution.

Lemma 2.2 *Let U_h be the solution of (7)–(9). Assume that the initial datum satisfies $\varphi_i < 1$, $0 \leq i \leq I-1$. Then, we have*

$$U_i(t) < 1, \quad 0 \leq i \leq I-1, \quad t \in (0, T_q^h).$$

Proof: Let $t_0 \in (0, T_q^h)$ be the first time $t \in (0, T_q^h)$ such that $U_i(t) < 1$ for $0 \leq i \leq I-1$, $t \in (0, t_0)$, but $U_j(t_0) = 1$ for a certain $j \in \{0, \cdots, I-1\}$. We have

$$\frac{dU_j(t_0)}{dt} = \lim_{k \to 0} \frac{U_j(t_0) - U_j(t_0 - k)}{k} \geq 0, \tag{18}$$

$$\delta^2 U_j(t_0) = \frac{2U_1(t_0) - 2U_0(t_0)}{h^2} \leq \quad \text{if} \quad j = 0, \tag{19}$$

$$\delta^2 U_j(t_0) = \frac{U_{j+1}(t_0) - 2U_j(t_0) + U_{j-1}(t_0)}{h^2} \leq 0 \quad \text{if} \quad 1 \leq j \leq I-1, \tag{20}$$

which implies that

$$\frac{dU_j(t_0)}{dt} - \delta^2 U_j(t_0) + \beta_j f(U_j(t_0)) > 0.$$

But, this contradicts (7) and the proof is complete. □

Another version of the maximum principle for semidiscrete equations is the following comparison lemma.

Lemma 2.3 *Let $f \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. If V_h, $W_h \in C^1([0,T), \mathbb{R}^{I+1})$ are such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + g(V_i(t), t) < \frac{dW_i(t)}{dt} - \delta^2 W_i(t) + g(W_i(t), t), \tag{21}$$

$$0 \leq i \leq I-1, \quad t \in (0, T),$$

$$V_I(t) < W_I(t), \quad t \in (0, T), \tag{22}$$

$$V_i(0) < W_i(0), \quad 0 \leq i \leq I, \quad t \in (0, T), \tag{23}$$

then $V_i(t) < W_i(t)$, $0 \leq i \leq I$, $t \in (0, T)$.

Proof: Let $Z_h(t) = W_h(t) - V_h(t)$ and let t_0 be the first $t \in (0, T)$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$, $0 \leq i \leq I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \cdots, I\}$. If $i_0 = I$, then we have a contradiction because of (22).

If $i_0 \in \{0, \cdots, I-1\}$, then we obtain

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq \quad \text{if} \quad i_0 = 0, \tag{24}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + g(W_{i_0}(t_0), t_0) - g(V_{i_0}(t_0), t_0) \leq 0.$$

But, this inequality contradicts (21), which ends the proof. □

The following results show some properties of the semidiscrete solution.

Lemma 2.4 *Let U_h be the solution of (7)–(9) such that the initial datum satisfies*

$$\varphi_{i+1} > \varphi_i, \quad 0 \leq i \leq I-1. \tag{25}$$

Then, we have for $t \in (0, T_q^h)$,

$$U_{i+1}(t) > U_i(t), \quad 0 \leq i \leq I-1. \tag{26}$$

Proof: Invoking Lemma 2.2, we know that

$$U_i(t) < 1, \quad 0 \leq i \leq I-1, \quad t \in (0, T_q^h).$$

Let t_1 be the first $t \in (0, T_q^h)$ such that $U_{i+1}(t) > U_i(t)$ for $t \in (0, t_1)$, $0 \leq i \leq I-1$, but

$$U_{k+1}(t_1) = U_k(t_1) \quad \text{for a certain} \quad k \in \{0, \cdots, I-1\}. \tag{27}$$

Without loss of generally, we may suppose that k is the smallest integer which verifies (27).

If $k = I-1$, then $U_{I-1}(t_1) = U_I(t_1) = 1$, which contradicts the fact that $U_{I-1}(t_1) < 1$.

If $k = 1, \cdots, I-2$, then we observe that

$$\frac{d(U_{k+1} - U_k)(t_1)}{dt} = \lim_{\sigma \to 0} \frac{(U_{k+1} - U_k)(t_1) - (U_{k+1} - U_k)(t_1 - \sigma)}{\sigma} \leq 0,$$

and

$$\delta^2(U_{k+1}-U_k)(t_1)=\frac{(U_{k+2}-U_{k+1})(t_1)-2(U_{k+1}-U_k)(t_1)+(U_k-U_{k-1})(t_1)}{h^2}>0,$$

which implies that

$$\frac{d(U_{k+1}-U_k)(t_1)}{dt}-\delta^2(U_{k+1}-U_k)(t_1)+\beta_{k+1}f(U_{k+1}(t_1))-\beta_k f(U_k(t_1))<0.$$

But, this contradicts (7).

If $k=0$, then we get

$$\delta^2(U_{k+1}-U_k)(t_1)=\frac{(U_{k+2}-U_{k+1})(t_1)-3(U_{k+1}-U_k)(t_1)}{h^2}<0.$$

Thanks to the above inequality, it is easy to see that

$$\frac{d(U_{k+1}-U_k)(t_1)}{dt}-\delta^2(U_{k+1}-U_k)(t_1)+\beta_{k+1}f(U_{k+1}(t_1))-\beta_k f(U_k(t_1))<0,$$

which contradicts (7). This ends the proof. □

Remark 2.1 The above lemma says that, if the initial datum of the semidiscrete solution is increasing in space, then the semidiscrete solution also satisfies this property. This result will be used later to show that the semidiscrete solution attains its minimum at the first node.

To end this section, let us give some properties of the operator δ^2.

Lemma 2.5 *Let V_h and $U_h\in\mathbb{R}^{I+1}$. If $\delta^+(U_0)\delta^+(V_0)\geq 0$ and*

$$\delta^+(U_i)\delta^+(V_i)\geq 0,\quad \delta^-(U_i)\delta^-(V_i)\geq 0,\quad 1\leq i\leq I-1, \tag{28}$$

then

$$\delta^2(U_iV_i)\geq U_i\delta^2V_i+V_i\delta^2U_i,\quad 0\leq i\leq I-1,$$

where $\delta^+(U_i)=\frac{U_{i+1}-U_i}{h}$ and $\delta^-(U_i)=\frac{U_{i-1}-U_i}{h}$.

Proof: A straightforward computation reveals that

$$\delta^2(U_0V_0)=2\delta^+(U_0)\delta^+(V_0)+U_0\delta^2V_0+V_0\delta^2U_0,$$

$$\delta^2(U_iV_i)=\delta^+(U_i)\delta^+(V_i)+\delta^-(U_i)\delta^-(V_i)+U_i\delta^2V_i+V_i\delta^2U_i,\quad 1\leq i\leq I-1.$$

Taking into account the assumptions of the lemma, we obtain the desired result. □

Lemma 2.6 *Let $U_h \in \mathbb{R}^{I+1}$ be such that $U_h > 0$. Then, the following estimates hold*

$$\delta^2 f(U_i) \geq f'(U_i)\delta^2 U_i, \quad 0 \leq i \leq I-1.$$

Proof: Applying Taylor's expansion, we get

$$\delta^2 f(U_0) = f'(U_0)\delta^2 U_0 + f''(\theta_0)\frac{(U_1 - U_0)^2}{h^2},$$

$$\delta^2 f(U_i) = f'(U_i)\delta^2 U_i + f''(\theta_i)\frac{(U_{i+1} - U_i)^2}{2h^2} + f''(\eta_i)\frac{(U_{i-1} - U_i)^2}{2h^2},$$

$$1 \leq i \leq I-1,$$

where θ_i is an intermediate value between U_i and U_{i+1}, and η_i the one between U_{i-1} and U_i. Use the fact that $U_h > 0$ to complete the rest of the proof. □

3 Quenching in the semidiscrete problem

In this section, under some assumptions, we show that the solution U_h of (7)–(9) quenches in a finite time and estimate its semidiscrete quenching time.

Our result is the following.

Theorem 3.1 *Let U_h be the solution of (7)–(9), and assume that there exists a constant $A \in (0, 1]$ such that the initial datum satisfies*

$$\delta^2 \varphi_i - \beta_i f(\varphi_i) \leq -A \sin(ih\pi) f(\varphi_i), \quad 1 \leq i \leq I-1, \tag{29}$$

$$1 - \frac{2\pi^2}{A}\int_0^{\varphi_{hmin}} \frac{d\sigma}{f(\sigma)} > 0. \tag{30}$$

Under the assumptions of Lemma 2.4, the solution U_h quenches in a finite time T_q^h, and the following estimation holds

$$T_q^h \leq -\frac{1}{\pi^2}\ln\left(1 - \frac{2\pi^2}{A}\int_0^{\varphi_{hmin}} \frac{d\sigma}{f(\sigma)}\right).$$

Proof: Since $(0, T_q^h)$ is the maximal time interval of existence of the solution U_h our aim is to show that T_q^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ defined as follows

$$J_i(t) = \frac{dU_i(t)}{dt} + C_i(t) f(U_i(t)), \quad 0 \leq i \leq I, \quad t \in [0, T_q^h),$$

where $C_i(t) = Ae^{-\lambda_h t}\cos(ih\frac{\pi}{2})$, $0 \le i \le I$, with $\lambda_h = \frac{2-2\cos(ih\frac{\pi}{2})}{h^2}$. A straightforward computation reveals that

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) = \frac{d}{dt}\left(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)\right) + C_i(t)f'(U_i(t))\frac{dU_i(t)}{dt}$$

$$-\delta^2(C_i(t)f(U_i(t))) + \frac{dC_i(t)}{dt}f(U_i(t)), \quad 0 \le i \le I-1, \quad t \in (0, T_q^h).$$

We observe that

$$\frac{dC_i(t)}{dt} - \delta^2 C_i(t) = 0, \quad C_{i+1}(t) < C_i(t), \quad 0 \le i \le I-1,$$

and due to Lemma 2.4, we find that $\delta^+(f(U_0))\delta^+(C_0) \ge 0$, and $\delta^+(f(U_i))\delta^+(C_i) \ge 0$, $\delta^-(f(U_i))\delta^-(C_i) \ge 0$, $0 \le i \le I-1$. It follows from Lemmas 2.5 and 2.6 that

$$\delta^2(C_i(t)f(U_i(t))) \ge C_i(t)f'(U_i(t))\delta^2 U_i(t) + f(U_i(t))\delta^2 C_i(t), \quad 0 \le i \le I-1.$$

Using the above estimates, we discover that

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \le \frac{d}{dt}\left(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)\right) + C_i(t)f'(U_i(t))\left(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)\right)$$

$$+f(U_i(t))\left(\frac{dC_i(t)}{dt} - \delta^2 C_i(t)\right), \quad 0 \le i \le I-1, \quad t \in (0, T_q^h).$$

With the help of (7), we derive the following estimates

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \le -\beta_i(t)f'(U_i(t))\frac{dU_i(t)}{dt} - \beta_i(t)C_i(t)f'(U_i(t))f(U_i(t)), \quad 0 \le i \le I-1.$$

Taking into account the expression of $J_i(t)$, we arrive at

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \le -\beta_i(t)f'(U_i(t))J_i(t), \quad 0 \le i \le I-1.$$

Obviously, we note that

$$J_I(t) = \frac{dU_I(t)}{dt} + C_I(t)f(U_I(t)) = 0,$$

and due to inequalities (29), we get $J_h(0) \le 0$. It follows from Lemma 2.1 that $J_h(t) \le 0$ for $t \in [0, T_h)$. This estimate may be rewritten in the following manner

$$\frac{dU_i(t)}{dt} \le -Ae^{-\lambda_h t}\cos\left(\frac{i\pi h}{2}\right) f(U_i(t)), \quad 0 \le i \le I, \quad t \in (0, T_q^h). \tag{31}$$

At the first node, we have

$$\frac{dU_0(t)}{dt} \leq -Ae^{-\lambda_h t} f(U_0(t)), \quad t \in (0, T_q^h). \tag{32}$$

Apply Taylor's expansion to obtain

$$\cos\left(\frac{\pi h}{2}\right) = 1 - \frac{\pi^2 h^2}{4} + \frac{\pi^3 h^3}{48} \sin\left(\frac{\pi h}{2}\theta\right),$$

where $\theta \in [0, 1]$. This implies that $\lambda_h \leq \frac{\pi^2}{2}$. Therefore using (32), we discover that

$$\frac{dU_0(t)}{dt} \leq -Ae^{-\frac{\pi^2}{2}t} f(U_0(t)), \quad t \in (0, T_q^h).$$

After a little transformation, this inequality becomes

$$\frac{dU_0}{f(U_0)} \leq -Ae^{-\frac{\pi^2}{2}t} dt, \quad t \in (0, T_q^h). \tag{33}$$

Integrate the above estimate over $(0, T_q^h)$ to arrive at

$$T_q^h \leq -\frac{2}{\pi^2} \ln\left(1 - \frac{\pi^2}{2A} \int_0^{U_0(0)} \frac{d\sigma}{f(\sigma)}\right). \tag{34}$$

We know from Lemma 2.4 that $U_0(t) = U_{hmin}(t)$ for $t \in (0, T_q^h)$, which implies that $U_0(0) = U_{hmin}(0) = \varphi_{hmin}$. Taking into account the above inequalities and (34), it is not difficult to check that

$$T_q^h \leq -\frac{2}{\pi^2} \ln\left(1 - \frac{\pi^2}{2A} \int_0^{\varphi_{hmin}} \frac{d\sigma}{f(\sigma)}\right). \tag{35}$$

Use the fact that the quantity on the right hand side of the above inequality is finite to complete the rest of the proof. □

Remark 3.1 Let $t_0 \in (0, T_q^h)$. Integrate the estimate (33) over (t_0, T_q^h) and use the fact that $U_{hmin}(t_0) = U_0(t_0)$ to obtain

$$T_q^h - t_0 \leq -\frac{2}{\pi^2} \ln\left(1 - \frac{\pi^2}{2A} e^{\frac{\pi^2}{2}t_0} \int_0^{U_{hmin}(t_0)} \frac{d\sigma}{f(\sigma)}\right). \tag{36}$$

The theorem below gives a lower bound of the semidiscrete quenching time.

Theorem 3.2 *Let U_h be the solution of (7)–(9). Assume that U_h quenches at the time T_q^h. Then, we have the following estimate*

$$T_q^h \geq \frac{1}{\|\beta_h\|_\infty} \int_0^{\varphi_{hmin}} \frac{ds}{f(s)}.$$

Proof: Let $\alpha(t)$ be the solution of the following differential equation

$$\alpha'(t) = -\|\beta_h\|_\infty f(\alpha(t)),\ t > 0,\ \alpha(0) = \varphi_{hmin},$$

and let $W_h(t)$ be the vector such that $W_i(t) = \alpha(t),\ 0 \leq i \leq I$. After a little transformation, it not hard to see that $\alpha(t)$ quenches in a finite time at the time $T_h = \frac{1}{\|\beta_h\|_\infty} \int_0^{\varphi_{hmin}} \frac{ds}{f(s)}$. Setting $Z_h(t) = W_h(t) - U_h(t)$, a straightforward computation reveals that

$$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) + \beta_i f'(\xi_i(t)) Z_i(t) \leq 0, \quad 0 \leq i \leq I-1, \quad t \in (0, T_h^*),$$

$$Z_I(t) \leq 0, \quad t \in (0, T_h^*),$$

$$Z_i(0) \leq 0, \quad 0 \leq i \leq I,$$

where $T_h^* = \min\{T_h, T_q^h\}$, and $\xi_i(t)$ is an intermediate value between $U_i(t)$ and $W_i(t)$. Invoking Lemma 2.1, we derive the following estimate $W_h(t) \leq U_h(t)$ for $t \in (0, T_h^*)$. Making use of the expression of W_h, we discover that

$$U_{hmin}(t) \geq \alpha(t) \quad \text{for} \quad t \in (0, T_h^*).$$

This implies that if $t < \frac{1}{\|\beta_h\|_\infty} \int_0^{\varphi_{hmin}} \frac{ds}{f(s)}$, then $U_{hmin}(t) > 0$. Therefore $T_q^h \geq \frac{1}{\|\beta_h\|_\infty} \int_0^{\varphi_{hmin}} \frac{ds}{f(s)}$, and the proof is complete. □

4 Convergence of the semidiscrete quenching time

In this section, under some assumptions, we prove that the semidiscrete quenching time converges to the real one when the mesh size goes to zero.

Firstly, we show that, in the interval $[0, T-\tau]$ with $\tau \in (0, T)$ where the continuous solution u obeys $u_{\min}(t) > 0$, the semidiscrete solution U_h approximates u when the mesh parameter h goes to zero. This result is stated in the following theorem.

Theorem 4.1 *Assume that (1)–(3) has a solution $u \in C^{4,1}([0,1] \times [0, T-\tau])$ such that $\min_{t\in[0,T]} u_{\min}(t) = \rho > 0$ with $\tau \in (0, T)$. Suppose that the potential and the initial datum of the problem (7)–(9) satisfy*

$$\|\beta_h - b_h\|_\infty = o(1) \quad as \quad h \to 0, \tag{37}$$

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0, \tag{38}$$

respectively, where $u_h(t) = (u(x_0, t), \cdots, u(x_I, t))^T$. Then, for h sufficiently small, the problem (7)–(9) has a unique solution $U_h \in C^1([0, T_q^h), \mathbb{R}^{I+1})$ such that

$$\max_{0 \leq t \leq T-\tau} \|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + \|\beta_h - b_h\|_\infty + h^2) \quad as\ h \to 0.$$

Proof: The problem (7)–(9) has for each h, a unique solution $U_h \in C^1([0,T_q^h),\mathbb{R}^{I+1})$. Let $t(h)$ the greatest value of $t>0$ such that

$$\|U_h(t)-u_h(t)\|_\infty < \frac{\rho}{2} \quad \text{for} \quad t\in(0,t(h)). \tag{39}$$

The relation (38) implies that $t(h)>0$ for h sufficiently small. Let $t^*(h)=\min\{t(h),T-\tau\}$. An application of the triangle inequality gives

$$U_{hmin}(t) \geq u_{\min}(t) - \|U_h(t)-u_h(t)\|_\infty \quad \text{for} \quad t\in(0,t^*(h)),$$

which implies that

$$U_{hmin}(t) \geq \rho - \frac{\rho}{2} = \frac{\rho}{2} \quad \text{for} \quad t\in(0,t^*(h)). \tag{40}$$

Apply Taylor's expansion to obtain

$$\delta^2 u(x_i,t) = u_{xx}(x_i,t) + \frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t), \quad 0\leq i\leq I-1.$$

Exploiting the above equalities, we arrive at

$$\frac{du(x_i,t)}{dt} = \delta^2 u(x_i,t) - b(x_i)f(u(x_i,t)) - \frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t), \quad 0\leq i\leq I-1.$$

Introduce the error of discretization

$$e_h(t) = U_h(t)-u_h(t), \quad t\in[0,t^*(h)).$$

Invoking the mean value theorem, we find that

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) = -\beta_i f'(\theta_i(t))e_i(t) - (\beta_i - b(x_i))f(u(x_i,t))$$

$$+\frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t), \quad 1\leq i\leq I-1, \quad t\in(0,t^*(h)), \tag{41}$$

where $\theta_i(t)$ is an intermediate value between $U_i(t)$ and $u(x_i,t)$. Let $M>0$ be such that

$$\frac{\|u_{xxxx}(\cdot,t)\|_\infty}{12} \leq M \text{ for } t\in[0,t^*(h)), \quad -\|\beta_h\|_\infty f'(\tfrac{\varrho}{2}) \leq M, \quad f(\frac{\rho}{2}) \leq M.$$

Making use of the above inequalities, it is not hard to see that

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) \leq M|e_i(t)| + Mh^2 + M\|\beta_h - b_h\|_\infty,$$

$$0\leq i\leq I-1, \quad t\in(0,t^*(h)).$$

Introduce the vector z_h such that

$$z_i(t) = e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + \|\beta_h - b_h\|_\infty + Mh^2), \quad 0 \le i \le I, \quad t \in [0,T].$$

A straightforward computation yields

$$\frac{dz_i(t)}{dt} - \delta^2 z_i(t) > M|z_i(t)| + Mh^2 + M\|\beta_h - b_h\|_\infty,$$

$$0 \le i \le I-1, \quad t \in (0, t^*(h)),$$

$$z_I(t) > e_I(t), \quad t \in (0, t^*(h)),$$

$$z_i(0) > e_i(0), \quad 0 \le i \le I.$$

It follows from Lemma 2.3 that

$$z_i(t) > e_i(t), \quad t \in (0, t^*(h)), \quad 0 \le i \le I.$$

In the same way, we also prove that

$$z_i(t) > -e_i(t), \quad t \in (0, t^*(h)), \quad 0 \le i \le I,$$

which implies that

$$\|U_h(t) - u_h(t)\|_\infty \le e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + \|\beta_h - b_h\|_\infty + Mh^2), \quad t \in (0, t^*(h)).$$

Let us show that $t^*(h) = T - \tau$. Suppose that $T - \tau > t(h)$. From (39), we obtain

$$\frac{\rho}{2} = \|U_h(t(h)) - u_h(t(h))\|_\infty \le e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + \|\beta_h - b_h\|_\infty + Mh^2).$$

Since the term on the right hand side of the above inequality goes to zero as h tends to zero, we deduce that $\frac{\rho}{2} \le 0$, which is impossible. Consequently $t^*(h) = T - \tau$, and the proof is complete. □

Now, we are in a position to prove the main result of this section.

Theorem 4.2 *Suppose that the solution u of (1)–(3) quenches in a finite time T such that $u \in C^{4,1}([0,1] \times [0,T))$. Assume that the potential and the initial datum of the problem (7)–(9) satisfy the conditions (34) and (35), respectively. Under the assumptions of Theorem 3.1, the problem (7)–(9) admits a unique solution U_h which quenches in a finite time T_q^h, and the following relation holds*

$$\lim_{h \to 0} T_q^h = T.$$

Proof: Let $0 < \varepsilon \le T/2$. There exists a constant $R > 0$ such that

$$-\frac{1}{\pi^2}\ln\left(1-\frac{2\pi^2}{A}e^{\pi^2 T}\int_0^x \frac{ds}{f(s)}\right) < \frac{\varepsilon}{2} \text{ for } x \in [0, R]. \tag{42}$$

Due to the fact that the solution u quenches in a finite time T, there exists $T_1 \in (T - \frac{\varepsilon}{2}, T)$ such that $0 < u_{\min}(t) < \frac{R}{2}$ for $t \in [T_1, T)$. Let $T_2 = \frac{T_1+T}{2}$. Obviously $0 < u_{\min}(t) < \frac{R}{2}$ for $t \in [0, T_2]$. Exploiting Theorem 4.1, we know that the problem (7)–(9) admits a unique solution U_h which obeys the following estimate $\|U_h(t) - u_h(t)\|_\infty < \frac{R}{2}$ for $t \in [0, T_2]$, which implies that $\|U_h(T_2) - u_h(T_2)\|_\infty < \frac{R}{2}$. An application of the triangle inequality gives

$$U_{hmin}(T_2) \le \|U_h(T_2) - u_h(T_2)\|_\infty + u_{\min}(T_2) \le \frac{R}{2} + \frac{R}{2} = R.$$

Taking into account Theorem 3.1, we note that $U_h(t)$ quenches in a finite time T_q^h. We infer from Remark 3.1 that

$$|T_q^h - T_2| \le -\frac{1}{\pi^2}\ln\left(1-\frac{2\pi^2}{A}e^{\pi^2 T_2}\int_0^{U_{hmin}(T_2)} \frac{ds}{f(s)}\right). \tag{43}$$

Since the function $s \longrightarrow -\ln(1-s)$ is an increasing function for positive values of s, it is not hard to see that the term on the right hand side of the above inequality is bounded from above by $-\frac{1}{\pi^2}\ln(1-\frac{2\pi^2}{A}e^{\pi^2 T}\int_0^{U_{hmin}(T_2)}\frac{ds}{f(s)})$. We deduce from (42) and (43) that $|T_q^h - T_2| \le \frac{\varepsilon}{2}$, which implies that

$$|T_q^h - T| \le |T_q^h - T_2| + |T_2 - T| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof. □

5 Numerical experiments

In this section, we give some computational experiments to confirm the theory given in the previous section. We consider the problem (1)–(3) in the case where $f(u) = u^{-p}$, $b(x) = 2 + x^2$, $u_0(x) = \frac{1+x^2}{2}$ with $p > 0$. We start by the construction of an adaptive scheme as follows. Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \le i \le I$, and approximate the solution u of (1)–(3) by the solution $U_h^{(n)} = (U_0^{(n)}, \cdots, U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} - \beta_0 (U_0^{(n)})^{-p},$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} - \beta_i (U_i^{(n)})^{-p}, \quad 1 \le i \le I-1,$$

$$U_I^{(n)} = 1,$$

$$U_i^{(0)} = \varphi_i, \quad 0 \le i \le I,$$

where $\varphi_i = \frac{1+(ih)^2}{2} - \varepsilon \frac{(\sin(\frac{i\pi h}{2})+1)}{5}$, $\beta_i = 1 + (ih)^2 - \epsilon \sin(i\pi h)$ with $\epsilon \in [0, 1]$. In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = \min \left\{ \frac{(1-h^2)h^2}{2}, h^2 (U_{hmin}^{(n)})^{p+1} \right\}.$$

Let us notice that the restriction on the time step ensures the positivity of the discrete solution. We also approximate the solution u of (1)–(3) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} - \beta_0 (U_0^{(n)})^{-p-1} U_0^{(n+1)},$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} - \beta_i (U_i^{(n)})^{-p-1} U_i^{(n+1)}, \quad 1 \le i \le I-1,$$

$$U_I^{(n+1)} = 1,$$

$$U_i^{(0)} = \varphi_i, \quad 0 \le i \le I.$$

As in the case of the explicit scheme, here, we also choose $\Delta t_n = h^2 (U_{hmin}^{(n)})^{p+1}$. Let us again remark that for the above implicit scheme, existence and positivity of the discrete solution are also guaranteed using standard methods (see, for instance [6]).

We need the following definition.

Definition 5.1 *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if $\lim_{n\to\infty} U_{hmin}^{(n)} = 0$, and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical quenching time of the discrete solution $U_h^{(n)}$.*

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |t_{n+1} - t_n| \le 10^{-16}.$$

The order (s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $p = 1$

First case: $\epsilon = 0$

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.0802018	1960	1.3	-
32	0.0800499	7550	5.3	-
64	0.0800131	28956	51	2.04
128	0.0800040	110721	1140	2.01

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.0808371	1975	2.8	-
32	0.0802084	7564	10.6	-
64	0.0800526	28969	164	2.01
128	0.0800014	110732	4620	1.61

Second case: $\epsilon = 1$

Table 3: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.0176645	1856	1.3	-
32	0.0177477	7167	5.4	-
64	0.0177705	27512	45	1.86
128	0.0177765	105213	1097	1.92

Table 4: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.0178058	1870	1.8	-
32	0.0177826	7179	11.4	-
64	0.0177791	27523	150	2.72
128	0.0177787	105223	3604	3.12

Third case: $\epsilon = 1/100$

Table 5: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.0794108	1960	1.2	-
32	0.0792633	7551	5.4	-
64	0.0792276	28962	81	2.72
128	0.0792188	110749	1380	2.02

Table 6: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.0800397	1975	1.8	-
32	0.0794202	7565	11.5	-
64	0.0792667	28974	193	2.01
128	0.0792286	110760	4260	2.01

Remark 5.1 Tables 1 and 2 provide us the results of the numerical quenching time when $\varepsilon = 0$. We observe that the numerical quenching time in this case is approximately equal to 0.08. It is worth noting that both explicit and implicit schemes give practically the same results, and we also observe that the variation of the different meshes has no important effects on the numerical quenching time. It is also important to point out that, when we look at Tables 5 and 6, we see that the numerical quenching times when $\epsilon \in (0, 1)$ is small enough, are slightly equal to that which corresponds to $\epsilon = 0$. On the other hand, when one examines Tables 1, 2, 3 and 4, one sees that an important perturbation on the potential and the initial datum has a meaningful impact on the numerical quenching time.

In the following, we also give some plots to illustrate our analysis. In the figures below, we can see that the discrete solution quenches in a finite time at the first node. Here, all schemes are highly consistent.

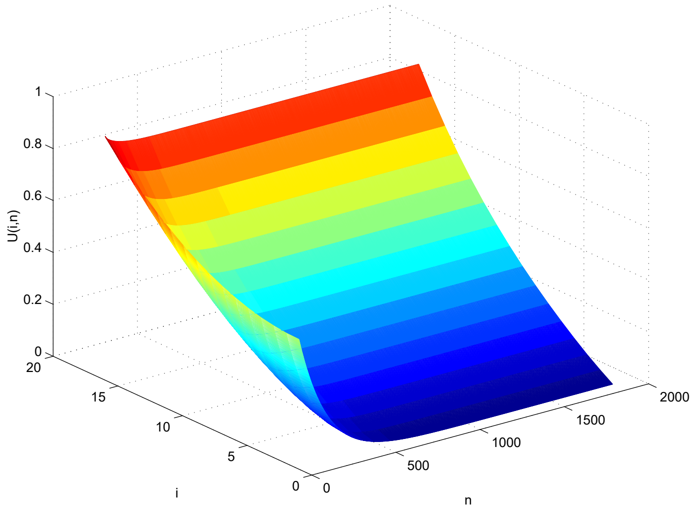

Figure 1: Evolution of the discrete solution $U_h^{(n)}$, $I = 16$, $\varepsilon = 0,\ f(s) = s^2$.

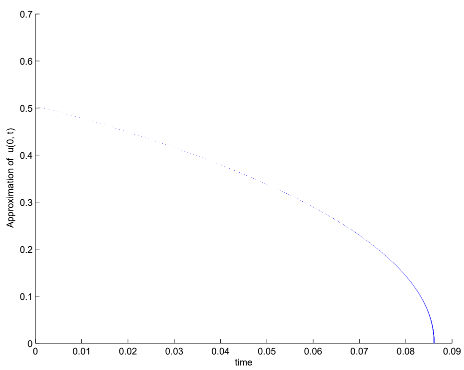

Figure 2: Approximation of $u(0,t)$, $I = 16$, $\varepsilon = 0,\ f(s) = s^2$.

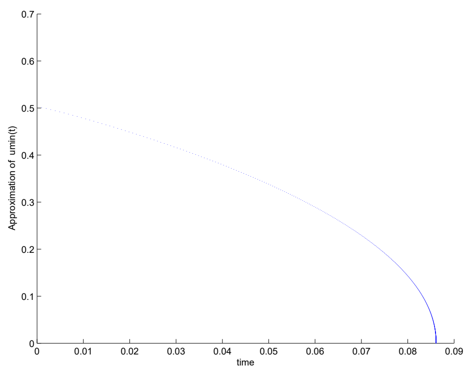

Figure 3: Approximation of $u_{min}(t)$, $I = 16$, $\varepsilon = 0,\ f(s) = s^2$.

6 Conclusion

In the present paper, we have considered a semilinear heat equation with a potential subject to Dirichlet-Neumann boundary conditions. We have constructed a semidiscrete scheme, and have shown that the solution of the semidiscrete scheme quenches in a finite time, and its semidiscrete quenching time converges to the continuous one when the mesh size tends to zero. We have studied in passing the continuity of the semidiscrete quenching time as a function of the potential and the initial datum. Our study can take into account some works where Dirichlet boundary conditions are considered. In fact, one knows that if a solution $u(x,t)$ is symmetric in $(-1,1)\times(0,T)$, then $u_x(0,t)=0$. This can permit to treat a problem where Dirichlet boundary condition is taken considering the problem developed in this paper. In the works to come, it will be better to consider the problem described in (1)–(3) using a full discrete scheme.

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Authors:

Théodore K. Boni
Institut National Polytechnique
Houphouët-Boigny de Yamoussoukro,
BP 1093 Yamoussoukro,
(Côte d'Ivoire)

e-mail: theokboni@yahoo.fr

Thibaut K. Kouakou
Université d'Abobo-Adjamé,
UFR-SFA,
Département de Mathématiques
et Informatiques,
02 BP 801 Abidjan 02,
(Côte d'Ivoire)

e-mail: kkthibaut@yahoo.fr

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Maciej Kalkowski, Michał Karoński, Florian Pfender

Vertex coloring edge weightings with integer weights at most 6

ABSTRACT. A weighting of the edges of a graph is called neighbor distinguishing if the weighted degrees of the vertices yield a proper coloring of the graph. In this note we show that such a weighting is possible from the weight set $\{1,2,3,4,5,6\}$ for all graphs not containing components with exactly 2 vertices.

All graphs in this note are finite and simple. For notation not defined here we refer the reader to [3].

For some $k \in \mathbb{N}$, let $f : E(G) \to \{1,2,\dots,k\}$ be an integer weighting of the edges of a graph G. This weighting is called vertex coloring if the weighted degrees $w(v) = \sum_{u \in N(v)} w(uv)$ of the vertices yield a proper coloring of the graph. It is easy to see that for every graph which does not have a component isomorphic to K^2, there exists such a weighting for some k.

In 2002, Karoński, Łuczak and Thomason (see [6]) conjectured that such a weighting with $k = 3$ is possible for all such graphs ($k = 2$ is not sufficient as seen for instance in complete graphs and cycles of length not divisible by 4). A first constant bound of $k = 30$ was proved by Addario-Berry et al. in 2007 [1], which was later improved to $k = 16$ in [2] and to $k = 13$ in [7].

In this note we show a completely different approach that improves the bound to $k = 6$. We were able to further improve the bound to $k = 5$ in [5], but this current note exhibits some interesting ideas with their own merit which were not used in the other paper.

Consider a related result by the first author using a total weighting, i.e. we add weights to the vertices as well.

Lemma 1 ([4]) *For any connected graph G with $|G| \geq 3$, there is an edge weighting $f : E(G) \to \{1,2,3\}$, and a vertex weighting $f' : V(G) \to \{0,1\}$ such that the total weight $w(v) := f'(v) + \sum_{w \in N(v)} f(vw)$ gives a proper coloring of $V(G)$.*

With the help of this result, the first author was able to reduce the bound to $k = 10$ after tripling all weights and adjusting some of the resulting edge weights by 1 with a Kempe chain type argument to get a neighbor distinguishing edge weighting.

In this note, we use similar ideas to get down to $k = 6$ in the original problem. We start with a slight generalization of Lemma 1. The proof is almost identical but is included here for completeness.

Lemma 2 *Let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$. Then, for any connected graph G with $|G| \geq 3$, and for any spanning tree T, there is an edge weighting $f : E(G) \to \{\alpha-\beta, \alpha, \alpha+\beta\}$, and a vertex weighting $f' : V(G) \to \{0, \beta\}$ such that the total weight $w(v) := f'(v) + \sum_{w \in N(v)} f(vw)$ gives a proper coloring of $V(G)$. Further, we can choose f in a way that $f(e) = \alpha$ for all edges $e \in E(T)$.*

Proof: We order the vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that for $k \geq 2$, every v_k has exactly one edge in T to a vertex in $\{v_1, v_2, \ldots, v_{k-1}\}$. We start by assigning the weight α to every edge of G, and then adjust this edge weight at most once to assign a total weight to every v_k in order, which then remains unchanged.

The vertex v_1 gets weight $\alpha d_G(v_1)$. Let us assume for some $k \geq 2$ that we have adjusted edge weights f on $E(G[\{v_1, \ldots, v_{k-1}\}]) \setminus E(T)$ and vertex weights f' on $\{v_1, \ldots, v_{k-1}\}$ so that the first $k-1$ vertices have their final total weight $w(v_i)$.

For v_k, we can adjust the weights of all edges $E(v_k, \{v_1, \ldots, v_{k-1}\}) \setminus E(T)$, by β: If $v_k v_i \in E(G) \setminus E(T)$ and $f'(v_i) = 0$, we can choose between $(f(v_k v_i) = \alpha, f'(v_i) = 0)$ and $(f(v_k v_i) = \alpha - \beta, f'(v_i) = \beta)$ without changing $w(v_i)$. Similarly, if $v_k v_i \in E(G) \setminus E(T)$ and $f'(v_i) = \beta$, we can choose between $(f(v_k v_i) = \alpha, f'(v_i) = \beta)$ and $(f(v_k v_i) = \alpha + \beta, f'(v_i) = 0)$ without changing $w(v_i)$. Finally, we can choose $f'(v_k)$. This gives us a total of $|E(v_k, \{v_1, \ldots, v_{k-1}\}) \setminus E(T)| + 2 = |E(v_k, \{v_1, \ldots, v_{k-1}\})| + 1$ different possibilities for $w(v_k)$, and we may pick one that is different from all weights in $N(v_k) \cap \{v_1, \ldots, v_{k-1}\}$.

Continuing in this fashion, we can find the desired weighting. □

Now we are ready to proof the main result of this note.

Theorem 3 *For every graph G without components isomorphic to a K^2, there is a weighting $\omega : E(G) \to \{1, 2, \ldots, 6\}$, such that the induced vertex weights $\omega(v) := \sum_{u \in N(v)} \omega(uv)$ properly color $V(G)$.*

Proof: We may assume that G is connected as we can treat every component separately. Start with any spanning tree T and consider the weighting (f, f', w) from Lemma 2 with parameters $\alpha = 4$ and $\beta = -2$. At this point, all edges and vertices have even weights. In the remainder of the proof we will adjust f and f', but $w(v)$ will remain unchanged (and even) for every vertex $v \in V(G)$.

Let $H = G[\{v \in V(G) \mid f'(v) = -2\}]$, and find a maximal spanning subgraph H_1 of H with maximum degree at most 2. Add -1 to the weights $f(e)$ of all edges in H_1, and adjust $f'(v)$ accordingly for all vertices in $V(H_1)$ to keep $w(v)$ unchanged. Now all vertices $v \in V(G)$ have $f'(v) \in \{0, -1, -2\}$, all edges $e \in E(G)$ have $f(e) \in \{1, 2, \ldots 6\}$, and all edges $e \in E(T)$ have $f(e) \in \{3, 4\}$.

For $i \in \{0, 1, 2\}$ let $S_i := \{v \in V(G) \mid f'(v) = -i\}$ and $s_i := |S_i|$. Note that all vertices $v \in S_0 \cup S_2$ have even weights $w(v) - f'(v)$, and vertices in S_1 have odd weights. By the maximality of H_1, all edges uv with $u, v \in S_1 \cup S_2$ have $u, v \in S_1$ and $uv \in E(H_1)$. In particular, $w(u) - f'(u) \neq w(v) - f'(v)$ for the end vertices of these edges. Let us denote the set of these edges by E^*.

If $s_2 = 0$, we are done by setting $\omega = f$. If $s_2 = 1$ and $s_1 = 0$, let $u \in S_2$. Note that all edges e incident to u have weights $f(e) \in \{2, 4, 6\}$. If u has a neighbor v with $w(u) + 2 \neq w(v)$, subtract 1 on the edge uv and we are done by setting $\omega = f$ (note that u and v are the only vertices with odd weight ω). If for all neighbors $v \in N(u)$ we have $w(u) + 2 = w(v)$ and $|N(u)| \geq 2$, subtract 1 on two different edges incident to u. Again, this leads to a proper weighting ω. Finally, if the only neighbor $v \in N(u)$ has $w(u) + 2 = w(v)$, we can find a vertex $x \in N_T(v) \setminus \{u\}$, subtract 1 from $f(uv)$ and add 1 to $f(xv)$, and again this leads to a proper weighting ω.

If $s_2 = 1$ and $s_1 \geq 1$, take a T-path between $u \in S_2$ and $v \in S_1$, and, in the manner of a Kempe chain argument, add and subtract 1 in turn to all edges of this path, making sure that we subtract 1 on the edge incident to v. This leads to a proper weighting ω.

If $s_2 \geq 2$, the following inductive argument shows that we can find $\lceil s_2/2 \rceil$ paths in T such that the set of ends of the paths is exactly S_2, and every edge of T is used at most twice. For $2 \leq s_2 \leq 3$, these paths are easy to find. For $s_2 \geq 4$, find an edge $e \in E(T)$ so that both components of $T - e$ contain at least 2 vertices in S_2 and at least one of the components contains an even number of vertices in S_2. Now apply induction on the two components to find the desired paths.

In the manner of a Kempe chain argument, add and subtract 1 in turn to all edges of each of these paths, such that only the weights of the end vertices are affected, and adjust f' for these vertices accordingly. If a vertex $u \in S_2$ is end vertex of two paths (i.e., if s_2 is odd), we make sure to subtract 1 on the edges incident to u of both paths so that we end up with $f'(u) = 0$. Note that we only use edges in $E(T)$, and therefore we do not introduce edge weights less than 1 or greater than 6. After this process, all vertices previously in S_2 now have weights $f'(v) \in \{-3, -1, 0\}$. If we set $\omega = f$, we see that $\omega = w$ for all vertices with $w(v)$ even, and the only edges between vertices with odd weight $\omega(v)$ are in E^*, and therefore their end vertices have different weights. Thus, ω is as desired. □

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Authors:

Maciej Kalkowski
Adam Mickiewicz University
Collegium Mathematicum
Umultowska 87
61-614 Poznań
Poland

e-mail: kalkos@amu.edu.pl

Michał Karoński
Adam Mickiewicz University
Collegium Mathematicum
Umultowska 87
61-614 Poznań
Poland

e-mail: karonski@amu.edu.pl

Florian Pfender
Universität Rostock,
Institut für Mathematik,
18051 Rostock,
Germany

e-mail: florian.pfender@uni-rostock.de

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A. Georgieva, S. Kostadinov

L_p - equivalence between two nonlinear impulse differential Equations with unbounded linear Parts and its application for partial impulse differential equations

ABSTRACT. An L_p-equivalence of two impulse differential equations with unbounded linear parts is proved by means of the Schauder-Tychonoff's fixed point theorem. An example of the theory of the partial impulse differential equations of parabolic type is given.

KEY WORDS AND PHRASES. Impulse differential equations, L_p-equivalence, Partial impulse differential equations of parabolic type.

1 Introduction

We study an equivalence in L_p $(1 \leq p < \infty)$ of two ordinary impulse differential equations with a possibly unbounded linear part. This means that to every bounded solution of the first equation there corresponds a bounded solution of the second equation such that their difference is in L_p and vice versa. In Theorem 1 we prove the L_p-equivalence making use of the Schauder-Tychonoff's fixed point principle. Further we give an example with an important application in physics. We consider two partial impulse differential equations with elliptic linear parts and reduce them to two ordinary impulse differential equations. These equations satisfy the conditions of Theorem 1 and are therefore L_p-equivalent. In this case, we establish "L_p-dependence" between the solutions of two partial equations.

2 Statement of the problem

Let X be a Banach space with norm $\|.\|$ and identity I. By $D(T) \subset X$ we will denote the domain of the operator $T : D(T) \to X$. We consider the following two impulse differential equations

$$\frac{du_i}{dt} = A_i(t)u_i + f_i(t, u_i) \quad \text{for} \;\; t \neq t_n \tag{1}$$

$$u_i(t_n^+) = Q_n^i(u_i(t_n)) + h_n^i(u_i(t_n)) \quad \text{for} \ \ n = 1, 2, ..., \tag{2}$$

where $A_i(t) : D(A_i(t)) \to X$ $(t \in \mathbb{R}_+)$ and $Q_n^i : D(Q_n^i) \to D(A_i(t_n))$ $(i = 1, 2)$ are linear (possibly unbounded) operators. The sets $D(A_i(t))$ and $D(Q_n^i)$ $(i = 1, 2; \ t \geq 0, \ n = 1, 2, ...)$ are dense in X. The functions $f_i(.,.) : \mathbb{R}_+ \times X \to X$ and $h_n^i : X \to X (n = 1, 2, ...)$ are continuous. The points of jump t_n satisfy the following conditions $0 = t_o < t_1 < ... < t_n < ...,$ $\lim\limits_{n \to \infty} t_n = \infty$. We set $Q_0^i = I, \ h_0^i(u) = 0 \ \ (i = 1, 2, \ u \in X)$.

Furthermore, we assume that all considered functions are left continuous.

Let $U_i(t, s)$ $(i = 1, 2; \ 0 \leq s \leq t)$ be Cauchy operators of the linear ordinary equations

$$\frac{du_i}{dt} = A_i(t) u_i \quad (i = 1, 2). \tag{3}$$

It is easy to prove that the functions $u_i(t) = V_i(t, s)\xi_i$ for $\xi_i \in D(A_i(s))$ $(i = 1, 2)$ with

$$V_i(t, s) = U_i(t, t_n) Q_n^i U_i(t_n, t_{n-1}) Q_{n-1}^i ... Q_k^i U_i(t_k, s) \tag{4}$$

$(0 \leq s \leq t_k \leq t_n < t)$ satisfy the linear impulse Cauchy problems

$$\frac{du_i}{dt} = A_i(t) u_i \quad \text{for} \ \ t \neq t_n \tag{5}$$

$$u_i(t_n^+) = Q_n^i(u_i(t_n)) \quad \text{for} \ \ n = 1, 2, ... \tag{6}$$

$$u_i(s) = \xi_i \quad (i = 1, 2). \tag{7}$$

Let us note that the operators $V_i(t, s)$ $(i = 1, 2)$ are bounded if one of the following conditions holds

1. $Q_n^i U_i(t_n, t_{n-1})$ are bounded operators $(i = 1, 2; \ n = 1, 2, ...)$.

2. $U_i(t_{n+1}, t_n) Q_n^i$ are bounded operators $(i = 1, 2; \ n = 1, 2, ...)$.

Definition 1 *The solutions of integral equations*

$$u_i(t) = V_i(t, s)\xi_i + \int_s^t V_i(t, \tau) f_i(\tau, u_i(\tau)) d\tau + \sum_{s < t_n < t} V_i(t, t_n^+) h_n^i(u_i(t_n)) \tag{8}$$

for $0 \leq s \leq t, \ \xi_i \in D(A_i(s)), \ u_i(s) = \xi_i$ *are called solutions of the impulse equations* (1), (2) $(i = 1, 2)$.

By $L_p(X)$, $1 \leq p < \infty$ we denote the space of all functions $u : \mathbb{R}_+ \to X$ for which $\int\limits_0^\infty \|u(t)\|^p dt < \infty$, with the norm $\|u\|_p = (\int\limits_0^\infty \|u(t)\|^p dt)^{\frac{1}{p}}$. Set $B_r = \{u \in X : \ \|u\| \leq r\}$.

Definition 2 *The equation (1), (2) for $i = 2$ is called L_p-equivalent to the equation (1), (2) for $i = 1$ in the ball B_r, if there exists $\rho > 0$ such that for any solution $u_1(t)$ of (1), (2) $(i = 1)$ lying in the ball B_r there exists a solution $u_2(t)$ of (1), (2) $(i = 2)$ lying in the ball $B_{r+\rho}$ and satisfying the relation $u_2(t) - u_1(t) \in L_p(X)$. If equation (1), (2) $(i = 2)$ is L_p-equivalent to equation (1), (2) $(i = 1)$ in the ball B_r and vice versa, we shall say that equations (1), (2) $(i = 1)$ and (1), (2) $(i = 2)$ are L_p-equivalent in the ball B_r.*

The paper aims at finding sufficiently conditions for the existence of L_p-equivalence between the impulse equations (1), (2) $(i = 1, 2)$.

3 Main results

3.1 L_p-equivalent impulse equations

Let us set

$$v(t) = u_2(t) - u_1(t),$$

where $u_i(t)$ $(i = 1, 2)$ being defined by (8).

Then the function $v(t)$ is a solution of the integral equation

$$v(t) = T(u_1, v)(t),$$

where

$$\begin{aligned} T(u_1, v)(t) &= V_2(t, 0)(u_1(0) + v(0)) - V_1(t, 0)u_1(0) + \\ &+ \int_0^t \{V_2(t, \tau)f_2(\tau, u_1(\tau) + v(\tau)) - V_1(t, \tau)f_1(\tau, u_1(\tau))\}d\tau + \\ &+ \sum_{0<t_n<t} \{V_2(t, t_n^+)h_n^2(u_1(t_n) + v(t_n)) - V_1(t, t_n^+)h_n^1(u_1(t_n))\} \end{aligned} \tag{9}$$

We shall prove that for each solution $u_1(t)$ of equation (1), (2) $(i = 1)$ lying in the ball B_r the operator $T(u_1, v)$ has a fixed point $v(t)$ such that $u_1(t) + v(t) \in B_{r+\rho}$ for some $\rho > 0$ and which is in $L_p(X)$.

Let $S(\mathbb{R}_+, X)$ be linear set of all functions which are continuous for $t \neq t_n$ $(n = 1, 2, ...)$, have in a both left and right limits at points t_n and are left continuous. The set $S(\mathbb{R}_+, X)$ is a locally convex space w.r.t. the metric

$$\rho(u, v) = \sup_{0<T<\infty} (1 + T)^{-1} \frac{\max_{0\leq t\leq T} \|u(t) - v(t)\|}{1 + \max_{0\leq t\leq T} \|u(t) - v(t)\|}.$$

The convergence w.r.t this metric coincides with the uniform convergence on each bounded interval. For this space an analog of Arzella-Ascoli's theorem is valid.

Lemma 1 [1] *The set* $M \subset S(\mathbb{R}_+, X)$*is relatively compact if and only if the intersections* $M(t) = \{m(t) : \ m \in M\}$ *are relatively compact for* $t \in \mathbb{R}_+$ *and* M *is equicontinuous on each interval* $(t_n, t_{n+1}]$ $(n = 0, 1, 2, ...)$.

Proof: We apply Arzella-Ascoli theorem to each interval $(t_n, t_{n+1}](n = 0, 1, 2, ...)$ and constitute a diagonal line sequence, which is converging on each of them. □

Lemma 2 [1] *Let the continuous compact operator* T *transform the set*

$$C(\rho) = \{v \in S(\mathbb{R}_+, X) : \ v(t) \in B_\rho, \ t \in \mathbb{R}_+\}$$

onto itself. Then T *has a fixed point in* $C(\rho)$.

3.2 Conditions for L_p-equivalence

Theorem 1 *Let the following conditions be fulfilled.*

1. *There exist positive functions* $K_i(t, s)$ $(i = 1, 2)$ *such that*

$$\|V_i(t, s)\xi\| \leq K_i(t, s)\|\xi\| \quad (0 \leq s \leq t, \ \xi \in D(A_i(s))),$$

where the functions $K_i(t, 0)$ $(i = 1, 2)$ *satisfy the following condition*:

There exist constants $r, \rho > 0$ *such that*

$$K_1(t, 0)\|\xi\| + K_2(t, 0)\|\eta\| \leq \chi_{r,\rho}(t) \quad (t \in \mathbb{R}_+, \ \eta \in B_{r+\rho}, \ \xi \in B_r),$$

where $\chi_{r,\rho}(t) \in L_p(\mathbb{R}_+)$.

2. *The functions* $f_i(t, u)$ *and* $K_i(t, s)$ $(i = 1, 2)$ *satisfy the conditions*:

2.1 $\sup\limits_{\|u\| \leq r} \int\limits_0^t K_1(t, \tau)\|f_1(\tau, u)\|d\tau + \sup\limits_{\|w\| \leq r+\rho} \int\limits_0^t K_2(t, \tau)\|f_2(\tau, w)\|d\tau \leq \psi_{r,\rho}(t)$, *where* $\psi_{r,\rho}(t)$ *is continuous and* $\psi_{r,\rho}(t) \in L_p(\mathbb{R}_+)$.

2.2 $\int\limits_0^t V_2(t, \tau)f_2(\tau, u_1(\tau) + v(\tau))d\tau \in K(t)$
$(v \in B_\rho, \ u_1 \in B_r, \ u_1 -$ *fixed*), *where for any fixed* $t \in \mathbb{R}_+$ $K(t)$ *is a compact subset of* X.

3. *The functions* $h_n^i(u)$ *and* $K_i(t, s)$ $(i = 1, 2)$ *satisfy the conditions*

3.1 $\sup\limits_{\|u\| \leq r} \sum\limits_{0<t_n<t} K_1(t, t_n^+)\|h_n^1(u)\| + \sup\limits_{\|w\| \leq r+\rho} \sum\limits_{0<t_n<t} K_2(t, t_n^+)\|h_n^2(w)\| \leq \varphi_{r,\rho}(t)$, *where* $\varphi_{r,\rho}(t) \in L_p(\mathbb{R}_+)$.

3.2 $\sum\limits_{0<t_n<t} V_2(t,t_n^+)h_n^2(u_1(t_n)+v(t_n)) \in K_n$
$(v \in B_\rho,\ u_1 \in B_r,\ u_1 - \mathit{fixed})$, *where for any fixed* $n = 1, 2, ...,$ K_n *is a compact subset of* X.

4. *The function* $f_2(t,w)$ *satisfies the condition*

$$\sup_{\|w\|\le r+\rho} K_2(t,\tau)\|f_2(\tau,w)\| \le \Phi_{r,\rho}(t,\tau),$$

where $\int\limits_0^t \Phi_{r,\rho}(t,\tau)d\tau < \infty$ *for any fixed* $t \in \mathbb{R}_+$.

5. *The inequality*

$$\chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t) \le \rho$$

holds for each $t \in \mathbb{R}_+$.

Then the equation (1), (2) *for* $i = 2$ *is* L_p*-equivalent to the equation* (1), (2) *for* $i = 1$ *in the ball* B_r.

Proof: We shall show that for any function $u_1(t) \in B_r$ $(t \in \mathbb{R}_+)$ the operator $T(u_1,v)$ defined by equality (9) maps the set

$$C(\rho) = \{v \in S(\mathbb{R}_+,X):\ v(t) \in B_\rho,\ t \in \mathbb{R}_+\}$$

into itself.

Let $u_1(t) \in B_r$ $(t \in \mathbb{R}_+)$ and let $v \in C(\rho)$. Then, making use of (9), we obtain the estimate

$$\begin{aligned}
&\|T(u_1,v)(t)\| \le \|V_2(t,0)(u_1(0)+v(0))\| + \|V_1(t,0)u_1(0)\| + \\
&+ \int\limits_0^t \|V_2(t,\tau)f_2(\tau,u_1(\tau)+v(\tau))\|d\tau + \int\limits_0^t \|V_1(t,\tau)f_1(\tau,u_1(\tau))\|d\tau + \\
&+ \sum_{0<t_n<t} \|V_2(t,t_n^+)h_n^2(u_1(t_n)+v(t_n))\| + \sum_{0<t_n<t} \|V_1(t,t_n^+)h_n^1(u_1(t_n))\| \\
&\le K_2(t,0)\|u_1(0)+v(0)\| + K_1(t,0)\|u_1(0)\| + \\
&+ \sup_{\|w\|\le r+\rho} \int\limits_0^t K_2(t,\tau)\|f_2(\tau,w)\|d\tau + \sup_{\|u\|\le r} \int\limits_0^t K_1(t,\tau)\|f_1(\tau,u)\|d\tau + \\
&+ \sup_{\|w\|\le r+\rho} \sum_{0<t_n<t} K_2(t,t_n^+)\|h_n^2(w)\| + \sup_{\|u\|\le r} \sum_{0<t_n<t} K_1(t,t_n^+)\|h_n^1(u)\| \\
&\le \chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t) \le \rho
\end{aligned}$$

for each $t \in \mathbb{R}_+$.

Let $M = \{m(t) = T(u_1,v)(t):\ \|v\| \le \rho,\ t \in \mathbb{R}_+\}$.

We shall show the equicontinuity of the functions of the set M. Let $t' > t''$ and $t', t'' \in (t_n, t_{n+1}]$. It is easily seen that

$$\begin{aligned}
&\|m(t') - m(t'')\| \leq \\
&\leq \|V_2(t',0)u_2(0) - V_2(t'',0)u_2(0)\| + \|V_1(t',0)u_1(0) - V_1(t'',0)u_1(0)\| + \\
&+ \sup_{\|w\|\leq r+\rho} \int_0^{t''} \|V_2(t',\tau)f_2(\tau,w) - V_2(t'',\tau)f_2(\tau,w)\|d\tau + \\
&+ \sup_{\|u\|\leq r} \int_0^{t''} \|V_1(t',\tau)f_1(\tau,u) - V_1(t'',\tau)f_1(\tau,u)\|d\tau + \\
&+ \sup_{\|w\|\leq r+\rho} \int_{t''}^{t'} K_2(t',\tau)\|f_2(\tau,w)\|d\tau + \sup_{\|u\|\leq r} \int_{t''}^{t'} K_1(t',\tau)\|f_1(\tau,u)\|d\tau + \\
&+ \sup_{\|w\|\leq r+\rho} \sum_{0<t_n<t''} \|V_2(t',t_n^+)h_n^2(w) - V_2(t'',t_n^+)h_n^2(w)\| + \\
&+ \sup_{\|u\|\leq r} \sum_{0<t_n<t''} \|V_1(t',t_n^+)h_n^1(u) - V_1(t'',t_n^+)h_n^1(u)\|.
\end{aligned}$$

The continuity of functions $V_i(t,\tau)$ $(i = 1,2)$ on $(t_n, t_{n+1}]$ and condition 2.1 of Theorem 1 imply the equicontinuity of the set M.

It follows from conditions 2.2, 3.2 and (9) that the sections $M(t) = \{m(t) : \ m \in M\}$ are compact for any $t \in \mathbb{R}_+$. Consequently, Lemma 1 implies the compactness of the set M.

Now, we shall show that the operator $T(u_1, v)$ is continuous in $S(\mathbb{R}_+, X)$.

Let the sequence $\{v_k(t)\} \subset C(\rho)$ be convergent in the metric of the space $S(\mathbb{R}_+, X)$ to the function $v(t) \in C(\rho)$. Then, for $t \in \mathbb{R}_+$ the sequence $f_2(t, u_1(t) + v_k(t))$ converges to $f_2(t, u_1(t) + v(t))$. Utilizing condition 4 of Theorem 1, we obtain that the convergent sequence of functions $V_2(t,\tau)f_2(\tau, u_1(\tau) + v_k(\tau))$ is majorized by the integrable function $\Phi_{r,\rho}(t,\tau)$. Therefore, we may pass to the limit in the formula.

$$\begin{aligned}
&T(u_1, v_k)(t) = V_2(t,0)(u_1(0) + v_k(0)) - V_1(t,0)u_1(0) + \\
&+ \int_0^t \{V_2(t,\tau)f_2(\tau, u_1(\tau) + v_k(\tau)) - V_1(t,\tau)f_1(\tau, u_1(\tau))\}d\tau + \\
&+ \sum_{0<t_n<t} \{V_2(t,t_n^+)h_n^2(u_1(t_n) + v_k(t_n)) - V_1(t,t_n^+)h_n^1(u_1(t_n))\}
\end{aligned}$$

Hence, $T(u_1, v_k)(t)$ tends to $T(u_1, v)(t)$ for $t \in \mathbb{R}_+$.

From Lemma 2 it follows that for any $u_1 \in B_r$ the operator $T(u_1, v)$ has a fixed point v in $C(\rho)$ i.e., $v = T(u_1, v)$.

We shall show that this fixed point $v(t)$ lies in $L_p(X)$.

$$\|v(t)\| \leq K_2(t,0)\|u_1(0)+v(0)\| + K_1(t,0)\|u_1(0)\|+$$
$$+\sup_{\|w\|\leq r+\rho}\int_0^t K_2(t,\tau)\|f_2(\tau,w)\|d\tau + \sup_{\|u\|\leq r}\int_0^t K_1(t,\tau)\|f_1(\tau,u)\|d\tau+$$
$$+\sup_{\|w\|\leq r+\rho}\sum_{0<t_n<t} K_2(t,t_n^+)\|h_n^2(w)\| + \sup_{\|u\|\leq r}\sum_{0<t_n<t} K_1(t,t_n^+)\|h_n^1(u)\|$$
$$\leq \chi_{r,\rho}(t)+\psi_{r,\rho}(t)+\varphi_{r,\rho}(t)$$
$$\|v\|_p \leq (\int_0^\infty |\chi_{r,\rho}(t)+\psi_{r,\rho}(t)+\varphi_{r,\rho}(t)|^p dt)^{\frac{1}{p}}$$
$$\leq \|\chi_{r,\rho}\|_p + \|\psi_{r,\rho}\|_p + \|\varphi_{r,\rho}\|_p$$

Hence, this fixed point belongs to the space $S(\mathbb{R}_+,X)$ i.e., equation (1), (2) for $i=2$ is L_p-equivalent to the equation (1), (2) for $i=1$ in the ball B_r.

Theorem 1 is proved. □

We shall illustrate Theorem 1 by an example of the qualitative theory of the nonlinear partial impulse differential equations.

Example In the example we consider two partial impulse differential equations and reduce them to two ordinary impulse differential equations. For these ordinary impulse differential equations, the conditions of Theorem 1 are fulfilled. Many notations and results for ordinary differential equations are taken from capite $5-7$ of [4]. The short introduction in the general theory of nonlinear partial impulse differential equations follows [2].

Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n, Q=(0,\infty)\times\Omega$ and $\Gamma=(0,\infty)\times\partial\Omega$.

We denote

$$P_n=\{(t_n,x):\ x\in\Omega\},\quad P=\bigcup_{n=1}^{\infty}P_n,$$

$$\Lambda_n=\{(t_n,x):\ x\in\partial\Omega\},\quad \Lambda=\bigcup_{n=1}^{\infty}\Lambda_n.$$

Consider the impulse nonlinear parabolic initial value problems

$$\frac{\partial u_i}{\partial t}=\tilde{A}_i(t,x,D)u_i+\tilde{f}_i(t,x,u_i),\ \ (t,x)\in Q\setminus P \tag{10}$$

$$D^\alpha u_i(t,x)=0,\ \ |\alpha|<m,\ (t,x)\in\Gamma\setminus\Lambda \tag{11}$$

$$u_i(0,x)=v_i(x),\ \ x\in\Omega \tag{12}$$

$$u_i(t_n^+,x)=\tilde{Q}_n^i(u_i(t_n,x))+\tilde{h}_n^i(u_i(t_n,x)),\ \ x\in\overline{\Omega},\ n=1,2,..., \tag{13}$$

where

$$\tilde{A}_i(t,x,D) = \sum_{|\alpha|\le 2m} a^i_\alpha(t,x)D^\alpha,$$

$\tilde{Q}^i_n : D(\tilde{Q}^i_n) \to D(\tilde{A}_i(t_n,x,D))$ $(n=1,2,...;i=1,2)$ are linear operators, $\tilde{f}_i(.,.,.) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $\tilde{h}^i_n : \mathbb{R} \to \mathbb{R}$ are continuous functions.

Let $X = L_p(\Omega,\mathbb{R})$ $(1<p<\infty)$, where

$$L_p(\Omega,\mathbb{R}) = \{v : \Omega \to \mathbb{R}; \int_\Omega |v(x)|^p dx < \infty\}$$

with norm $|v|_p = (\int_\Omega |v(x)|^p dx)^{\frac{1}{p}}$.

With the family $\tilde{A}_i(t,x,D)$, $t \in \mathbb{R}_+$, $(i=1,2)$ of strongly elliptic operators we associate a family of linear operators $A_i(t)$, $t \in \mathbb{R}_+$, $(n=1,2)$ acting in X by

$$A_i(t)u_i = \tilde{A}_i(t,x,D)u_i, \ \text{ for } u_i \in D.$$

This is done as follows $D = D(A_i(t)) = W^{2m,p}(\Omega) \bigcap W^{m,p}_0(\Omega)$, $(i=1,2;\ t\in\mathbb{R}_+)$.

Let $v_i \in X$. We set

$$f_i(t,u_i)(x) = \tilde{f}_i(t,x,u_i(t,x)),\ \ u_i \in X,\ t\in\mathbb{R}_+,\ x\in\overline{\Omega}\ (i=1,2),$$
$$Q^i_n(u_i(t_n))(x) = \tilde{Q}^i_n(u_i(t_n,x)),\ \ h^i_n(u_i(t_n))(x) = \tilde{h}^i_n(u_i(t_n,x)),$$

where $Q^i_n : D(Q^i_n) \to D$ $(D(Q^i_n) \subset X$ lie dense in X $(i=1,2))$ are linear operators, $f^i_n : \mathbb{R}_+ \times X \to X$ and $h^i_n : X \to X$ are continuous functions.

We shall prove the L_p-equivalence between the equations (1), (2) $(i=1,2)$.

Let $U_i(t,s)$ $(i=1,2)$ are the Cauchy operators of the equations

$$\frac{du_i}{dt} = A_i(t)u_i$$

Sufficient conditions for the validity of the estimates

$$|U_i(t,s)|_{p\to p} \le C_i e^{-k_i(t-s)}\ \ (0\le s\le t;\ C_i,k_i>0 \text{ constants},\ i=1,2)$$

are given in [4].

We shall consider the concrete case when $t_n = n$ $(n=1,2,...)$,

$$\tilde{f}_1(t,x,u_1) = e^{\gamma_1 t}\tfrac{u_1^2(t,x)}{1+u_1^2(t,x)},\ \ \tilde{f}_2(t,x,u_2) = e^{\gamma_2 t}\sin u_2(t,x),$$

$$\tilde{Q}^1_n\xi_1 = \tfrac{k_1 n}{C_1(1+n^2)e^{C_1+k_1}}\xi_1,\ \ \tilde{Q}^2_n\xi_2 = \tfrac{k_2 n}{C_2(1+n^2)e^{C_2+k_2}}\xi_2,$$

$$\tilde{h}^1_n(u_1(t_n,x)) = e^{\alpha_1 n}2^{-u_1(t_n,x)},\ \ \tilde{h}^2_n(u_2(t_n,x)) = e^{\alpha_2 n}\tfrac{1}{1+u_2^2(t_n,x)}$$

where $-1 < \gamma_i + k_i < 0$ and $\alpha_i + k_i < \ln \frac{1}{2}$ $(i = 1, 2)$.

Then

$$f_1(t, u_1) = e^{\gamma_1 t} \tfrac{u_1^2(t)}{1+u_1^2(t)}, \quad f_2(t, u_2) = e^{\gamma_2 t} \sin u_2(t),$$

$$Q_n^1 \xi_1 = \tfrac{k_1 n}{C_1(1+n^2)e^{C_1+k_1}} \xi_1, \quad \tilde{Q}_n^2 \xi_2 = \tfrac{k_2 n}{C_2(1+n^2)e^{C_2+k_2}} \xi_2,$$

$$h_n^1(u_1(t_n)) = e^{\alpha_1 n}.2^{-u_1(t_n)}, \quad h_n^2(u_2(t_n)) = e^{\alpha_2 n} \tfrac{1}{1+u_2^2(t_n)}$$

Let $V_i(t, s)$ $(i = 1, 2;\ 0 \le s \le t)$ are the Cauchy operators of the linear impulse equations

$$\frac{du_i}{dt} = A_i(t) u_i \ \text{ for } t \ne t_n$$

$$u_i(t_n^+) = Q_n^i(u_i(t_n)) \ \text{ for } n = 1, 2, ...$$

Then for $0 < s \le k < n < t$, $\xi \in D$ the following estimates are valid

$$|V_1(t, s)\xi|_p = |U_1(t, t_n) Q_n^1 ... Q_k^1 U_1(t_k, s)\xi|_p$$

$$\le C_1 e^{-k_1(t-n)} \tfrac{k_1 n}{C_1(1+n^2)e^{C_1+k_1}} \cdots \tfrac{k_1 k}{C_1(1+k^2)e^{C_1+k_1}} C_1 e^{-k_1(k-s)} |\xi|_p$$

$$\le \tfrac{C_1}{e^{C_1(n-k+1)}} \tfrac{k_1^{n-k}}{e^{k_1(n-k+1)}} k_1 n e^{-k_1(t-s)} |\xi|_p \le k_1 t e^{-k_1(t-s)} |\xi|_p.$$

Similarly

$$|V_2(t, s)\xi|_p \le k_2 t e^{-k_2(t-s)} |\xi|_p.$$

We set

$$k_i(t, s) = k_i t e^{-k_i(t-s)} \ \ (i = 1, 2)$$

Let $r > 0$ and

$$\rho > \frac{2}{e-1}(r + 2(\mu(\Omega))^{\frac{1}{p}}) \tag{14}$$

We shall show that the conditions of Theorem 1 are fulfilled. For any $\xi \in B_r$, $\eta \in B_{r+\rho}$, $t \in \mathbb{R}_+$ we obtain

$$K_1(t, 0)|\xi|_p + K_2(t, 0)|\eta|_p = k_1 t e^{-k_1 t} |\xi|_p + k_2 t e^{-k_2 t} |\eta|_p \le$$

$$\le k_1 t e^{-k_1 t} r + k_2 t e^{-k_2 t}(r + \rho).$$

Let us set

$$\chi_{r,\rho}(t) = k_1 t e^{-k_1 t} r + k_2 t e^{-k_2 t}(r + \rho).$$

We shall show that condition 2.1 of Theorem 1 is fulfilled.

$$\sup_{|u|_p \le r} \int_0^t K_1(t, \tau) |f_1(\tau, u)|_p d\tau + \sup_{|w|_p \le r+\rho} \int_0^t K_2(t, \tau) |f_2(\tau, w)|_p d\tau$$

$$= \sup_{|u|_p \le r} \int_0^t k_1 t e^{-k_1(t-\tau)} e^{\gamma_1 \tau} \left|\tfrac{u^2(\tau)}{1+u^2(\tau)}\right|_p d\tau +$$

$$+ \sup_{|w|_p \le r+\rho} \int_0^t k_2 t e^{-k_2(t-\tau)} e^{\gamma_2 \tau} |\sin w(\tau)|_p d\tau \le$$

$$\le k_1te^{-k_1t}(\mu(\Omega))^{\frac{1}{p}}\int\limits_0^t e^{(k_1+\gamma_1)\tau}d\tau+$$
$$+k_2te^{-k_2t}(\mu(\Omega))^{\frac{1}{p}}\int\limits_0^t e^{(k_2+\gamma_2)\tau}d\tau\le$$
$$\le k_1te^{-k_1t}\tfrac{(\mu(\Omega))^{\frac{1}{p}}}{-(k_1+\gamma_1)}+k_2te^{-k_2t}\tfrac{(\mu(\Omega))^{\frac{1}{p}}}{-(k_2+\gamma_2)}$$

Let us set

$$\psi_{r,\rho}(t)=k_1te^{-k_1t}\frac{(\mu(\Omega))^{\frac{1}{p}}}{-(k_1+\gamma_1)}+k_2te^{-k_2t}\frac{(\mu(\Omega))^{\frac{1}{p}}}{-(k_2+\gamma_2)}.$$

We shall prove condition 3.1 of Theorem 1.

$$\sup_{|u|_p\le r}\sum_{0<n<t}K_1(t,t_n^+)|h_n^1(u(t_n))|_p+\sup_{|w|_p\le r+\rho}\sum_{0<n<t}K_2(t,t_n^+)|h_n^2(w(t_n))|_p=$$
$$=\sup_{|u|_p\le r}\sum_{0<n<t}k_1te^{-k_1(t-n)}e^{\alpha_1 n}|2^{-u(t_n)}|_p+$$
$$+\sup_{|w|_p\le r+\rho}\sum_{0<n<t}k_2te^{-k_2(t-n)}e^{\alpha_2 n}|\tfrac{1}{1+w^2(t_n)}|_p\le$$
$$\le k_1te^{-k_1t}(\mu(\Omega))^{\frac{1}{p}}\sum_{0<n<t}e^{(k_1+\alpha_1)n}+$$
$$+k_2te^{-k_2t}(\mu(\Omega))^{\frac{1}{p}}\sum_{0<n<t}e^{(k_2+\alpha_2)n}\le$$
$$\le k_1te^{-k_1t}(\mu(\Omega))^{\frac{1}{p}}\tfrac{e^{\alpha_1+k_1}}{1-e^{\alpha_1+k_1}}+k_2te^{-k_2t}(\mu(\Omega))^{\frac{1}{p}}\tfrac{e^{\alpha_2+k_2}}{1-e^{\alpha_2+k_2}}$$

Set

$$\varphi_{r,\rho}(t)=k_1te^{-k_1t}(\mu(\Omega))^{\frac{1}{p}}\frac{e^{\alpha_1+k_1}}{1-e^{\alpha_1+k_1}}+k_2te^{-k_2t}(\mu(\Omega))^{\frac{1}{p}}\frac{e^{\alpha_2+k_2}}{1-e^{\alpha_2+k_2}}.$$

It is not hard to check if the functions $\chi_{r,\rho}(t)$, $\psi_{r,\rho}(t)$ and $\varphi_{r,\rho}(t)$ lie in the space $L_p(\mathbb{R}_+)$.

Condition 4 of Theorem 1 is fulfilled with

$$\Phi_{r,\rho}(t,\tau)=k_2te^{-k_2t}(\mu(\Omega))^{\frac{1}{p}}e^{(k_2+\gamma_2)\tau}\in L_1(\mathbb{R}_+)$$

for any fixed $t\in\mathbb{R}_+$.

We shall show that condition 5 of Theorem 1 holds

$$\chi_{r,\rho}(t)+\psi_{r,\rho}(t)+\varphi_{r,\rho}(t)=$$
$$=k_1te^{-k_1t}(r-(\mu(\Omega))^{\frac{1}{p}}\tfrac{1}{k_1+\gamma_1}+(\mu(\Omega))^{\frac{1}{p}}\tfrac{e^{\alpha_1+k_1}}{1-e^{\alpha_1+k_1}})+$$
$$+k_2te^{-k_2t}(r+\rho-(\mu(\Omega))^{\frac{1}{p}}\tfrac{1}{k_2+\gamma_2}+(\mu(\Omega))^{\frac{1}{p}}\tfrac{e^{\alpha_2+k_2}}{1-e^{\alpha_2+k_2}}).$$

From condition (14) we obtain

$$\chi_{r,\rho}(t)+\psi_{r,\rho}(t)+\varphi_{r,\rho}(t)\le\rho\ \text{ for each } t\in\mathbb{R}_+.$$

By means of a compactness criterion from [3] we shall prove condition 2.2.

The set

$$M(t) = \{m(t) = \int_0^t V_2(t,\tau) f_2(\tau, u_1(\tau) + v(\tau)) d\tau : \ |v|_p \le \rho\}$$

is a compact subset of X for any fixed t.

Indeed,

$$|m(t)(x)| \le k_2 t e^{-k_2 t} \int_0^t e^{(k_2+\gamma_2)\tau} |\sin(v(\tau)(x) + u_1(\tau)(x))| d\tau$$
$$\le \int_0^t e^{(k_2+\gamma_2)\tau} d\tau = \tfrac{1}{k_2+\gamma_2}(e^{(k_2+\gamma_2)t} - 1), \text{ i.e.}$$
$$(\int_\Omega |m(t)(x)|^p dx)^{\frac{1}{p}} \le \tfrac{1}{k_2+\gamma_2}(e^{(k_2+\gamma_2)t} - 1)(\mu(\Omega))^{\frac{1}{p}}$$

and hence $|m(t)(x)|_p \le N$ (N-constant).

We show that

$$|m(t)(x+h) - m(t)(x)|_p \to 0 \ \ (h \to 0).$$

This follows from the relations below

$$|m(t)(x+h) - m(t)(x)| \le$$
$$\le \int_0^t e^{(k_2+\gamma_2)\tau} |\sin(v(\tau)(x+h) + u_1(\tau)(x+h)) - \sin(v(\tau)(x) + u_1(\tau)(x))| d\tau$$
$$\le \int_0^t e^{(k_2+\gamma_2)\tau} |v(\tau)(x+h) - v(\tau)(x)| d\tau + \int_0^t e^{(k_2+\gamma_2)\tau} |u(\tau)(x+h) - u(\tau)(x)| d\tau$$

In a similar way, we show the validity of condition 3.2. The conditions of Theorem 1 are fulfilled and hence the ordinary equations (1), (2) ($i = 1, 2$) are in B_r L_p-equivalent. Hence, every solution $u_1(t,x)$ of (10)-(13) ($i = 1$) induces a solution $u_2(t,x)$ of (10)-(13) ($i = 2$) such that the function $\alpha_1(t) = |u_1(t,x) - u_2(t,x)|$ lies in $L_p(\mathbb{R}_+)$ for any $x \in \Omega$.

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Authors:

A. Georgieva
University of Food Technologies
26 Maritza Blvd.
4002 Plovdiv
BULGARIA

S. Kostadinov
University of Plovdiv
Department of Mathematics and Informatics
24 Tsar Asen St.
4000 Plovdiv
BULGARIA

e-mail: stepanko@mail.bg

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Manfred Krüppel

De Rham's singular function, its partial derivatives with respect to the parameter and binary digital sums

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ABSTRACT. In this paper we show connections between sums related to the binary sum-of-digits function and the function of de Rham $R_a(x)$, and its partial derivatives with respect to the parameter. Starting point is a formula from [3] for the calculation of $R_a(x)$ for dyadic rational x. From this we derive an exact formula for exponential sums of digital sums, and by means of usual differentiations we find exact expressions for some digital sums. In particular, we get the well known result of Trollope-Delange concerning the sum-of-digits function and the formula of Coquet for power sums of digital sums.

KEY WORDS. De Rham's singular function, Takagi's nowhere differentiable function, digital sums, sum-of-digits function.

1 Introduction

In 1956 G. de Rham [13] proved that for a fixed parameter $a \in (0,1)$ the system of functional equations

$$\left.\begin{aligned} f\left(\tfrac{x}{2}\right) &= af(x), \\ f\left(\tfrac{x+1}{2}\right) &= a+(1-a)f(x) \end{aligned}\right\} \qquad (x \in [0,1]) \tag{1.1}$$

has a unique bounded solution $f = R_a(x)$ with $R_a(0) = 0$ and $R_a(1) = 1$. It is $R_{1/2}(x) = x$, but for $a \neq \frac{1}{2}$ de Rham's function $R_a(x)$ is a strictly singular function with the property

$$\int_0^1 R_a(x)dx = a. \tag{1.2}$$

In the literature this function is also called Lebesgue's singular function, cf. e.g. [16] or [1]. For $x \in [0,1]$ with the binary expansion

$$x = 0, d_1 d_2 \ldots \qquad (d_k \in \{0,1\}) \tag{1.3}$$

it holds

$$R_a(x) = \sum_{k=1}^{\infty} d_k a^{k-s_k}(1-a)^{s_k} \tag{1.4}$$

where $s_1 = 0$ and $s_k = d_1 + \ldots + d_{k-1}$ for $k \geq 1$, cf. [11]. In [2] it was shown that for $\ell \in \mathbb{N}$ and $n = 0, 1, \ldots, 2^\ell - 1$ de Rham's function satisfies the equations

$$R_a\left(\frac{n+x}{2^\ell}\right) = R_a\left(\frac{n}{2^\ell}\right) + a^\ell q^{s(n)} R_a(x) \qquad (x \in [0,1]) \tag{1.5}$$

where $q = \frac{1-a}{a}$ and where $s(n)$ denotes the number of ones in the binary representation of n. Let us mention that $s(n) = s_k$ for $n = [2^k x]$. Moreover, for $n = 0, 1, \ldots, 2^\ell$ it holds the formula

$$R_a\left(\frac{n}{2^\ell}\right) = a^\ell \sum_{k=0}^{n-1} q^{s(k)} \tag{1.6}$$

which is starting point of this paper.

In [3] it was considered de Rham's function in connection with two-scale-difference equations. In this paper we uncover connections between de Rham's function $R_a(x)$, its partial derivatives with respect to a and several binary digital sums. The fundamental result in the theory of digital sums is the well known Trollope-Delange formula. It expressed the sum-of-digits function

$$S(N) = \sum_{n=0}^{N-1} s(n) \tag{1.7}$$

by the exact formula ([15], [5])

$$S(N) = \frac{N \log_2 N}{2} + N F(\log_2 N) \tag{1.8}$$

where $F(u)$ is an 1-periodic function given by

$$F(u) = \frac{1-u}{2} - \frac{1}{2^u} T\left(\frac{1}{2^{1-u}}\right) \qquad (0 \leq u \leq 1)$$

with Takagi's function $T(x)$ defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \Delta(2^n x)$$

where $\Delta(x) = \operatorname{dist}(x, \mathbb{Z})$. The function F is continuous but nowhere differentiable. Formula (1.8) was proved in [7] by means of Mellin transforms and in [10] it was shown that

$$F(u) = -\frac{u}{2} - \frac{1}{2^{u+1}} T(2^u) \qquad (u \leq 0). \tag{1.9}$$

Digital sum problems are investigated by many authors, cf. e.g. [6], [4], [12]. For a historical survey see [14], [7].

In this paper we also investigate some digital sums. At first we show by means of formula (1.6) that the exponential sum

$$E_q(N) = \sum_{n=0}^{N-1} q^{s(n)} \tag{1.10}$$

with $q > 0$ can be expressed by the exact formula

$$E_q(N) = N^\alpha G_q(\log_2 N) \tag{1.11}$$

where $\alpha = \log_2(1+q)$ and where $G_q(u)$ is a continuous, 1-periodic function which is connected with de Rham's function by

$$G_q(u) = a^u R_a(2^u) \qquad (u \leq 0)$$

where $a = \frac{1}{1+q}$. The case $q = 1$ is trivial, namely $E_1(N) = N$. However for $q \neq 1$ the function $G_q(u)$ has the interesting property that it is differentiable almost everywhere and the local maxima and minima are dense in $\mathbb{R}$, Theorem 2.1. Let us mention that $E_2(N)$ is equal to the number of odd binomial coefficients in the first N rows of Pascal's triangle and that $E_2(N)$ was already investigated by many authors, cf. e.g. [14], [7].

Furthermore, for $m \in \mathbb{N}$ we investigate the partial derivative $\frac{\partial^m}{\partial a^m} R_a(x)$ which is a continuous function with respect to x, cf. [16], [1]. In particular we obtain the well known connection

$$\left.\frac{\partial R_a(x)}{\partial a}\right|_{a=1/2} = 2T(x) \tag{1.12}$$

between de Rham's function R_a and Takagi's function T, cf. [16]. Moreover, we calculate the functions $T_m(x) = \frac{\partial^m}{\partial a^m} R_a(x)|_{a=1/2}$ at all dyadic rational $x \in [0, 1]$, Proposition 4.2. These functions are used for the representation of some digital sums.

For the binomial sum

$$B_m(N) = \sum_{n=0}^{N-1} \binom{s(n)}{m} \tag{1.13}$$

with $m \geq 1$ it holds the exact formula

$$B_m(N) = \frac{N}{m!}\left(\frac{\log_2 N}{2}\right)^m + N \sum_{\ell=0}^{m-1} (\log_2 N)^\ell F_{m,\ell}(\log_2 N)$$

where $F_{m,\ell}(u)$ are 1-periodic, continuous functions which can be expressed by the functions $T_1(x), \dots, T_m(x)$, Theorem 5.3 and Proposition 5.1. In particular for $m = 1$ we get the Trollope-Delange formula (1.8).

Finally we investigate the digital power sum

$$S_m(N) = \sum_{n=0}^{N-1} s(n)^m \tag{1.14}$$

with $m \in \mathbb{N}$. Stolarsky [14] has proved the asymptotic formula

$$S_m(N) = N\left(\frac{\log_2 N}{2}\right)^m + \mathcal{O}\{N(\log N)^{m-1})\}$$

which is optimal in the sense that for the values

$$\alpha_m = \limsup \frac{S_m(N) - N\left(\frac{\log_2 N}{2}\right)^m}{N(\log N)^{m-1}} \tag{1.15}$$

and β_m, the corresponding lim inf, it holds: $-\infty < \beta_m < \alpha_m < \infty$. In particular $\alpha_1 = 0$, $\beta_1 = (\log 3/\log 4) - 1$. Coquet [4] obtained the precise formula

$$S_m(N) = N\left(\frac{\log_2 N}{2}\right)^m + N\sum_{\ell=0}^{m-1}(\log_2 N)^\ell G_{m,\ell}(\log_2 N) \tag{1.16}$$

where $G_{m,\ell}(u)$ are certain 1-periodic functions, and in [6] it was shown by means of binomial measures that these functions are continuous, cf. also [12].

We also prove the formula (1.16) of Coquet with explicit representations for the functions $G_{m,\ell}(u)$, Theorem 6.1, and determine the values α_m and β_m, Proposition 6.2.

2 Exponential sums of digital sums

For the exponential sum (1.10) with given $q > 0$ it follows from (1.6) that for $N \le 2^\ell$ it holds

$$E_q(N) = \frac{1}{a^\ell} R_a\left(\frac{N}{2^\ell}\right) \tag{2.1}$$

where R_a is de Rham's function corresponding to $a = \frac{1}{1+q}$. In particular $E_q(2^\ell) = (q+1)^\ell$. In order to obtain a formula independent of ℓ we note that with

$$\alpha = -\log_2 a = \log_2(1+q) \tag{2.2}$$

the first equation of (1.1) with $f = R_a$ yields

$$\frac{R_a(\frac{x}{2})}{(\frac{x}{2})^\alpha} = \frac{R_a(x)}{x^\alpha} \qquad (0 < x \le 1). \tag{2.3}$$

Hence the function

$$g_q(x) = \frac{R_a(x)}{x^\alpha} \qquad (0 < x \le 1) \tag{2.4}$$

has for $0 < x < \frac{1}{2}$ the property $g_q(2x) = g_q(x)$ so that it can be extended for all $x > 0$ by

$$g_q(2x) = g_q(x). \tag{2.5}$$

According to (2.5) the function

$$G_q(u) = g_q(2^u) \qquad (u \in \mathbb{R}) \tag{2.6}$$

is periodic with period 1.

Theorem 2.1 *For $q > 0$ the exponential sum $E_q(N)$ from (1.10) can be expressed by the exact formula*

$$E_q(N) = N^\alpha G_q(\log_2 N) \tag{2.7}$$

where $\alpha = \log_2(1+q)$ and where $G_q(u)$ is an 1-periodic function which is connected with de Rham's function R_a corresponding to $a = \frac{1}{1+q}$ by

$$G_q(u) = a^u R_a(2^u) \qquad (u \leq 0). \tag{2.8}$$

G_q is continuous and differentiable almost everywhere. For $q \neq 1$ the maxima and minima of G_q are lying dense in $\mathbb{R}$.

Proof: For given $N \in \mathbb{N}$ we choose ℓ such that $2^\ell > N$. For the exponential sum (1.10) we get from (1.6)

$$E_q(N) = \frac{1}{a^\ell} R_a\left(\frac{N}{2^\ell}\right) = N^\alpha \left(\frac{2^\ell}{N}\right)^\alpha R_a\left(\frac{N}{2^\ell}\right)$$

where we have used that $a = \frac{1}{1+q}$ and $\frac{1}{a} = 2^\alpha$ according to (2.2). In view of (2.4) and (2.5) we get

$$E_q(N) = N^\alpha g_q(N) \tag{2.9}$$

and it follows (2.7) with the 1-periodic function $G_q(u)$ given by (2.6). For $u \leq 0$ we have $0 < 2^u \leq 1$ and from (2.6) and (2.4) with $x = 2^u$ we get in view of $\frac{1}{2^\alpha} = a$

$$G_q(u) = \frac{R_a(2^u)}{2^{u\alpha}} = a^u R_a(2^u)$$

i.e. the representation (2.8).

In order to obtain properties of G_q we investigate g_q. Obviously, g_q is continuous in $[\frac{1}{2}, 1]$ and hence at any $x > 0$. Moreover, g_q is differentiable almost everywhere in $(0, \infty)$ since R_a is increasing. For $q \neq 1$ we have $a = \frac{1}{1+q} \neq \frac{1}{2}$. Since for fixed $a \neq \frac{1}{2}$ the function R_a is singular there are points $x \in (0,1)$ where the derivative $R_a'(x) = +\infty$, and equation (1.5) implies that such points lie dense in $(0,1)$. This is valid also for the function g_q so that there is no interval where it can be decreasing. On the other side, since R_a is singular, it follows for such x with $R_a'(x) = 0$ that

$$\left(\frac{R_a(x)}{x^\alpha}\right)' = \frac{R_a'(x) x^\alpha - \alpha x^{\alpha-1} R_a(x)}{x^{2\alpha}} = -\frac{\alpha R_a(x)}{x^{\alpha+1}}.$$

Hence, the function g_q is differentiable almost everywhere in $(0, \infty)$ with $g_q'(x) = -\frac{\alpha}{x} g_q(x) < 0$ so that there is no interval where g_q can be increasing. By (2.6) it follow the assertions for G_q. □

3 Partial derivative of de Rham's function

It is known that $R_a(x)$ is differentiable with respect to a, cf. [16]. Here we show the differentiability by means of a method as in [2], Proposition 2.3.

Proposition 3.1 *For fixed $x \in [0,1]$ with the dyadic representation (1.3) de Rham's function R_a is differentiable with respect to a and the derivative $\frac{\partial}{\partial a} R_a(x)$ is a continuous function with respect to x which has the representation*

$$\frac{\partial}{\partial a} R_a(x) = \sum_{k=1}^{\infty} d_k (k(1-a) - s_k) a^{k-s_k-1} (1-a)^{s_k-1} \tag{3.1}$$

with $s_1 = 0$ and $s_k = d_1 + \ldots + d_{k-1}$ for $k \geq 1$.

Proof: We denote the formal derivative of (1.4) with respect to a by $\varphi(x,a)$ which reads

$$\varphi(x,a) = \sum_{k=1}^{\infty} d_k (k - s_k) a^{k-s_k-1} (1-a)^{s_k} - \sum_{k=1}^{\infty} d_k s_k a^{k-s_k} (1-a)^{s_k-1}. \tag{3.2}$$

We show that for fixed x this function is an analytic function with respect to a in the complex domain $D = \{a : |a| < 1, |1-a| < 1\}$. Namely, for fixed $\varepsilon > 0$ choosing $\max(|a|, |1-a|) \leq 1 - \varepsilon$ we obtain for $x = \sum d_k 2^{-k}$ the estimate

$$|\varphi(x,a)| \leq 2 \sum_{k=1}^{\infty} k(1-\varepsilon)^{k-1} = \frac{2}{\varepsilon^2}$$

in view of $0 \leq s_k \leq k-1$. This implies that the series (3.2) of polynomials in a is uniformly convergent in every compact subset of the domain $D = \{a : |a| < 1, |1-a| < 1\}$. Consequently, in this domain $\varphi(x,a)$ is a continuous function with respect to a and it is the derivative of R_a, cf. [9, p. 353]. Hence, from (3.2) we get the derivative (3.1) of R_a with respect to a. □

In the following we denote the first partial derivative of de Rham's function with respect to a by

$$D_1(x,a) = \frac{\partial}{\partial a} R_a(x). \tag{3.3}$$

Differentiation of (1.1) with respect to a yields the following system of functional equations:

$$\left.\begin{array}{rcl} f\left(\frac{x}{2}\right) &=& R_a(x) + af(x), \\ f\left(\frac{x+1}{2}\right) &=& 1 - R_a(x) + (1-a)f(x) \end{array}\right\} \quad (x \in [0,1]). \tag{3.4}$$

By a result of Girgensohn [8] this system has exactly one continuous solution f. Hence, we have $f(x) = D_1(x,a)$.

Proposition 3.2 *For fixed a the first partial derivative (3.3) satisfies $D_1(0,a) = D_1(1,a) = 0$ and $D_1(x,a) > 0$ for $0 < x < 1$ with $D_1(\frac{1}{2},a) = 1$. Moreover, it holds*

$$D_1(1-x,a) = D_1(x,1-a) \qquad (x \in [0,1]) \tag{3.5}$$

and

$$\int_0^{1/2} D_1(x,a)dx = a, \qquad \int_{1/2}^1 D_1(x,a)dx = 1-a. \tag{3.6}$$

Proof: The property (3.5) follows from $R_a(1-x) = 1 - R_{1-a}(x)$, cf. [2, Proposition 2.3]. Equation (3.4) for $x = 0$ yields $D_1(0,a) = 0$ for each $a \in (0,1)$, and hence it holds $D_1(1,a) = 0$ in view of (3.5). The second equation (3.4) for $x = 0$ yields $D_1(\frac{1}{2},a) = 1$. In order to show that $D_1(x,a) > 0$ for $0 < x < 1$ we remark that $D_1(x_0,a) > 0$ for a certain x_0 implies $D_1(\frac{x_0}{2},a) > 0$, since (3.4), and also $D_1(1-\frac{x_0}{2},a) > 0$ in view of (3.5). Hence, $D_1(\frac{1}{2},a) = 1$ implies $D_1(x,a) > 0$ for all dyadic rational $x \in (0,1)$, and in view of the continuity of $D_1(x,a)$ with respect to x it follows $D_1(x,a) \geq 0$ for all $0 < x < 1$. Equation (3.4) implies $D_1(x,a) > 0$ for $0 < x \leq \frac{1}{2}$, and therefore $D_1(x,a) > 0$ for $0 < x < 1$ according to (3.5).

From (3.4) with $f(x) = D_1(x,a)$ we get

$$D_1\left(\frac{x}{2},a\right) + D_1\left(\frac{x+1}{2},a\right) = D_1(x,a) + 1,$$

and by integration we obtain

$$\int_0^1 D_1(x,a)dx = 1.$$

Hence and in view of (1.2) we get by integration of the first equation in (3.4) that

$$2\int_0^{1/2} D_1(x,a)dx = \int_0^1 R_a(x)dx + a\int_0^1 D_1(x,a)dx = 2a,$$

i.e. the first relation in (3.6), and the second relation follows in view of (3.5). □

We are especially interested in the case $a = \frac{1}{2}$ where the system (3.4) with $g = 2f$ attains the form

$$\left.\begin{array}{rcl} g\left(\frac{x}{2}\right) &=& \frac{x}{2} + \frac{1}{2}g(x), \\ g\left(\frac{x+1}{2}\right) &=& \frac{1-x}{2} + \frac{1}{2}g(x) \end{array}\right\} \qquad (x \in [0,1]) \tag{3.7}$$

which has the unique solution $g(x) = \frac{1}{2}D_1(x,\frac{1}{2})$, i.e.

$$g(x) = \frac{1}{2}\frac{\partial}{\partial a}R_a(x)\bigg|_{a=1/2}. \tag{3.8}$$

It is known that also Takagi's function T satisfies the system (3.7), cf. [10]. Hence, we have the interesting connection

$$\frac{\partial}{\partial a}R_a(x)\Big|_{a=1/2} = 2T(x) \tag{3.9}$$

which is proved in [16] by means of Schauder expansion of $R_a(x)$. According to Proposition 3.1 for the Takagi function T we find the representation

$$T(x) = \sum_{k=1}^{\infty} d_k \frac{k-2s_k}{2^k} \tag{3.10}$$

with x from (1.3).

4 Partial derivatives of higher order

We investigate the derivatives of higher order with respect to a of de Rham's function $R_a(x)$, cf. [16]. The existence of

$$D_m(x,a) = \frac{\partial^m}{\partial a^m} R_a(x) \tag{4.1}$$

follows from the fact that R_a is holomorphic in a. From (1.1) we find for $m \geq 2$ by repeated differentiation with respect to a the functional equations

$$\left.\begin{array}{rcl} D_m\left(\frac{x}{2},a\right) &=& mD_{m-1}(x,a)+aD_m(x,a) \\ D_m\left(\frac{x+1}{2},a\right) &=& -mD_{m-1}(x,a)+(1-a)D_m(x,a) \end{array}\right\} \quad (x\in[0,1]) \tag{4.2}$$

and by a result of Girgensohn [8] the functions D_m are continuous with respect to x.

Proposition 4.1 *The function D_m $(m \in \mathbb{N})$ has the property*

$$D_m(1-x,a) = (-1)^{m+1} D_m(x,1-a). \tag{4.3}$$

For $m \geq 2$ it holds

$$\int_0^1 D_m(x,a)dx = 0. \tag{4.4}$$

Proof: The symmetry property (4.3) follows from $R_a(1-x) = 1 - R_{1-a}(x)$, cf. [2, Proposition 2.3]. For $m \geq 2$ we get from (4.2)

$$D_m\left(\frac{x}{2},a\right) + D_m\left(\frac{x+1}{2},a\right) = D_m(x,a).$$

From this equation we find by induction on n that

$$\sum_{\nu=0}^{2^n-1} D_m\left(\frac{x+\nu}{2^n},a\right) = D_m(x,a).$$

Dividing by 2^n we obtain as $n \to \infty$ equation (4.4). □

In particular we use the derivatives at $a = \frac{1}{2}$, i.e.

$$T_m(x) = \frac{\partial^m}{\partial a^m} R_a(x)\Big|_{a=1/2}. \tag{4.5}$$

Thus $T_0(x) = x$ and $T_1(x) = 2T(x)$ where T is Takagi's function. These functions were investigated already in [1] where it was shown that for $m \geq 1$ they are continuous but nowhere differentiable. In particular the extreme values of T_2 and T_3 were studied.

Proposition 4.1 implies that for $m \geq 1$ the function T_m has the symmetry property

$$T_m(1-x) = (-1)^{m+1} T_m(x) \tag{4.6}$$

and according to (4.2) for $m \geq 2$ it satisfies the functional equations

$$\left.\begin{aligned} T_m\left(\tfrac{x}{2}\right) &= mT_{m-1}(x) + \tfrac{1}{2}T_m(x) \\ T_m\left(\tfrac{x+1}{2}\right) &= -mT_{m-1}(x) + \tfrac{1}{2}T_m(x) \end{aligned}\right\} \quad (x \in [0,1]). \tag{4.7}$$

Hence, T_m is a so-called Knopp-function

$$T_m(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} g_m(2^n x)$$

with the generating function g_m given by $g_m(x) = mT_{m-1}(2x)$ for $0 \leq x \leq \frac{1}{2}$ as well as $g_m(1-x) = -g_m(x)$ and $g_m(1+x) = g_m(x)$ for $x \in \mathbb{R}$.

Proposition 4.2 *For $m \geq 1$ the derivatives (4.5) of de Rham's function R_a satisfy the functional relations*

$$T_m\left(\frac{n+x}{2^\ell}\right) = T_m\left(\frac{n}{2^\ell}\right) + \sum_{\mu=0}^{m} a_\mu T_\mu(x) \tag{4.8}$$

where $\ell \in \mathbb{N}$, $n = 0, 1, \ldots, 2^\ell - 1$, $x \in [0,1]$, $T_0(x) = x$ and where a_μ are the constants

$$a_\mu = \binom{m}{\mu} \frac{\partial^{m-\mu}}{\partial a^{m-\mu}} a^{\ell - s(n)} (1-a)^{s(n)}\Big|_{a=1/2}. \tag{4.9}$$

Moreover, for $n = 0, 1, \ldots, 2^\ell$ it holds

$$T_m\left(\frac{n}{2^\ell}\right) = \frac{m!}{2^{\ell-m}} \sum_{j=0}^{n-1} \sum_{r=0}^{m} (-1)^r \binom{s(j)}{r} \binom{\ell - s(j)}{m-r}. \tag{4.10}$$

Proof: By m differentiations of (1.5) with respect to a we find

$$D_m\left(\frac{n+x}{2^\ell}\right) = D_m\left(\frac{n}{2^\ell}\right) + \sum_{\mu=0}^{m} \binom{m}{\mu} D_\mu(x) \frac{\partial^{m-\mu}}{\partial a^{m-\mu}} a^{\ell-s(n)}(1-a)^{s(n)}$$

so that for $a = \frac{1}{2}$ it follows (4.8) with the constants (4.9). Next we want to determine a_0. For this reason we compute

$$\frac{\partial^m}{\partial a^m}(1-a)^{s(n)}a^{\ell-s(n)} = \sum_{r=0}^{m}\binom{m}{r}(-1)^r \frac{s(n)!(1-a)^{s(n)-r}}{(s(n)-r)!}\frac{(\ell-s(n))!a^{\ell-s(n)-m+r}}{(\ell-s(n)-m+r)!}$$

so that for $a = \frac{1}{2}$ we obtain

$$a_0 = \frac{m!}{2^{\ell-m}}\sum_{r=0}^{m}(-1)^r\binom{s(n)}{r}\binom{\ell-s(n)}{m-r}.$$

In view of $T_0(1) = 1$ and $T_\mu(1) = 0$ for $\mu > 1$ equation (4.8) with j instead of n yields

$$T_m\left(\frac{j+1}{2^\ell}\right) = T_m\left(\frac{j}{2^\ell}\right) + \frac{m!}{2^{\ell-m}}\sum_{r=0}^{m}(-1)^r\binom{s(j)}{r}\binom{\ell-s(j)}{m-r}$$

and equation (4.10) follows by summation. □

In particular for $m = 1$ we obtain

$$T_1\left(\frac{n}{2^\ell}\right) = \frac{n\ell}{2^{\ell-1}} - \frac{1}{2^{\ell-2}}\sum_{j=0}^{n-1}s(j)$$

in accordance with $T_1(x) = 2T(x)$, cf. [10, formula (2.2)]. Equation (4.10) for $n = 1$ yields

$$T_m\left(\frac{1}{2^\ell}\right) = \frac{m!}{2^{\ell-m}}\binom{\ell}{m},$$

i.e. $T_m(\frac{1}{2^m}) = m!$ and $T_m(\frac{1}{2^\ell}) = 0$ for $\ell = 0, \ldots, m-1$.

5 Binomial sums of digital sums

Before we investigate the power sum (1.14) first we consider the binomial sum (1.13). By Theorem 2.1 it holds

$$B_m(N) = \frac{1}{m!}\frac{\partial^m}{\partial q^m}N^\alpha G_q(\log_2 N)\bigg|_{q=1} \tag{5.1}$$

where $q = q(a) = \frac{a}{1-a}$. In order to obtain an explicit expression for the formula (5.1) we use (2.8). So for $u \le 0$ we get

$$\frac{\partial^m}{\partial q^m}N^\alpha G_q(u)\bigg|_{q=1} = \frac{\partial^m}{\partial q^m}N^\alpha a^u R_a(2^u)\bigg|_{q=1} \tag{5.2}$$

where $a = a(q) = \frac{1}{1+q}$. Since the function $u \mapsto a^u R_a(2^u)$ for $u \le 0$ is continuous and of period 1 also the functions $F_k(u)$ defined by

$$F_k(u) = \frac{\partial^k}{\partial q^k} a^u R_a(2^u)\Big|_{q=1} \qquad (u \le 0) \tag{5.3}$$

are continuous and of period 1 so that they can be extended by $F_k(u+1) = F_k(u)$ for all $u \in \mathbb{R}$. In particular, in view of $a(1) = \frac{1}{2}$ and $R_{1/2}(x) = x$ we find

$$F_0(u) = a^u R_a(2^u)\big|_{q=1} = 1. \tag{5.4}$$

In order to express $F_k(u)$ for $k \ge 1$ we introduce polynomials

$$P_{k,\ell}(u) = (-1)^k \frac{k!}{\ell!} \binom{u+k-1}{k-\ell} \qquad (0 \le \ell \le k). \tag{5.5}$$

Obviously, $P_{k,\ell}(u)$ is a polynomial of degree $k-\ell$. In particular

$$P_{k,0}(u) = (-1)^k u(u+1)\cdots(u+k-1), \qquad P_{k,k}(u) = (-1)^k. \tag{5.6}$$

It is easy to see that for $1 \le \ell \le k$ we have

$$P_{k+1,\ell}(u) = -P_{k,\ell-1}(u) - (u+k+\ell)P_{k,\ell}(u). \tag{5.7}$$

Proposition 5.1 *For $k \ge 1$ the 1-periodic function $F_k(u)$ has the representation*

$$F_k(u) = (-1)^k \frac{u(u+1)\cdots(u+k-1)}{2^k} + \frac{1}{2^{u+k}} \sum_{\ell=1}^{k} \frac{P_{k,\ell}(u)}{2^\ell} T_\ell(2^u) \qquad (u \le 0) \tag{5.8}$$

with the polynomials $P_{k,\ell}(u)$ from (5.5) and the partial derivatives T_ℓ from (4.5).

Proof: At first we show by induction on k that for $u \le 0$

$$\frac{\partial^k}{\partial q^k} a^u R_a(2^u) = \sum_{\ell=0}^{k} P_{k,\ell}(u) a^{u+k+\ell} \frac{\partial^\ell}{\partial a^\ell} R_a(2^u).$$

This is valid for $k=1$, then by $\frac{\partial a}{\partial q} = \frac{-1}{(1+q)^2} = -a^2$ we have for $u \le 0$

$$\begin{aligned} \frac{\partial}{\partial q} a^u R_a(2^u) &= \frac{\partial}{\partial a} a^u R_a(2^u) \frac{\partial a}{\partial q} \\ &= -u a^{u+1} R_a(2^u) - a^{u+2} \frac{\partial}{\partial a} R_a(2^u). \end{aligned}$$

If this is true for a fixed k then it follows in view of $\frac{da}{dq} = -a^2$

$$\begin{aligned} &\frac{\partial^{k+1}}{\partial q^{k+1}} a^u R_a(2^u) = -a^2 \frac{\partial}{\partial a} \left(\frac{\partial^k}{\partial q^k} a^u R_a(2^u) \right) \\ &= -\sum_{\ell=0}^{k} (u+k+\ell) P_{k,\ell}(u) a^{u+k+\ell+1} \frac{\partial^\ell}{\partial a^\ell} R_a(2^u) - \sum_{\ell=0}^{k} P_{k,\ell}(u) a^{u+k+\ell+2} \frac{\partial^{\ell+1}}{\partial a^{\ell+1}} R_a(2^u) \\ &= \sum_{\ell=0}^{k+1} P_{k+1,\ell}(u) a^{u+k+1+\ell} \frac{\partial^\ell}{\partial a^\ell} R_a(2^u). \end{aligned}$$

According to (5.3) we have to take $q = 1$, i.e. $a(1) = \frac{1}{2}$, and applying (4.5), $R_{1/2}(x) = x$ as well as (5.6) it follows the assertion. □

Remark In particular, formula (5.8) for $k = 1$ yields

$$F_1(u) = -\frac{u}{2} - \frac{1}{2^{u+2}} T_1(2^u) \qquad (u \leq 0),$$

and it follows by (1.9) that

$$F_1(u) = F(u) \tag{5.9}$$

in the formula of Trollope-Delange.

Now for $k \in \mathbb{N}$ and $\ell = 0, \ldots, k$ we introduce numbers $a_{k,\ell}$ as follows:

$$\left.\begin{array}{lcll} a_{k,k} & = & 1 & \text{for} \quad k \geq 0 \\ a_{k,0} & = & 0 & \text{for} \quad k \geq 1 \\ a_{k+1,\ell} & = & a_{k,\ell-1} - k a_{k\ell} & \text{for} \quad k \geq 1, \ \ 1 \leq \ell \leq k \end{array}\right\}. \tag{5.10}$$

The numbers can be represented in a modified Pascal triangle, cf. Figure 1.

$$\begin{array}{ccccccccccccl} & & & & & 1 & & & & & & & k=0 \\ & & & & 0 & & 1 & & & & & & k=1 \\ & & & 0 & & -1 & & 1 & & & & & k=2 \\ & & 0 & & 2 & & -3 & & 1 & & & & k=3 \\ & 0 & & -6 & & 11 & & -6 & & 1 & & & k=4 \\ 0 & & 24 & & -50 & & 25 & & -10 & & 1 & & k=5 \end{array}$$

Figure 1: The first numbers $a_{k,\ell}$.

It is easy to show by induction that for $k \geq 1$ we have:

$$a_{k,1} = (-1)^{k-1}(k-1)!, \qquad a_{k,k-1} = -\binom{k}{2} \tag{5.11}$$

and for $k \geq 2$

$$a_{k,k-2} = \binom{k}{3} \frac{3k-1}{4}.$$

Lemma 5.2 *For integer $k \geq 1$ it holds*

$$\frac{\partial^k}{\partial q^k} N^\alpha = \frac{N^\alpha}{(1+q)^k} \sum_{\ell=1}^{k} (\log_2 N)^\ell a_{k,\ell} \tag{5.12}$$

with the coefficients $a_{k,\ell}$ from (5.10).

Proof: From $\alpha(q) = \log_2(1+q)$ and $\frac{\partial \alpha}{\partial q} = \frac{1}{(1+q)\log 2}$ we get

$$\frac{\partial}{\partial q} N^\alpha = N^\alpha \frac{\log_2 N}{1+q}$$

so that (5.12) is true for $k = 1$ since $a_{1,1} = 1$. Assume that (5.12) is valid for a fixed $k \geq 1$ then in view of $a_{k,0} = 0$ we get

$$\begin{aligned} \frac{\partial^{k+1}}{\partial q^{k+1}} N^\alpha &= \frac{N^\alpha}{(1+q)^{k+1}} \sum_{\ell=1}^{k} (\log_2 N)^{\ell+1} a_{k,\ell} - \frac{k N^\alpha}{(1+q)^{k+1}} \sum_{\ell=1}^{k} (\log_2 N)^\ell a_{k,\ell} \\ &= \frac{N^\alpha}{(1+q)^{k+1}} \sum_{\ell=1}^{k+1} (\log_2 N)^\ell a_{k+1,\ell} \end{aligned}$$

on account of (5.10) and $a_{k+1,k+1} = a_{k,k} = 1$. So (5.12) is proved by induction. □

Theorem 5.3 *For the binary binomial sum (1.14) with integer $m \geq 1$ we have the exact formula*

$$\frac{1}{N} B_m(N) = \frac{1}{m!} \left(\frac{\log_2 N}{2} \right)^m + \frac{1}{m!} \sum_{\ell=0}^{m-1} (\log_2 N)^\ell F_{m,\ell}(\log_2 N) \tag{5.13}$$

where $F_{m,\ell}(u)$ are continuous functions of period 1 which have the representations

$$F_{m,\ell}(u) = \sum_{k=0}^{m-\ell} \binom{m}{k} \frac{a_{m-k,\ell}}{2^{m-k}} F_k(u) \tag{5.14}$$

with $a_{k,\ell}$ from (5.10) and $F_k(u)$ from (5.3).

Proof: We apply (5.1) with the 1-periodic function $G_q(u)$. In order to express the term

$$\left. \frac{\partial^m}{\partial q^m} N^\alpha G_q(u) \right|_{q=1}$$

as 1-periodic function we use (5.2) for $u \leq 0$. By Leibniz's formula

$$\frac{\partial^m}{\partial q^m} N^\alpha a^u R_a(2^u) = \sum_{k=0}^{m} \binom{m}{k} \frac{\partial^k}{\partial q^k} N^\alpha \frac{\partial^{m-k}}{\partial q^{m-k}} a^u R_a(2^u).$$

Now, $\alpha(q) = \log_2(1+q)$ yields $\alpha(1) = 1$ and by Lemma 5.2 we get

$$\left. \frac{\partial^k}{\partial q^k} N^\alpha \right|_{q=1} = \frac{N}{2^k} \sum_{\ell=0}^{k} (\log_2 N)^\ell a_{k,\ell}$$

with $a_{k,\ell}$ from (5.10). Using (5.3) we obtain for $u \leq 0$

$$\left. \frac{\partial^m}{\partial q^m} N^\alpha a^u R_a(2^u) \right|_{q=1} = N \sum_{k=0}^{m} \binom{m}{k} \frac{1}{2^k} \sum_{\ell=0}^{k} a_{k,\ell} (\log_2 N)^\ell F_{m-k}(u).$$

Since the right-hand side is 1-periodic for all $u \in \mathbb{R}$, it follows by (5.2)

$$\frac{\partial^m}{\partial q^m} N^\alpha G_q(u)\bigg|_{q=1} = \sum_{\ell=0}^{m} \sum_{k=\ell}^{m} \binom{m}{k} \frac{a_{k,\ell}}{2^k} (\log_2 N)^\ell F_{m-k}(u)$$

and by (5.1) we get

$$\frac{1}{N} B_m(N) = \frac{1}{m!} \sum_{\ell=0}^{m} \sum_{k=\ell}^{m} \binom{m}{k} \frac{a_{k,\ell}}{2^k} (\log_2 N)^\ell F_{m-k}(\log_2 N).$$

Replacing k by $m-k$ it follows

$$\frac{1}{N} B_m(N) = \frac{1}{m!} \sum_{\ell=0}^{m} (\log_2 N)^\ell F_{m,\ell}(\log_2 N)$$

with the functions (5.14). In particular, $F_{m,m}(u) = \frac{a_{m,m}}{2^m} F_0(u) = \frac{1}{2^m}$ since $F_0(u) = 1$, cf. (5.4). This completes the proof. □

Remarks **1.** In case $m = 1$ we have $B_1(N) = S(N)$, cf. (1.7). Formula (5.14) yields $F_{1,0}(u) = F_1(u)$ and (5.13) simplifies to the Trollope-Delange formula (1.8) with $F(u) = F_1(u)$, cf. (5.9).

2. Note

$$\begin{aligned} F_{m,m-1}(u) &= \frac{a_{m,m-1}}{2^m} F_0(u) + \frac{m a_{m-1,m-1}}{2^{m-1}} F_1(u) \\ &= -\frac{1}{2^m} \binom{m}{2} + \frac{m}{2^{m-1}} F(u) \end{aligned}$$

where we have used (5.11), $F_0(u) = 1$ and (5.9). So (5.13) yields the asymptotic formula

$$\frac{1}{N} B_m(N) = \frac{1}{m!} \left(\frac{L}{2}\right)^m + \frac{1}{m!} \left(\frac{L}{2}\right)^{m-1} \left\{ -\frac{1}{2} \binom{m}{2} + m F(L) \right\} + \mathcal{O}\left(L^{m-2}\right) \tag{5.15}$$

with $L = \log_2 N$.

3. Further,

$$\begin{aligned} F_{m,m-2}(u) &= \frac{a_{m,m-2}}{2^m} F_0(u) + m \frac{a_{m-1,m-2}}{2^{m-1}} F_1(u) + \binom{m}{2} \frac{a_{m-2,m-2}}{2^{m-2}} F_2(u) \\ &= \frac{3m-1}{2^{m+2}} \binom{m}{3} - \frac{m}{2^{m-1}} \binom{m}{2} F_1(u) + \frac{1}{2^{m-2}} \binom{m}{2} F_2(u). \end{aligned}$$

Thus (5.13) yields for $m = 2$ the precise formula

$$\frac{1}{N} B_2(N) = \frac{1}{2} \left(\frac{L}{2}\right)^2 + \frac{L}{4} \left\{ -\frac{1}{2} + 2F_1(L) \right\} - \frac{1}{2} F_1(L) + \frac{1}{2} F_2(L)$$

again with $L = \log_2 N$ and $F_1(u) = F(u)$, cf. (5.9). Compare with (5.15) for $m = 2$.

6 Power sums of digital sums

In order to obtain formulas for the power sums (1.14) we need the Stirling numbers of second kind $s_{m,n}$ defined by

$$x^m = \sum_{n=0}^{m} s_{m,n} n! \binom{x}{n} . \tag{6.1}$$

These numbers are nonnegative integers. In particular, $s_{m,m} = 1$ for $m \geq 0$ and $s_{m,0} = 0$ for $m \geq 1$. Now we prove the formula (1.16) of Coquet.

Theorem 6.1 *For the power sum (1.14) it holds the formula of Coquet*

$$\frac{1}{N} S_m(N) = \left(\frac{\log_2 N}{2} \right)^m + \sum_{\ell=0}^{m-1} (\log_2 N)^\mu G_{m,\ell}(\log_2 N) \tag{6.2}$$

where $G_{m,\ell}(u)$ are continuous, 1-peridic functions given by

$$G_{m,\ell}(u) = \sum_{k=0}^{m-\ell} \sum_{n=\ell+k}^{m} \binom{n}{k} \frac{a_{n-k,\ell}}{2^{n-k}} s_{m,n} F_k(u) \tag{6.3}$$

with the coefficients from (5.10) and (6.1) and the functions $F_k(u)$ from (5.3).

Proof: According to (6.1) we have

$$S_m(N) = \sum_{n=0}^{m} s_{m,n} n! B_n(N)$$

and by Theorem 5.3 we find with $L = \log_2 N$

$$\begin{aligned} \frac{1}{N} S_m(N) &= \sum_{n=0}^{m} s_{m,n} \sum_{\ell=0}^{n} L^\ell F_{n,\ell}(L) \\ &= \sum_{\ell=0}^{m} L^\ell \sum_{n=\ell}^{m} s_{m,n} F_{n,\ell}(L) \\ &= \sum_{\ell=0}^{m} L^\ell G_{m,\ell}(L) \end{aligned}$$

where in view of (5.14) with n instead of m

$$G_{m,\ell}(u) = \sum_{n=\ell}^{m} s_{m,n} \sum_{k=0}^{n-\ell} \binom{n}{k} \frac{a_{n-k,\ell}}{2^{n-k}} F_k(u)$$

which implies (6.3). In particular, we have

$$G_{m,m}(u) = \frac{a_{m,m}}{2^m} s_{m,m} F_0(u) = \frac{1}{2^m}$$

which completes the proof. □

Proposition 6.2 *For α_m from (1.15) and β_m the correspond* lim inf *it holds*

$$\alpha_m = \frac{m(m-1)}{2^{m+1}(\log 2)^{m-1}}, \qquad \beta_m = \alpha_m - \frac{m}{(2\log 2)^{m-1}}\left(1 - \frac{\log 3}{\log 4}\right). \tag{6.4}$$

For $m > 1$ it holds $0 < \beta_m < \alpha_m$ and $\alpha_m \to 0$ as $m \to \infty$.

Proof: First we compute α_m. For α_m from (1.15) we get by (6.2)

$$\begin{aligned}\alpha_m &= \limsup \frac{1}{(\log N)^{m-1}} \sum_{\mu=0}^{m-1} (\log_2 N)^\mu G_{m,\mu}(\log_2 N) \\ &= \frac{1}{(\log 2)^{m-1}} \max G_{m,m-1}(u).\end{aligned}$$

By (6.3) we get

$$\begin{aligned}G_{m,m-1}(u) &= \frac{a_{m-1,m-1}}{2^{m-1}} s_{m,m-1} F_0(u) + \frac{a_{m,m-1}}{2^m} s_{m,m} F_0(u) + m\frac{a_{m-1,m-1}}{2^{m-1}} s_{m,m} F_1(u) \\ &= \frac{1}{2^{m-1}}\binom{m}{2} - \frac{1}{2^m}\binom{m}{2} + \frac{m}{2^{m-1}} F(u) \\ &= \frac{1}{2^m}\binom{m}{2} + \frac{m}{2^{m-1}} F(u)\end{aligned}$$

where we have used that $s_{m,m-1} = \binom{m}{2}$, $F_0(u) = 1$ and $F_1(u) = F(u)$ in the formula of Trollope-Delange, cf. (5.9). We know $\max F(u) = 0$ and $\min F(u) = (\log 3/\log 4) - 1$, cf. e.g. [14]. Hence

$$\max G_{m,m-1}(u) = \frac{1}{2^m}\binom{m}{2}$$

which yields the value for α_m where $\alpha_m \to 0$ as $m \to \infty$. For β_m we get

$$\beta_m = \frac{1}{(\log 2)^{m-1}} \min G_{m,m-1}(u)$$

which yields

$$\beta_m = \frac{m}{(2\log 2)^{m-1}}\left(\frac{m-1}{4} - 1 + \frac{\log 3}{\log 4}\right)$$

and (6.4). Obviously $\beta_m < \alpha_m$. Now $\beta_2 > 0$ and hence $\beta_m > 0$ for all $m \geq 2$. □

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Author:

Manfred Krüppel
Universität Rostock
Institut für Mathematik
Ulmenstr. 69, Haus 3
18057 Rostock

e-mail: manfred.krueppel@uni-rostock.de

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Weerayuth Nilsrakoo, Satit Saejung[1]

A new strong convergence Theorem for non-Lipschitzian Mappings in a uniformly convex Banach Space

ABSTRACT. In the present paper, we establish new strong convergence theorems of the modified Mann and the modified Ishikawa iterative scheme with errors for a mapping which is asymptotically nonexpansive in the intermediate sense in a uniformly convex Banach space. Our theorems significantly extend and improve Kim and Kim's results. The results in the paper even in the case of asymptotically nonexpansive mappings are new.

KEY WORDS AND PHRASES. asymptotically nonexpansive in the intermediate sense, asymptotically nonexpansive, Mann-type iteration, Ishikawa-type iteration, uniformly convex Banach space.

1 Introduction

Let X be a real Banach space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $x, y \in C$ and all $n \geq 1$. If $k_n \equiv 1$, then T is known as a nonexpansive mapping. The mapping T is called *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in C$ and all $n \geq 1$. The mapping T is called *asymptotically nonexpansive in the intermediate sense* ([1]) provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} \big(\|T^n x - T^n y\| - \|x - y\|\big) \leq 0.$$

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From the above definitions, it follows that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian. Furthermore, if $T : C \to C$ is asymptotically nonexpansive and C is bounded, then T is asymptotically nonexpansive in the intermediate sense. There is a mapping which is asymptotically nonexpansive in the intermediate sense but is not Lipschitzian as the following example shows.

Example 1.(see [5]) Let $X = \mathbb{R}$, $C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in C$, we define

$$Tx = \begin{cases} k \sin \frac{1}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

The concept of asymptotic nonexpansiveness was introduced by Goebel and Kirk [4] in 1972. They proved that every asymptotically nonexpansive self-mapping of a bounded closed convex subset of a uniformly convex Banach space has a fixed point. Several authors have studied methods for the iterative approximation of fixed points of mappings which are asymptotically nonexpansive and asymptotically nonexpansive in the intermediate sense (see for example [2, 3, 5, 7]). In [7], Schu introduced the modified Mann and the modified Ishikawa iterative schemes. Recently, Kim and Kim [5] considered the modified Mann and the modified Ishikawa iterative schemes with errors in the sense of Xu [11] of a mapping which is asymptotically nonexpansive in the intermediate sense in a uniformly convex Banach space. The scheme is defined as follows.

Let C be a nonempty convex subset of a Banach space X and $T : C \to C$ be a mapping.
Algorithm 1. For a given $x_1 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad n \geq 1, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$ and $\{\gamma'_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$, and $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C. The iterative scheme (1.1) is called the *modified Ishikawa iterative scheme with errors* in the sense of Xu.

If $\beta'_n = \gamma'_n \equiv 0$ and $\alpha'_n \equiv 1$, then Algorithm 1 reduces to
Algorithm 2. For a given $x_1 \in C$, compute the sequence $\{x_n\}$ by the iterative scheme

$$x_{n+1} = \alpha_n x_n + \beta_n T^n x_n + \gamma_n u_n, \quad n \geq 1, \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, and $\{u_n\}$ is a bounded sequence in C. The iterative scheme (1.2) is called the *modified Mann iterative scheme with errors* in the sense of Xu.

If $\gamma_n = \gamma'_n \equiv 0$, then Algorithm 1 reduces to modified Ishikawa iterative scheme, while setting $\beta'_n = \gamma_n = \gamma'_n \equiv 0$ and $\alpha'_n \equiv 1$, reduces to modified Mann iterative scheme.

The purpose of this paper is to establish several strong convergence theorems of the Ishikawa iterative scheme with errors for mappings of asymptotically nonexpansive in the intermediate sense in a uniformly convex Banach space.

2 Auxiliary Lemmas

For convenience, we use the notations $\lim_n \equiv \lim_{n\to\infty}$, $\liminf_n \equiv \liminf_{n\to\infty}$, and $\limsup_n \equiv \limsup_{n\to\infty}$. In the sequel, we shall need the following lemmas.

Lemma 2.1 **([9], Lemma 1)** *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $a_{n+1} \le a_n + b_n$ for all $n \ge 1$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_n a_n$ exists.*

Lemma 2.2 **([6], Lemma 2.2)** *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of nonnegative numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n\mu_n < \infty$. Then $\liminf_n \mu_n = 0$.*

Lemma 2.3 *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative numbers such that $\sum_{n=1}^{\infty} \min\{a_n, a_{n+1}\} = \infty$ and $\sum_{n=1}^{\infty} a_n b_n < \infty$. Then there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_j b_{n_j} = 0$ and $\lim_j b_{n_j+1} = 0$.*

Proof: We first observe that

$$\begin{aligned}\sum_{n=1}^{\infty} \min\{a_n, a_{n+1}\}(b_n + b_{n+1}) &= \sum_{n=1}^{\infty} \min\{a_n, a_{n+1}\}b_n + \sum_{n=1}^{\infty} \min\{a_n, a_{n+1}\}b_{n+1} \\ &\le \sum_{n=1}^{\infty} a_n b_n + \sum_{n=1}^{\infty} a_{n+1}b_{n+1} < \infty.\end{aligned}$$

Since $\sum_{n=1}^{\infty} \min\{a_n, a_{n+1}\} = \infty$ and Lemma 2.2, we have $\liminf_n(b_n + b_{n+1}) = 0$. Then there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_j(b_{n_j} + b_{n_j+1}) = 0$. It follows from $b_n \ge 0$ that $\lim_j b_{n_j} = 0$ and $\lim_j b_{n_j+1} = 0$. □

Lemma 2.4 *Let C be a nonempty convex subset of a Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set $F(T) := \{x \in C : x = Tx\}$ is not empty, and $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

$$\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \beta_n\gamma'_n < \infty, \text{ and } \sum_{n=1}^{\infty} \beta_n c_n < \infty,$$

where $c_n := \max\left\{0, \sup_{x,y\in C}\left(\|T^n x - T^n y\| - \|x - y\|\right)\right\}$ for each $n \ge 1$. Then we have the following conclusions.

(i) $\lim_n \|x_n - p\|$ *exists for any* $p \in F(T)$.

(ii) $\lim_n d(x_n, F(T))$ *exists, where* $d(x, F(T))$ *denotes the distance from* x *to the fixed-point set* $F(T)$.

Proof: Let $p \in F(T)$. We note that $\{u_n - p\}$ and $\{v_n - p\}$ are two bounded sequences in C. Let

$$M := \sup\{\|u_n - p\|, \|v_n - p\| : n \geq 1\}.$$

By using (1.1), we have

$$\begin{aligned}\|y_n - p\| &\leq \alpha'_n\|x_n - p\| + \beta'_n\|T^n x_n - p\| + \gamma'_n\|v_n - p\| \\ &\leq (\alpha'_n + \beta'_n)\|x_n - p\| + \beta'_n(\|T^n x_n - p\| - \|x_n - p\|) + \gamma'_n M \\ &\leq \|x_n - p\| + \beta'_n c_n + \gamma'_n M,\end{aligned}$$

and so

$$\begin{aligned}&\|x_{n+1} - p\| \\ &\leq \alpha_n\|x_n - p\| + \beta_n\|T^n y_n - p\| + \gamma_n\|u_n - p\| \\ &\leq \alpha_n\|x_n - p\| + \beta_n\|y_n - p\| + \beta_n(\|T^n y_n - p\| - \|y_n - p\|) + \gamma_n M \\ &\leq (\alpha_n + \beta_n)\|x_n - p\| + \beta_n\beta'_n c_n + \beta_n\gamma'_n M + \beta_n c_n + \gamma_n M \\ &\leq \|x_n - p\| + 2\beta_n c_n + \beta_n\gamma'_n M + \gamma_n M. \qquad (2.1)\end{aligned}$$

Then

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + 2\beta_n c_n + \beta_n\gamma'_n M + \gamma_n M.$$

Consequently, the conclusions of the lemma follow from Lemma 2.1. This completes the proof. □

By Xu's inequality [10, Theorem 2], we have the following lemma.

Lemma 2.5 **([3], Lemma 1.4)** *Let* X *be a uniformly convex Banach space and* $B_r := \{x \in X : \|x\| \leq r\}, r > 0$. *Then there exists a continuous strictly increasing convex function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that*

$$\|\lambda x + \mu y + \xi z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 - \lambda\mu g(\|x - y\|),$$

for all $x, y, z \in B_r$ *and* $\lambda, \mu, \xi \in [0, 1]$ *with* $\lambda + \mu + \xi = 1$.

The following lemmas are the important ingredients for proving our main results in the next section.

Lemma 2.6 *Let C be a nonempty convex subset of a uniformly convex Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set $F(T)$ is not empty, and $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

$$\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \beta_n \gamma_n' < \infty, \text{ and } \sum_{n=1}^{\infty} \beta_n c_n < \infty.$$

Then

$$\sum_{n=1}^{\infty} \beta_n(1-\beta_n) g(\|T^n y_n - x_n\|) < \infty. \tag{2.2}$$

Proof: Let $p \in F(T)$, it follows from Lemma 2.4 that $\{x_n - p\}$, $\{T^n x_n - p\}$, $\{y_n - p\}$, $\{T^n y_n - p\}$, $\{u_n - p\}$, and $\{v_n - p\}$ are all bounded. We may assume that such sequences belong to B_r where $r > 0$. By Lemma 2.5, we have

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\le \alpha_n' \|x_n - p\|^2 + \beta_n' \|T^n x_n - p\|^2 + \gamma_n' \|v_n - p\|^2 - \alpha_n' \beta_n' g(\|T^n x_n - x_n\|) \\
&\le (\alpha_n' + \beta_n') \|x_n - p\|^2 + \beta_n' (\|T^n x_n - p\|^2 - \|x_n - p\|^2) + \gamma_n' r^2 \\
&\le \|x_n - p\|^2 + \beta_n' (\|T^n x_n - p\| - \|x_n - p\|)(\|T^n x_n - p\| + \|x_n - p\|) + \gamma_n' r^2 \\
&\le \|x_n - p\|^2 + 2r\beta_n' c_n + \gamma_n' r^2 \\
&\le \|x_n - p\|^2 + (c_n + \gamma_n')M,
\end{aligned}$$

and

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + \beta_n \|T^n y_n - p\|^2 + \gamma_n \|u_n - p\|^2 - \alpha_n \beta_n g(\|T^n y_n - x_n\|) \\
&\le \alpha_n \|x_n - p\|^2 + \beta_n \|y_n - p\|^2 + \beta_n (\|T^n y_n - p\|^2 - \|y_n - p\|^2) + \gamma_n r^2 \\
&\quad + \gamma_n \beta_n g(\|T^n y_n - x_n\|) - (1-\beta_n)\beta_n g(\|T^n y_n - x_n\|) \\
&\le \alpha_n \|x_n - p\|^2 + \beta_n \|y_n - p\|^2 + 2r\beta_n c_n + \gamma_n r^2 + \gamma_n \beta_n g(2r) \\
&\quad - (1-\beta_n)\beta_n g(\|T^n y_n - x_n\|) \\
&\le (\alpha_n + \beta_n) \|x_n - p\|^2 + \beta_n (c_n + \gamma_n')M + (\beta_n c_n + 2\gamma_n)M \\
&\quad - (1-\beta_n)\beta_n g(\|T^n y_n - x_n\|) \\
&\le \|x_n - p\|^2 + (2\beta_n c_n + \beta_n \gamma_n' + 2\gamma_n)M - (1-\beta_n)\beta_n g(\|T^n y_n - x_n\|),
\end{aligned}$$

where $M = \max\{2r, r^2, g(2r)\}$. This implies that

$$(1-\beta_n)\beta_n g(\|T^n y_n - x_n\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (2\beta_n c_n + \beta_n \gamma_n' + 2\gamma_n)M.$$

By Lemma 2.4(i), we obtain (2.2) and the proof is finished. □

Lemma 2.7 *Let C be a nonempty convex subset of a Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set is not empty, and $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the restrictions that $\lim_n \gamma_n = \lim_n \gamma'_n = 0$. If there exists a subsequence $\{n_j\}$ of $\{n\}$ such that*

$$\lim_j \|T^{n_j} x_{n_j} - x_{n_j}\| = 0 = \lim_j \|T^{n_j+1} x_{n_j+1} - x_{n_j+1}\|, \tag{2.3}$$

then $\lim_j \|T x_{n_j} - x_{n_j}\| = 0$.

Proof: We note that

$$\lim_n c_n = \lim_n \max\left\{0, \sup_{x,y\in C} \left(\|T^n x - T^n y\| - \|x - y\|\right)\right\} = 0. \tag{2.4}$$

Using (1.1), we see that

$$\begin{aligned}
&\|T^{n_j} y_{n_j} - x_{n_j}\| \\
&\le (\|T^{n_j} y_{n_j} - T^{n_j} x_{n_j}\| - \|y_{n_j} - x_{n_j}\|) + \|y_{n_j} - x_{n_j}\| + \|T^{n_j} x_{n_j} - x_{n_j}\| \\
&\le c_{n_j} + \|y_{n_j} - x_{n_j}\| + \|T^{n_j} x_{n_j} - x_{n_j}\| \\
&\le c_{n_j} + \beta'_{n_j} \|T^{n_j} x_{n_j} - x_{n_j}\| + \gamma'_{n_j} \|v_{n_j} - x_{n_j}\| + \|T^{n_j} x_{n_j} - x_{n_j}\| \to 0,
\end{aligned}$$

and so,

$$\|x_{n_j+1} - x_{n_j}\| \le \beta_{n_j} \|T^{n_j} y_{n_j} - x_{n_j}\| + \gamma_{n_j} \|u_{n_j} - x_{n_j}\| \to 0. \tag{2.5}$$

Thus

$$\|T^{n_j+1} x_{n_j+1} - T^{n_j+1} x_{n_j}\| \le c_{n_j+1} + \|x_{n_j+1} - x_{n_j}\| \to 0. \tag{2.6}$$

Finally, we have

$$\begin{aligned}
\|x_{n_j} - T x_{n_j}\| &\le \|x_{n_j+1} - x_{n_j}\| + \|x_{n_j+1} - T^{n_j+1} x_{n_j+1}\| \\
&\quad + \|T^{n_j+1} x_{n_j+1} - T^{n_j+1} x_{n_j}\| + \|T^{n_j+1} x_{n_j} - T x_{n_j}\| \to 0,
\end{aligned}$$

since (2.5), (2.3), (2.6) and uniform continuity of T. We reach the desired conclusion. □

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set $F(T)$ is not empty, and $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $\sum_{n=1}^{\infty} \min\{\beta_n(1 - \beta_n), \beta_{n+1}(1 - \beta_{n+1})\} = \infty$,

(ii) $\limsup_n \beta'_n < 1$,

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \beta_n \gamma'_n < \infty$, $\lim_n \gamma'_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n c_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Let $\{u_n\}$ be a given sequence in C. Recall that a mapping $T : C \to C$ with the nonempty fixed-point set $F(T)$ in C satisfies *Condition (A) with respect to the sequence* $\{u_n\}$ ([8]) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$f(d(u_n, F(T))) \leq \|Tu_n - u_n\|, \text{ for all } n \geq 1.$$

Proof: By (2.2) and Lemma 2.3, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_j g(\|T^{n_j} y_{n_j} - x_{n_j}\|) = 0 = \lim_j g(\|T^{n_j+1} y_{n_j+1} - x_{n_j+1}\|).$$

From g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that

$$\lim_j \|T^{n_j} y_{n_j} - x_{n_j}\| = 0 = \lim_j \|T^{n_j+1} y_{n_j+1} - x_{n_j+1}\|. \tag{3.1}$$

By using (1.1), we have

$$\begin{aligned} \|T^n x_n - x_n\| &\leq \|T^n y_n - T^n x_n\| + \|T^n y_n - x_n\| \\ &\leq (\|T^n y_n - T^n x_n\| - \|y_n - x_n\|) + \|y_n - x_n\| + \|T^n y_n - x_n\| \\ &\leq c_n + \beta'_n \|T^n x_n - x_n\| + \gamma'_n \|v_n - x_n\| + \|T^n y_n - x_n\|. \end{aligned}$$

This together with (2.4), (3.1) and $\lim_n \gamma'_n = 0$ gives

$$\lim_j (1 - \beta'_{n_j}) \|T^{n_j} x_{n_j} - x_{n_j}\| = 0 = \lim_j (1 - \beta'_{n_j+1}) \|T^{n_j+1} x_{n_j+1} - x_{n_j+1}\|.$$

As $\liminf_n (1 - \beta'_n) = 1 - \limsup_n \beta'_n > 0$, we have

$$\lim_j \|T^{n_j} x_{n_j} - x_{n_j}\| = 0 = \lim_j \|T^{n_j+1} x_{n_j+1} - x_{n_j+1}\|.$$

By Lemma 2.7, we have

$$\lim_j \|Tx_{n_j} - x_{n_j}\| = 0. \tag{3.2}$$

Let f be a nondecreasing function corresponding to Condition (A) with respect to $\{x_n\}$. Then

$$f(d(x_{n_j}, F(T))) \leq \|Tx_{n_j} - x_{n_j}\| \to 0.$$

Next, we prove that $\lim_n d(x_n, F(T)) = 0$. By Lemma 2.4(ii), suppose that $\lim_n d(x_n, F(T)) = b > 0$. Then there exists $K \in \mathbb{N}$ such that

$$0 < \frac{b}{2} \leq d(x_{n_j}, F(T)) \quad \text{for all } j \geq K.$$

By the definition of f, we obtain

$$0 < f(\frac{b}{2}) \leq f(d(x_{n_j}, F(T))) \quad \text{for all } j \geq K.$$

Therefore $\lim_j f(d(x_{n_j}, F(T))) \geq f(\frac{b}{2}) > 0$, this is a contradiction. Hence

$$\lim_n d(x_n, F(T)) = 0.$$

Let $a_n = 2\beta_n c_n + \beta_n \gamma'_n M + \gamma_n M$ for all $n \in \mathbb{N}$ and $p \in F(T)$. Then, by (2.1),

$$\|x_{n+1} - p\| \leq \|x_n - p\| + a_n,$$

and hence

$$\|x_m - p\| \leq \|x_n - p\| + \sum_{i=n}^{m-1} a_i, \tag{3.3}$$

for all $m \geq n$. We now prove that $\{x_n\}$ is a Cauchy sequence in C. Let $\varepsilon > 0$. Since

$$\lim_n d(x_n, F(T)) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} a_n < \infty,$$

there exists a positive integer N such that

$$d(x_N, F(T)) < \frac{\varepsilon}{4} \quad \text{and} \quad \sum_{i=N}^{\infty} a_i \leq \frac{\varepsilon}{4}.$$

There must exist $q \in F(T)$ such that

$$\|x_N - q\| = d(x_N, q) < \frac{\varepsilon}{4}.$$

From (3.3), it follows that, for all $m, n \geq N$,

$$\begin{aligned}
\|x_m - x_n\| &\leq \|x_m - q\| + \|x_n - q\| \\
&\leq 2\|x_N - q\| + \sum_{i=N}^{n-1} a_i + \sum_{i=N}^{m-1} a_i \\
&\leq 2\|x_N - q\| + 2\sum_{i=N}^{\infty} a_i \\
&< 2 \cdot \frac{\varepsilon}{4} + 2 \cdot \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in C. In virtue of the completeness of C, we may assume that $x_n \to q'$ as $n \to \infty$ where $q' \in C$. By the continuity of T and (3.2), we have $Tq' = q'$, so q' is a fixed point of T. This completes the proof. □

Letting $\beta'_n = \gamma'_n \equiv 0$ and $\alpha'_n \equiv 1$ in Theorem 3.1, we obtain the following Mann-type convergence.

Theorem 3.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set is not empty, and $\{x_n\}$be a sequence in C defined by Algorithm 2 with the following restrictions:*

(i) $\sum_{n=1}^{\infty} \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\} = \infty$,

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, *and* $\sum_{n=1}^{\infty} \beta_n c_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Corollary 3.3 **([5], Theorem 1)** *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set is not empty, and $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \varepsilon \le \beta_n \le 1-\varepsilon < 1$,

(ii) $0 < \varepsilon' \le \alpha_n$,

(iii) $\limsup_n \beta'_n < 1$,

(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, *and* $\sum_{n=1}^{\infty} c_n < \infty$.

If T is completely continuous, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof: By (i), we have $\varepsilon^2 \le \beta_n(1-\beta_n)$. Then

$$0 < \varepsilon^2 \le \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\}.$$

Therefore $\sum_{n=1}^{\infty} \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\} = \infty$. Moreover, since T is completely continuous, T satisfies Condition (A) with respect to the sequence $\{x_n\}$ (see [3, Corollary 2.5]). The proof is finished, by using Theorem 3.1. □

Remark 3.4 Condition (ii) is superfluous because (ii) is exactly implied by (i).

Corollary 3.5 **([5], Theorem 2)** *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $T : C \to C$ be a mapping which is asymptotically nonexpansive in the intermediate sense and the fixed-point set is not empty, and $\{x_n\}$ be a sequence in C defined by Algorithm 2 with the following restrictions:*

(i) $0 < \varepsilon \leq \beta_n \leq 1 - \varepsilon < 1$,

(ii) $0 < \varepsilon' \leq \alpha_n$,

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, *and* $\sum_{n=1}^{\infty} c_n < \infty$.

If T is completely continuous, then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3.6 Theorem 3.1 extends and improves Theorem 1 of [5] in the following ways:

(i) The condition $\sum_{n=1}^{\infty} \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\} = \infty$ is *strictly* weaker than the condition $0 < \varepsilon \leq \beta_n \leq 1 - \varepsilon < 1$. In fact, our result is applicable to the case of $\beta_n = 1/n$ while such chosen parameters are not satisfied the requirement of [5, Theorem 1].

(ii) The condition $0 < \varepsilon' \leq \alpha_n$ is removed.

(iii) The restrictions $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$ are weakened and replaced by $\sum_{n=1}^{\infty} \beta_n \gamma'_n < \infty$, $\lim_n \gamma'_n = 0$, and $\sum_{n=1}^{\infty} \beta_n c_n < \infty$.

(iv) The complete continuity imposed on T is replaced by the more general Condition (A) with respect to $\{x_n\}$.

Since every asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediate sense whenever C is bounded, we have the following theorems.

Theorem 3.7 *Let C be a nonempty closed convex and bounded subset of a uniformly convex Banach space X, $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ of real numbers, and $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $\sum_{n=1}^{\infty} \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\} = \infty$,

(ii) $\limsup_n \beta'_n < 1$,

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \beta_n \gamma'_n < \infty$, $\lim_n \gamma'_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n(k_n - 1) < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 3.8 *Let C be a nonempty closed convex and bounded subset of a uniformly convex Banach space X, $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ of real numbers, and $\{x_n\}$ be a sequence in C defined by Algorithm 2 with the following restrictions:*

(i) $\sum_{n=1}^{\infty} \min\{\beta_n(1-\beta_n), \beta_{n+1}(1-\beta_{n+1})\} = \infty$,

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, *and* $\sum_{n=1}^{\infty} \beta_n(k_n - 1) < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3.9 Theorem 3.7 and 3.8 extend and improve Theorem 1.1 and 1.2 of Chang [2], respectively.

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Authors:

Weerayuth Nilsrakoo
Department of Mathematics
Statistics and Computer,
Ubon Rajathanee University,
Ubon Ratchathani 34190,
Thailand

e-mail: nweerayuth@sci.ubu.ac.th

Satit Saejung
Department of Mathematics,
Khon Kaen University,
Khon Kaen 40002,
Thailand

e-mail: saejung@kku.ac.th

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Klaus – Dieter Drews

Nochmals zur Astronomischen Uhr in Rostocks St.-Marien – klärende Mathematik für die Sonnenaufgangszeiten des neuen Kalendariums

Auf der großen Kalenderscheibe im unteren Teil der Astronomischen Uhr in der Rostocker St.-Marien-Kirche befinden sich für Tage und Jahre mannigfache Angaben, darunter die uns hier interessierenden Sonnenaufgangszeiten. Sie bilden den vierten Ring von außen auf der weißen Scheibe, nach den Ringen der Tagesdaten, -buchstaben und -heiligen. Ihr Schriftbild umfaßt (unter Auslassung des 29.2.) jeweils zwei Tage, gesondert den 31.12., die Zeiten werden hier immer als dem ersten der beiden Tage zugehörig interpretiert.

Astronomische Uhr in St. Marien, Kalenderscheibe, 7. Dezember

Die Scheibe vollführt in 365 Tagen einen Umlauf, mitternachts rückt sie um ein Tagesdatum vor. Auf dieses weist der links positionierte „Kalendermann“, dagegen ist der aus dem Zentrum ragende Zeiger auf die Sonnenaufgangszeit gerichtet.

Astronomische Uhr in St. Marien, 7.12.2009 14 Uhr

Die Gültigkeitszeitspanne der jahresbezogenen Angaben des jetzigen Kalendariums reicht von 1885 bis 2017, ab 2018 muß die Scheibe vor allem wegen der angegebenen Ostertermine (sie stehen im innersten Kreis der weißen Scheibe) fortgeschrieben werden, rechnerische Vorbereitungen hierzu für wieder 133 Jahre von 2018 bis 2150 waren schon lange getätigt [8].

Vor einigen Monaten aber entspann sich eine Diskussion darüber, ob nicht bei Neubeschriftung der Scheibe auch die verzeichneten Sonnenaufgangszeiten revidiert werden sollten. In der Literatur wird nämlich berichtet, daß diese Zeiten für die Beschriftung der Kalenderscheibe zur 1885 erfolgten Wiederinbetriebnahme der Rostocker Uhr aus der Lübecker Marienkirche von der dortigen Astronomischen Uhr übernommen wurden ([4], [6], [7]). Ja, wenn es sich bisher um Lübecker Zeiten, Lübecker Ortszeiten handelte, so müßten sie den Rostocker Gegebenheiten angepaßt werden, aber der Tradition entsprechend dann doch nun mit Rostocker *Ortszeit*, so die Position in erster und beinahe schon entscheidender Instanz, bevor Verf. von dem Vorhaben erfuhr. Es bedurfte jedoch einiger genauerer Analysen und vor allem auch präziserer Vorstellungen von den Begriffen Ortszeit oder Mitteleuropäischer Zeit, um überzeugend zu einer angemessenen Entscheidung zu gelangen. Argumente dafür werden in den vorliegenden Ausführungen zusammengetragen, eingedenk auch der Präferenz, die unsere Generation besitzt, künftigen Generationen der nächsten fast 1½ Jahrhunderte eine erneuerte Scheibe zu präsentieren. Immerhin gehören die Sonnenaufgangszeiten zu denjenigen Eintragungen, die speziell auf Rostock bezogen sind.

Für die Sonnenaufgangszeiten an der Rostocker Astronomischen Uhr (notiert in den Spalten Z_{1855} der Gesamtaufstellung Tabelle 1 auf den S. 99 f.) gilt jedenfalls sofort:

> Als Lübecker Zeiten aus den Jahren vor 1885 können die jetzigen Angaben auf der Kalenderscheibe keine Zeiten sein, die ausdrücklich an der Mitteleuropäischen Zeit orientiert sind.

Diese wurde erst am 1.4.1893 als verbindlich für das ganze Deutschland eingeführt.

1 Erläuterungen zur mittleren und zur wahren Ortszeit sowie zur Südstellung der Sonne um 12 Uhr

Die Zeitangabe in Mitteleuropäischer Zeit (MEZ) ist die *mittlere Ortszeit*, welche exakt für den geographischen Längenkreis 15° östlicher Länge gilt, auf ihm liegt ziemlich genau Görlitz, sie ist heutzutage als *Zonenzeit* für eine übernationale Region gültig. Die Weltzeit oder Westeuropäische Zeit, das ist die mittlere Ortszeit der geographischen Länge 0° (Greenwich), hat konstant 1 h Rückstand zur MEZ, es ist erst 11 Uhr Weltzeit bei 12 Uhr MEZ.

> Jede *mittlere* Zeit läuft gleichförmig ab, mit der Maßeinheit Sekunde, wie sie im Internationalen Einheitensystem definiert ist.
>
> (Armband- oder Taschenuhren sollten so gehen.)

Zu jedem Längenkreis, zu jedem Ort kann seine eigene mittlere Ortszeit (MOZ) herangezogen werden, und wir verwenden folgende

Bezeichnungen: Wir kennzeichnen Zeiten an bestimmten Orten durch entsprechende Anfügungen, wie $\mathrm{MOZ}_{\mathrm{Rostock}}$, $\mathrm{MOZ}_{\mathrm{Lübeck}}$, speziell ist $\mathrm{MOZ}_{15°\mathrm{O}} = \mathrm{MEZ}$. Auch bestimmte Ereignisse, wie Südstellung der Sonne (abgekürzt Sst.) oder Sonnenaufgang (SA), kennzeichnen wir analog, und wir verwenden zur Angabe der Uhrzeit eines Ereignisses die übliche Funktionsschreibweise, z.B. $\mathrm{MEZ}(\mathrm{Sst.}_{\mathrm{Rostock}})$. Schon hier sei vermerkt, daß bez. der noch zu betrachtenden wahren Ortszeit (WOZ) ebenso verfahren werden soll.

Nun ist die Zeitangabe, wie bereits angemerkt, in westlicher Richtung 60 min zurück pro 15 Längengraden, d.h. 4 min/°. Für Rostock (12,07°O) oder Lübeck (10,75°O) sind 2,93° bzw. 4,25° gegenüber 15°O zu berücksichtigen, d.h. $2{,}93 \cdot 4\,\mathrm{min}$ $(= 11{,}72\,\mathrm{min} \approx 12\,\mathrm{min})$ bzw. $4{,}25 \cdot 4\,\mathrm{min}$ $(= 17\,\mathrm{min})$ sind je zu subtrahieren:

Die $\mathrm{MOZ}_{\mathrm{Rostock}}$ ist somit in jedem festen Moment 12 min zurück gegenüber der MEZ:

$$\mathrm{MOZ}_{\mathrm{Rostock}} = \mathrm{MEZ} - 12\,\mathrm{min} \quad (\mathrm{MEZ} - 11{,}72\,\mathrm{min})\,. \tag{1}$$

Es ist z.B. noch 11:48 $\mathrm{MOZ}_{\mathrm{Rostock}}$ um 12 MEZ, 12:00 $\mathrm{MOZ}_{\mathrm{Rostock}}$ wäre erst um 12:12 MEZ.

Analog gilt

$$\mathrm{MOZ}_{\mathrm{Lübeck}} = \mathrm{MEZ} - 17\,\mathrm{min}\,, \tag{2}$$

12:00 $\mathrm{MOZ}_{\mathrm{Lübeck}}$ wäre z.B. erst um 12:17 MEZ.

Allgemein halten wir fest:

Jede mittlere Ortszeit hat eine konstante Differenz zur MEZ. (3)

Von der mittleren Ortszeit ist die wahre Ortszeit zu unterscheiden.

> Die *wahre Ortszeit* (WOZ) wird durch den wahren Stand der Sonne bestimmt.
>
> Sie wird auf (adäquat eingerichteten) Sonnenuhren angezeigt, Sonnensüdstellung ist stets um 12:00 WOZ. Ihr Ablauf ist Schwankungen gegenüber dem gleichförmigen Ablauf der MOZ unterworfen.

Die Differenz der WOZ zur MOZ setzt sich aus Anteilen zweier Einflüsse zusammen, deren Ursachen beiläufig erläutert seien.

Zum einen legt die Erde auf ihrer elliptischen Bahn nach dem zweiten KEPLERschen Gesetz in Umgebung der Sonnenferne (Aphel, Anfang Juli) pro Tag einen kürzeren Weg zurück als in Umgebung der Sonnennähe (Perihel, Anfang Januar). Damit entstehen von Sonnensüdstellung zu Südstellung jedes Meridians unterschiedliche Drehwinkel der Erdachse. Zwar vergehen hierfür stets 24 h WOZ, aber beim Aphel für kleineren Drehwinkel weniger, beim Perihel für größeren Drehwinkel mehr als 24 h MOZ. Der sich so ergebende erste Anteil der Differenz zwischen WOZ und MOZ nimmt in Umgebung des Aphels also täglich zu, in Umgebung des Perihels dagegen täglich ab. Infolgedessen gibt es zwischen beiden Apsiden jeweils extreme Differenzen, und zwar eine positive und eine negative.

Diese Jahresperiode wird zum zweiten überlagert durch eine halbjährige Periode, die von der zur Ebene der Erdumlaufbahn um 23,5° geneigten Erdrotationsachse herrührt. Man denke sich dazu modellmäßig die Erdkugel auf ihrer Bahn bei Frühlings- und Sommeranfang sowie bei einer Zwischenposition, und zwar zunächst mit nichtgeneigter Achse. Dabei betrachte man einen Meridian jeweils in Sonnensüdstellung und je zugehöriger WOZ und MOZ. Wird sodann überall die Achse sachgemäß geneigt, so bleiben bei den beiden erstgenannten Positionen die Sonnensüdstellungen des Meridians und damit die Uhrzeiten unbeeinflußt. In der Zwischenposition aber hat der Meridian jetzt die Sonnensüdstellung bereits überschritten (was anschaulicher wird bei drastischer Übertreibung der Neigung), d.h., die WOZ ist bereits vorausgeeilt. Dieser zweite Anteil der Differenz von WOZ und MOZ, der bei Frühlings- und Sommeranfang Null ist, wächst zwischen beiden Positionen bis zu einem Maximum, um danach wieder zu Null abzufallen. Von Sommer- bis Herbstanfang kehrt sich der Effekt um, zwischenzeitlich wird ein extremer negativer Wert erreicht, und danach wiederholt sich alles halbjährig.

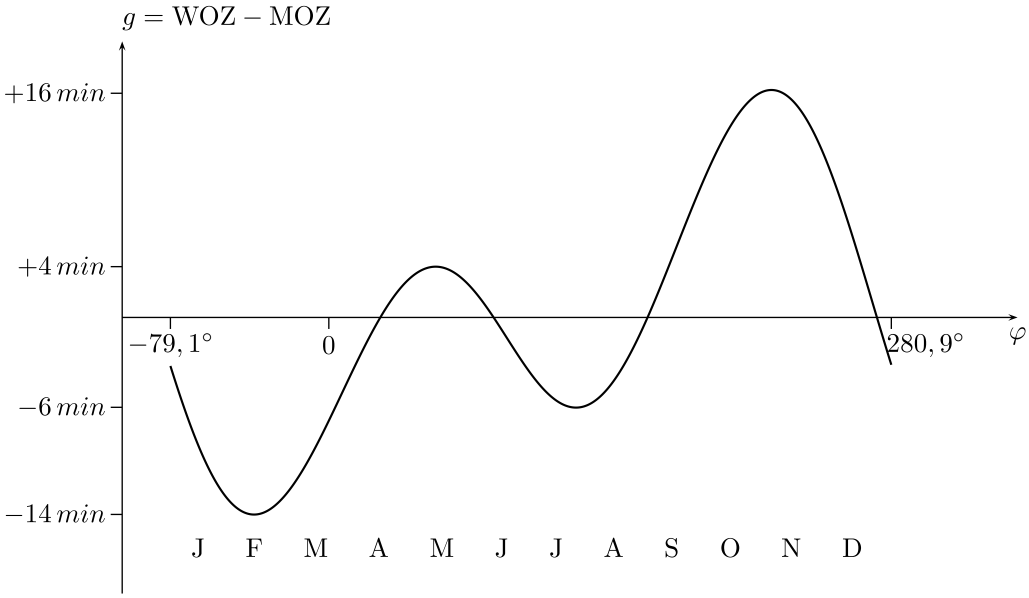
In der Summe beider Anteile ergibt sich die sogenannte

$$\textit{Zeitgleichung}: \quad g = \mathrm{WOZ} - \mathrm{MOZ}.$$

Zum numerischen Verlauf der Differenz g findet sich in der Arbeit [1] eine elementar zugängliche formelmäßige Herleitung. Das Ergebnis ist:

$$g = -460{,}7\,s \cdot \sin(\varphi + 76{,}9^\circ) + 590{,}5\,s \cdot \sin 2\varphi - 12{,}2\,s \cdot \sin 4\varphi\,,$$

worin g in Sekunden angegeben wird; die auftretenden Konstanten sind auf eine Dezimale gerundet. Der Winkel φ bezieht sich auf den scheinbaren Umlauf des Leitstrahls der Sonne, mit $\varphi = 0^\circ$ in Richtung des Frühlingspunktes (20. oder 21. März, hier 21. im Normaljahr) und einem 360°-Umlauf im *Tropischen Jahr* von 365,2422 Tagen. Am ersten Summanden der Formel ist die soeben qualitativ erläuterte Jahresperiode mit Nullstellen bei Perihel und Aphel zu erkennen, die beiden weiteren Summanden beschreiben die sich überlagernde Halbjahresperiode mit Nullstellen an den Grenzen der Jahreszeiten.



Zeitgleichung, nach [1]

Vom 21. März vergehen bis zum 31. Dezember 285 Tage, es wird $\varphi = 285 \cdot \frac{360°}{365{,}2422} \approx 280{,}9°$. Allerdings verläuft φ, wie bereits aus den gerade wieder angesprochenen qualitativen Erläuterungen hervorgeht, nicht gleichförmig mit der Zeit, der mittleren Zeit, ab, die Zeitskala des Jahres ist also bez. der φ-Achse unproportionalen, wenngleich für unser Anliegen unerheblichen, Dehnungen unterworfen, simpel auch schon von Normal- oder Schaltjahr abhängig. Wir ziehen von g nur wenige spezielle Werte und ansonsten den qualitativen Verlauf heran. Dabei übergehen wir auch geringe Veränderungen von g etwa innerhalb eines Tages.

Kommentar 1 über die Rundung der Zeitangaben: Es erscheint für unser Vorhaben als völlig ausreichende Genauigkeit und wird im folgenden so gehandhabt, sämtliche Uhrzeiten auf volle Minuten zu runden.

Wir wollen einen Blick auf extreme Werte der Zeitgleichung und im Zusammenhang damit zunächst auf Südstellungen der Sonne werfen. Aus der Zeitgleichung folgt

$$\mathrm{MOZ} = \mathrm{WOZ} - g\,,$$

speziell bei Südstellungen der Sonne gilt daher für jeden Ort (und selbstredend je abhängig vom Datum)

$$\mathrm{MOZ}_{\mathrm{Ort}}(\mathrm{Sst.}_{\mathrm{Ort}}) = \mathrm{WOZ}_{\mathrm{Ort}}(\mathrm{Sst.}_{\mathrm{Ort}}) - g = 12{:}00 - g\,. \tag{4}$$

Bei 15° östlicher Länge ist speziell $\mathrm{MOZ}_{15°\mathrm{O}} = \mathrm{MEZ}$, folglich

$$\mathrm{MEZ}(\mathrm{Sst.}_{15°\mathrm{O}}) = 12{:}00 - g\,.$$

Für Südstellungen der Sonne bei 15°O in MEZ ergeben sich aus dem Verlauf von g die folgenden markanten Daten und Uhrzeiten:

12. 2.	$g = -14$ min (Minimum),	Sst. erst 12:14 MEZ,
16. 4.	$g = 0$ min,	Sst. 12:00, bisher später, künftig früher,
14. 5.	$g = +4$ min (Maximum),	Sst. schon 11:56 MEZ,
15. 6.	$g = 0$ min,	Sst. 12:00, bisher früher, künftig später,
26. 7.	$g = -6$ min (Minimum),	Sst. erst 12:06 MEZ,
2. 9.	$g = 0$ min,	Sst. 12:00, bisher später, künftig früher,
3.11.	$g = +16$ min (Maximum),	Sst. schon 11:44 MEZ,
26.12.	$g = 0$ min,	Sst. 12:00, bisher früher, künftig später.

> In MEZ erfolgt die Südstellung der Sonne bei 15°O ‚lediglich an vier Tagen' im Jahr ‚genau' um 12 Uhr.

Kommentar 2 über die Rundung der Zeitangaben: Die genaueren relativen Extremwerte für g (aus [3]) sind $-14{,}3$ min, $+3{,}8$ min, $-6{,}2$ min bzw. $+16{,}4$ min. Bei Rundung auf volle Minuten gelten die angegebenen Uhrzeiten der Südstellungen dann auch für benachbarte Tage der notierten Daten, und ferner tritt die genaue Südstellung in mittlerer Ortszeit nicht unbedingt zur vollen Minute ein; auf diese Unschärfe sollten die soeben verwendeten einfachen Anführungszeichen deuten, die wir entsprechend auch künftig setzen.

Wird Gleichung (4) speziell auf Rostock bezogen, so folgt

$$\mathrm{MOZ}_{\mathrm{Rostock}}(\mathrm{Sst.}_{\mathrm{Rostock}}) = 12{:}00 - g\,.$$

Die **obige Aufstellung** der markanten Daten und Uhrzeiten für 15°O gilt darum **ebenso für Rostock**, wenn nur die $\mathbf{MOZ_{Rostock}}$ statt der MEZ genommen wird, speziell gilt ebenso:

> In $\mathrm{MOZ}_{\mathrm{Rostock}}$ erfolgt die Südstellung der Sonne in Rostock ‚lediglich an vier Tagen' im Jahr ‚genau' um 12 Uhr.

Zieht man nun schließlich für Rostock die Angabe in geläufiger MEZ heran, so gilt nach (1)

$$\mathrm{MEZ} = \mathrm{MOZ}_{\mathrm{Rostock}} + 12\,\mathrm{min}\,.$$

Für Südstellungen in Rostock ergibt das

$$\mathrm{MEZ}(\mathrm{Sst.}_{\mathrm{Rostock}}) = \mathrm{MOZ}_{\mathrm{Rostock}}(\mathrm{Sst.}_{\mathrm{Rostock}}) + 12\,\mathrm{min}\,.$$

Hierin ist rechts $\mathrm{MOZ}_{\mathrm{Rostock}}(\mathrm{Sst.}_{\mathrm{Rostock}}) = \mathrm{MOZ}_{15°\mathrm{O}}(\mathrm{Sst.}_{15°\mathrm{O}}) = \mathrm{MEZ}(\mathrm{Sst.}_{15°\mathrm{O}})$, weil Gleichung (4) für jeden Ort gilt, und man erhält

$$\mathrm{MEZ}(\mathrm{Sst.}_{\mathrm{Rostock}}) = \mathrm{MEZ}(\mathrm{Sst.}_{15°\mathrm{O}}) + 12\,\mathrm{min}\,.$$

Für Südstellungen der Sonne in Rostock bei MEZ-Angabe sind demnach alle Uhrzeiten aus obiger zu 15°O gehörenden Aufstellung um 12 min (11,72 min) zu erhöhen. Folgende Angaben sind dann bemerkenswert:

12. 2.		Sst. erst 12:26 MEZ,
14. 5.		Sst. „schon“ 12:08 MEZ,
26. 7.		Sst. erst 12:18 MEZ,
	4.10.	Sst. 12:00 MEZ, bisher später, künftig früher,
3.11.		Sst. schon 11:55 MEZ (bez. der Minutenabweichung von 12:00 ist nämlich genauer $-16{,}4 + 11{,}72 = -4{,}68$, s. Kommentar 2 und Formel (1)),
	29.11.	Sst. 12:00 MEZ, bisher früher, künftig später.

Die Daten 4.10 und 29.11. wurden gesondert ermittelt.

> In MEZ erfolgt die Südstellung der Sonne in Rostock ‚lediglich an zwei Tagen‘ im Jahr ‚genau‘ um 12 Uhr.

(Die aufgelisteten Erscheinungen wie auch der Gesamtverlauf der Südstellungen lassen sich bei einmal erkannter Südrichtung leicht selbst beobachten. Zu denken ist jedoch an die Sommerzeit, in ihr scheidet z.B. auch die Sst. 12:00 ‚am 4.10.‘ aus.)

Für Lübeck gilt (s. (2)) $\mathrm{MEZ} = \mathrm{MOZ}_{\mathrm{Lübeck}} + 17\,\mathrm{min}$,

und somit müssen die in der Aufstellung für 15°O notierten Werte für die Sst. in Lübeck und MEZ um 17 min erhöht werden. Selbst die früheste Uhrzeit der Sst., nämlich am 3.11., lautet dann schon 12:01.

> In MEZ erfolgt die Südstellung der Sonne in Lübeck stets erst nach 12 Uhr.

Zusammengefaßt ist folgendes zur Kenntnis zu nehmen:

> Würden die SA-Zeiten Rostocks, für die wir uns ja vorrangig interessieren wollen, in $\mathrm{MOZ}_{\mathrm{Rostock}}$ oder in MEZ angegeben, so wären diese Angaben keinesfalls daran orientiert, daß um 12:00 entsprechender Zeitmessung die Sonne genau im Süden stünde (letzteres geschieht nur an wenigen Tagen im Jahr).

2 Zu den mittleren und wahren Ortszeiten für die Sonnenaufgänge in Rostock und in Lübeck

Es sei also davon ausgegangen, daß die Daten der Beschriftung für die Rostocker Uhr damals 1885 von der Uhr aus der Lübecker Marienkirche übernommen worden sind. Darum stellt sich die Frage, wie groß denn die Abweichung der SA-Zeiten zwischen Rostock und Lübeck in Wirklichkeit ist, d.h. welche Fehler allein hierdurch in den derzeitigen SA-Zeiten auf unserer Scheibe enthalten sind. Zu dieser Beurteilung erstellen wir zunächst Aussagen über das Verhältnis der MOZ und WOZ im Vergleich beider Orte. Wir verwenden als geographische Koordinaten: Rostock $(12,07°\mathrm{O},\ 54,08°\mathrm{N})$, Lübeck $(10,75°\mathrm{O},\ 53,83°\mathrm{N})$.

Zuerst zu mittleren Ortszeiten.

Auf dem Breitenkreis Rostocks liege mit der geographischen Länge Lübecks der Ort R'. Wir ziehen die Gleichungen (1) und (2) für die Ereignisse $\mathrm{SA}_{\mathrm{Rostock}}$ bzw. $\mathrm{SA}_{R'}$ heran, und können dabei (2) statt auf Lübeck auch auf R' beziehen:

$$\mathrm{MOZ}_{\mathrm{Rostock}}(\mathrm{SA}_{\mathrm{Rostock}}) = \mathrm{MEZ}(\mathrm{SA}_{\mathrm{Rostock}}) - 12\,\mathrm{min}\,,$$
$$\mathrm{MOZ}_{R'}(\mathrm{SA}_{R'}) = \mathrm{MEZ}(\mathrm{SA}_{R'}) - 17\,\mathrm{min}\,.$$

Andererseits verstreichen zwischen den beiden SA-Ereignissen von Rostock und R' wegen der Differenz der geographischen Längen stets 5min der Weltzeit oder der MEZ:

$$\mathrm{MEZ}(\mathrm{SA}_{R'}) = \mathrm{MEZ}(\mathrm{SA}_{\mathrm{Rostock}}) + 5\,\mathrm{min}\,.$$

In der Zwischenzeit ist allerdings die Erde, bzw. scheinbar die Sonne, auf ihrer Umlaufbahn ein wenig weitergewandert, und damit könnte die MEZ für den SA in R' gegenüber Rostock zusätzlich etwas geändert sein. Nun aber ändert sich in Rostock die MEZ des SA in einem ganzen Tag, d.i. im Prinzip nach einer Erdachsendrehung von 360°, stets um weniger als 3 min (s. Tabelle 1 S. 111 f., in Schritten von immer zwei Tagen sind 5 min die höchste Änderung). Für einen Ort auf demselben Breitenkreis wie Rostock wird die Differenz seiner SA-Zeit zu derjenigen von Rostock am selben Tag demnach weniger als

$$(\text{Längendifferenz zu Rostock}) \cdot \frac{3\,\mathrm{min}}{360°}$$

betragen; für Rostocker und Lübecker Länge $(12,07° - 10,75° = 1,32°)$ ergibt dies vernachlässigbare 0,011 min, die letzte obige Gleichung bleibt.

Aus den drei Gleichungen folgt dann (z.B. Einsetzen der letzten in die vorletzte)

$$\mathrm{MOZ}_{R'}(\mathrm{SA}_{R'}) = \mathrm{MOZ}_{\mathrm{Rostock}}(\mathrm{SA}_{\mathrm{Rostock}}). \tag{5}$$

[Evtl. auch einfach so: $\mathrm{MOZ}_{R'}$ ist stets 5 min geringer als $\mathrm{MOZ}_{\mathrm{Rostock}}$, aber vom $\mathrm{SA}_{\mathrm{Rostock}}$ bis zum $\mathrm{SA}_{R'}$ vergehen ebendiese 5 min.]

Uns interessiert jedoch Lübeck statt R', und zwischen beiden differieren die geographischen Breiten geringfügig. Es steht zu vermuten, daß dieser Einfluß ebenso nicht wesentlich ist. Wir wollen dennoch im Anhang am Ende dieses Teilabschnittes eine Abschätzung bringen, denn andererseits haben größere Differenzen in der geographischen Breite durchaus beträchtlichen Einfluß auf die SA-Zeiten (man denke z.B. an die Polregionen im Vergleich zu hiesigen).

Für unsere Zwecke halten wir vorab schon fest:

> Die mittleren Ortszeiten des Sonnenaufgangs für Rostock und Lübeck, d.h. $\mathrm{MOZ}_{\mathrm{Rostock}}(\mathrm{SA}_{\mathrm{Rostock}})$ und $\mathrm{MOZ}_{\mathrm{Lübeck}}(\mathrm{SA}_{\mathrm{Lübeck}})$, können als ‚praktisch identisch' angesehen werden. (6)

Zweitens nun zu wahren Ortszeiten.

Diese hängen über die Zeitgleichung mit den mittleren Ortszeiten zusammen, es gilt

$$g = \mathrm{WOZ}_{\mathrm{Rostock}}(\mathrm{SA}_{\mathrm{Rostock}}) - \mathrm{MOZ}_{\mathrm{Rostock}}(\mathrm{SA}_{\mathrm{Rostock}}) \quad \text{und}$$
$$g = \mathrm{WOZ}_{\mathrm{Lübeck}}(\mathrm{SA}_{\mathrm{Lübeck}}) - \mathrm{MOZ}_{\mathrm{Lübeck}}(\mathrm{SA}_{\mathrm{Lübeck}}).$$

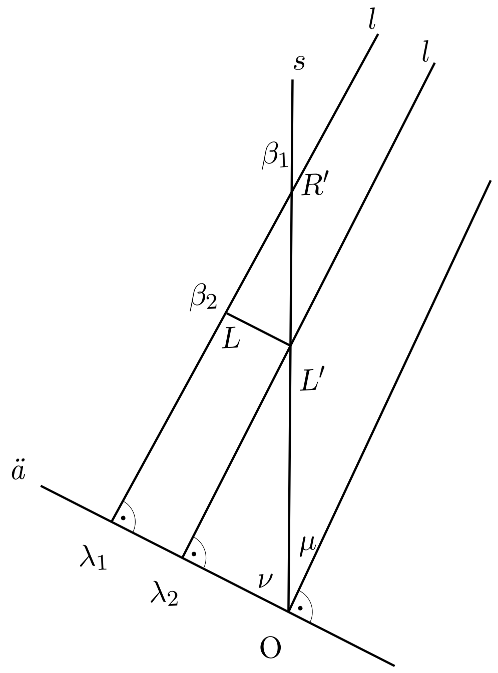
Je an demselben Tag können wir nach einer bei der Einführung der Zeitgleichung oben gemachten Bemerkung den Wert von g im kleinen Intervall von $\mathrm{SA}_{\mathrm{Rostock}}$ bis $\mathrm{SA}_{\mathrm{Lübeck}}$ als konstant ansehen. Dann folgt aus der soeben festgestellten Übereinstimmung der mittleren Ortszeiten:

> Auch die wahren Ortszeiten des Sonnenaufgangs für Rostock und Lübeck, d.h. $\mathrm{WOZ}_{\mathrm{Rostock}}(\mathrm{SA}_{\mathrm{Rostock}})$ und $\mathrm{WOZ}_{\mathrm{Lübeck}}(\mathrm{SA}_{\mathrm{Lübeck}})$, können als ‚praktisch identisch' angesehen werden. (7)

(Wir bleiben bei diesen Formulierungen, obwohl laut Anhang Differenzen um $\pm 1^1/_2$ min zu bestimmten Jahreszeiten festzustellen sind.)

> Die SA-Zeiten von 1885 könnten also evtl. weiterhin für Rostock akzeptiert werden, wenn sie wahre oder mittlere Zeiten von Lübeck wären.

Anhang: Zur Abhängigkeit der SA-Zeiten von der geographischen Breite.



Wir denken uns in der Skizze die Erdkugel von rechts bestrahlt, wobei sich die Schattengrenze s abzeichne, und betrachten Orte R' und L auf gemeinsamem Längenkreis l mit den Breiten β_1, β_2, dabei liege R' auf s, L noch im Schatten. Punkt O liege auf s und dem Äquator *ä*, bez. O seien die Längengrade λ_1, λ_2 gemessen. Bei Weiterdrehung der Erdkugel gelangt L in die Lage L' auf s (unter Vernachlässigbarkeit der geringfügigen zwischenzeitlichen Verlagerung von s, s. vor (5)).

In der Stellung von l bei O wird der Neigungswinkel μ sichtbar, μ ist der momentane, zur Sonne gerichtete Anteil der Erdachsenneigung, im Sommerhalbjahr also $0° \leq \mu \leq 23{,}5°$.

Man erkennt rechtwinklige sphärische Dreiecke mit den Hypotenusen OR' bzw. OL' und gemeinsamem Innenwinkel ν. In ihnen gelten die Gleichungen

$$\sin\lambda_i = \cot\nu \cdot \tan\beta_i = \cot(90° - \mu) \cdot \tan\beta_i, \text{ somit}$$
$$\sin\lambda_i = \tan\mu \cdot \tan\beta_i\,.$$

Gewünscht wird eine Aussage über $\lambda_1 - \lambda_2$, insbesondere über das Maximum dieser Differenz, und es leuchtet anschaulich ein, daß dieses für das größtmögliche $\mu = 23{,}5°$ angenommen wird (wir übergehen eine analytische Bestätigung).

Die Erddrehung benötigt 4min MOZ pro Längengrad. Um $(\lambda_1 - \lambda_2) \cdot 4\,\text{min}$ wäre der SA in L später als in R', dies in gemeinsamer MOZ auf demselben Längenkreis.

Mit $\beta_1 = 54{,}08°$, $\beta_2 = 53{,}83°$ (den Breiten von Rostock und Lübeck) sowie $\mu = 23{,}5°$ errechnet man für R' und L(übeck) $0{,}38° < \lambda_1 - \lambda_2 < 0{,}4°$, und damit

$$1{,}5\,\text{min} < (\lambda_1 - \lambda_2) \cdot 4\,\text{min} < 1{,}6\,\text{min}\,.$$

In Rostock und R' aber sind nach (5) die MOZ für den jeweiligen SA identisch, ebenso dann die WOZ (Zeitgleichung!), d.h. im Ergebnis:

> Um die Sommersonnenwende sind in Lübeck die MOZ und die WOZ des SA etwa $1^1/_2$ min später als die entsprechenden Ortszeiten in Rostock. Dieser Differenzwert ist der maximale für das ganze Jahr. (In MEZ erfolgt der SA in Lübeck etwa $(5 + 1^1/_2)$ min später als in Rostock.)

Für $\mu = 0°$ (bei Frühlings- und Herbstanfang) ist $\lambda_1 = \lambda_2 = \lambda_1 - \lambda_2 = 0$, die Zeiten differieren praktisch nicht. (In MEZ sind es etwa 5 min.)

Für $-23,5° \leq \mu \leq 0°$ (im Winterhalbjahr) müßte die Skizze an der Schattengrenze gespiegelt werden, L läge dann schon im Licht, SA wäre in L eher(!) als in R'. Mit $\mu = -23,5°$ ändern sich in obiger Rechnung lediglich die Vorzeichen, und so sind um die Wintersonnenwende in Lübeck die MOZ und WOZ des SA etwa $1^1/_2$ min eher(!) als die entsprechenden Ortszeiten in Rostock. (Die MEZ des SA ist etwa $(5 - 1^1/_2)$ min später(!).)

3 Versuch einer Einordnung der Sonnenaufgangzeiten von 1885, d.h. von 1855, Entscheidung über die Neubeschriftung

Die Beschriftung der Rostocker Kalenderscheibe erfolgte zwar 1885 mit Übernahme der SA-Zeiten von der alten Lübecker Uhr, aber die Zeitspanne der Lübecker Kalenderscheibe reichte von 1855 bis 1999 [9], die Daten müssen daher schon vor dieser Zeit gewonnen worden sein. Die Uhr wurde 1942 bei der Bombardierung von Lübecks St. Marien vernichtet.

Gehen wir jetzt davon aus, daß uns heutzutage für Rostock die Zeiten des SA in MEZ sowie in WOZ bekannt sind, d.h. $\mathrm{MEZ}(\mathrm{SA}_{\mathrm{Rostock}})$ und $\mathrm{WOZ}_{\mathrm{Rostock}}(\mathrm{SA}_{\mathrm{Rostock}})$. Sie finden sich in den entsprechenden Spalten der Tabelle 1 auf S. 99 f., Ausführungen zu ihrer Ermittlung folgen im nächsten Abschnitt. Jene Tabelle enthält auch die heutigen Eintragungen auf der Rostocker Kalenderscheibe, die für die Formalien mit $\mathrm{Z}_{1855}(\mathrm{SA}_{\mathrm{Lübeck}})$ bezeichnet seien.

Wir wollen beurteilen, ob diese Eintragungen den SA für Lübeck in $\mathrm{MOZ}_{\mathrm{Lübeck}}$ angeben. Dazu wurde Δ_1 gebildet, s. entsprechende Spalte in Tabelle 1, und zwar

$$\Delta_1 = \mathrm{Z}_{1855}(\mathrm{SA}_{\mathrm{Lübeck}}) - \mathrm{MEZ}(\mathrm{SA}_{\mathrm{Rostock}})\,.$$

Weil nach (1) $\mathrm{MEZ} = \mathrm{MOZ}_{\mathrm{Rostock}} + 12\,\mathrm{min}$ gilt, wird

$$\Delta_1 = \mathrm{Z}_{1855}(\mathrm{SA}_{\mathrm{Lübeck}}) - \mathrm{MOZ}_{\mathrm{Rostock}}(\mathrm{SA}_{\mathrm{Rostock}}) - 12\,\mathrm{min}\,.$$

Wären nun die Zeiten $\mathrm{Z}_{1855}(\mathrm{SA}_{\mathrm{Lübeck}})$ als $\mathrm{MOZ}_{\mathrm{Lübeck}}(\mathrm{SA}_{\mathrm{Lübeck}})$ interpretierbar, so wäre

$$\Delta_1 = \mathrm{MOZ}_{\mathrm{Lübeck}}(\mathrm{SA}_{\mathrm{Lübeck}}) - \mathrm{MOZ}_{\mathrm{Rostock}}(\mathrm{SA}_{\mathrm{Rostock}}) - 12\,\mathrm{min}\,.$$

Tabelle 1: Gesamtaufstellung der SA-Zeiten für Rostock, an der Astronomischen Uhr jetzt (Z_{1855}), in MEZ sowie in WOZ; $\Delta_1 = Z_{1855} - \text{MEZ}$, $\Delta_2 = Z_{1855} - \text{WOZ}$ (je in Minuten).

	Z_{1855}	MEZ	Δ_1	WOZ	Δ_2
1.1.	8:27	8:31	- 4	8:16	+11
3.1.	8:25	8:31	- 6	8:15	+10
5.1.	8:23	8:30	- 7	8:13	+10
7.1.	8:21	8:29	- 8	8:11	+10
9.1.	8:19	8:28	- 9	8:09	+10
11.1.	8:16	8:26	-10	8:07	+ 9
13.1.	8:14	8:25	-11	8:05	+ 9
15.1.	8:12	8:23	-11	8:02	+10
17.1.	8:09	8:21	-12	7:59	+10
19.1.	8:06	8:19	-13	7:56	+10
21.1.	8:02	8:16	-14	7:53	+ 9
23.1.	7:59	8:13	-14	7:50	+ 9
25.1.	7:55	8:11	-16	7:47	+ 8
27.1.	7:51	8:08	-17	7:43	+ 8
29.1.	7:48	8:04	-16	7:39	+ 9
31.1.	7:44	8:01	-17	7:36	+ 8
2.2.	7:40	7:58	-18	7:32	+ 8
4.2.	7:36	7:54	-18	7:28	+ 8
6.2.	7:32	7:50	-18	7:24	+ 8
8.2.	7:27	7:47	-20	7:21	+ 6
10.2.	7:23	7:43	-20	7:17	+ 6
12.2.	7:18	7:39	-21	7:13	+ 5
14.2.	7:12	7:34	-22	7:08	+ 4
16.2.	7:09	7:30	-21	7:04	+ 5
18.2.	7:05	7:26	-21	7:00	+ 5
20.2.	7:00	7:22	-22	6:56	+ 4
22.2.	6:56	7:17	-21	6:52	+ 4
24.2.	6:52	7:13	-21	6:48	+ 4
26.2.	6:46	7:08	-22	6:43	+ 3
28.2.	6:41	7:03	-22	6:39	+ 2
2.3.	6:38	6:58	-20	6:34	+ 4
4.3.	6:35	6:53	-18	6:29	+ 6
6.3.	6:29	6:49	-20	6:26	+ 3
8.3.	6:25	6:44	-19	6:21	+ 4
10.3.	6:20	6:39	-19	6:17	+ 3
12.3.	6:15	6:34	-19	6:12	+ 3
14.3.	6:10	6:29	-19	6:08	+ 2
16.3.	6:06	6:24	-18	6:03	+ 3
18.3.	6:02	6:19	-17	5:59	+ 3
20.3.	6:00	6:14	-14	5:55	+ 5
22.3.	5:56	6:10	-14	5:51	+ 5
24.3.	5:50	6:05	-15	5:46	+ 4
26.3.	5:45	6:00	-15	5:42	+ 3
28.3.	5:42	5:55	-13	5:38	+ 4
30.3.	5:37	5:50	-13	5:33	+ 4

	Z_{1855}	MEZ	Δ_1	WOZ	Δ_2
1.4.	5:31	5:45	-14	5:29	+ 2
3.4.	5:27	5:40	-13	5:25	+ 2
5.4.	5:23	5:35	-12	5:20	+ 3
7.4.	5:19	5:30	-11	5:16	+ 3
9.4.	5:16	5:25	- 9	5:11	+ 5
11.4.	5:12	5:20	- 8	5:07	+ 5
13.4.	5:08	5:16	- 8	5:03	+ 5
15.4.	5:04	5:11	- 7	4:59	+ 5
17.4.	5:00	5:06	- 6	4:54	+ 6
19.4.	4:54	5:02	- 8	4:50	+ 4
21.4.	4:51	4:57	- 6	4:46	+ 5
23.4.	4:47	4:52	- 5	4:42	+ 5
25.4.	4:43	4:48	- 5	4:38	+ 5
27.4.	4:39	4:44	- 5	4:34	+ 5
29.4.	4:35	4:39	- 4	4:30	+ 5
1.5.	4:28	4:35	- 7	4:26	+ 2
3.5.	4:24	4:31	- 7	4:22	+ 2
5.5.	4:20	4:27	- 7	4:18	+ 2
7.5.	4:16	4:23	- 7	4:14	+ 2
9.5.	4:12	4:19	- 7	4:11	+ 1
11.5.	4:08	4:16	- 8	4:07	+ 1
13.5.	4:05	4:12	- 7	4:04	+ 1
15.5.	4:02	4:09	- 7	4:00	+ 2
17.5.	3:59	4:05	- 6	3:57	+ 2
19.5.	3:56	4:02	- 6	3:54	+ 2
21.5.	3:52	3:59	- 7	3:51	+ 1
23.5.	3:49	3:56	- 7	3:48	+ 1
25.5.	3:46	3:54	- 8	3:45	+ 1
27.5.	3:44	3:51	- 7	3:42	+ 2
29.5.	3:42	3:49	- 7	3:40	+ 2
31.5.	3:40	3:47	- 7	3:37	+ 3
2.6.	3:38	3:45	- 7	3:35	+ 3
4.4.	3:35	3:43	- 8	3:33	+ 2
6.6.	3:34	3:42	- 8	3:32	+ 2
8.6.	3:32	3:41	- 9	3:30	+ 2
10.6.	3:32	3:40	- 8	3:29	+ 3
12.6.	3:31	3:39	- 8	3:28	+ 3
14.6.	3:31	3:39	- 8	3:27	+ 4
16.6.	3:30	3:38	- 8	3:26	+ 4
18.6.	3:30	3:38	- 8	3:25	+ 5
20.6.	3:29	3:38	- 9	3:25	+ 4
22.6.	3:29	3:39	-10	3:25	+ 4
24.6.	3:30	3:39	- 9	3:25	+ 5
26.6.	3:30	3:40	-10	3:25	+ 5
28.6.	3:31	3:41	-10	3:26	+ 5
30.6.	3:32	3:43	-11	3:27	+ 5

	Z_{1855}	MEZ	Δ_1	WOZ	Δ_2		Z_{1855}	MEZ	Δ_1	WOZ	Δ_2
2.7.	3:33	3:44	-11	3:28	+ 5	2.10.	6:23	6:15	+ 8	6:14	+ 9
4.7.	3:35	3:46	-11	3:30	+ 5	4.10.	6:27	6:19	+ 8	6:19	+ 8
6.7.	3:37	3:48	-11	3:31	+ 6	6.10.	6:31	6:23	+ 8	6:23	+ 8
8.7.	3:39	3:50	-11	3:33	+ 6	8.10.	6:35	6:27	+ 8	6:28	+ 7
10.7.	3:40	3:52	-12	3:35	+ 5	10.10.	6:39	6:30	+ 9	6:32	+ 7
12.7.	3:42	3:55	-13	3:38	+ 4	12.10.	6:43	6:34	+ 9	6:36	+ 7
14.7.	3:44	3:57	-13	3:40	+ 4	14.10.	6:48	6:38	+10	6:40	+ 8
16.7.	3:46	4:00	-14	3:42	+ 4	16.10.	6:53	6:42	+11	6:45	+ 8
18.7.	3:48	4:03	-15	3:45	+ 3	18.10.	6:58	6:45	+13	6:48	+10
20.7.	3:51	4:05	-14	3:47	+ 4	20.10.	7:04	6:49	+15	6:53	+11
22.7.	3:54	4:08	-14	3:50	+ 4	22.10.	7:09	6:53	+16	6:57	+12
24.7.	3:57	4:12	-15	3:54	+ 3	24.10.	7:13	6:57	+16	7:01	+12
26.7.	4:00	4:15	-15	3:57	+ 3	26.10.	7:17	7:01	+16	7:05	+12
28.7.	4:04	4:18	-14	4:00	+ 4	28.10.	7:21	7:05	+16	7:10	+11
30.7.	4:08	4:21	-13	4:03	+ 5	30.10.	7:25	7:09	+16	7:14	+11
1.8.	4:11	4:25	-14	4:07	+ 4	1.11.	7:29	7:13	+16	7:18	+11
3.8.	4:15	4:28	-13	4:11	+ 4	3.11.	7:33	7:17	+16	7:22	+11
5.8.	4:19	4:32	-13	4:14	+ 5	5.11.	7:37	7:20	+17	7:25	+12
7.8.	4:23	4:35	-12	4:18	+ 5	7.11.	7:41	7:24	+17	7:29	+12
9.8.	4:26	4:39	-13	4:22	+ 4	9.11.	7:45	7:28	+17	7:33	+12
11.8.	4:30	4:42	-12	4:26	+ 4	11.11.	7:48	7:32	+16	7:36	+12
13.8.	4:35	4:46	-11	4:30	+ 5	13.11.	7:51	7:36	+15	7:40	+11
15.8.	4:39	4:49	-10	4:33	+ 6	15.11.	7:54	7:40	+14	7:44	+10
17.8.	4:43	4:53	-10	4:37	+ 6	17.11.	7:57	7:44	+13	7:47	+10
19.8.	4:47	4:56	- 9	4:41	+ 6	19.11.	8:01	7:47	+14	7:50	+11
21.8.	4:51	5:00	- 9	4:45	+ 6	21.11.	8:04	7:51	+13	7:53	+11
23.8.	4:55	5:04	- 9	4:50	+ 5	23.11.	8:08	7:55	+13	7:56	+12
25.8.	5:00	5:07	- 7	4:54	+ 6	25.11.	8:11	7:58	+13	7:59	+12
27.8.	5:04	5:11	- 7	4:58	+ 6	27.11.	8:15	8:01	+14	8:02	+13
29.8.	5:08	5:14	- 6	5:02	+ 6	29.11.	8:18	8:05	+13	8:05	+13
31.8.	5:12	5:18	- 6	5:06	+ 6	1.12.	8:20	8:08	+12	8:07	+13
2.9.	5:16	5:21	- 5	5:10	+ 6	3.12.	8:22	8:11	+11	8:09	+13
4.9.	5:20	5:25	- 5	5:15	+ 5	5.12.	8:24	8:14	+10	8:11	+13
6.9.	5:24	5:29	- 5	5:19	+ 5	7.12.	8:26	8:16	+10	8:13	+13
8.9.	5:28	5:32	- 4	5:23	+ 5	9.12.	8:28	8:19	+ 9	8:15	+13
10.9.	5:33	5:36	- 3	5:27	+ 6	11.12.	8:29	8:21	+ 8	8:16	+13
12.9.	5:37	5:39	- 2	5:32	+ 5	13.12.	8:29	8:23	+ 6	8:17	+12
14.9.	5:41	5:43	- 2	5:36	+ 5	15.12.	8:30	8:25	+ 5	8:18	+12
16.9.	5:46	5:47	- 1	5:40	+ 6	17. 12	8:30	8:27	+ 3	8:19	+11
18.9.	5:52	5:50	+ 2	5:44	+ 8	19.12.	8:31	8:28	+ 3	8:19	+12
20.9.	5:57	5:54	+ 3	5:49	+ 8	21.12.	8:30	8:29	+ 1	8:19	+11
22.9.	6:02	5:57	+ 5	5:53	+ 9	23.12.	8:30	8:30	0	8:19	+11
24.9.	6:06	6:01	+ 5	5:57	+ 9	25.12.	8:29	8:31	- 2	8:19	+10
26.9.	6:11	6:04	+ 7	6:02	+ 9	27.12.	8:29	8:31	- 2	8:19	+10
28.9.	6:15	6:08	+ 7	6:06	+ 9	29.12.	8:28	8:31	- 3	8:18	+10
30.9.	6:19	6:12	+ 7	6:10	+ 9	31.12.	8:28	8:31	- 3	8:17	+11

Ende Tabelle 1

Aber $\text{MOZ}_{\text{Lübeck}}(\text{SA}_{\text{Lübeck}})$ und $\text{MOZ}_{\text{Rostock}}(\text{SA}_{\text{Rostock}})$ sind nach (6) ‚praktisch identisch', womit Δ_1 annähernd konstant (etwa -12 min) sein müßte.

Davon kann allerdings nicht im entferntesten die Rede sein, Δ_1 variiert von -22 min (Ende Februar) zunächst bis -4 min (Ende April), dann wieder bis -15 min (Ende Juli) aber anschließend bis +17 min (Anfang November).

> Die SA-Zeiten von 1855 sind keine Angaben in mittlerer Ortszeit Lübecks. Sie sind nach (3) auch keine Angaben in MOZ eines anderen Ortes.

Nun zum Standpunkt über das Vorliegen wahrer Ortszeiten in den Angaben von 1855. Dazu wurde in der Tabelle 1 die Differenz Δ_2 gebildet:

$$\Delta_2 = \text{Z}_{1855}(\text{SA}_{\text{Lübeck}}) - \text{WOZ}_{\text{Rostock}}(\text{SA}_{\text{Rostock}})\,.$$

Die Aussage, daß die Zeiten $\text{Z}_{1855}(\text{SA}_{\text{Lübeck}})$ als $\text{WOZ}_{\text{Lübeck}}(\text{SA}_{\text{Lübeck}})$ interpretierbar sind, geht nach (7) einher mit der Feststellung, daß Δ_2 ‚praktisch konstant gleich Null' ist. Obwohl dies numerisch nicht durchaus behauptet werden kann, entnehmen wir dennoch dem Werteverlauf von Δ_2 Argumente für den geäußerten Standpunkt, abgesehen davon, daß im Grunde gar keine andere Auswahl bleibt.

Man beachte als erstes die ruhig wirkenden Änderungen beim Werteverlauf von Δ_2, ganz gegensätzlich zu Δ_1. Ferner liegt Δ_2 während des langen Zeitabschnitts vom 8.2. bis zum 16.9. im Intervall von +1 min bis +6 min (es gelte als akzeptabel). Dabei gibt es im Mai zeitweise die minimale Differenz +1 min, und danach übersteigen die Differenzen bis Anfang August nur selten den Wert +4 min. (Denkt man hier an den Anhang vom 2. Abschnitt zurück, so sind im Sommer zeitweise die $\text{WOZ}_{\text{Lübeck}}(\text{SA}_{\text{Lübeck}})$ gegenüber den $\text{WOZ}_{\text{Rostock}}(\text{SA}_{\text{Rostock}})$ um etwa 1,5 min zu erhöhen. Es ist also ganz natürlich, daß in dieser Jahreszeit die Angaben $\text{Z}_{1855}(\text{SA}_{\text{Lübeck}})$, eben als $\text{WOZ}_{\text{Lübeck}}(\text{SA}_{\text{Lübeck}})$, die $\text{WOZ}_{\text{Rostock}}(\text{SA}_{\text{Rostock}})$ um etwa solchen Wert übersteigen. Aber eine zu penible Beurteilung der Unterschiede verbietet sich, weil die SA-Zeiten von Jahr zu Jahr mindestens im Sekundenbereich differieren, über längere Zeitspannen evtl. um Minuten, s. im 4. Abschnitt.)

Die größeren Abweichungen im Spätherbst und Frühwinter stören allerdings, es sei jedoch daran erinnert, daß es sich um Werte handelt, die vor 1855 ermittelt wurden. Erfolgte dies damals vielleicht durch unmittelbare Beobachtung, verbunden mit den Widrigkeiten insbesondere der kalten Jahreszeiten und möglicherweise sich ändernden geographischen Gegebenheiten, sowohl im Standort als auch im Horizontverlauf? –

Zusammengefaßt ergaben sich für die Neubeschriftung der Rostocker Kalenderscheibe mit SA-Zeiten nachfolgende Positionen, grundsätzlich unter der Prämisse, daß Änderungen jeglicher Art an den jetzigen Angaben stets nur unter Rücksicht auf die Historie der Astronomischen Uhr erfolgen werden:

1. Neue MOZ für Rostock, die vorab favorisiert wurden, kommen nicht in Frage.
2. (a) Die bisherigen Werte werden beibehalten, denn sie kommen über weite Zeiten des Jahres den WOZ für Rostock nahe; jedenfalls wird ihre Qualität für Rostock kaum dadurch gemindert, daß sie einst für Lübeck bestimmt wurden.
   (b) Neue, also verbesserte WOZ für Rostock werden aufgebracht.
   (c) Die Rostocker Aufgangszeiten werden in MEZ aufgebracht.

Zu den unter 2. aufgelisteten Varianten bildete sich die Meinung: Neue und besser zutreffende WOZ bringen keinen erstrebenwerten Zugewinn; der Historie verpflichtet könnten dann eher die bisherigen Eintragungen beibehalten werden. Andererseits es ist aber wohl doch angebracht, für die Angaben eine Uhrzeit zu verwenden, die allen Besuchern (und Stadtführern) sofort verständlich ist.

So fiel die Entscheidung, die SA für Rostock in MEZ aufzutragen.

Verf. meint, man könnte durchaus äußern, es sind dies die lokalen SA-Zeiten, zu deutsch hieße es nichts anderes als Ortszeiten, um dieses leicht Zuspruch erheischende Wort zu bedienen, nämlich für den Ort Rostock gültig in heute am Ort gebräuchlicher Zeitangabe. Und wie anders sollte die Absicht auch bei früheren Beschriftungen der Kalenderscheiben unter den damaligen Gegebenheiten gewesen sein, als es vielleicht noch verbreiteter war, sich am wahren Stand der Sonne, folglich der WOZ, zu orientieren?

Zögernd nur wurde bei der Entscheidung davon abgesehen, die Werte von April bis Oktober in Sommerzeit (MESZ) umzurechnen, dies in der Hoffnung, daß deren Einführung nicht von endgültiger Dauer sein wird.

4 Zur Ermittlung der verwendeten Zeitangaben

Die Mitteleuropäischen Zeiten des Sonnenaufgangs in Rostock (MEZ(SA_{Rostock})) wurden bestimmt mit dem Astronomieprogramm Cyber Sky 3.3.1, welches die oben schon herangezogenen Koordinaten 12,07°O, 54,08°N verwendet. Als zusätzliche Kontrolle dienten Werte des „Image Pressezentrum München“, die auch weiterhin in hiesiger Tagespresse veröffentlicht werden. Die dort verwendeten Koordinaten sind 12,09°O, 54,06°N, so daß geringfügige Abweichungen zwischen beiden Angaben nicht ausgeschlossen sind. Diese traten gelegentlich auf, betrugen dann rundungsbedingt 1min, niemals mehr, was als Bestätigung für die eingangs genannten Zeiten gelte.

Zusammen mit den ebenso gewonnenen Sonnenuntergangszeiten erhält man in der Mitte von SA und SU die MEZ für die Südstellung der Sonne (MEZ($Sst._{\text{Rostock}}$)). Wird nun die Differenz

MEZ(Sst.$_{\text{Rostock}}$) − 12:00 von der MEZ aller Angaben des betreffenden Tages subtrahiert, so erhält man die wahre Ortszeit der jeweiligen Ereignisse, speziell die uns interessierenden WOZ$_{\text{Rostock}}$(SA$_{\text{Rostock}}$). (Genaugenommen ist bei zunehmender Tageslänge der Nachmittag schon etwas gedehnt gegenüber dem Vormittag, bei abnehmender Tageslänge etwas gekürzt; in diesem Sinne wurden die verwendeten MEZ(Sst.$_{\text{Rostock}}$) nötigenfalls gerundet.)

> Letztlich wurden unsere Zeiten aus den Rostocker MEZ(SA) der Vierjahreszeitspanne von 2004 bis 2007 als arithmetische Mittel bestimmt. Erforderliche Rundung geschah entsprechend den Tendenzen (8), s. etwas weiter unten.

Dieses Vorgehen war zunächst lediglich eine praktisch realisierbare Variante, die Anfang 2009 vom Verf. in die Diskussion eingebracht werden konnte, als ihm unverhofft bekannt wurde, daß der Auftrag für die Beschriftung einer neuen Deckplatte der Kalenderscheibe bereits erteilt sei und darin als SA-Zeiten neue mittlere Ortszeiten vorgesehen seien. Wir erläutern nun die Hintergründe für die Ermittlung vertretbarer Zeitangeben.

Das *Tropische Jahr*, als Zeitspanne zwischen zwei Durchgängen der Sonne durch den Frühlingspunkt, hat (auch schon gerundet) die Länge 365,2422 d. Unser *Normaljahr* von 365 d ist 0,2422 d zu kurz, die Sonne hat im Tierkreis nach Ablauf genau dieser Tage noch nicht wieder die Stellung des Vorjahres erreicht, ihre Stellung ist in der Tendenz je zum vorhergehenden Datum des Vorjahres verschoben. Dementsprechend ändern sich die MEZ(SA), der SA erfolgt in Phasen seiner täglichen Verfrühung noch etwas später, in Phasen täglicher Verspätung noch etwas eher als im Vorjahr.

Ab 1.3.2004 vergehen bis zum 1.3.2007 drei Normaljahre, die Zeitspanne ist 0,7266 d ($= 3 \cdot 0,2422\,\text{d}$) zu kurz; um beinahe ³⁄₄ der um den 1.3. aktuellen täglichen Änderung ist die MEZ(SA) am 1.3.2007 größer als am 1.3.2004. Diese Tatsache gilt mutatis mutandis für alle Datenpaare bis hin zum 28.2. der Jahre 2005 und 2008. Danach folgt ein Schalttag.

Nach *vier Jahren*, dem Ablauf von drei zu kurzen Normaljahren und einem *Schaltjahr* von 366 d, welches 0,7578 d zu lang ist gegenüber dem Tropischen Jahr, ist die Zeitspanne wegen $(3 \cdot -0,2422 + 0,7578)\,\text{d} = 0,0312\,\text{d}$ etwas zu lang gegenüber vier Tropischen Jahren. Die Stellung der Sonne ist nach Ablauf dieser vier Jahre in der Tendenz schon etwas zum nachfolgenden Datum von damals verschoben. Das bedeutet,

> nach vier Jahren ist die MEZ(SA) in Phasen täglicher Verfrühung etwas kleiner, (8)
> in Phasen täglicher Verspätung schon etwas größer als seinerzeit.

Alle jährlichen Veränderungen liegen für das einzelne Datum im Sekundenbereich, bei Rundung äußert sich das gelegentlich auch als Minute.

Tabelle 2: Beispiele sekundengenauer Rostocker Zeiten in MEZ

	SA 1.3.	SA 2.3.
2004	6:59:48	6:57:27
2005	7:00:22	6:58:01
2006	7:00:57	6:58:36
2007	7:01:31	6:59:10
2008	6:59:44	6:57:23

Kommentar 3 über die Rundung der Zeitangaben: Die verwendeten MEZ und WOZ des SA für Rostock sind arithmetische Mittel von minutengerundeten Zeiten aus den Jahren 2004 bis 2007, und diese Mittel wurden ihrerseits gegebenenfalls im Sinne der Tendenzen (8) gerundet. Dadurch, daß sich die MEZ(SA) nach drei Jahren je um 0,7266 der täglichen Veränderungen ändert, treten hier gerundet bis zu 2 min auf; die vierjährigen Mittel differieren dann (zumindest zunächst) von den korrekten MEZ(SA) um höchstens 1 min.

Über die gesamte Zeitspanne der neuen Kalenderscheibe wird diese Abweichung höchstens 2 min betragen, s. hierzu Anhang b).

Um Penibilität in Grenzen zu halten, wurden zur Bestimmung der Mittelwerte keine sekundengenauen Werte herangezogen. Solche dienen hier nur als zusätzliche Illustration (sie sind ja ebenfalls gerundet).

Anhang: a) Ein Detail zur täglichen Abfolge der SA-Zeiten während des Jahres.

Aus (1) und der Zeitgleichung folgt die Beziehung

$$\mathrm{MEZ}(\mathrm{SA}_{\mathrm{Rostock}}) = \mathrm{WOZ}_{\mathrm{Rostock}}(\mathrm{SA}_{\mathrm{Rostock}}) - g + 12\,\mathrm{min}\,.$$

Danach stimmen in der Abfolge der Kalenderdaten die Änderungstendenzen der MEZ(SA) und der WOZ(SA) überein, wenn die Tendenzen der WOZ(SA) und von g entgegengesetzt sind, die Änderung der MEZ(SA) wird dann gegenüber derjenigen der WOZ(SA) sogar verstärkt. Nun variiert die WOZ(SA) mit dem Stand der Sonne, sie steigt vom Sommeranfang (z.Zt. 21.6.) bis zum Winteranfang (z.Zt. 21. oder 22.12.) und fällt dann wieder bis zum Sommeranfang. Von Mitte Februar etwa bis Anfang Mai, wenn g im Gegensatz zur WOZ(SA) steigt, gibt es eine kräftige Verfrühung der MEZ(SA), stets mindestens 4 min in zwei Tagen, oftmals 5 min.

Schon ab Mitte Mai aber fällt g, und das dann verstärkt, auch die WOZ(SA) fällt noch, aber dann immer weniger. Dies führt dazu, daß die MEZ(SA) ihr Minimum, den frühesten SA, schon ‚am 18.6.' erreicht. Andererseits fällt am Kalenderjahresende die WOZ(SA) ab dem Winteranfang wieder, aber g fällt zunächst noch stärker, so daß die MEZ(SA) ihr Maximum, den spätesten SA, erst ‚am 29.12.' hat. (Dual hierzu ist in MEZ der späteste Sonnenuntergang erst ‚am 24.6.' (Johannistag), der früheste dagegen schon ‚am 13.12.'.) –

Anhang: b) Begutachtung der gewählten SA-Zeiten gegenüber den tatsächlichen bis 2150.

Der oben zuletzt angesprochene Zyklus von vier Jahren läuft im *Gregorianischen Kalender* vom 1.3.2004 bis zum 1.3.2096 23mal ab, wegen $23 \cdot 0{,}0312\,\mathrm{d} = 0{,}7176\,\mathrm{d}$ ist die Sonne beinahe um einen $^3/_4$ Tag ab dem 1.3. im Jahr 2096 gegenüber 2004 voraus. Aber vom 1.3.2096 bis zum 1.3.2103 laufen 7 Normaljahre ab (den 29.2.2100 gibt es nicht); 2103 ist die Sonne ab dem 1.3. wegen $0{,}7176\,\mathrm{d} + 7 \cdot -0{,}2422\,\mathrm{d} = -0{,}9778\,\mathrm{d}$ um fast einen Tag gegenüber 2004 zurück.

Tabelle 3: Beispiele sekundengenauer Rostocker Zeiten in MEZ

	SA 1.3.	SA 2.3.		SA 1.3.	SA 2.3.
2096	6:58:02	6:55:40	2100	7:00:19	6:57:58
2097	6:58:36	6:56:14	2101	7:00:53	6:58:32
2098	6:59:10	6:56:48	2102	7:01:27	6:59:06
2099	6:59:45	6:57:24	2103	7:02:01	6:59:40
			2104	7:00:15	6:57:53

Das Jahr 2104 ist mit 366 d um 0,7578 d zu lang, ab 1.3. ist die Sonne wegen $-0{,}9778\,\mathrm{d} + 0{,}7578\,\mathrm{d} = -0{,}2200\,\mathrm{d}$ um etwa $^1/_4$ Tag gegenüber 2004 zurück.

Bis zum 1.3.2148 folgen 11 Vierjahreszeiträume, in denen die Sonne je 0,0312 d gewinnt, $-0{,}2200\,\mathrm{d} + 11 \cdot 0{,}0312\,\mathrm{d} = +0{,}1232\,\mathrm{d}$ zeigt den Stand ab 1.3. im Jahr 2148. Schließlich folgen noch zwei Normaljahre, d.h., ab 1.3. im Jahr 2150 gilt gegenüber 2004 $+0{,}1232\,\mathrm{d} + 2 \cdot -0{,}2422\,\mathrm{d} = -0{,}3612\,\mathrm{d}$.

Das Durchschnittsniveau bez. 2004 der von uns herangezogenen Jahre 2004 - 2007 liegt bei $3 \cdot -0{,}2422\,\mathrm{d}/2 = -0{,}3633\,\mathrm{d}$. In dieser Größe ist es angesiedelt im soeben bis 2150 ermittelten Abweichungsspektrum bez. 2004 mit dessen Extrema +0,7176 d und −0,9778 d in den Jahren 2096 bzw. 2103. Maximal um etwa eine Tagesänderung der MEZ(SA) (denn $0{,}7176 - (-0{,}3633) = 1{,}0809$) werden 2096 unsere Durchschnittswerte von den korrekten abweichen. Nun verändert sich die MEZ(SA) täglich um weniger als 2,5 min (in annähernd dieser Größe ist dies nur in der unter a) erwähnten Phase von Februar bis Mai der Fall). Minutengerundete Abweichungen übersteigen daher 2 min nicht, das gilt bis 2150.

Schluß

Die Astronomische Uhr aus dem Jahre 1472, also 110 Jahre vor der Gregorianischen Kalenderreform erstellt, ist eine der besonderen Kostbarkeiten von St. Marien. Ihr fortdauernder adäquater Gang, gesichert durch Engagement und handwerkliches Geschick, bedarf dennoch gelegentlicher Korrekturen [2] und jetzt im Kalendarium nach Ablauf von 133 Jahren einer Erneuerung. Obige Ausführungen liefern hierzu Abwägungen und Begründungen für die aufzubringenden Rostocker Sonnenaufgangszeiten, zweitäglich angegeben in der nunmehr bereits seit 1893 üblichen Mitteleuropäischen Zeit. Erfreulichen Umständen war es zu verdanken, daß bereits am 25. September 2009 das neue Kalendarium feierlich übergeben wurde, restaurativ gefertigt als Deckplatte für die jetzige Scheibe, vorläufig zur Aufstellung in der Kirche.

Jüngst trat eine weitere Kostbarkeit der Marienkirche ebenfalls besonders ins Blickfeld, es ist die große Bronzeglocke, die 1290 in Rostock gegossen wurde [5]. In fatalen Kriegswirren hatte sie schwer gelitten und war seit langem leider gesprungen. Aber am 31. August 2009 wurde sie abgeholt zu jetzt endlich möglicher Restauration, zusammen mit drei weiteren mittelalterlichen Glocken des früheren Geläuts von St. Marien.

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Autor:

Klaus-Dieter Drews, i. R.
Universität Rostock
Institut für Mathematik
18051 Rostock
Germany

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