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Results on partial Derivatives of the incomplete Beta Function

ABSTRACT. The incomplete Beta function $B(a, b; x)$ is defined by

$$B(a,b;x) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt\,,$$

for $a, b > 0$ and $0 < x < 1$. This definition was extended to negative integer values of a and b by Özçağ et al. Partial derivatives of the incomplete Beta function $B(a, b; x)$ for negative integer values of a and b were then evaluated. In the following, it is proved that

$$B_{0,1}(-1,1;x) = -\ln\frac{x}{1-x} - \frac{\ln(1-x)}{x} - 1$$

and

$$nB_{0,1}(-n,1;x) = -\ln\frac{x}{1-x} - \frac{\ln(1-x)}{x^n} - n^{-1} + \sum_{i=1}^{n-1}\frac{x^{-i}}{i}\,,$$

for $n = 2, 3, \dots,$ where

$$\frac{\partial^{m+n}}{\partial a^m \partial b^n} B(a,b;x) = B_{m,n}(a,b;x)\,.$$

Further results are also given.

KEY WORDS. Beta function, incomplete Beta function, neutrix, neutrix limit

1 INTRODUCTION

In a change of notation, the incomplete Beta function $B(a, b; x)$ is defined by

$$B(a,b;x) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt, \qquad a, b > 0, \quad 0 < x < 1$$

see Özçağ et al [6].

The following definitions were given by van der Corput [4].

Definition 1.1 *A neutrix N is defined as a commutative additive group of functions $\nu(\xi)$ defined on a domain N' with values in an additive group N'', where further, if for some $\nu \in N$, $\nu(\xi) = \gamma$ for all $\xi \in N'$, then $\gamma = 0$. The functions in N are called negligible functions.*

Definition 1.2 *Let N' be a set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function in N' with values in N'' and it is possible to find a constant c such that $f(\xi) - c \in N$, then c is called the neutrix limit of f as ξ tends to b and we write $\mathrm{N}{-}\lim_{\xi\to b} f(\xi) = c$.*

Note that if f tends to c in the normal sense as ξ tends to b, then it converges to c in the neutrix sense.

Now let N be the neutrix having domain $N' = (0, x) \quad (0 < x < 1)$ and range N'' the real numbers, with the negligible functions finite linear sums of the functions

$$\epsilon^\lambda \ln^{r-1} \epsilon, \quad \ln^r \epsilon \qquad (\lambda < 0, \quad r = 1, 2, \ldots)$$

and all functions which converge to zero in the normal sense as ϵ tends to zero.

It was proved, see Özçağ et al. [6] and [7] that

$$B(a, b; x) = \mathrm{N}{-}\lim_{\epsilon\to 0} \int_\epsilon^x t^{a-1}(1-t)^{b-1}\, dt$$

for all values of a and b and in general

$$\begin{aligned} \frac{\partial^{m+n}}{\partial a^m \partial b^n} B(a, b; x) &= B_{m,n}(a, b; x) \\ &= \mathrm{N}{-}\lim_{\epsilon\to 0} \int_\epsilon^x t^{a-1} \ln^m t\, (1-t)^{b-1} \ln^n(1-t)\, dt \end{aligned}$$

for $m, n = 0, 1, 2, \ldots$ and all values of a and b.

Note that $B_{m,n}(a, b; x)$ is not necessarily equal to $B_{m,n}(b, a; x)$.

Note also that if $a > 0$, then

$$B_{m,n}(a, b; x) = \lim_{\epsilon\to 0} \int_\epsilon^x t^{a-1} \ln^m t\, (1-t)^{b-1} \ln^n(1-t)\, dt$$

for $m, n = 0, 1, 2, \ldots$.

The following results were proved in [2]:

$$B(0, 0; x) = \ln \frac{x}{1-x}, \tag{1}$$

$$B(n, 0; x) = -\ln(1-x) - \sum_{i=1}^{n-1} \frac{x^i}{i}, \quad n = 1, 2, \ldots, \tag{2}$$

the sum being empty when $n = 1$ and

$$B(-n, 0; x) = \ln \frac{x}{1-x} - \sum_{i=1}^{n} \frac{x^{-i}}{i}, \quad n = 1, 2, \ldots. \tag{3}$$

2 MAIN RESULTS

We now prove the following theorem:

Theorem 2.1

$$B(0, -n; x) = \ln \frac{x}{1-x} + \sum_{i=1}^{n} \frac{(1-x)^{-i}}{i} - \phi(n)\,, \tag{4}$$

for $n = 1, 2, \ldots,$ *where*

$$\phi(n) = \sum_{i=1}^{n} \frac{1}{i}$$

is the $n-th$ *harmonic number.*

Proof. We have

$$\begin{aligned}
\int_{\epsilon}^{x} t^{-1}(1-t)^{-n-1}\,dt &= \int_{1-x}^{1-\epsilon} t^{-n-1}(1-t)^{-1}\,dt \\
&= \int_{1-x}^{1-\epsilon} \left[(1-t)^{-1} + \sum_{i=1}^{n+1} t^{-i} \right] dt \\
&= \ln x - \ln \epsilon - \ln(1-x) + \ln(1-\epsilon) - \sum_{i=1}^{n} \frac{1}{i} \left[\frac{1}{(1-\epsilon)^i} - \frac{1}{(1-x)^i} \right]
\end{aligned}$$

and it follows that

$$\begin{aligned}
B(0, -n; x) &= \mathrm{N}\!-\!\lim_{\epsilon \to 0} \int_{\epsilon}^{x} t^{-1}(1-t)^{-n-1}\,dt \\
&= \ln \frac{x}{1-x} - \mathrm{N}\!-\!\lim_{\epsilon \to 0} [\ln \epsilon - \ln(1-\epsilon)] - \lim_{\epsilon \to 0} \sum_{i=1}^{n} \frac{1}{i} \left[\frac{1}{(1-\epsilon)^i} - \frac{1}{(1-x)^i} \right] \\
&= \ln \frac{x}{1-x} - \sum_{i=1}^{n} \frac{1}{i} \left[1 - \frac{1}{(1-x)^i} \right].
\end{aligned}$$

Equation (4) follows. □

Equation (4) corrects a result given in [6].

Theorem 2.2

$$B_{1,0}(1,-1;x) = (1-x)^{-1}\ln x - \ln\frac{x}{1-x} \tag{5}$$

and

$$nB_{1,0}(1,-n;x) = (1-x)^{-n}\ln x - \ln\frac{x}{1-x} - \sum_{i=1}^{n-1}\frac{(1-x)^{-i}}{i} + \phi(n-1) \tag{6}$$

for $n = 2, 3, \ldots$.

Proof. We have

$$\begin{aligned} n\int_\epsilon^x \ln t(1-t)^{-n-1}\,dt &= \int_\epsilon^x \ln t\,d(1-t)^{-n} \\ &= (1-x)^{-n}\ln x - (1-\epsilon)^{-n}\ln\epsilon - \int_\epsilon^x t^{-1}(1-t)^{-n}\,dt \end{aligned}$$

and it follows that

$$\begin{aligned} nB_{1,0}(1,-n;x) &= \mathrm{N}-\lim_{\epsilon\to 0} n\int_\epsilon^x \ln t(1-t)^{-n-1}\,dt \\ &= (1-x)^{-n}\ln x - \mathrm{N}-\lim_{\epsilon\to 0}(1-\epsilon)^{-n}\ln\epsilon - \mathrm{N}-\lim_{\epsilon\to 0}\int_\epsilon^x t^{-1}(1-t)^{-n}\,dt \\ &= (1-x)^{-n}\ln x - B(0,-n+1;x)\,. \end{aligned}$$

Equation (5) now follows on using equation (1) and equation (6) follows on using equation (4) for $n = 2, 3, \ldots$. □

Theorem 2.3

$$B_{0,1}(-1,1;x) = -\ln\frac{x}{1-x} - \frac{\ln(1-x)}{x} - 1 \tag{7}$$

and

$$nB_{0,1}(-n,1;x) = -\ln\frac{x}{1-x} - \frac{\ln(1-x)}{x^n} - n^{-1} + \sum_{i=1}^{n-1}\frac{x^{-i}}{i}\,, \tag{8}$$

for $n = 2, 3, \ldots$.

Proof. We have

$$\begin{aligned} n\int_\epsilon^x t^{-n-1}\ln(1-t)\,dt &= -\int_\epsilon^x \ln(1-t)\,dt^{-n} \\ &= \epsilon^{-n}\ln(1-\epsilon) - x^{-n}\ln(1-x) - \int_\epsilon^x t^{-n}(1-t)^{-1}\,dt \end{aligned}$$

and it follows that

$$\begin{aligned} nB_{0,1}(-n,1;x) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0} n\int_\epsilon^x t^{-n-1}\ln(1-t)\,dt \\ &= -n^{-1} - x^{-n}\ln(1-x) - B(-n+1,0;x)\,. \end{aligned}$$

Equation (7) now follows on using equation (1) and equation (8) follows on using equation (3) for $n = 2, 3, \ldots$. □

Theorem 2.4

$$B_{0,1}(-n,r+1;x) = \sum_{i=0}^{r} \frac{(-1)^{i-1}}{n-i}\binom{r}{i}\left[x^{i-n}\ln(1-x) + \ln\frac{x}{1-x} - \sum_{k=1}^{n-i-1}\frac{x^{-k}}{k} + \frac{1}{n-i}\right] \tag{9}$$

for $n = 1, 2, \ldots$ and $r = 0, 1, 2, \ldots, n-1$, the sum $\sum_{k=1}^{n-i-1}\frac{x^{-k}}{k}$ being empty when $i = n-1$,

$$B_{0,1}(-n,n+1;x) = \sum_{i=0}^{n-1} \frac{(-1)^{i-1}}{n-i}\binom{n}{i}\left[x^{i-n}\ln(1-x) + \ln\frac{x}{1-x} - \sum_{k=1}^{n-i-1}\frac{x^{-k}}{k} + \frac{1}{n-i}\right] - (-1)^n\sum_{i=1}^{\infty}\frac{x^i}{i^2}\,, \tag{10}$$

for $n = 1, 2, \ldots$, the sum $\sum_{k=1}^{n-i-1}\frac{x^{-k}}{k}$ being empty when $i = n-1$ and

$$\begin{aligned} B_{0,1}(-n,r+1;x) = {} & \sum_{i=0}^{n-1} \frac{(-1)^{i-1}}{n-i}\binom{r}{i}\left[x^{i-n}\ln(1-x) + \ln\frac{x}{1-x} - \sum_{k=1}^{n-i-1}\frac{x^{-k}}{k} + \frac{1}{n-i}\right] + (-1)^n\binom{r}{n}\sum_{i=1}^{\infty}\frac{x^i}{i^2} \\ & + \sum_{i=n+1}^{r}\frac{(-1)^i}{i-n}\binom{r}{i}\left[x^{i-n}\ln(1-x) - \ln(1-x) - \sum_{k=1}^{i-n}\frac{x^k}{k}\right] \end{aligned} \tag{11}$$

for $n = 1, 2, \ldots$ and $r = n+1, n+2, \ldots$.

Proof. Integrating by parts, we have

$$\begin{aligned} \int_\epsilon^x t^{-n-1}(1-t)^r\ln(1-t)\,dt &= \sum_{i=0}^{r}(-1)^i\binom{r}{i}\int_\epsilon^x t^{i-n-1}\ln(1-t)\,dt \\ = \sum_{i=0}^{r}\frac{(-1)^i}{i-n}\binom{r}{i} & \left[x^{i-n}\ln(1-x) - \epsilon^{i-n}\ln(1-\epsilon) + \int_\epsilon^x t^{i-n}(1-t)^{-1}\,dt\right], \end{aligned} \tag{12}$$

for $r = 0, 1, 2, \ldots, n-1$.

Since

$$\epsilon^{i-n} \ln(1-\epsilon) = -\sum_{j=1}^{\infty} \frac{\epsilon^{i+j-n}}{j},$$

it follows that

$$\mathrm{N}\!-\!\lim_{\epsilon\to 0} \epsilon^{i-n} \ln(1-\epsilon) = \begin{cases} -(n-i)^{-1}, & 0 \le i \le n-1, \\ 0, & i \ge n. \end{cases} \tag{13}$$

It now follows from equations (12) and (13) that

$$\begin{aligned} B_{0,1}(-n, r+1; x) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0} \int_{\epsilon}^{x} t^{-n-1} \ln(1-t)(1-t)^r \, dt \\ &= \sum_{i=0}^{r} \frac{(-1)^i}{i-n} \binom{r}{i} \left[x^{i-n} \ln(1-x) + B(-n+i+1, 0; x) - \frac{1}{i-n} \right] \\ &= \sum_{i=0}^{r} \frac{(-1)^{i-1}}{n-i} \binom{r}{i} \left[x^{i-n} \ln(1-x) + \ln\frac{x}{1-x} - \sum_{k=1}^{n-i-1} \frac{x^{-k}}{k} + \frac{1}{n-i} \right], \end{aligned}$$

on using equation (3), proving equation (9).

For the case $r = n$, equation (12) has to be replaced by the equation

$$\begin{aligned} \int_{\epsilon}^{x} t^{-n-1}(1-t)^n \ln(1-t)\, dt &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \int_{\epsilon}^{x} t^{i-n-1} \ln(1-t)\, dt \\ &= \sum_{i=0}^{n-1} \frac{(-1)^i}{i-n} \binom{n}{i} \left[x^{i-n} \ln(1-x) - \epsilon^{i-n} \ln(1-\epsilon) + \int_{\epsilon}^{x} t^{i-n}(1-t)^{-1}\, dt \right] \\ &\quad + (-1)^n \int_{\epsilon}^{x} t^{-1} \ln(1-t)\, dt\,. \end{aligned} \tag{14}$$

It now follows from equations (13) and (14) that

$$\begin{aligned} B_{0,1}(-n, n+1; x) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0} \int_{\epsilon}^{x} t^{-n-1} \ln(1-t)(1-t)^n \, dt \\ &= \sum_{i=0}^{n-1} \frac{(-1)^i}{i-n} \binom{n}{i} \left[x^{i-n} \ln(1-x) + B(-n+i+1, 0; x) - \frac{1}{i-n} \right] \\ &\quad + (-1)^n B_{0,1}(0, 1; x)\,. \end{aligned} \tag{15}$$

Now

$$\int_{\epsilon}^{x} t^{-1} \ln(1-t)\, dt = -\sum_{i=1}^{\infty} \int_{\epsilon}^{x} \frac{t^{i-1}}{i}\, dt = -\sum_{i=1}^{\infty} \frac{x^i - \epsilon^i}{i^2}$$

and so

$$\mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^x t^{-1}\ln(1-t)\,dt = -\sum_{i=1}^\infty \frac{x^i}{i^2} = B_{0,1}(0,1;x)\,. \tag{16}$$

It now follows from equations (15) and (16) that

$$B_{0,1}(-n,n+1;x) = \sum_{i=0}^{n-1}\frac{(-1)^{i-1}}{n-i}\binom{n}{i}\left[x^{i-n}\ln(1-x)+\ln\frac{x}{1-x}\right.$$
$$\left.-\sum_{k=1}^{n-i-1}\frac{x^{-k}}{k}+\frac{1}{n-i}\right]-(-1)^n\sum_{i=1}^\infty\frac{x^i}{i^2}\,,$$

proving equation (10).

When $r>n$, equation (12) has to be replaced by

$$\int_\epsilon^x t^{-n-1}(1-t)^r\ln(1-t)\,dt = \sum_{i=0}^r(-1)^i\binom{r}{i}\int_\epsilon^x t^{i-n-1}\ln(1-t)\,dt$$
$$=\sum_{i=0}^{n-1}\frac{(-1)^i}{i-n}\binom{r}{i}\left[x^{i-n}\ln(1-x)-\epsilon^{i-n}\ln(1-\epsilon)+\int_\epsilon^x t^{i-n}(1-t)^{-1}\,dt\right]$$
$$+(-1)^n\binom{r}{n}\int_\epsilon^x t^{-1}\ln(1-t)\,dt$$
$$+\sum_{i=n+1}^r\frac{(-1)^i}{i-n}\binom{r}{i}\left[x^{i-n}\ln(1-x)-\epsilon^{i-n}\ln(1-\epsilon)+\int_\epsilon^x t^{i-n}(1-t)^{-1}\,dt\right]\,. \tag{17}$$

It now follows from equations (16) and (17) that

$$B_{0,1}(-n,r+1;x) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^x t^{-n-1}(1-t)^r\ln(1-t)\,dt$$
$$=\sum_{i=0}^{n-1}\frac{(-1)^{i-1}}{n-i}\binom{r}{i}\left[x^{i-n}\ln(1-x)+\ln\frac{x}{1-x}-\sum_{k=1}^{n-i-1}\frac{x^{-k}}{k}+\frac{1}{n-i}\right]$$
$$-(-1)^n\binom{r}{n}\sum_{i=1}^\infty\frac{x^i}{i^2}+\sum_{i=n+1}^r\frac{(-1)^i}{i-n}\binom{r}{i}\left[x^{i-n}\ln(1-x)-\ln(1-x)-\sum_{k=1}^{i-n}\frac{x^k}{k}\right],$$

since it was proved in [6] that

$$B(n,0;x) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^x t^{n-1}(1-t)^{-1}\,dt = -\ln(1-x)-\sum_{k=1}^{n-1}\frac{x^k}{k}$$

for $n=1,2,\ldots$. Equation (11) is now proved. □

For further related results see [1], [2], [3] and [5].

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Laure Cardoulis

Existence of solutions for a system involving the (2,q)-Laplacian operator in a bounded domain

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ABSTRACT. In this paper we study the existence of a non trivial weak solution for a system involving the Laplacian operator and the q-Laplacian operator in a bounded domain Ω of $\mathbb{R}^N$ with sufficiently smooth boundary.

KEY WORDS. (2,q)-Laplacian operator, system, existence of solutions

1 Introduction

We consider in this paper the following system for $i = 1, \cdots, m$,

$$\begin{cases} -\Delta u_i - \Delta_q u_i + w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = g_i(., u_1, \cdots, u_m) \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega. \end{cases} \quad (S, q, g)$$

where Ω is a bounded domain with sufficiently smooth boudary, $\Omega \subset \mathbb{R}^N$.
We recall that the q-Laplacian operator is defined by $\Delta_q \phi = div(|\nabla\phi|^{q-2}\nabla\phi)$ and we suppose $q > 2$ in the whole paper. We study the existence of a weak non-trivial solution $u = (u_1, \cdots, u_m) \in W$ for the system (S, q, g) where the variational space is $W = (W_0^{1,q}(\Omega))^m$, $W_0^{1,q}(\Omega)$ being the usual Sobolev space endowed with the norm $\|\phi\|_0^{1,q}(\Omega) = (\int_\Omega |\nabla\phi|^q)^{1/q}$. We also denote $H = (W_0^{1,2}(\Omega))^m$ and $\|.\|_W$, $\|.\|_H$, the norms on W and H ($\|u\|_W = (\sum_{i=1}^m \|u_i\|^q_{W_0^{1,q}(\Omega)})^{1/q}$).

We assume throughout all the paper that the bounded functions a_{ij}, w_i (for $i, j = 1, \cdots, m$) satisfy the following hypothesis

Assumption 1.1 **i)** *$a_{ij}, w_i \in L^\infty(\Omega)$, $a_{ii} \geq 0$, $w_i \geq 0$ a. e. on Ω.*

ii) *The matrix $A = (a_{ij})$ is symmetric and satisfies ${}^t\xi A\xi \geq 0$ for all ${}^t\xi = (\xi_1, \cdots, \xi_m) \in \mathbb{R}^m$.*

Note that the above Assumption 1.1ii) is satisfied when the matrix A is a positive definite one. Introduce now the following functionals for $u = (u_1, \cdots, u_m) \in W$

$$H_1(u) = \sum_{i=1}^{m} \int_{\Omega} (|\nabla u_i|^2 + a_{ii} u_i^2 + \sum_{j=1, i \neq j}^{m} a_{ij} u_j u_i), \tag{1.1}$$

and

$$H_2(u) = \sum_{i=1}^{m} \int_{\Omega} (|\nabla u_i|^q + w_i |u_i|^q). \tag{1.2}$$

Since A is symmetric then $H_1(u) = \sum_{i=1}^{m} \int_{\Omega} (|\nabla u_i|^2 + a_{ii} u_i^2 + 2 \sum_{j=1, i<j}^{m} a_{ij} u_j u_i)$.
Note that $(H_1(u))^{1/2}$ and $(H_2(u))^{1/q}$ define norms on H and W equivalent to the norms $\|.\|_H$ and $\|.\|_W$ respectively.

We consider different cases for the functions g_i : in the second section we deal with $g_i(., u_1, \cdots, u_m) := h_i \in W^{-1,q'}(\Omega)$ the dual space of $W_0^{1,q}(\Omega)$ with $\frac{1}{q} + \frac{1}{q'} = 1$. In the third section, we define $g_i(., u_1, \cdots, u_m) := m_i |u_i|^{q-2} u_i$ where the functions m_i are bounded and indefinite. In the fourth section we consider the case $g_i(., u_1, \cdots, u_m) := \lambda f_i |u_i|^{\gamma-2} u_i$ where the functions f_i are still bounded and undefinite, λ is a positive real parameter and the coefficient γ satisfies some hypotheses in which $\gamma < q$.

In each of the precedent cases, the system (S, q, g) will be rewritten under a variational form with $I(u)$ an adapted Euler functional defined in W and the existence of weak solutions for the system (S, q, g) will be equivalent to the existence of critical points for this functional I. In the second and third sections, we will mimimize the Euler functional I using either standard arguments (cf. Theorem I.1.2 in [18]) or the Moutain-Pass Theorem. In the third section, we will use the principal eigenvalue $\lambda_{1,q,\rho}$ of the q-Laplacian operator associated with a weight ρ whereas in the fourth section we will define a characteristic value λ_1^+ (see (4.7)).

Equations and systems with the p-Laplacian have been widely studied for the existence of solutions or the maximum and antimaximum principles (see for examples [3, 9–13], see also [14] for the fibering procedure). These last few years, equations with the (p,q)-Laplacian have been studied (see for examples [4, 6, 15, 19, 21] in a bounded domain and [5] in $\mathbb{R}^N$). Authors study the existence of solutions (sometimes the sign of these solutions and generalized eigenvalue problems) mainly by minimization of the energy functional either by standard arguments or the mountain-pass geometry, also by using the method of sub- and super-solutions. The case of the (2,q)-Laplacian arises in quantum physics (see [2]). A few systems with two equations have been studied (see for example [16] for a system with two equations, one with the p-Laplacian and the other one with the q-Laplacian ; see also [20] for a system of two equations with the (p,q)-Laplacian with critical nonlineariries) but as far as we know, there is no system with n equations for the (2,q)-Laplacian studied yet.

This paper is organised as follows: in section 2, we use standard arguments for minimizing the functional I when we consider the case where $g_i(.,u_1,\cdots,u_m) := h_i \in W^{-1,q'}(\Omega)$. In section 3 (in the case of $g_i(.,u_1,\cdots,u_m) := m_i|u_i|^{q-2}u_i$ and $q < 2^*$ where $2^* = \frac{2N}{N-2}$ if $N > 2$ and $2^* = \infty$ if $N \leq 2$), first we recall some results of the existence of the principal eigenvalue for the q-Laplacian operator associated with a bounded weight (and the existence of a positive eigenfunction associated with). Then we use the Mountain-Pass Theorem in order to get the existence of a non-trivial solution for our system. Finally in section 4 (when $g_i(.,u_1,\cdots,u_m) := \lambda f_i|u_i|^{\gamma-2}u_i$ with $2 < \gamma < q$ and $\gamma < 2^*$ where $2^* = \frac{2N}{N-2}$ if $N > 2$ and $2^* = \infty$ if $N \leq 2$), first we follow a method introduced by Cherfils-Il'Yasov in [7] for one equation involving the (p-q)-Laplacian operator to define a characteristic value λ_1^+. Then we get the existence of a non-trivial solution by means of global minimization of the Euler functional.

2 First case: $g_i(.,u_1,\cdots,u_m) := h_i \in W^{-1,q'}(\Omega)$

In this case the system (S,q,g) is rewritten under the following form

$$\begin{cases} -\Delta u_i - \Delta_q u_i + w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = h_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

with $h_i \in W^{-1,q'}(\Omega)$ for each $i = 1,\cdots,m$. Recall that $-\Delta_q$ may be seen acting from $W_0^{1,q}(\Omega)$ into $W^{-1,q'}(\Omega)$ with $\frac{1}{q} + \frac{1}{q'} = 1$ by

$$< -\Delta_q \phi, \psi >_{q',q} = \int_\Omega |\nabla\phi|^{q-2}\nabla\phi\cdot\nabla\psi \text{ for all } \phi,\psi \in W_0^{1,q}(\Omega)$$

(see [8, 17]) where $< .,. >_{q',q}$ denotes the duality mapping between $W^{-1,q'}(\Omega)$ and $W_0^{1,q}(\Omega)$. Therefore the Euler functional is, for $u = (u_1,\cdots,u_m) \in W$,

$$I(u) = \frac{1}{2}H_1(u) + \frac{1}{q}H_2(u) - \sum_{i=1}^m < h_i, u_i >_{q',q} . \tag{2.2}$$

The result of the existence of solution for the system (2.1) is the following.

Theorem 2.1 *Assume that Assumption 1.1 is satisfied and that $h_i \in W^{-1,q'}(\Omega)$ for each $i = 1,\cdots,m$. Then the system (2.1) has a unique solution.*

Proof. The functional $I : W \to \mathbb{R}$ defined by (2.2) is weakly lower semi-continuous by the compactness of the embedding of W to $(L^q(\Omega))^m$ and $(L^2(\Omega))^m$ and of class C^1 on W. Moreover this functional I is also coercive. Indeed by the Young's inequality we have

$$|< h_i, u_i >_{q',q}| \leq \|h_i\|_{W^{-1,q'}(\Omega)}\|u_i\|_{W_0^{1,q}(\Omega)} \leq \frac{1}{2q}\|u_i\|^q_{W_0^{1,q}(\Omega)} + C\|h_i\|^{q'}_{W^{-1,q'}(\Omega)}$$

with $C > 0$, C independent of u. And since $H_1(u) \geq 0$ and $H_2(u) \geq \|u\|_W$ we get that

$$I(u) \geq \frac{1}{2q}\|u\|_W - C\sum_{i=1}^m \|h_i\|^{q'}_{W^{-1,q'}(\Omega)}.$$

Therefore the functional I has a gobal minimizer (cf.[18, Theorem I.1.2]) and the system (2.1) has a solution.
Let us prove now the uniqueness of the solution. Suppose on the contrary that there exist two distinct solutions $u = (u_1, \cdots, u_m) \in W$ and $v = (v_1, \cdots, v_m) \in W$ for (2.1), so there exists k such that $u_k \neq v_k$. Since

$$(I'(u) - I'(v)) \cdot (u - v) = I'(u) \cdot u - I'(v) \cdot u - I'(u) \cdot v + I'(v) \cdot v = 0,$$

we have

$$\sum_{i=1}^m \int_\Omega |\nabla u_i|^2 + \sum_{i,j=1}^m \int_\Omega a_{ij}u_ju_i + \sum_{i=1}^m \int_\Omega (|\nabla u_i|^q + w_i|u_i|^q)$$
$$-\sum_{i=1}^m \int_\Omega \nabla v_i \cdot \nabla u_i - \sum_{i,j=1}^m \int_\Omega a_{ij}v_ju_i - \sum_{i=1}^m \int_\Omega (|\nabla v_i|^{q-2}\nabla v_i \cdot \nabla u_i + w_i|v_i|^{q-2}v_iu_i) = 0$$

and on the other hand

$$\sum_{i=1}^m \int_\Omega \nabla u_i \cdot \nabla v_i + \sum_{i,j=1}^m \int_\Omega a_{ij}u_jv_i + \sum_{i=1}^m \int_\Omega (|\nabla u_i|^{q-2}\nabla u_i \cdot \nabla v_i + w_i|u_i|^{q-2}u_iv_i)$$
$$-\sum_{i=1}^m \int_\Omega |\nabla v_i|^2 - \sum_{i,j=1}^m \int_\Omega a_{ij}v_jv_i - \sum_{i=1}^m \int_\Omega (|\nabla v_i|^q + w_i|v_i|^q) = 0.$$

So we get

$$\sum_{i=1}^m \int_\Omega \nabla u_i \cdot (\nabla u_i - \nabla v_i) + \sum_{i,j=1}^m \int_\Omega a_{ij}u_j(u_i - v_i) + \sum_{i=1}^m \int_\Omega |\nabla u_i|^{q-2}\nabla u_i \cdot (\nabla u_i - \nabla v_i)$$
$$+\sum_{i=1}^m \int_\Omega w_i|u_i|^{q-2}u_i(u_i - v_i) - \sum_{i=1}^m \int_\Omega \nabla v_i \cdot (\nabla u_i - \nabla v_i) - \sum_{i,j=1}^m \int_\Omega a_{ij}v_j(u_i - v_i)$$
$$-\sum_{i=1}^m \int_\Omega |\nabla v_i|^{q-2}\nabla v_i \cdot (\nabla u_i - \nabla v_i) - \sum_{i=1}^m \int_\Omega w_i|v_i|^{q-2}v_i(u_i - v_i) = 0.$$

Thus

$$\sum_{i=1}^{m}\int_{\Omega}|\nabla u_i-\nabla v_i|^2+\sum_{i=1}^{m}\int_{\Omega}(|\nabla u_i|^{q-2}\nabla u_i-|\nabla v_i|^{q-2}\nabla v_i)\cdot(\nabla u_i-\nabla v_i)$$

$$+\sum_{i,j=1}^{m}\int_{\Omega}a_{ij}(u_j-v_j)(u_i-v_i)+\sum_{i=1}^{m}\int_{\Omega}w_i(|u_i|^{q-2}u_i-|v_i|^{q-2}v_i)(u_i-v_i)=0.$$

The last equality can be rewritten under the following form with the duality product $<.,.>_{q',q}$

$$\sum_{i=1}^{m}<-\Delta u_i+\Delta v_i,u_i-v_i>_{2,2}+\sum_{i=1}^{m}<-\Delta_q u_i+\Delta_q v_i,u_i-v_i>_{q',q}$$

$$+\sum_{i,j=1}^{m}<a_{ij}(u_j-v_j),u_i-v_i>_{2,2}+\sum_{i=1}^{m}<w_i(|u_i|^{q-2}u_i-|v_i|^{q-2}v_i),u_i-v_i>_{q',q}=0.$$

Moreover a consequence of the strict convexity of the spaces $W_0^{1,2}(\Omega)$ and $W_0^{1,q}(\Omega)$ is that the duality mappings $-\Delta$ and $-\Delta_q$ are strictly monotone. So from $u_k\neq v_k$ we get

$$<-\Delta u_k+\Delta v_k,u_k-v_k>_{2,2}>0,$$

and

$$<-\Delta_q u_k+\Delta_q v_k,u_k-v_k>_{q',q}\geq(\|u_k\|^{q-1}_{W_0^{1,q}(\Omega)}-\|v_k\|^{q-1}_{W_0^{1,q}(\Omega)})(\|u_k\|_{W_0^{1,q}(\Omega)}-\|v_k\|_{W_0^{1,q}(\Omega)})\geq 0$$

since $x\mapsto x^{q-1}$ is increasing on $[0,\infty)$ (and even $<-\Delta_q u_k+\Delta_q v_k,u_k-v_k>_{q',q}>0$ from [8, Proposition 1]).
Thus

$$\sum_{i=1}^{m}<-\Delta u_i+\Delta v_i,u_i-v_i>_{2,2}+\sum_{i=1}^{m}<-\Delta_q u_i+\Delta_q v_i,u_i-v_i>_{q',q}>0.$$

Furthermore, since the function $x\mapsto|x|^{q-2}x$ is increasing and $w_i\geq 0$, we have

$$\sum_{i=1}^{m}<w_i(|u_i|^{q-2}u_i-|v_i|^{q-2}v_i),u_i-v_i>_{q',q}\geq 0.$$

Finally from Assumption 1.1,

$$\sum_{i,j=1}^{m}<a_{ij}(u_j-v_j),u_i-v_i>_{2,2}\geq 0.$$

Therefore we get a contradiction. □

Remark: We can generalize Theorem 2.1 replacing the 2-Laplacian operator by the p-Laplacian with $2 < p < q$, that for the following system

$$\begin{cases} -\Delta_p u_i - \Delta_q u_i + w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = h_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

and even for

$$\begin{cases} -\Delta_p u_i - \Delta_q u_i + b_i|u_i|^{p-2}u_i + w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = h_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

under the additional hypothesis that the bounded functions b_i, $i = 1, \cdots, m$ are non-negative.

3 Second case: $g_i(., u_1, \cdots, u_m) := m_i|u_i|^{q-2}u_i$

In this section we assume that

Assumption 3.1 *$q < 2^*$ where $2^* = \frac{2N}{N-2}$ if $N > 2$ and $2^* = \infty$ if $N \leq 2$,*

and we rewrite the system (S, q, g) under the following form:
for $i = 1, \cdots, m$,

$$\begin{cases} -\Delta u_i - \Delta_q u_i + w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = m_i|u_i|^{q-2}u_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega. \end{cases} \quad (3.1)$$

Note that the decomposition with the weights $c_i := m_i - w_i$ does not necessarily coincide with the decomposition $c_i = c_{i+} - c_{i-}$ where $c_{i+} = \max(c_i, 0)$ and $c_{i-} = max(-c_i, 0)$. Define now for $u = (u_1, \cdots, u_m) \in W$ the functional

$$M(u) = \sum_{i=1}^m \int_\Omega m_i|u_i|^q. \quad (3.2)$$

The Euler functional associated with (3.1) is consequently for $u = (u_1, \cdots, u_m) \in W$,

$$I(u) = \frac{1}{2}H_1(u) + \frac{1}{q}H_2(u) - \frac{1}{q}M(u). \quad (3.3)$$

First let us recall the usual weighted eigenvalue problem for the q-Laplacian:

$$\begin{cases} -\Delta_q u = \lambda\rho|u|^{q-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \quad (3.4)$$

with a bounded weight function ρ and a real parameter λ. It is said that λ is an eigenvalue of the q-Laplacian associated with the weight ρ if (3.4) has a non-trivial solution u which is called an eigenfunction associated with λ. It is well known (see [1]) that if the Lebesgue measure of $\{x \in \Omega, \rho(x) > 0\}$ is positive, then the first positive eigenvalue $\lambda_{1,q,\rho}$ of $-\Delta_q$ with weight function ρ is obtained by the Rayleight quotient

$$\lambda_{1,q,\rho} = \inf\{\frac{\int_\Omega |\nabla u|^q}{\int_\Omega \rho|u|^q}; u \in W_0^{1,q}(\Omega), \int_\Omega \rho|u|^q > 0\}. \tag{3.5}$$

Moreover, $\lambda_{1,q,\rho}$ has a positive eigenfunction $\phi_{1,q,\rho} \in C_0^{1,\alpha_q}(\overline{\Omega})$ (for some $\alpha_q \in (0,1)$). Assume in this section that

Assumption 3.2 **i)** *For all* $i = 1, \cdots, m$, $m_i \in L^\infty(\Omega)$,

ii) *For all* $i = 1, \cdots, m$, *the real* 1 *is not an eigenvalue of the q-Laplacian with the weight* $m_i - w_i$.

Assume also in this section that either Assumption 3.3 or Assumption 3.4 holds

Assumption 3.3 *There exists* $k \in \{1, \cdots, m\}$ *such that:*

$$meas\{x \in \Omega, (m_k - w_k)(x) > 0\} \neq 0 \text{ and } \lambda_{1,q,m_k - w_k} < 1.$$

Assumption 3.4 *There exist* $k, l \in \{1, \cdots, m\}$, $k \neq l$ *such that:*

$$meas\{x \in \Omega, (m_k - w_k)(x) > 0\} \neq 0 \text{ and } \lambda_{1,q,m_k - w_k} + \int_\Omega (w_l - m_l)|\phi_{1,q,m_k - w_k}|^q < 0$$

with $\phi_{1,q,m_k - w_k}$ *the normalized eigenfunction associated with* $\lambda_{1,q,m_k - w_k}$.

Note that Assumption 3.4 is satisfed when $\lambda_{1,q,m_k - w_k}(m_k - w_k) + w_l - m_l < 0$ a. e. in Ω. Our aim is to study the existence of a weak solution for the system (3.1) by minimizing the functional I defined by (3.3). As in section 2, the functional I is weakly lower semi-continous on W but may be no more coercive so we cannot use standard arguments for minimizing I. First, we prove that any Palais-Smale sequence is bounded in W and has a strong convergent subsequence. Then we are able to apply the Mountain-Pass Lemma and Assumptions 3.3 or 3.4 allow us to get a non-trivial solution.

We say that $(u_n) \subset W$, $u_n = (u_{1n}, \cdots, u_{mn})$, is a Palais-Smale sequence if it satisfies the following conditions

$$|I(u_n)| \leq D \text{ for all } n \in \mathbb{N} \text{ and } \|I'(u_n)\|_{W^*} \to 0 \text{ as } n \to \infty \tag{3.6}$$

with some constant $D > 0$, W^* being the dual space of W.

Lemma 3.1 *Assume that Assumptions 1.1 and 3.2 are satisfied. If* $(u_n) \subset W$, $u_n = (u_{1n}, \cdots, u_{mn})$, *is a Palais-Smale sequence, then* (u_n) *is bounded in* W.

Proof. Let $(u_n) \subset W$, $u_n = (u_{1n}, \cdots, u_{mn})$, be a Palais-Smale sequence. We want to prove that $(\|u_n\|_W)_n$ is bounded or equivalently that $(H_2(u_n))_n$ is bounded. But

$$\frac{1}{q}H_2(u_n) = I(u_n) - \frac{1}{2}H_1(u_n) + \frac{1}{q}M(u_n) \le D + \frac{1}{q}M(u_n) \le D + C\|u_n\|^q_{(L^q(\Omega))^m} \tag{3.7}$$

with C a positive constant, C independent of u_n (since the functions m_i are bounded in the functional $M(u)$ defined by (3.2)). So it is sufficient to show that $(\|u_n\|_{(L^q(\Omega))^m})$ is bounded. We adapt ideas from [19]. Assume on the contrary that $\alpha_n := \|u_n\|_{(L^q(\Omega))^m} \to_{n\to\infty} \infty$ (for a subsequence) and denote $v_n = \frac{1}{\alpha_n}u_n = (v_{1n}, \cdots, v_{mn})$. From (3.7), we deduce that $(\|v_n\|_W)$ is bounded and from the compact embedding of W into $(L^q(\Omega))^m$ we get the existence of $v_0 = (v_{01}, \cdots, v_{0m}) \in W$ such that (v_n) converges to v_0, strongly in $(L^q(\Omega))^m$ and weakly in W (for a subsequence).

• Now we prove that (v_n) converges strongly to v_0 in W. Indeed by taking $\phi_n := \frac{1}{\alpha_n^{q-1}}(v_n - v_0)$, we obtain

$$\begin{aligned} I'(u_n).\phi_n &= \frac{1}{\alpha_n^{q-1}}\sum_{i=1}^m \int_\Omega (\nabla u_{in}.\nabla(v_{in} - v_{0i}) + a_{ii}u_{in}(v_{in} - v_{0i})) \\ &+\frac{1}{\alpha_n^{q-1}}\sum_{i=1}^m \int_\Omega (|\nabla u_{in}|^{q-2}\nabla u_{in}.\nabla(v_{in} - v_{0i}) + w_i|u_{in}|^{q-2}u_{in}(v_{in} - v_{0i})) \\ &+ \frac{1}{\alpha_n^{q-1}}\sum_{i,j;i\ne j}\int_\Omega a_{ij}u_{jn}(v_{in} - v_{0i}) - \frac{1}{\alpha_n^{q-1}}\sum_{i=1}^m\int_\Omega m_i|u_{in}|^{q-2}u_{in}(v_{in} - v_{0i}). \end{aligned} \tag{3.8}$$

But $u_n = \alpha_n v_n$ so (3.8) becomes

$$\begin{aligned} I'(u_n).\phi_n &= \frac{1}{\alpha_n^{q-2}}\sum_{i=1}^m \int_\Omega (\nabla v_{in}.\nabla(v_{in} - v_{0i}) + a_{ii}v_{in}(v_{in} - v_{0i})) \\ &+\sum_{i=1}^m \int_\Omega (|\nabla v_{in}|^{q-2}\nabla v_{in}.\nabla(v_{in} - v_{0i}) + w_i|v_{in}|^{q-2}v_{in}(v_{in} - v_{0i})) \\ &+ \frac{1}{\alpha_n^{q-2}}\sum_{i,j;i\ne j}\int_\Omega a_{ij}v_{jn}(v_{in} - v_{0i}) - \sum_{i=1}^m\int_\Omega m_i|v_{in}|^{q-2}v_{in}(v_{in} - v_{0i}). \end{aligned} \tag{3.9}$$

Note that $|I'(u_n).\phi_n| \le \|I'(u_n)\|_{W^*}\|\phi_n\|_W = \|I'(u_n)\|_{W^*}\frac{1}{\alpha_n^{q-1}}\|v_n - v_0\|_W$ so $I'(u_n).\phi_n \to_{n\to\infty} 0$ from (3.6), $\alpha_n \to_{n\to\infty} \infty$ and $(\|v_n\|_W)$ bounded. Moreover, since the functions a_{ij}, w_i, m_i are bounded there exists a positive constant, denoting C at each step, such that

$$|\int_\Omega a_{ij}v_{jn}(v_{in} - v_{0i}))| \le C\|v_{jn}\|_{L^2(\Omega)}\|v_{in} - v_{0i}\|_{L^2(\Omega)} \le C\|v_n\|_W\|v_n - v_0\|_{(L^q(\Omega))^m}$$

and therefore

$$\int_\Omega a_{ij} v_{jn}(v_{in} - v_{0i})) \to 0 \text{ as } n \to \infty. \tag{3.10}$$

By the same way, for $b_i = w_i$ or $b_i = m_i$,

$$|\int_\Omega b_i |v_{in}|^{q-2} v_{in}(v_{in} - v_{0i}))| \leq C(\int_\Omega |v_{in}|^q)^{\frac{q-1}{q}} (\int_\Omega |v_{in} - v_{0i}|^q)^{1/q} \leq C\|v_n\|_W^{q-1} \|v_n - v_0\|_{(L^q(\Omega))^m}$$

so

$$\int_\Omega b_i |v_{in}|^{q-2} v_{in}(v_{in} - v_{0i})) \to 0 \text{ as } n \to \infty. \tag{3.11}$$

Recall that $< .,. >_{q',q}$ is the duality product between $W^{-1,q'}(\Omega)$ and $W_0^{1,q}(\Omega)$ with $\frac{1}{q} + \frac{1}{q'} = 1$. From (3.9), (3.10), (3.11), we deduce that

$$\frac{1}{\alpha_n^{q-2}} \sum_{i=1}^m < -\Delta v_{in}, v_{in} - v_{0i} >_{2,2} + \sum_{i=1}^m < -\Delta_q v_{in}, v_{in} - v_{0i} >_{q',q} \to 0 \text{ as } n \to \infty. \tag{3.12}$$

Moreover we have (see also the proof of Theorem 2.1)

$$< -\Delta_q v_{in} + \Delta_q v_{0i}, v_{in} - v_{0i} >_{q',q} \geq (\|v_{in}\|^{q-1}_{W_0^{1,q}(\Omega)} - \|v_{0i}\|^{q-1}_{W_0^{1,q}(\Omega)})(\|v_{in}\|_{W_0^{1,q}(\Omega)} - \|v_{0i}\|_{W_0^{1,q}(\Omega)}) \geq 0 \tag{3.13}$$

and

$$< -\Delta v_{in} + \Delta v_{0i}, v_{in} - v_{0i} >_{2,2} = \|v_{in} - v_{0i}\|^2_{W_0^{1,2}(\Omega)} \geq (\|v_{in}\|_{W_0^{1,2}(\Omega)} - \|v_{0i}\|_{W_0^{1,2}(\Omega)})^2. \tag{3.14}$$

From (3.13) and (3.14) we get

$$\begin{aligned}
0 \leq \sum_{i=1}^m &(\|v_{in}\|^{q-1}_{W_0^{1,q}(\Omega)} - \|v_{0i}\|^{q-1}_{W_0^{1,q}(\Omega)})(\|v_{in}\|_{W_0^{1,q}(\Omega)} - \|v_{0i}\|_{W_0^{1,q}(\Omega)}) \\
&+ \frac{1}{\alpha_n^{q-2}} \sum_{i=1}^m (\|v_{in}\|_{W_0^{1,2}(\Omega)} - \|v_{0i}\|_{W_0^{1,2}(\Omega)})^2 \\
\leq \frac{1}{\alpha_n^{q-2}} &\sum_{i=1}^m < -\Delta v_{in}, v_{in} - v_{0i} >_{2,2} + \sum_{i=1}^m < -\Delta_q v_{in}, v_{in} - v_{0i} >_{q',q} \\
&+ \sum_{i=1}^m < \Delta_q v_{0i}, v_{in} - v_{0i} >_{q',q} + \frac{1}{\alpha_n^{q-2}} \sum_{i=1}^m < \Delta v_{0i}, v_{in} - v_{0i} >_{2,2} .
\end{aligned}$$

Because the right-hand side of the above estimate tends to 0 as n tends to infinity (from (3.12) and the weak convergence of (v_n) to v_0 in W) we obtain that for $i = 1, \cdots, m$, $\|v_{in}\|_{W_0^{1,q}(\Omega)} \to \|v_{0i}\|_{W_0^{1,q}(\Omega)}$ as $n \to \infty$ and therefore (v_n) strongly converges to v_0 in W.

• Finally, we prove that v_{0i} is a non-trivial solution of the eigenvalue problem of the q-Laplacian with weight $m_i - w_i$ for at least one i.

Let $\phi = (\phi_1, \cdots, \phi_m) \in W$. Taking $\frac{1}{\alpha_n^{q-1}}\phi$ as a test function, since $u_n = \alpha_n v_n$, we have

$$I'(u_n).\frac{1}{\alpha_n^{q-1}}\phi = \frac{1}{\alpha_n^{q-2}} \sum_{i=1}^m \int_\Omega (\nabla v_{in}.\nabla \phi_i + \sum_{j=1}^m \int_\Omega a_{ij} v_{jn} \phi_i)$$

$$+\sum_{i=1}^{m}\int_{\Omega}(|\nabla v_{in}|^{q-2}\nabla v_{in}.\nabla\phi_i + w_i|v_{in}|^{q-2}v_{in}\phi_i - m_i|v_{in}|^{q-2}v_{in}\phi_i).$$

Letting $n\to\infty$, we see that for each $i=1,\cdots,m$,

$$\begin{cases} -\Delta_q v_{0i} + w_i|v_{0i}|^{q-2}v_{0i} = m_i|v_{0i}|^{q-2}v_{0i} \text{ in } \Omega \\ v_{0i}=0 \text{ on } \partial\Omega \end{cases}. \tag{3.15}$$

Since $\|v_n\|_{(L^q(\Omega))^m} = 1$ and (v_n) converges strongly to v_0 in W we get that $\|v_0\|_W \geq 1$. Therefore there exists i such that v_{0i} is a weak solution to (3.15). This contradicts Assumption 3.2. □

Lemma 3.2 *Assume that Assumptions 1.1 and 3.2 are satisfied. If $(u_n)\subset W$, $u_n = (u_{1n},\cdots,u_{mn})$, is a Palais-Smale sequence, then (u_n) has a strong convergent subsequence in W.*

Proof. Let (u_n) be a Palais-Smale sequence in W, $u_n=(u_{1n},\cdots,u_{mn})$. By Lemma 3.1, the sequence (u_n) is bounded in W. From the compact embedding of $W^{1,q}(\Omega)$ into $L^q(\Omega)$ we get the existence of $u_0=(u_{01},\cdots,u_{0m})\in W$ such that (u_n) converges to u_0 strongly in $(L^q(\Omega))^m$ and weakly in W (for a subsequence still denoted by (u_n)). We want to prove that $\|u_n\|_W\to\|u_0\|_W$ as $n\to\infty$ and we proceed as in the proof of Lemma 3.1.
Since $|I'(u_n).(u_n-u_0)|\leq\|I'(u_n)\|_{W^*}(\|u_n\|_W+\|u_0\|_W)$ we deduce that

$$I'(u_n).(u_n-u_0)\to 0 \text{ as } n\to\infty. \tag{3.16}$$

But

$$I'(u_n).(u_n-u_0)=\sum_{i=1}^{m}\int_{\Omega}(\nabla u_{in}\cdot\nabla(u_{in}-u_{0i})+\sum_{j=1}^{m}a_{ij}u_{jn}(u_{in}-u_{0i}))$$

$$+\sum_{i=1}^{m}\int_{\Omega}(|\nabla u_{in}|^{q-2}\nabla u_{in}.\nabla(u_{in}-u_{0i})+(w_i-m_i)|u_{in}|^{q-2}u_{in}(u_{in}-u_{0i})).$$

As in Lemma 3.1, denoting b_i either w_i or m_i, we have for $i,j=1,\cdots,m,$,

$$\int_{\Omega} b_i|u_{in}|^{q-2}u_{in}(u_{in}-u_{0i}))\to 0 \text{ as } n\to\infty, \int_{\Omega} a_{ij}u_{jn}(u_{in}-u_{0i}))\to 0 \text{ as } n\to\infty. \tag{3.17}$$

From (3.16) and (3.17), we get that

$$\sum_{i=1}^{m} < -\Delta u_{in}, u_{in}-u_{0i} >_{2,2} + \sum_{i=1}^{m} < -\Delta_q u_{in}, u_{in}-u_{0i} >_{q',q} \to 0 \text{ as } n\to\infty.$$

Moreover we have

$$0\leq\sum_{i=1}^{m}(\|u_{in}\|_{W_0^{1,q}(\Omega)}^{q-1}-\|u_{0i}\|_{W_0^{1,q}(\Omega)}^{q-1})(\|u_{in}\|_{W_0^{1,q}(\Omega)}-\|u_{0i}\|_{W_0^{1,q}(\Omega)})$$

$$+\sum_{i=1}^{m}(\|u_{in}\|_{W_0^{1,2}(\Omega)}-\|u_{0i}\|_{W_0^{1,2}(\Omega)})^2$$

$$\leq \sum_{i=1}^{m} < -\Delta u_{in}, u_{in}-u_{0i} >_{2,2} + \sum_{i=1}^{m} < -\Delta_q u_{in}, u_{in}-u_{0i} >_{q',q}$$

$$+\sum_{i=1}^{m} < \Delta_q u_{0i}, u_{in}-u_{0i} >_{q',q} + \sum_{i=1}^{m} < \Delta u_i, u_{in}-u_{0i} >_{2,2} .$$

As in Lemma 3.1 we deduce that for $i=1,\cdots,m$, $\|u_{in}\|_{W_0^{1,q}(\Omega)} \to \|u_{0i}\|_{W_0^{1,q}(\Omega)}$ as $n\to\infty$ and therefore (u_n) strongly converges to u_0 in W. □

So we can state the main result of this section

Theorem 3.1 *Assume that Assumptions 1.1, 3.1 and 3.2 are satisfied. Assume also that either Assumption 3.3 or 3.4 holds. Then the system (3.1) has a non-trivial solution in W.*

Proof. The C^1-functional I satisfies the Palais-Smale conditions and $I(0)=0$.
• First, we claim that there exist positive constants $\rho^*>0$ and $\delta>0$ such that $I(u)\geq\delta$ for any $u=(u_1,\cdots,u_m)\in W$ satisfying $\|u\|_W=\rho^*$.
Let $u=(u_1,\cdots,u_m)\in W$. Put $\rho=\|u\|_W$ and note that $H_1(u)\geq\|u\|_H^2$ and $H_2(u)\geq\rho^q$.
Moreover, since $q<2^*$, for $i=1,\cdots,m$,

$$|\int_\Omega m_i|u_i|^q|\leq(\int_\Omega|m_i|^r)^{1/r}(\int_\Omega|u_i|^{qt})^{1/t} \text{ with } \frac{1}{r}+\frac{1}{t}=1 \text{ and } s:=qt<2^*.$$

From the continous embedding of $W^{1,2}(\Omega)\subset L^s(\Omega)$ we deduce the existence of a positive constant C_1 such that $|\int_\Omega m_i|u_i|^q|\leq C_1\|u_i\|^q_{W^{1,2}(\Omega)}$. Thus

$$|M(u)|\leq C_1\|u\|_H^q$$

and

$$I(u)\geq\frac{1}{q}\rho^q+\frac{1}{2}\|u\|_H^2(1-\frac{2C_1}{q}\|u\|_H^{q-2}).$$

Recall also that there exists a positive constant $C_2>0$ such that $\|u\|_H\leq C_2\|u\|_W$ for all $u\in W$.
Therefore if $\rho\leq\rho^*:=\frac{1}{C_2}\left(\frac{q}{2C_1}\right)^{\frac{1}{q-2}}$, then $1-\frac{2C_1}{q}\|u\|_H^{q-2}\geq 1-\frac{2C_1}{q}(C_2\rho)^{q-2}\geq 0$ and

$$I(u)\geq\frac{1}{q}\rho^q:=\delta.$$

• Assume here that Assumption 3.3 is satisfied with $k=1$ for simplicity.
Let ϕ_{1,q,m_1-w_1} be the normalized eigenfunction associated with λ_{1,q,m_1-w_1} (i. e. be such that $\int_\Omega(m_1-w_1)|\phi_{1,q,m_1-w_1}|^q=1$, we may choose such ϕ_{1,q,m_1-w_1} because the equation (3.4) is

homogeneous). Denote $\Phi_q = (\phi_{1,q,m_1-w_1}, 0, \cdots, 0)$ and take R sufficiently large such that $\|R\Phi_q\|_W > \rho^*$. We have from (3.4) and (3.5)

$$I(R\Phi_q) = \frac{R^2}{2}H_1(\Phi_q) + \frac{R^q}{q}\int_\Omega (|\nabla\phi_{1,q,m_1-w_1}|^q + (w_1 - m_1)|\phi_{1,q,m_1-w_1}|^q)$$

$$= \frac{R^2}{2}H_1(\Phi_q) + \frac{R^q}{q}\int_\Omega (\lambda_{1,q,m_1-w_1}(m_1 - w_1) + w_1 - m_1)|\phi_{1,q,m_1-w_1}|^q.$$

So, since $\lambda_{1,q,m_1-w_1} < 1$,

$$I(R\Phi_q) = \frac{R^2}{2}H_1(\Phi_q) + \frac{R^q}{q}(\lambda_{1,q,m_1-w_1} - 1) < 0$$

for R sufficiently large.Therefore we can apply the mountain-pass theorem to deduce that I has a non-trivial critical point which is a non-trivial weak solution of the system (3.1).

• Assume now that Assumption 3.4 is satisfied with $k = 1$ and $l = 2$ for simplicity.
Denote again ϕ_{1,q,m_1-w_1} the normalized eigenfunction associated with λ_{1,q,m_1-w_1} such that $\int_\Omega (m_1 - w_1)|\phi_{1,q,m_1-w_1}|^q = 1$ and denote here $\Psi_q = (0, \phi_{1,q,m_1-w_1}, 0, \cdots, 0)$. Take R sufficiently large such that $\|R\Psi_q\|_W > \rho^*$. We have here

$$I(R\Psi_q) = \frac{R^2}{2}H_1(\Psi_q) + \frac{R^q}{q}\int_\Omega (|\nabla\phi_{1,q,m_1-w_1}|^q + (w_2 - m_2)|\phi_{1,q,m_1-w_1}|^q)$$

$$= \frac{R^2}{2}H_1(\Psi_q) + \frac{R^q}{q}\int_\Omega (\lambda_{1,q,m_1-w_1}(m_1 - w_1) + w_2 - m_2)|\phi_{1,q,m_1-w_1}|^q.$$

From Assumption 3.4, we get that $I(R\Psi_q) < 0$ for R sufficiently large.Therefore, as in the precedent case, we apply the mountain-pass theorem and deduce that I has a non-trivial critical point. □

Remark: As in section 2, we can generalize Theorem 3.1 replacing the 2-Laplacian operator by the p-Laplacian with $2 \le p < q$ for the following system

$$\begin{cases} -\Delta_p u_i - \Delta_q u_i + b_i|u_i|^{p-2}u_i + \lambda w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = \lambda m_i|u_i|^{q-2}u_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

under the additional hypotheses that the bounded functions b_i, $i = 1, \cdots, m$ are non-negative and λ is a real parameter. Then the hypothesis ii) in Assumption 3.2 is replaced by λ is not an eigenvalue of $-\Delta_q$ associated with $m_i - w_i$ for each i. Moreover the hypothesis $\lambda_{1,q,m_k-w_k} < 1$ in Assumption 3.3 is replaced by $\lambda_{1,q,m_k-w_k} < \lambda$.

4 Third case: $g_i(., u_1, \cdots, u_m) := \lambda f_i|u_i|^{\gamma-2}u_i$

In this section we rewrite the system (S, q, g) under the following form:
for $i = 1, \cdots, m$,

$$\begin{cases} -\Delta u_i - \Delta_q u_i + w_i|u_i|^{q-2}u_i + \sum_{j=1}^m a_{ij}u_j = \lambda f_i|u_i|^{\gamma-2}u_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

We assume throughout all this section that the indefinite bounded functions f_i and the coefficients γ and q satisfy the following hypotheses

Assumption 4.1 **i)** $2 < \gamma < q$,

ii) $\gamma < 2^*$ *where* $2^* = \frac{2N}{N-2}$ *if* $2 < N$ *and* $2^* = \infty$ *if* $2 \geq N$,

iii) *For each* $i = 1, \cdots, m$, $f_i \in L^\infty(\Omega)$ *and* $meas\{x \in \Omega, f_i(x) > 0\} \neq 0$.

We also define the functionals

$$F(u) = \sum_{i=1}^m \int_\Omega f_i|u_i|^\gamma \tag{4.2}$$

and

$$I_\lambda(u) = \frac{1}{2}H_1(u) + \frac{1}{q}H_2(u) - \frac{\lambda}{\gamma}F(u) \tag{4.3}$$

where H_1 and H_2 are respectively defined by (1.1) and (1.2). We recall that we study here the existence of a weak non-trivial solution $u = (u_1, \cdots, u_m) \in W$ for the system (4.1) with respect to the real positive parameter λ and that the existence of weak solutions for the system (4.1) is equivalent to the existence of critical points for the Euler functional I_λ. The main result is the existence of a weak non-trivial solution for the system (4.1) associated with $\lambda > \lambda_1^+$ where λ_1^+ is defined by (4.7). For the first part of this section we follow a method developed by Cherfils-Il'Yasov in [7] for one equation with the (p,q)-Laplacian operator. This method is based on proving the existence of solution for $\lambda = \lambda_1^+$ then on applying the mountain-pass theorem for $\lambda > \lambda_1^+$. Although we also could apply the mountain-pass theorem for our case, we will use in fact standard arguments to minimize the functional I_λ.

In section 4.1 we present some preliminary results: we define λ_1^+ and we prove the existence of a solution for the system (4.1) for $\lambda = \lambda_1^+$. The section 4.2 is devoted to the main theorem of the existence of a solution for the system (4.1) associated with $\lambda > \lambda_1^+$.

4.1 Some preliminaries results

As in [7] we define for $\lambda > 0$, $t > 0$ and $u \in W$, $\tilde{I}_\lambda(t, u) = I_\lambda(tu)$.

Lemma 4.1 *Assume that Assumptions 1.1, 4.1 i), 4.1 iii) are satisfied. For given u in $W, u \neq 0$ such that $F(u) \neq 0$, the unique solution $(t(u), \lambda(u))$ of the system* $\begin{cases} \frac{\partial}{\partial t}\tilde{I}_\lambda(t,u) = 0 \\ \frac{\partial^2}{\partial t^2}\tilde{I}_\lambda(t,u) = 0 \end{cases}$ *is given by*

$$t(u) = \left(\frac{\gamma - 2}{q - \gamma}\right)^{\frac{1}{q-2}} \left(\frac{H_1(u)}{H_2(u)}\right)^{\frac{1}{q-2}} > 0, \quad \lambda(u) = C_{q,\gamma}\frac{H_1(u)^\alpha H_2(u)^{1-\alpha}}{F(u)} \tag{4.4}$$

with

$$\alpha = \frac{q-\gamma}{q-2}, \quad C_{q,\gamma} = \frac{q-2}{(q-\gamma)^\alpha(\gamma-2)^{1-\alpha}}. \tag{4.5}$$

Proof. The system (S) $\begin{cases} \frac{\partial}{\partial t}\tilde{I}_\lambda(t,u) = 0 \\ \frac{\partial^2}{\partial t^2}\tilde{I}_\lambda(t,u) = 0 \end{cases}$ is equivalent to the system

$$\begin{cases} tH_1(u) + t^{q-1}H_2(u) - \lambda t^{\gamma-1}F(u) = 0 \\ H_1(u) + (q-1)t^{q-2}H_2(u) - \lambda(\gamma-1)t^{\gamma-2}F(u) = 0 \end{cases}$$

and to the following system

$$\begin{cases} H_1(u) + t^{q-2}H_2(u) - \lambda t^{\gamma-2}F(u) = 0 \\ H_1(u) + (q-1)t^{q-2}H_2(u) - \lambda(\gamma-1)t^{\gamma-2}F(u) = 0 \end{cases}.$$

Therefore

$$(q-2)t^{q-2}H_2(u) - \lambda(\gamma-2)t^{\gamma-2}F(u) = 0. \tag{4.6}$$

Note that the system (S) is not solvable in the case where $u \in W$, $u \neq 0$ satisfies $F(u) = 0$ (since if $u \neq 0$, then $H_2(u) \neq 0$ and from (4.6) we deduce $F(u) \neq 0$).
We deduce that

$$\lambda = \frac{(q-2)t^{q-2}H_2(u)}{(\gamma-2)t^{\gamma-2}F(u)}.$$

Replacing λ by $\frac{(q-2)t^{q-2}H_2(u)}{(\gamma-2)t^{\gamma-2}F(u)}$ in $H_1(u) + t^{q-2}H_2(u) - \lambda t^{\gamma-2}F(u) = 0$, we get that $t^{q-2} = (\frac{\gamma-2}{q-\gamma})\frac{H_1(u)}{H_2(u)}$. And we obtain (4.4) associated with (4.5). □

Thus we can define the following characteristic points (recall that F is defined by (4.2))

$$\Lambda_1^+ = \inf\{\lambda(u), u \in W, F(u) > 0\} \text{ and } \lambda_1^+ = \frac{\gamma}{2^\alpha q^{1-\alpha}}\Lambda_1^+. \tag{4.7}$$

Lemma 4.2 *Assume that Assumptions 1.1 and 4.1 are satisfied.*
We have $0 < \Lambda_1^+ < \lambda_1^+$.

Proof. Let $u = (u_1, \cdots, u_m) \in W$ be such that $F(u) > 0$.
First from $\gamma < 2^*$, let (t, l) be such that $\gamma < t < 2^*$ and $\frac{1}{l} + \frac{\gamma}{t} = 1$. Since $W_0^{1,2}(\Omega) \subset L^t(\Omega)$ with a continuous embedding and since the functions f_i are bounded, there exist positive constants still denoting C at each step and depending on some Sobolev constants, such that for $i = 1, \cdots, m$

$$|\int_\Omega f_i |u_i|^\gamma| \leq (\int_\Omega |f_i|^l)^{1/l} (\int_\Omega |u_i|^t)^{\gamma/t} \leq C\|u_i\|^\gamma_{L^t(\Omega)} \leq C\|u_i\|^\gamma_{W_0^{1,2}(\Omega)}.$$

Then

$$F(u) \leq CH_1(u)^{\gamma/2}.$$

By the same way, from $\gamma < q$, let $s = \frac{q}{\gamma}$ and r be such that $\frac{1}{s} + \frac{1}{r} = 1$.
Then we have

$$|\int_\Omega f_i|u_i|^\gamma| \leq m(\int_\Omega |f_i|^r)^{1/r} (\int_\Omega |u_i|^{\gamma s})^{1/s} \leq C\|u_i\|^\gamma_{L^q(\Omega)} \leq C\|u_i\|^\gamma_{W_0^{1,q}(\Omega)}$$

and

$$F(u) \leq CH_2(u)^{\gamma/q}.$$

Therefore there exists a positive constant C', independent of u, such that

$$\lambda(u) = C_{q,\gamma} \frac{H_1(u)^\alpha H_2(u)^{1-\alpha}}{F(u)} \geq C' C_{q,\gamma} \frac{F(u)^{\frac{2\alpha}{\gamma}} F(u)^{\frac{q(1-\alpha)}{\gamma}}}{F(u)} = C' C_{q,\gamma}$$

since $\frac{2\alpha}{\gamma} + \frac{q(1-\alpha)}{\gamma} = 1$. Thus $\Lambda_1^+ > 0$.

Finally we prove that $\Lambda_1^+ < \lambda_1^+$.
Indeed note that $\lambda_1^+ > \Lambda_1^+ \Leftrightarrow \frac{\gamma}{2^\alpha q^{1-\alpha}} \Lambda_1^+ > \Lambda_1^+ \Leftrightarrow (\frac{\gamma}{2})^{q-2} > (\frac{q}{2})^{\gamma-2}$.
Denote $\mu = \frac{q-2}{2} > 0$ and $\eta = \frac{\gamma-2}{2} > 0$. Since $2 < \gamma < q$ we have $\mu > \eta$. Moreover the function f defined by $f(x) = (1+x)^{1/x}$, is strictly decreasing on $(0, \infty)$. Then $(1+\mu)^{1/\mu} < (1+\eta)^{1/\eta}$. And we get that $(\frac{q}{2})^{\gamma-2} < (\frac{\gamma}{2})^{q-2}$. So $\Lambda_1^+ < \lambda_1^+$. □

We obtain now the following result that will enable us to get the existence of a non-trivial solution for the system (4.1) associated with λ_1^+.

Proposition 4.1 *Assume that Assumptions 1.1 and 4.1 are satisfied. Assume that $u = (u_1, \cdots, u_m) \in W$ satisfies $F(u) \neq 0$ and $\lambda'(u) = 0$ (i.e. u is a critical point of $\lambda(u)$). Then $\tilde{u} = (\tilde{u_1}, \cdots, \tilde{u_m}) \in W$ is a non-trivial solution of the system (4.1) associated with $\tilde{\lambda} = \frac{\gamma}{2^\alpha q^{1-\alpha}} \lambda(u)$ where for all $i = 1, \cdots, m$, $\tilde{u_i} = \frac{1}{s} u_i$ and $\frac{1}{s} = (\frac{q}{2})^{\frac{1}{q-2}} t(u) > 0$. Moreover $I_{\tilde{\lambda}}(\tilde{u}) = 0$.*

Proof. Let $u=(u_1,\cdots,u_m)\in W$ which satisfies $F(u)\neq 0$ and $\lambda'(u)=0$.
For all test function ϕ, we have

$$\frac{\partial\lambda}{\partial u_1}(u)\cdot\phi=0.$$

So

$$2C_{q,\gamma}\alpha(H_1(u))^{\alpha-1}(H_2(u))^{1-\alpha}(F(u))^{-1}\int_\Omega(\nabla u_1\cdot\nabla\phi+a_{11}u_1\phi+\sum_{j=2}^m a_{1j}u_j\phi)$$
$$+qC_{q,\gamma}(1-\alpha)(H_1(u))^{\alpha}(H_2(u))^{-\alpha}(F(u))^{-1}\int_\Omega(|\nabla u_1|^{q-2}\nabla u_1\cdot\nabla\phi+w_1|u_1|^{q-2}u_1\phi)$$
$$-\gamma C_{q,\gamma}\alpha(H_1(u))^{\alpha}(H_2(u))^{1-\alpha}(F(u))^{-2}\int_\Omega f_1|u_1|^{\gamma-2}u_1\phi=0.$$

And

$$2C_{q,\gamma}\alpha\left(\frac{H_1(u)}{H_2(u)}\right)^{\alpha-1}\int_\Omega(\nabla u_1\cdot\nabla\phi+a_{11}u_1\phi+\sum_{j=2}^m a_{1j}u_j\phi)$$
$$+qC_{q,\gamma}(1-\alpha)\left(\frac{H_1(u)}{H_2(u)}\right)^{\alpha}\int_\Omega(|\nabla u_1|^{q-2}\nabla u_1\cdot\nabla\phi+w_1|u_1|^{q-2}u_1\phi)$$
$$-\lambda(u)\gamma\int_\Omega f_1|u_1|^{\gamma-2}u_1\phi=0.$$

Define $\tilde{u_i}=\frac{1}{s}u_i$ for $i=1,\cdots,m$, $s>0$ and $H(u)=\frac{H_1(u)}{H_2(u)}$. Then

$$2C_{q,\gamma}\alpha(H(u))^{\alpha-1}s\int_\Omega(\nabla\tilde{u_1}\cdot\nabla\phi+a_{11}\tilde{u_1}\phi+\sum_{j=2}^m a_{1j}\tilde{u_j}\phi)$$
$$+C_{q,\gamma}(1-\alpha)(H(u))^{\alpha}qs^{q-1}\int_\Omega(|\nabla\tilde{u_1}|^{q-2}\nabla\tilde{u_1}\cdot\nabla\phi+w_1|\tilde{u_1}|^{q-2}\tilde{u_1}\phi)$$
$$-\lambda(u)\gamma s^{\gamma-1}\int_\Omega f_1|\tilde{u_1}|^{\gamma-2}\tilde{u_1}\phi=0.$$

And equivalently

$$2C_{q,\gamma}\alpha(H(u))^{\alpha-1}s^{2-\gamma}\int_\Omega(\nabla\tilde{u_1}\cdot\nabla\phi+a_{11}\tilde{u_1}\phi+\sum_{j=2}^m a_{1j}\tilde{u_j}\phi)$$
$$+C_{q,\gamma}(1-\alpha)(H(u))^{\alpha}qs^{q-\gamma}\int_\Omega(|\nabla\tilde{u_1}|^{q-2}\nabla\tilde{u_1}\cdot\nabla\phi+w_1|\tilde{u_1}|^{q-2}\tilde{u_1}\phi)$$
$$-\lambda(u)\gamma\int_\Omega f_1|\tilde{u_1}|^{\gamma-2}\tilde{u_1}\phi=0.$$

Multiplying this last equation by $\frac{1}{2^\alpha q^{1-\alpha}}$ and denoting $\tilde{\lambda}=\frac{\gamma}{2^\alpha q^{1-\alpha}}\lambda(u)$ we get

$$\left(\frac{2(q-\gamma)}{q(\gamma-2)H(u)}\right)^{\frac{\gamma-2}{q-2}}s^{2-\gamma}\int_\Omega(\nabla\tilde{u_1}\cdot\nabla\phi+a_{11}\tilde{u_1}\phi+\sum_{j=2}^m a_{1j}\tilde{u_j}\phi)$$

$$+\left(\frac{q(\gamma-2)H(u)}{2(q-\gamma)}\right)^{\frac{q-\gamma}{q-2}} s^{q-\gamma}\int_\Omega (|\nabla\tilde{u_1}|^{q-2}\nabla\tilde{u_1}\cdot\nabla\phi + w_1|\tilde{u_1}|^{q-2}\tilde{u_1}\phi)$$
$$-\tilde{\lambda}\int_\Omega f_1|\tilde{u_1}|^{\gamma-2}\tilde{u_1}\phi = 0.$$

Choosing $s=\left(\frac{2(q-\gamma)}{q(\gamma-2)H(u)}\right)^{\frac{1}{q-2}}$ we obtain

$$\int_\Omega(\nabla\tilde{u_1}\cdot\nabla\phi + a_{11}\tilde{u_1}\phi + \sum_{j=2}^m a_{1j}\tilde{u_j}\phi) + \int_\Omega(|\nabla\tilde{u_1}|^{q-2}\nabla\tilde{u_1}\cdot\nabla\phi + w_1|\tilde{u_1}|^{q-2}\tilde{u_1}\phi)$$
$$-\tilde{\lambda}\int_\Omega f_1|\tilde{u_1}|^{\gamma-2}\tilde{u_1}\phi = 0.$$

Doing the same for $\tilde{u_i}$, $i=2,\cdots,m$, we get that $\tilde{u}=(\tilde{u_1},\cdots,\tilde{u_m})$ is a weak solution of (4.1) associated with $\tilde{\lambda}$.

Now we prove that $I_{\tilde{\lambda}}(\tilde{u})=0$.
Recall that $\tilde{u}=\frac{1}{s}u$ and $\tilde{\lambda}=\frac{\gamma}{2^\alpha q^{1-\alpha}}\lambda(u)$. Then we have

$$I_{\tilde{\lambda}}(\tilde{u}) = \frac{H_1(u)}{2s^2} + \frac{H_2(u)}{qs^q} - \frac{C_{q,\gamma}H_1(u)^\alpha H_2(u)^{1-\alpha}}{2^\alpha q^{1-\alpha}s^\gamma}.$$

Denoting $r=\frac{C_{q,\gamma}H_1(u)^\alpha H_2(u)^{1-\alpha}}{2^\alpha q^{1-\alpha}s^\gamma}$, since $\frac{1}{s}=(\frac{q}{2})^{\frac{1}{q-2}}t(u)>0$ and $\alpha=\frac{q-\gamma}{q-2}$ we get

$$I_{\tilde{\lambda}}(\tilde{u}) = r\left[\frac{1}{C_{q,\gamma}}\left(\frac{qH_1(u)}{2H_2(u)}\right)^{1-\alpha}\left(\frac{q}{2}\right)^{\alpha-1}(t(u))^{2-\gamma} + \frac{1}{C_{q,\gamma}}\left(\frac{2H_2(u)}{qH_1(u)}\right)^{\alpha}\left(\frac{q}{2}\right)^{\alpha}(t(u))^{q-\gamma} - 1\right].$$

But $t(u)=\left(\frac{\gamma-2}{q-\gamma}\right)^{\frac{1}{q-2}}\left(\frac{H_1(u)}{H_2(u)}\right)^{\frac{1}{q-2}}>0$ and $C_{q,\gamma}=\frac{q-2}{(q-\gamma)^\alpha(\gamma-2)^{1-\alpha}}$ so

$$I_{\tilde{\lambda}}(\tilde{u}) = r\left[\frac{q-\gamma}{q-2}+\frac{\gamma-2}{q-2}-1\right]=0.$$

□

Proposition 4.2 *Assume that Assumptions 1.1 and 4.1 are satisfied and* $0<\lambda<\Lambda_1^+$. *Then the system* (4.1) *has no non-trivial solution in* W *associated with* λ.

Proof. Assume that $0<\lambda<\Lambda_1^+$. Assume also that the system (4.1) has a non-trivial solution $u=(u_1,\cdots,u_m)\in W$ associated with λ. Then we have
$H_1(u)+H_2(u)=\lambda F(u)$. Note that this is impossible if $F(u)\le 0$.

Therefore assume now that $F(u) > 0$.
Recall that $\tilde{I}_\lambda(t,v) = I_\lambda(tv) = \frac{t^2}{2}H_1(v) + \frac{t^q}{q}H_2(v) - \frac{\lambda t^\gamma}{\gamma}F(v)$ for all $t > 0$ and $v \in W$. We have $\frac{\partial}{\partial t}\tilde{I}_\lambda(t,v) = tH_1(v) + t^{q-1}H_2(v) - \lambda t^{\gamma-1}F(v)$ and in particular, since u is a weak solution of (4.1), note that

$$\frac{\partial}{\partial t}\tilde{I}_\lambda(\|u\|, \frac{1}{\|u\|}u) = I'_\lambda(u) \cdot \frac{1}{\|u\|}u = 0.$$

Moreover we have $\frac{\partial}{\partial t}\tilde{I}_\lambda(t,v) = t^{\gamma-1}R_\lambda(t,v)$ with

$$R_\lambda(t,v) = t^{2-\gamma}H_1(v) + t^{q-\gamma}H_2(v) - \lambda F(v).$$

Let $v \in W$ be such that $v \neq 0$ and $F(v) > 0$. Note that from Lemma 4.1 we have $R_{\lambda(v)}(t(v),v) = 0$.
Moreover we can prove that $R_\lambda(t,v) \geq R_\lambda(t(v),v)$ for all $t > 0$.
Indeed let $f(t) = t^{2-\gamma}H_1(v) + t^{q-\gamma}H_2(v)$. The function f admits a global minimum on $t(v)$ on $(0,\infty)$ so $f(t) \geq f(t(v)) = \left(\frac{H_1(v)}{\alpha}\right)^\alpha \left(\frac{H_2(v)}{1-\alpha}\right)^{1-\alpha} > 0$. Therefore $R_\lambda(t,v) \geq R_\lambda(t(v),v)$ for all $t > 0$.
Finally since $\lambda < \Lambda_1^+ \leq \lambda(v)$, we get that $R_\lambda(t,v) > R_{\lambda(v)}(t,v)$ for all $t > 0$. Thus $R_\lambda(t,v) \geq R_\lambda(t(v),v) > R_{\lambda(v)}(t(v),v) = 0$ and $\frac{\partial}{\partial t}\tilde{I}_\lambda(t,v) = \frac{\partial}{\partial t}I_\lambda(tv) > 0$ for all $t > 0$.
So, choosing $t = \|u\|$ and $v = \frac{1}{\|u\|}u$, we get a contradiction since $\frac{\partial}{\partial t}\tilde{I}_\lambda(\|u\|, \frac{1}{\|u\|}u) = 0$. □

Now we obtain a minimizer for Λ_1^+.

Proposition 4.3 *Assume that Assumptions 1.1 and 4.1 are satisfied. There exists $v = (v_1, \cdots, v_m) \in W$ such that $\lambda(v) = \Lambda_1^+$.*

Proof. First note that $\lambda(tu) = \lambda(u)$ for all $t > 0$ and $u \in W$.
Define $\tilde{t}(u) = \frac{1}{((H_1(u))^\alpha (H_2(u))^{1-\alpha})^{\frac{1}{\gamma}}}$ for $u \in W \setminus \{0\}$ and note that

$$(H_1(\tilde{t}(u)u)^\alpha (H_2(\tilde{t}(u)))^{1-\alpha} = 1.$$

Therefore we can derive that

$$\Lambda_1^+ = \inf\{\lambda(u), u \in W \text{ such that } F(u) > 0 \text{ and } H_1(u)^\alpha H_2(u)^{1-\alpha} = 1\}.$$

Then we consider a minimizing sequence (v_n) of Λ_1^+.
We have $\gamma = 2\alpha + q(1-\alpha)$, so

$$\|v_n\|_H^\gamma = \|v_n\|_H^{2\alpha}\ \|v_n\|_H^{q(1-\alpha)}$$

and since $W \subset H$ with a continuous embedding, there exists a positive constant C such that

$$\|v_n\|_H^\gamma \leq C\|v_n\|_H^{2\alpha}\ \|v_n\|_W^{q(1-\alpha)}.$$

But H_1 and H_2 are equivalent norms respectively in H and W so we get that

$$\|v_n\|_H^\gamma \leq C(H_1(v_n))^\alpha (H_2(v_n))^{1-\alpha} = C$$

(for a positive constant C). We deduce that (v_n) is a bounded sequence in H. By the compact embedding $W_0^{1,2}(\Omega) \subset L^\gamma(\Omega)$ (for $\gamma < 2^*$), we get the existence of $v = (v_1, \cdots, v_m) \in H$ such that (v_n) converges to v, strongly in $(L^\gamma(\Omega))^m$ and weakly in H (for a subsequence).

Afterwards we prove that $F(v) > 0$, $v \in W$ and since H_1 and H_2 are weakly lower semi-continous in H and W respectively, we get that $\lambda(v) = \Lambda_1^+$.

Indeed, since F is a continuous function and $F(v_n) > 0$, $F(v_n) \to_{n\to\infty} F(v)$, we have $F(v) \geq 0$. Moreover, if $F(v) = 0$, then $\lambda(v_n) = \frac{C_{q,\gamma}}{F(v_n)} \to_{n\to\infty} \infty$. This contradicts $\lambda(v_n) \to_{n\to\infty} \Lambda_1^+$. So $F(v) > 0$ and $v \neq 0$.

Now we prove that $v \in W$. Recall that (v_n) is a bounded sequence in H and that (v_n) converges to $v \neq 0$ strongly in $(L^\gamma(\Omega))^m$. So there exists a positive constant C' such that $\|v_n\|_{(L^\gamma(\Omega))^m} \geq C' > 0$ for n large enough. Therefore, from the continuous embedding $H \subset (L^\gamma(\Omega))^m$, we get that $\|v_n\|_H \geq C' > 0$ for n large enough.
Finally from $\|v_n\|_H \geq C' > 0$ and $\|v_n\|_H^{2\alpha} \|v_n\|_W^{(1-\alpha)q} \leq C$ we obtain that (v_n) is a bounded sequence in W. Therefore (v_n) admits a subsequence, still denoted (v_n) such that (v_n) converges to v strongly in $(L^\gamma(\Omega))^2$ and weakly in W. Thus $v \in W$.

Finally we prove that $\lambda(v) = \Lambda_1^+$.
From the weakly semi-continuousness of H_1 and H_2 respectively on H and W we have

$$H_1(v) \leq \liminf H_1(v_n) \text{ and } H_2(v) \leq \liminf H_2(v_n).$$

But $\lambda(v_n) = \frac{C_{q,\gamma}}{F(v_n)} = \frac{C_{q,\gamma}(H_1(v_n))^\alpha (H_2(v_n))^{1-\alpha}}{F(v_n)} \to_{n\to\infty} \Lambda_1^+$. Passing to the limit inf as n tends to ∞ we get that $\Lambda_1^+ \geq \frac{C_{q,\gamma}(H_1(v))^\alpha (H_2(v))^{1-\alpha}}{F(v)} = \lambda(v)$. We deduce that

$$\lambda(v) = \Lambda_1^+.$$

This concludes the proof. □

Contrary to [7], we are not able to prove that the minimizer v is non-negative because of the coupling terms $a_{ij}v_j v_i$ in $H_1(v)$. Finally combining Propositions 4.1 and 4.3, since v (defined by Proposition 4.3) is a critical point of $\lambda(u)$, we derive the existence of a non-trivial weak solution $u^+ = (u_1^+, \cdots, u_m^+)$ for the system (4.1) associated with λ_1^+. This is the following result

Proposition 4.4 *Assume that Assumptions 1.1 and 4.1 are satisfied. There exists $u^+ = (u_1^+, \cdots, u_m^+) \in W$ a non-trivial solution for the system (4.1) associated with λ_1^+. Moreover $I_{\lambda_1^+}(u^+) = 0$ and $F(u^+) > 0$.*

Proof. From Proposition 4.3 we have $\lambda(v) = \Lambda_1^+ = \inf\{\lambda(u), u \in W, F(u) > 0\}$. Thus v is a critical point of the function λ on W. From Proposition 4.1 we derive that there exists a non-trivial solution $u^+ = (u_1^+, \cdots, u_m^+)$ of system (4.1) associated with $\frac{\gamma}{2^\alpha q^{1-\alpha}}\lambda(v) = \lambda_1^+$ where for all $i = 1, \cdots, m$, $u_i^+ = \frac{1}{s}v_i$ and $\frac{1}{s} = (\frac{q}{2})^{\frac{1}{q-2}}t(v) > 0$. Moreover from Proposition 4.1, $I_{\lambda_1^+}(u^+) = 0$ and from Proposition 4.3, $F(u^+) = \frac{1}{s^\gamma}F(v) > 0$. □

4.2 Main result

Theorem 4.1 *Assume that Assumptions 1.1 and 4.1 are satisfied. If $\lambda > \lambda_1^+$, then the system (4.1), associated with λ, admits a non-trivial solution in W.*

Proof. Even if we could follow [7] for proving this result using the mountain-pass theorem, we use here standard arguments by global minimization of the C^1-functional I_λ. Note that I_λ is weakly lower semi-continuous by the compact embedding of W into $(L^q(\Omega))^m$ and $(L^2(\Omega))^m$. Moreover I_λ is coercive: indeed for any $u \in W$,

$$I_\lambda(u) \geq \frac{1}{q}H_2(u) - \frac{\lambda}{\gamma}F(u).$$

Since $|F(u)| \leq C\|u\|_W^\gamma$ with C a positive constant, we get that

$$I_\lambda(u) \geq \frac{1}{q}\|u\|_W^q \left(1 - \frac{\lambda C q}{\gamma}\|u\|_W^{\gamma-q}\right).$$

Thus I_λ is coercive. Furthermore from Proposition 4.4, we have $I_{\lambda_1^+}(u^+) = 0$ and $F(u^+) > 0$. Finally from the hypothesis $\lambda > \lambda_1^+$, we get that $I_\lambda(u^+) < I_{\lambda_1^+}(u^+) = 0$. Therefore we deduce that I_λ has a non-trivial critical point which is a non-trivial weak solution of the system (4.1) associated with λ. □

Remarks: We can get the same results for a larger class of coefficients, assuming that $a_{ij}, w_i, f_i \in L^r(\Omega)$ for some $r > 1$ as in [7]. But we have not been able to adapt this method for a system with a (p,q)-Laplacian operator (with $p \neq 2$) and even for a non-symmetric system with a (2,q)-Laplacian operator. However in the particular case where the matrix A is not symmetric and has the following form: $A = (a_{ij})$ with $a_{j1} = K a_{1j}$ for $j = 2, \cdots, m$ for some positive constant $K > 0$ (K independent of j) and $a_{ij} = a_{ji}$ for $i, j \geq 2$, we can generalize all the above results. Indeed we introduce the diagonal matrix $D = (d_{ij})$ with

$d_{11} = K$, $d_{ii} = 1$ for $i = 2, \cdots, m$ and $d_{ij} = 0$ if $i \neq j$. We replace the functionals H_1, H_2 and F (defined before by (1.1),(1.2),(4.2)) by

$$H_2(u) = \sum_{i=1}^{m} d_{ii} \int_\Omega (|\nabla u_i|^q + w_i |u_i|^q), \; F(u) = \sum_{i=1}^{m} d_{ii} \int_\Omega f_i |u_i|^\gamma,$$

$$H_1(u) = \sum_{i=1}^{m} d_{ii} \int_\Omega (|\nabla u_i|^2 + a_{ii} u_i^2 + \sum_{j=1, i\neq j}^{m} a_{ij} u_j u_i),$$

$$H_1(u) = \sum_{i=1}^{m} d_{ii} \int_\Omega |\nabla u_i|^2 + \int_\Omega {}^tUDAU \text{ with} {}^tU = (u_1, \cdots, u_m).$$

Therefore if we assume that the matrix DA satisfies the following hypothesis ${}^t\xi DA\xi \geq 0$ for all ${}^t\xi = (\xi_1, \cdots, \xi_m) \in \mathbb{R}^m$, we still derive that the Euler functional I_λ defined by (4.3) (with the new functionals H_1, H_2 and F) is associated with the system (4.1) and the existence of weak solutions for the system (4.1) is equivalent to the existence of critical points for I_λ. Finally due to the coupling term of system (4.1), note that we just obtain the existence of a non-trivial solution in Theorem 4.1 contrary to [7] where a non-negative solution is obtained.

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Harry Poppe

Ascoli-Arzela-Theory based on continuous convergence in an (almost) non-Hausdorff setting 2

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1 Introduction

We start with the paper [1] and thus come back to continuous convergence and to the characterization of compactness with respect to this convergence structure for the space $C(X,Y)$ of continuous functions, where X and Y are topological spaces. More generally we can use for X, Y convergence spaces, as was done for instance in [11] and [15]. But in the first paper of this title X and Y were topological spaces and we will continue with this assumption.

What is the aim of our paper?

1. In the main theorems [15, (3.24), (3.27)], [1, 33] and corollary [11, 10] necessary and sufficient conditions were given to ensure that $H \subseteq C(X,Y)$ is relatively compact w. r. t. continuous convergence. Here, as a corollary, we characterize compactness of H.

2. In the papers [11], [1] not provided examples which show that the assumptions in our theorems (for instance that Y is Hausdorff) we cannot omit.

3. It is known for long time that the important notion of equicontinuity can be characterized using the canonical map as used in duality theory (embedding in a second dual) ([7], and [15, theorem 4.36]). In the paper [8] and especially in the book [12] this approach was extended to include even continuity too. But the two Ascoli-Arzela theorems ([12, (13.15), (13.21)]) based on this approach are not correct. We will show this by an instructive counter example. And we will give some comments for this situation.

2 Compactness in $(C(X,Y), c\text{-lim})$

We will use the following notion of relative compactness: Let X be a topological space, then $A \subset X$ is called relatively compact iff for each ultrafilter π on X,

$$A \in \pi \Longrightarrow \exists\, x \in X : \pi \longrightarrow x\,. \quad \text{(see [16], [3])}$$

We still need a lemma.

Lemma 2.1 *Let X be a topological space, Y a Hausdorff topological space; let η be a topology* (lim *a convergence structure*) *on $C(X,Y)$ with $\tau_p \leq \eta$ (τ_p-lim $\leq$ lim). If $H \subseteq C(X,Y)$ is η-compact, then H is τ_p-closed in Y^X.*

For a (simple) proof see lemma [3, theorem 3.1].

Now [1, theorem 33] states: If X, Y topological spaces, $H \subseteq C(X,Y)$, $H \neq \emptyset$ and we consider for H the two conditions:

(α) $\forall x \in X : H(x) = \{f(x) | f \in H\}$ is relatively compact.

(β) H is evenly continuous.

then the following holds:

1. Let X be Hausdorff; H relatively c-compact $\Longrightarrow (\alpha), (\beta)$.
2. Let X be a T_3-space, $(\alpha), (\beta) \Longrightarrow H$ is in $(C(X,Y), c\text{-lim})$ relatively compact.

Theorem 2.2 [Corollary of [1, theorem33]] *Let be X, Y topological spaces, $H \subseteq C(X,Y)$; for H we consider the conditions:*

(α) *$\forall x \in X : H(x)$ is relatively compact*

(β) *H is evenly continuous*

(γ) *H is in $Y^X \tau_p$-closed.*

Then hold:

1. Let Y be Hausdorff, H is c-lim-compact $\Longrightarrow (\alpha), (\beta), (\gamma)$.
2. $(\alpha), (\beta), (\gamma) \Longrightarrow H$ is in $(C(X,Y), c\text{-lim})$ compact.

Proof. 1. H c-lim-compact in $C(X,Y) \Longrightarrow H$ is c-lim-relatively compact; then follows $(\alpha), (\beta)$ by theorem 33. Now since $\tau_p \leq c$-lim holds in $C(X,Y)$, lemma 2.1 yields condition (γ) too.

2. By the Tychonoff theorem: $(\alpha) \Longrightarrow H$ is τ_p-relatively compact in Y^X, hence H is τ_p-compact in Y^X by (γ); $H \subseteq C(X,Y) \Longrightarrow H$ is in $C(X,Y)\tau_p$-compact too. Now let π be an ultrafilter on $C(X,Y)$ with $H \in \pi$: we find $g \in H : \pi \xrightarrow{\tau_p} g$, but then follows: $\pi \xrightarrow{c} g$ by (β) and by [1, theorem 31]. Hence H is in $C(X,Y)$ c-lim-compact.

3 Examples

For the construction of our examples we need a result of S. Mrowka which we found in [6] and a corollary of this result.

Proposition 3.1 *Let (X,τ) be a Hausdorff topologigal space, where of course τ means the system of all open sets of X; let $(A_i)_{i\in I}$ be a net in 2^X, 2^X is the set of closed sets of X. Then (A_i) has a subnet Kuratowski-Hausdorff-converging in 2^X.*

Proof. We know: a net(B_i) (from 2^X) converges iff $L_sB_i \subseteq L_iB_i$ holds, meaning:

$$\forall x \in X, \forall G \in \tau : x \in G \text{ and } G \cap B_i \neq \emptyset$$

for all i from a confinal set of I it follows that eventually $G \cap B_i \neq \emptyset$, since $\{G \in \tau | x \in G\}$ is a basis of the neighborhood filter $\underline{U}(x)$. Here by Ls, Li we denote the limit superior and limit inferior respectively.

Now we consider the two-point space $\{0,1\}$ provided with discrete topology.

$$\forall i \in I: \text{ let be } f_i \in \{0,1\}^\tau : \forall G \in \tau : f_i(G) = \begin{cases} 1, & A_i \cap G \neq \emptyset \\ 0, & A_i \cap G = \emptyset\,. \end{cases}$$

Obviously, the map $f_i \to A_i$ is injective.

$\{0,1\}^\tau$ with pointwise topology τ_p is compact by the Tychonoff theorem, and hence for (f_i) there exists a subnet (f_{i_k}) and a f from $\{0,1\}^\tau$ such that $f_{i_k} \xrightarrow{\tau_p} f$. Now we want to show $Ls(A_{i_k}) \subseteq Li(A_{i_k}) : \forall (x,G) \in X \times \tau : x \in Ls(A_{i_k})$ and $x \in G$: there exists a confinal subset

$$K_1 \subseteq K \text{ such that } \forall k \in K_1 : A_{i_k} \cap G \neq \emptyset$$

implying $\forall k \in K_1 : f_{i_k}(G) = 1$. We assume that $f(G) = 0$ holds, $\{0\}$ is open and for our net $(f_{i_k})_{k\in K}$ holds: $f_{i_k}(G) \longrightarrow f(G)$ implying eventually $f_{i_k}(G) = 0$, yielding a contradiction because K_1 is kofinal in K and $\forall k \in K_1 : f_{i_k}(G) = 1$.

Hence we have $f(G) = 1$; now $f_{i_k}(G) \longrightarrow f(G) = 1$ and $\{1\}$ is open implies: eventually $f_{i_k}(G) = 1$ and thus eventually $A_{i_k} \cap G \neq \emptyset$ showing that $x \in Li(A_{i_k})$.

Corollary 3.2 *Let X be a Hausdorff topological space, F the Sierpinski-space with open sets: $\emptyset, \{0\}, \{0,1\} = F$. Then $(C(X,F), c\text{-lim}) = C_c(X,F)$ is compact.*

Proof. Let χ_{A_i} be a net from $C(X,F)$, meaning that all A_i are closed sets in X, hence $\forall i \in I : A_i \in 2^X$. By the proposition 3.1 (A_i) has a subnet (A_{i_k}) converging to a set $A \in 2^X$. Hence we get $LsA_{i_k} = LiA_{i_k} = A$, $LsA_{i_k} = A$ shows:

$$\chi_{A_{i_k}} \xrightarrow{c} \chi_A \quad \text{in} \quad C(X,F)\,,$$

hence we found a subnet converging continuously to χ_A.

Thus $C_c(X,F)$ is compact.

At first we show that lemma 2.1 does not work if Y is not Hausdorff.

Example 3.3 Let be $X = \mathbb{R}$, the reals with Euclidian topology and F the Sierpinski-space. By corollary 3.2 $(C(\mathbb{R},F), c\text{-lim})$ is compact; the pointwise topology τ_p is splitting and thus $\tau_p\text{-lim} \leq c\text{-lim}$.

But by example [15, (2.16) (b)] $Cc(\mathbb{R},F)$ is not closed in F^X.

The basic result that for conjoining topologies the (relative) compactness of $H \subseteq C(X,Y)$ implies that H is evenly continuous is well-known ([10, chapt. 7, theorem 20]; [1, theorem 32]; [15, theorem 3.21]).

For a concrete formulation we take here [1, theorem 32]:

Let X be a topological space, Y a Hausdorff topological space and let $H \subseteq C(X,Y) \subseteq Y^X$. Let lim be a convergence structure for $C(X,Y)$ such that

1. H is in $(C(X,Y), \lim)$ relatively compact
2. lim is conjoining for $C(X,Y)$

Then H is evenly continuous.

In theorem [15, theorem (3.21)] X is a convergence space and Y is a Hausdorff pseudotopological convergence space.

Our next example shows that we cannot omit the assumption that Y is Hausdorff.

Example 3.4 We use the same space as in example 3.3. Again we have a space $Y = F$ which is not Hausdorff. We have here $H = C(\mathbb{R},F)$ and the convergence structure lim for $C(\mathbb{R},F)$ is the continuous convergence: $\lim = c\text{-lim}$; c-lim is conjoining for $C(\mathbb{R},F)$ and $C_c(\mathbb{R},F)$ is compact. We will show that $C(\mathbb{R},F)$ is not evenly continuous. $\mathbb{R}$ and F are first countable spaces and hence by [15, theorem 3.18] we can use sequences instead of filters or nets to characterize even continuity: For $n \in \mathbb{N}$, $n \geq 1$ let $A_n = \left[\frac{1}{n}, 1\right] \subseteq \mathbb{R}$ and χ_{A_n} denotes, as usual, the characteristic function of A_n. For $0 \in \mathbb{R}$ we find $\chi_{A_n} \to 0 \in F$, $\frac{1}{n} \to 0 \in \mathbb{R}$.

Now we assume that $C(\mathbb{R},F)$ is evenly continuous; then follows $\chi_{A_n}\left(\frac{1}{n}\right) \to 0 \in F$; since $\{0\}$ is open in F there exists $n_0 \in \mathbb{N}$: $\forall n \geq n_0$: $\chi_{A_n}\left(\frac{1}{n}\right) = 0 \in F$, but $\forall n \in \mathbb{N}, n \geq 1 : \chi_{A_n}\left(\frac{1}{n}\right) = 1 \in F$, a contradiction.

Remark 3.5 Example 3.4 of course works for assertion 1 of theorem 2.2 too. Since $Cc(\mathbb{R}, F)$ also is relatively compact and c-lim is conjoining our example shows that we cannot omit in assertion 1 of [1, theorem 33] that Y is Hausdorff.

We consider a nice topological space Y, meaning that Y is at least Hausdorff and a topology τ for $C(X, Y)$. The fact that $H \subseteq C(X, Y)$ is τ-compact must not imply that H is evenly continuous if τ is not conjoining for $C(X, Y)$. We will explain this situation by an example. As concrete topologies τ we consider the pointwise topology τ_p and the uniform topology τ_u.

Example 3.6 We use an example from classical analysis of a sequence of functions from

$$C([0,1], \mathbb{R}) : \forall (n, x) \in (\mathbb{N} - \{0\}) \times [0,1] : f_n : f_n(x) = \frac{nx}{1 + (nx)^2}, f_0 : \forall x \in [0,1] : f_0(x) = 0 \,.$$

then holds:

1. $f_n \xrightarrow{\tau_p} f_0$
2. (f_n) does not converge uniformly to f_0

Proof. 1. $\forall n, n \geq 1 : f_n(0) = 0 \to 0 = f_0(0)$;
$\forall x \in (x, 1] : \frac{(nx)^2}{1+(nx)^2} \leq 1 \Longrightarrow |f_n| = f_n = \frac{nx}{1+(nx)^2} \leq \frac{1}{x} \cdot \frac{1}{n} \to 0$, hence $|f_n(x) - f_0(x)| \to 0$ for $n \to +\infty$.

2. $\forall n \geq 1 : x = \frac{1}{n} \in (0, 1]$ and $f_n\left(\frac{1}{n}\right) = \frac{1}{2}$. But then (f_n) cannot converges uniformly to f_0 on $[0, 1]$.

Now let be $H = \{f_n | n \geq 1\} \cup \{f_0\} \subseteq C([0, 1], \mathbb{R})$.

Then holds:

1. H is τ_p-compact
2. H is not evenly continuous
3. τ_p is not conjoining for $C([0, 1], \mathbb{R})$.

Proof. 1. is obvious

2. (f_n) does not converge continuously to f_0 : $[0, 1]$ is compact (and Hausdorff) implying that then c-lim $= \tau_u$-lim, yielding that $f_n \to f_0$ uniformly, a contradiction.

   If we assume that H is evenly continuous then $f_n \xrightarrow{\tau_p} f_0 \Longrightarrow f_n \xrightarrow{c} f_0$ by the basis [1, theorem 31], a contradiction.

3. If τ_p is conjoining then c-lim $\leq \tau_p$-lim since c-lim is splitting (and conjoining) for $C([0, 1], \mathbb{R})$ implying $f_n \xrightarrow{\tau_p} f_0 \Longrightarrow f_n \xrightarrow{c} f_0$, a contradiction.

Finally, we will show that assertion 2 of theorem 2.2 is not true if condition (γ) is not fulfilled.

Example 3.7 Let be $X = Y = \mathbb{R}$ we consider the sequence (f_n), $n \in \mathbb{N}, n \geq 1 : f_n : f_n(x) = \frac{1}{n}x$, $\forall x \in \mathbb{R}$; again let be $f_0 : \forall x \in \mathbb{R} : f(x) = 0$, hence $\forall n \in \mathbb{N} : f_n \in C(\mathbb{R}, \mathbb{R})$.

$H = \{f_n | n \in \mathbb{N}, n \geq 1\}$; for fixed $x \in \mathbb{R} : H(x) = \left\{\frac{1}{n}x | n \in \mathbb{N}, n \geq 1\right\}$ is bounded and hence relatively compact in $Y = \mathbb{R}$. Thus condition (α) holds for H.

Moreover let be $x \in \mathbb{R}$;

$$\forall(\varepsilon, n) \in (0, +\infty) \times (\mathbb{N} - \{0\})$$

let be $\delta = \varepsilon$ and $y \in U_\delta(x) : |f_n(y) - f_n(x)| = |\frac{1}{n}y - \frac{1}{n}x| = \frac{1}{n}|y - x| \leq |y - x| < \varepsilon$; hence H is equicontinuous on $\mathbb{R}$ which implies that H is evenly continuous showing that condition (β) is fulfilled too.

We see at once that $f_n \xrightarrow{\tau_p} f_0$ and even $f_n \xrightarrow{c} f_0$ hold. $H \cup \{f_0\}$ is τ_p-compact; we have $f_0 \notin H$ and $f_m \neq f_n \forall (m, n) \in \mathbb{N} \times \mathbb{N}$, $m \neq n$, $m \geq 1$, $n \geq 1$, $f_n \xrightarrow{\tau_p} f_0$ in $C(\mathbb{R}, \mathbb{R}) \Longrightarrow f_n \xrightarrow{\tau_p} f_0$ in $\mathbb{R}^\mathbb{R}$: each τ_p-neighbourhood of f_0 in $\mathbb{R}^\mathbb{R}$ contains infinitively many functions f_n implying that f_0 is a τ_p-cluster point of H. Thus H is not τ_p-closed in $\mathbb{R}^\mathbb{R}$ and hence not τ_p-compact since $Y = \mathbb{R}$ is Hausdorff. $H \subseteq C(\mathbb{R}, \mathbb{R}) \Longrightarrow H$ is not τ_p-compact in $C(\mathbb{R}, \mathbb{R})$ implying that H is not c-lim-compact in $C(\mathbb{R}, \mathbb{R})$ since τ_p-lim $\leq c$-lim holds.

4 Duality and the Ascoli-Arzela theorems

In the introduction we mentioned that the equicontinuity of a subset $H \subseteq C(X, Y)$ can be characterized by embedding of X into a function space using the canonical map. In [8] this approach was extended to include even continuity and also the topological equicontinuity of Royden.

At length we find it in the book [12]. We want to consider here equicontinuity and even continuity. In [2], [4] and [5] R. Bartsch and I developed and studied a general duality system

$$(X, Y, X^d, X^{dd}, J : X \to X^{dd})$$

where X^d is the first dual space of X with respect to Y, X^{dd} is the second dual space of X w. r. t. Y and J denotes the canonical map as is known from classical duality examples.

And we can include these characterization of equicontinuity and even continuity into this general scheme:

Let X, Y be topological spaces and $H \subseteq C(X, Y)$. We can consider (H, τ_p) as the redefined first dual space of X w. r. t. Y according to [2, 4.3., p. 284]: $X^d = (H, \tau_p)$. by definition [2, 4.1.] we see that $H = X^d$ has no defect since in H there are no algebraic operations defined.

Hence by [2, definition 4.2.] and [4, definition 2.2.] the second dual space of X w. r. t. Y is $X^{dd} = C((H, \tau_p), Y)$. The canonical map

$$\begin{aligned} J : X \to C((H, \tau_p), Y), \\ \forall x \in X : Jx = \omega(x, \cdot)\,, \\ \omega(x, \cdot) : (H, \tau_p) \to Y\,; \\ \forall h \in H : \omega(x, \cdot)(h) = \omega(x, h) = h(x)\,. \end{aligned}$$

We now need the convergence structure of strict (strong) continuous convergence.

Generalizing a formulation, where sequences were used ([9]), in ([15, 2.25]) I defined:

Definition 4.1 *Let X, Y be topological spaces, Φ a filter in Y^X; we say that Φ converges strictly continuous to $f, \Phi \overset{\mathrm{str}\,c}{\longrightarrow} f$, iff for each $y \in Y$ and each filter φ on X : $f\varphi \to y \Longrightarrow \omega(\Phi \times \varphi) = \Phi(\varphi) \to y$.*

Remark 4.2

1. Of course, a net (f_i) from Y^X converges strictly continuous to $f \in Y^X$ iff for each $y \in Y$ and each net (x_k) from X holds: $f(x_k) \to y \Longrightarrow f_i(x_k) \to y$
2. $\mathrm{str}\, c$-lim is conjoining for $C(X, Y)$ since we see at once that c-lim $\leq \mathrm{str}\, c$-lim holds.
3. Strict continuous convergence has similar properties as of continuous convergence, especially $\mathrm{str}\, c$-lim is a pseudotopological convergence structure and if Y Hausdorff then $(C(X, Y), \mathrm{str}\, c\text{-lim})$ is Hausdorff too.
4. If X is compact and Hausdorff then c-lim $= \mathrm{str}\, c$-lim on $C(X, Y)$ (see [17], and also [13]).

Now we come to the characterizations of even/equi-continuity as already announced.

Proposition 4.3 *Let X, Y be topological spaces, $H \subseteq C(X, Y)$; equivalent are:*

(1) $J : X \to (C((H, \tau_p), Y), \mathrm{str}\, c\text{-lim})$ *is continuous*

(2) H *is evenly continuous*

Proof. (1)$\Longrightarrow$(2): $\forall (x, y) \in X \times Y$, for each net (x_k) in X s. th. $x_k \to x$, for each net (h_i) from H s. th. $h_i(x) \to y$ we want to show: $h_i(x_k) \to y$.

Now by (1) $x_k \to x \Longrightarrow Jx_k \to Jx$, meaning that $\omega(x_k, \cdot) \overset{\mathrm{str}\,c}{\longrightarrow} \omega(x, \cdot)$.

$$\forall k \in K : \omega(x_k, \cdot) : (H, \tau_p) \to Y$$

is continuous and $\omega(x, \cdot) : (H, \tau_p) \to Y$ is continuous by [2, lemma 4.1., (1)] and hence $\omega(x_k, \cdot), \omega(x, \cdot) \in C((H, \tau_p), Y)$.

By the definition of strict continuous convergence and since we know that $h_i(x) \to y$, which means $\omega(x,\cdot)(h_i) \to y$ we get at once:

$$\omega(x_k,\cdot)(h_i) = h_i(x_k) \to y\,.$$

Hence H is evenly continuous.

(2)$\Longrightarrow$(1): $\forall (x,y) \in X \times Y : \forall (x_k), (x_k)$ net from X s. th. $x_k \to x$, we will show: $Jx_k \to Jx$ w. r. t. str c-lim: $\omega(x_k,\cdot) \overset{\text{str}\,c}{\longrightarrow} \omega(x,\cdot)$: let (h_i) be a net from H such that $\omega(x,\cdot)(h_i) \to y$, hence $h_i(x) \to y$; now by (2): $x_k \to x$ and $h_i(x) \to y \Longrightarrow h_i(x_k) \to y$, meaning $\omega(x_k,\cdot)(h_i) \to y$. Thus $\omega(x_k,\cdot) \overset{\text{str}\,c}{\longrightarrow} \omega(x,\cdot)$ yielding that J is continuous.

Remark 4.4 Proposition 4.3 was proved in [12, theorem (13.16)]. But instead of strict continuity here was used the notion of Pettis-convergence:

[12, (13.7) Definition]. A net (f_i) from $H \subseteq C(X,Y)$ Pettis converges to f if for each $y \in Y$ and each neighborhood V of y there is a neighborhood W of y such that eventually $f_i(f^{-1}(W)) \subseteq V$.

But in [17] was shown that the two convergence structures are equivalent.

The following proposition was proved in [12, theorem (13.12)]. Our proof is somewhat more clear.

Proposition 4.5 *Let X be a topological and $(Y,\mathcal{A})$ an uniform space; let be $H \subseteq C(X,Y)$.*

Equivalent are:

(1) *H is equicontinuous*

(2) *$J : X \to (C((H,\tau_p),Y),\tau_u)$ is continuous*

Proof. (1)$\Longrightarrow$(2): $((x_k),x), (x_k)$ a net in $X, x \in X$; we want to show:

$$x_k \to x \Longrightarrow Jx_k = \omega(x_k,\cdot) \to \omega(x,\cdot) = Jx$$

w. r. t. the uniform topology $\tau_u : \forall V \in \mathcal{A}$, for (V,x) by (1) there exists a neighborhood

$$U \in \underline{U}(x) : \forall (y,h) \in U \times H : (h(y),h(x)) = (\omega(y,\cdot)(h), \omega(x,\cdot)(h)) \in V;$$
$$\exists k_0 \in K : \forall k \geq k_0 : x_k \in U\,.$$

Now we have:

$$\forall (k,h) \in \{k \in K | k \geq k_0\} \times H : x_k \in U \Longrightarrow (h(x_k),h(x)) = (\omega(x_k,h),\ w(x,h)) \in V$$

showing that $\omega(x_k,\cdot) \overset{\tau_u}{\longrightarrow} \omega(x,\cdot)$ holds.

(2)$\Longrightarrow$(1): $\forall(x,V) \in X \times \mathcal{A}$, $(H,V) = \{(p,q) \in C(H,Y) \times C(H,Y) \,|\, \forall h \in H : (p(h), q(h) \in V\}$; now $\omega(x,\cdot) \in C((H,\tau_p),Y)$; we consider

$$(H,V)(\omega(x,\cdot)) = \{p \in C(H,Y) | \forall h \in H : (p(h), \omega(x,\cdot)(h)) = (p(h), h(x)) \in V\}$$

is a τ_u-neighborhood of $\omega(x,\cdot)$. Hence $\exists U \in \underline{U}(x) : J(U) \subseteq (H,V)(\omega(x,\cdot))$ by (2) showing that holds:

$$\forall (y,h) \in \underline{U}(x) \times H \Longrightarrow (h(y), h(x)) = (\omega(y,\cdot)(h), \omega(x,\cdot)(h)) \in V\,,$$

since $\omega(y,\cdot) \in (H,V)(\omega(x,\cdot)$. But this means that H is equicontinuous.

A conjoining topology or convergence structure can be defined (or characterized) by the continuity of the evaluation map ω. And if we consider the definition of continuous convergence then it is nearby that a conjoining convergence structure also can be characterized in a suitable way using the embedding into the second dual.

This is our next result.

Proposition 4.6 *Let X, Y be topological spaces, let $H \subseteq C(X,Y)$ and let* lim *be a convergence structure on H (maybe also* lim *is defined on $C(X,Y)$ s. th. $(H, \lim)$ is a convergence space). We assume that τ_p-$\lim \leq \lim$ holds. Then are equivalent:*

(1) lim *is conjoining for H*

(2) $J : X \to (C((H,\lim),Y), c\text{-}\lim)$ *is continuous*

Proof. We know that lim is conjoining for H iff $\omega = \omega(\cdot,\cdot) : X \times (H,\lim) \to Y$ is continuous.

(1)$\Longrightarrow$(2): $\forall(x,(x_k)), x \in X, (x_k)$ a net from X s. th. $x_k \to x$. We will show:

$$Jx_k \xrightarrow{c} Jx, \text{ hence } \omega(x_{x_k},\cdot) \xrightarrow{c} \omega(x,\cdot)\,.$$

Since τ_p-$\lim \leq \lim$ holds:

$$\forall k \in K, \forall x \in X : \omega(x_k,\cdot), \omega(x,\cdot) \in C((H,\lim),Y)\,.$$

Let (h_i) a net from $H, h \in H$ and $h_i \xrightarrow{\lim} h$; now

$$x_k \to x, h_i \xrightarrow{\lim} h \Longrightarrow \omega(x_k,h_i) \to \omega(x,h)$$

since ω is continuous, hence

$$h_i(x_k) \to h(x) \Longrightarrow \omega(x_k,\cdot)(h_i) \to \omega(x,\cdot)(h)$$

showing that $Jx_k \xrightarrow{c} Jx$ wich means: J is continuous.

(2)$\Longrightarrow$(1): Let be (x_k) a net from X, $x_k \to x \in X$, (h_i) a net from H s.th. $h_i \xrightarrow{\lim} h \in H$; by (2): $x_k \to x \Longrightarrow \omega(x_k, \cdot) \xrightarrow{c} \omega(x, \cdot)$; but then

$$h_i \xrightarrow{\lim} h \Longrightarrow \omega(x_k, \cdot)(h_i) \longrightarrow \omega(x, \cdot)(h) \Longrightarrow \omega(x_k, h_i) \longrightarrow \omega(x, h)$$

yielding that lim is conjoining for H.

Corollary 4.7 *We use the assumptions of proposition 4.6*

1. *Let* $\lim = c\text{-}\lim$ *for* $C(X, Y)$*; since* $c\text{-}\lim$ *is conjoining for* $C(X, Y)$ *and hence for* $H \subseteq C(X, Y)$ *too we get:*

$$J : X \to (C((H, c\text{-}\lim), Y), c\text{-}\lim)$$

is continuous

Remark: *For* $H = C(X, Y)$ *this result was shown in* [11, *theorem* 3., 1.]

2. $\lim = \operatorname{str} c\text{-}\lim$ *is conjoining and hence we get:*

$$J : X \to (C((H, \operatorname{str} c\text{-}\lim), Y), c\text{-}\lim)$$

is continuous.

As already mentioned in our text [1, theorem 32] provides a necessary compactness criterion: for each conjoining topology or convergence structure: the compactness of $H \subseteq C(X, Y)$ implies that H is evenly continuous. But conversely we can't obtain a smooth sufficient criterion for an arbitrary conjoining convergence structure: We have a simple, but fundamental fact: pointwise convergence plus even continuity equals continuous convergence but not more. (see for instance [1, theorem 31]). And continuous convergence is the smallest conjoining convergence structure for $C(X, Y)$. Already in a paper from 1971 ([14, theorem 1]) I proved a necessary and sufficient compactness criterion for conjoining convergence structures. This criterion shows that one can't go beyond c-lim. With some slight improvements the original theorem reads as follows:

Theorem 4.8 *Let* X, Y *be topological spaces and* Y *is Hausdorff; let* $H \subseteq C(X, Y)$ *and* lim *be a pseudotopological convergence structure for* $C(X, Y)$*. We assume that* lim *is a conjoining convergence structure for* $C(X, Y)$*. Equivalent are:*

(1) H *is* lim-*compact*

(2) (α) $\forall x \in X$*:* $H(x)$ *is relatively compact*

(β) H *is evenly continuous*

(γ) H *is* τ_p*-closed in* Y^X

(δ) $\lim = c\text{-}\lim$ *on* H

Proof. (1)$\Longrightarrow$(2): Since lim is conjoining for $C(X,Y)$ we have c-lim $\le$ lim and hence H is also c-lim-compact. But then follow $(\alpha),(\beta),(\gamma)$ by theorem 2.2. We have c-lim $\le$ lim on H; now let be: $\forall(\psi,f)$, ψ ultrafilter on $C(X,Y)$, $f\in H$; let be $H\in\psi$ and $\psi\overset{c\text{-lim}}{\longrightarrow} f$; $(H,\lim)$ is compact and hence $\psi\overset{\lim}{\longrightarrow} g\in H$ and thus $\psi\overset{c\text{-lim}}{\longrightarrow} g$.

Y is Hausdorff by assumption and thus $(C(X,Y),c\text{-lim})$ is Hausdorff too implying: $g=f$. But then we see: $\psi\overset{c\text{-lim}}{\longrightarrow} f\Longrightarrow\psi\overset{\lim}{\longrightarrow} f$.since c-lim and lim are pseudotopological convergence spaces we get: lim $\le$ c-lim and hence lim $=$ c-lim on H.

(2)$\Longrightarrow$(1): Theorem 2.2 shows $(\alpha),(\beta) and (\gamma)\Longrightarrow H$ is c-lim compact in $C(X,Y)$; now $(H,c\text{-lim})$ compact and $(H,\lim)=(H,c\text{-lim})$ by (δ) implies that H is lim-compact too.

Concluding we will consider the two Ascoli-Arzela theorems in [12] (as announced in the introduction), where we (partially), use our notations:

Theorem [12, (13.15)] *Let X be a regular space and Y a uniform space. Then $H\subseteq C(X,Y)$ is compact w. r. t. a jointly continuous topology η if and only if*

(a) *H is η-closed*

(b) *$H(x)$ has compact closures for each $x\in X$*

(c) *the natural map*

$$J: X\to (C((H,\tau_p),Y),\tau_U)$$

is continuous.

By proposition 4.5 we know that condition (c) is equivalent to H being equicontinuous.

Now theorem 4.8 shows that in general (a), (b) and (c) of (13.15) do not imply the compactness of H for each conjoining topology η for $C(X,Y)$ (or for H). For instance, if X is not compact in general τ_u-lim is strictly stronger than c-lim. Look at our example 4.9. Thus the sufficient assertion of theorem (13.15) is wrong. Quite analogously we find that [12, theorem (13.21)] is not correct too.

Here we have even continuity instead of equicontinuity.

We come now to our last example.

Example 4.9 We consider again example 3.6. Now let be

$$H\subseteq C(\mathbb{R},\mathbb{R}),\ H=\left\{f_n:\forall x\in\mathbb{R}:f_n(x)=\frac{1}{n}x|n\in\mathbb{N}\right\}\cup\{f_0\}=\left\{\frac{1}{n}x|n\ge 1\right\}\cup\{f_0\},$$

where f_0 is the zerofunction on $\mathbb{R}$. We show that hold:

(1) H is equicontinuous and hence evenly continuous.

(2) $H(x)$ is compact for each $x\in\mathbb{R}$

(3) H is τ_p-compact

(4) H is in $\mathbb{R}^{\mathbb{R}}\tau_p$-closed

(5) H is c-lim-compact

(6) H is τ_{co}-compact

(7) H is τ_u-closed in $C(\mathbb{R},\mathbb{R})$

(8) H is not τ_u-compact

Proof. (1) For example 3.6 we showed that $H - \{f_0\}$ is equicontinuous, we show in the same manner that H is equicontinuous:

$$\forall (x,y) \in \mathbb{R}\times\mathbb{R},\ \forall n \geq 1 : |f_n(x) - f_n(y)| = \frac{1}{n}|x-y|;$$

$$\forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon) > 0,\ \ \delta := \varepsilon : \forall (x,y) \in \mathbb{R}\times\mathbb{R} : |x-y| < \delta \Longrightarrow \frac{1}{n}|x-y| \leq \frac{\delta}{n} \leq \varepsilon\,;$$

but also

$$|x-y| < \delta \Longrightarrow |f_0(x) - f_0(y)| = |0-0| \leq \varepsilon\,.$$

Thus H is uniformly equicontinuous and hence equicontinuous and evenly continuous.

(2) $\forall x \in \mathbb{R} : H(x) = \{f_n(x) | n \in \mathbb{N}\}$ is homeomorph to the compact set

$$\left\{\frac{1}{n} | n \in \mathbb{N},\ n \geq 1\right\} \cup \{0\} \subseteq \mathbb{R} = Y\,.$$

(3) $\forall x \in \mathbb{R} : f_n(x) = \frac{x}{n} \to 0$ showing $f_u \xrightarrow{\tau_p} f_0$ and hence $H = \{f_n | n \in \mathbb{N}\backslash\{0\}\} \cup \{f_0\}$ is τ_p-compact.

(4) H is τ_p-compact in $C(\mathbb{R},\mathbb{R}) \Longrightarrow H$ is τ_p-compact in $\mathbb{R}^{\mathbb{R}}$; $(\mathbb{R}^{\mathbb{R}}, \tau_p)$ is Hausdorff $\Longrightarrow H$ is in $\mathbb{R}^{\mathbb{R}}\tau_p$-closed.

(5) By theorem 2.2 from (1), (2) and (4) follows that H is in $C(\mathbb{R},\mathbb{R})$ c-lim-compact.

(6) $\mathbb{R} = X$ is locally compact and Hausdorff and thus τ_{co}-lim $= c$-lim, where τ_{co} is the compact-open topology. Then $(C(\mathbb{R},\mathbb{R}), c\text{-lim})$ is a topological space.

(7) The uniform topology τ_u in $\mathbb{R}^{\mathbb{R}}$ can be defined by the use of neighborhoods. And then we see that τ_u is first countable. Hence we can work with sequences.

We assume that H has a τ_u-accumulation point $g \in C(\mathbb{R},\mathbb{R})$; $g \notin H \Longrightarrow g \neq f_0$ on $\mathbb{R}$. Then there exists a sequence (f_n) from H s. th. $f_n \xrightarrow{\tau_u} g$; then holds $f_n \xrightarrow{\tau_p} g$ too. Otherwise $\forall n \in \mathbb{N} : f_n \in H$ and (f_n) cannot be a constant sequence. Hence we find a subsequence (f_{n_k}) s. th. $f_{n_k} \xrightarrow{\tau_p} f_0$ implying that $f_{n_k} \xrightarrow{\tau_p} g$; but then $g = f_0$ because $(C(\mathbb{R},\mathbb{R}), \tau_p)$ is Hausdorff; $g = f_0$ yields a contradiction.

Thus H is τ_u-closed.

(8) We assume that H is τ_u-compact; since H consists of a sequence there exists a subsequence (g_{n_k}) of (f_n) and a $g \in H$ s. th. $g_{n_k} \xrightarrow{\tau_u} g$ yielding $g_{n_k} \xrightarrow{\tau_p} g$ too. But then we know from the proof of (7) that $g = f_0$ holds.

Now $\{g_{n_k} | k \in \mathbb{N}\}$ is an infinite set of unbounded functions on $\mathbb{R}$ showing that $g_{n_k} \xrightarrow{\tau_u} f_0$ is not possible, a contradiction. Hence H is not τ_u-compact.

Remarks 1. Here we have again an concrete example which shows that in general does not hold: (f_n) is converging pointwise, (f_n) is equicontinuous implies that $f_n)$ converges uniformly.

2. What is the result of example 4.9?

The uniform topology τ_u (for $C(\mathbb{R}, \mathbb{R})$) is conjoining. By assertions (1), (2), (7) of 4.9 we see that the assumptions of [12, theorem (13.15)] are fullfilled.

Thus this theorem asserts that H is τ_u-compact, but this contradicts assertion (8) of 4.9 which states that H is not τ_u-compact.

Since H is evenly continuous too our example also works for [12, theorem (13.21)] yielding that the sufficient assertion of this theorem also is wrong.

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Applications of Carleman inequalities for a two-by-two parabolic system in an unbounded guide

ABSTRACT. In this article we consider the inverse problem of determining some of the coefficients of a two-by-two parabolic system defined on an unbounded guide. Using an adapted Carleman estimate, we establish local stability results for at least two coefficients of this system in any finite portion of the guide. These estimates are obtained with data of the solution at a fixed time and boundary measurements for observations.

KEY WORDS. inverse problems, Carleman inequalities, heat operator, system, unbounded guide

1 Introduction

Let ω be a bounded connex domain in $\mathbb{R}^{n-1}$, $n \geq 2$ with C^2 boundary. Denote $\Omega := \mathbb{R} \times \omega$, $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$. We consider the following system

$$\begin{cases} \partial_t u - \Delta u + au + bv = g_1 \ \text{ in } \ Q, \\ \partial_t v - \Delta v + cu + dv = g_2 \ \text{ in } \ Q, \\ u = h_1 \text{ and } v = h_2 \ \text{ on } \ \Sigma, \\ u(x, 0) = u_0(x) \text{ and } v(x, 0) = v_0(x) \ \text{ in } \ \Omega, \end{cases} \tag{1.1}$$

where a, b, c, d are bounded coefficients defined on Ω such that

$$a, b, c, d \in \Lambda(M_0) := \{f \in L^\infty(\Omega), \|f\|_{L^\infty(\Omega)} \leq M_0\} \text{ for some } M_0 > 0.$$

Our inverse problem is to estimate at least two coefficients between a, b, c, d from the data of the solution (u, v) at $T/2$ and the measurement of (u, v) on a part of the boundary.
We will consider (u, v) (resp. $(\widetilde{u}, \widetilde{v})$) a solution of (1.1) associated with (a, b, c, d, u_0, v_0, g_1, g_2, h_1, h_2) (resp. ($\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$, $\widetilde{u_0}$, $\widetilde{v_0}$, g_1, g_2, h_1, h_2)) and two positive reals l, L such that $l < L$. Denote

$$\Omega_L = (-L, L) \times \omega \text{ and } \Omega_l = (-l, l) \times \omega.$$

The first result of this paper gives a Hölder result (3.3) for the coefficients b and c in the case where $\widetilde{a} = a$, $\widetilde{d} = d$ and is the following (see Theorem 3.1)

$$\|b-\widetilde{b}\|^2_{L^2(\Omega_l)} + \|c-\widetilde{c}\|^2_{L^2(\Omega_l)} \leq K\left(\|(u-\widetilde{u})(.,\frac{T}{2})\|^2_{H^2(\Omega_L)} + \|(v-\widetilde{v})(.,\frac{T}{2})\|^2_{H^2(\Omega_L)}\right.$$
$$\left. + \int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u-\tilde{u}))|^2 + |\partial_\nu(\partial_t^k(v-\tilde{v}))|^2)\,d\sigma\,dt\right)^\kappa$$

where K is a positive constant, $\kappa \in (0,1)$, γ_L is a part of the boundary (see (2.2)), and assuming that the hypothesis (3.2) is satisfied.

The second result (3.15) of this paper is also a Hölder stability result for the four coefficients a, b, c, d (see Theorem 3.2)

$$\|a-\widetilde{a}\|^2_{L^2(\Omega_l)} + \|b-\widetilde{b}\|^2_{L^2(\Omega_l)} + \|c-\widetilde{c}\|^2_{L^2(\Omega_l)} + \|d-\widetilde{d}\|^2_{L^2(\Omega_l)}$$
$$\leq K\left(\|\sum_{k=0}^{1}\partial_t^k(u-\widetilde{u})(.,\frac{T}{2})\|^2_{H^2(\Omega_L)} + \|\sum_{k=0}^{1}\partial_t^k(v-\widetilde{v})(.,\frac{T}{2})\|^2_{H^2(\Omega_L)}\right.$$
$$\left. + \int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u-\tilde{u}))|^2 + |\partial_\nu(\partial_t^k(v-\tilde{v}))|^2)\,d\sigma\,dt\right)^\kappa$$

with stronger hypotheses (3.13) and (3.14) than those in Theorem 3.1 (see (3.2)).

The third theorem of this paper gives a Hölder stability result (3.34) (see Theorem 3.3) for the following reaction-diffusion system

$$\begin{cases} \partial_t u - \Delta u + au + bv + A_1\cdot\nabla u + A_2\cdot\nabla v = g_1 & \text{in } Q,\\ \partial_t v - \Delta v + cu + dv + A_3\cdot\nabla u + A_4\cdot\nabla v = g_2 & \text{in } Q,\\ u = h_1 \text{ and } v = h_2 & \text{on } \Sigma,\\ u(x,0) = u_0(x) \text{ and } v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where all the coefficients $a, b, c, d, A_1, A_2, A_3, A_4$ are bounded ($a, b, c, d \in \Lambda(M_0)$ and $A_1, A_2, A_3, A_4 \in \Lambda(M_0)^n \cap H^1(\Omega)^n$). We obtain a stability result for the coefficients b and A_3 (assuming A_3 has the form $A_3 = \nabla g$) with the same kind of observations in the right-hand side of (3.34) as we have obtained in (3.3) or (3.15). Assuming that the Assumptions (3.32) and (3.33) hold, we get the following result

$$\|b-\widetilde{b}\|^2_{L^2(\Omega_l)} + \|A_3-\widetilde{A_3}\|^2_{(L^2(\Omega_l))^n}$$
$$\leq K\left(\|\sum_{k=0}^{1}\partial_t^k(u-\widetilde{u})(.,\frac{T}{2})\|^2_{H^2(\Omega_L)} + \|\sum_{k=0}^{1}\partial_t^k(v-\widetilde{v})(.,\frac{T}{2})\|^2_{H^3(\Omega_L)}\right.$$
$$\left. + \int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u-\tilde{u}))|^2 + |\partial_\nu(\partial_t^k(v-\tilde{v}))|^2)\,d\sigma\,dt\right)^\kappa.$$

Of course each of these above stability results implies an uniqueness result.
Up to our knowledge, there are few results concerning the simultaneous identification of more than one coefficient in each equation (see for example [1] and also [5] where the authors give a stability result for the diffusion coefficient a and the potential b of the Schrödinger operator $i\partial_t q + a\Delta q + bq$). In previous papers, stability results have been obtained for parabolic systems but, as far as we know, these papers have investigated the case of bounded domains and have given results with observations on a subdomain of their domain ([1, 7]...). Furthermore, there is no result for a two-by-two parabolic system with only one observation on a part of the boundary and without any data of the solution at a fixed time even in a bounded domain. We will use here the global Carleman estimate (2.5) for one equation given in [3] based on a classical Carleman estimate given in [12, 13]. Our choice of weight functions is adapted for this unbounded domain but will give us Hölder, and not Lipschitz, estimates of the coefficients. Recall that the method using Carleman estimates for solving inverse problems has been initiated by [2]. Our results extend to a system previous results for one equation defined on an unbounded guide (see [3] for the heat operator $\partial_t u - \Delta u + qu$ and [4] for the heat operator $\partial_t u - \nabla \cdot (c\nabla u)$ where stability results are given either for the potential q or for the diffusion coefficient c).
This Paper is organized as follows. In section 2, we specify the weight functions used for our Carleman estimate (cf (2.1), (2.3)) and due to the particular symmetric form of these weight functions with respect to x_1 and $t - T/2$ we recall from [3] the inequality (2.4), crucial for our final estimates (3.3), (3.15) and (3.34). Then in section 3 we state and prove our stability results, first for the coefficients b, c, after for a, b, c, d and finally for b, A_3.

2 Carleman estimate

Denote $Q_L = \Omega_L \times (0,T) = (-L,L) \times \omega \times (0,T)$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $x' = (x_2, \cdots, x_n)$ and define the operator

$$Au = \partial_t u - \Delta u.$$

Let $l > 0$, following [3] in this section, we consider some positive real $L > l$ and choose $a \in \mathbb{R}^n \setminus \Omega$ such that if

$$\tilde{d}(x) = |x' - a'|^2 - x_1^2 \text{ for } x \in \Omega_L, \text{ then } \tilde{d} > 0 \text{ in } \Omega_L, \ |\nabla \tilde{d}| > 0 \text{ in } \overline{\Omega_L}. \tag{2.1}$$

Moreover define

$$\Gamma_L = \{x \in \partial\Omega_L, \ < x - a, \nu(x) > \geq 0\} \text{ and } \gamma_L = \Gamma_L \cap \partial\Omega. \tag{2.2}$$

Here $< .,. >$ denotes the usual inner product in $\mathbb{R}^n$ and $\nu(x)$ the outward unit normal vector to $\partial\Omega_L$ at x. Notice that γ_L does not contain any cross section of the guide. From [12] we consider weight functions as follows: for $t \in (0,T)$, if $M_1 > \sup_{0<t<T}(t-T/2)^2 = (T/2)^2$,

$$\psi(x,t) = \tilde{d}(x) - \left(t - \frac{T}{2}\right)^2 + M_1, \quad \text{and } \phi(x,t) = e^{\lambda\psi(x,t)}. \tag{2.3}$$

The constant $\lambda > 0$ will be set in Proposition 2.2 and is usually used as a large parameter in Carleman inequalities. Since we will not use it, we will consider λ fixed in the article. We recall from [3] the following result.

Proposition 2.1 *There exists $T > 0$, $L > l$, $a \in \mathbb{R}^2 \setminus \Omega$ and $\tilde{\epsilon} > 0$ such that (2.1) holds and, setting*

$$O_{L,\tilde{\epsilon}} = (\Omega_L \times ((0, 2\tilde{\epsilon}) \cup (T - 2\tilde{\epsilon}, T))) \cup (((-L, -L+2\tilde{\epsilon}) \cup (L - 2\tilde{\epsilon}, L)) \times \omega \times (0,T)),$$

we have

$$d_1 < d_0 < d_2 \tag{2.4}$$

where

$$d_0 = \inf_{\Omega_l} \phi(., \frac{T}{2}), \;\; d_1 = \sup_{O_{L,\tilde{\epsilon}}} \phi \text{ and } d_2 = \sup_{\overline{\Omega_L}} \phi(., \frac{T}{2}).$$

We will use the following notations: Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index with $\alpha_i \in \mathbb{N} \cup \{0\}$. We set $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and define

$$H^{2,1}(Q_L) = \{u \in L^2(Q_L), \partial_x^\alpha \partial_t^{\alpha_{n+1}} u \in L^2(Q_L), |\alpha| + 2\alpha_{n+1} \leq 2\}$$

endowed with its norm

$$\|u\|^2_{H^{2,1}(Q_L)} = \sum_{|\alpha|+2\alpha_{n+1}\leq 2} \|\partial_x^\alpha \partial_t^{\alpha_{n+1}} u\|^2_{L^2(Q_L)}.$$

We recall here a global Carleman-type estimate proved in [3], based on a classical Carleman estimate (see Yamamoto [12, Theorem 7.3]).

Proposition 2.2 *There exist a value of $\lambda > 0$ and positive constants s_0 and $C = C(\lambda, s_0)$ such that*

$$I(u) := \int_{Q_L} \left(\frac{1}{s\phi}(|\partial_t u|^2 + |\Delta u|^2) + s\phi\, |\nabla u|^2 + s^3\phi^3 |u|^2 \right) e^{2s\phi} dx\, dt$$

$$\leq C\|e^{s\phi} A u\|^2_{L^2(Q_L)} + Cs^3 e^{2sd_1} \|u\|^2_{H^{2,1}(Q_L)} + Cs \int_{\gamma_L \times (0,T)} |\partial_\nu u|^2 e^{2s\phi} d\sigma\, dt, \tag{2.5}$$

for all $s > s_0$ and all $u \in H^{2,1}(Q_L)$ satisfying $u(.,0) = u(.,T) = 0$ in Ω_L, $u = 0$ on $\partial\Omega_L \times (0,T)$. We denote $\partial_\nu u = \nu \cdot \nabla u$.

In fact the above Proposition 2.2 is still valid for a more general function u: we can replace the condition $u = 0$ on $\partial\Omega_L \times (0,T)$ in Proposition 2.2 by $u = 0$ on $(\partial\Omega \cap \partial\Omega_L) \times (0,T)$. Since the method of Carleman estimates requires several time differentiations, we assume in the following that u, v (resp. $\widetilde{u}, \widetilde{v}$) belong to $\mathcal{H} = H^3(0,T,H^3(\Omega))$ satisfying the a-priori bound

$$\|u\|_{\mathcal{H}} < M_2 \text{ and } \|v\|_{\mathcal{H}} < M_2 \text{ for given } M_2 > 0.$$

From now on, we use the notation $w(\frac{T}{2}) = w(., \frac{T}{2})$ for any function w.

3 Inverse problems

3.1 The first result

Consider here (u, v) (resp. $(\widetilde{u}, \widetilde{v})$) a strong solution of (1.1) associated with $(a,\ b,\ c,\ d,\ u_0,\ v_0,\ g_1,\ g_2,\ h_1,\ h_2)$ (resp. $(a,\ \widetilde{b},\ \widetilde{c},\ d,\ \widetilde{u_0},\ \widetilde{v_0},\ g_1,\ g_2,\ h_1,\ h_2)$). Assume that all the coefficients $a, b, c, d, \widetilde{b}, \widetilde{c}$ belong to $\Lambda(M_0)$. From [8, Lemma 4.2], we derive the following result, also used in [3]

Lemma 3.1 *There exist some positive constants C, s_1 such that*

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}|z(T/2)|^2 dx \le Cs\int_{Q_L} e^{2s\phi}\phi^2|z|^2 dx\ dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}|\partial_t z|^2 dx\ dt,$$

for all $s \ge s_1$ and $z \in H^1(0,T;L^2(\Omega_L))$.

For the sake of completeness, we recall its proof.

Proof. Consider η defined by (3.4) and any $w \in H^1(0,T;L^2(\Omega_L))$. Since $\eta(\frac{T}{2}) = 1$ and $\eta(0) = 0$, we have

$$\int_{\Omega_L} w(x,T/2)^2 dx = \int_{\Omega_L} (\eta(T/2)w(x,T/2))^2 dx = \int_{\Omega_L}\int_0^{T/2} \partial_t(\eta^2(t)|w(x,t)|^2) dt\ dx$$

$$= 2\int_0^{T/2}\int_{\Omega_L} \eta^2(t)w(x,t)\partial_t w(x,t) dx\ dt + 2\int_0^{T/2}\int_{\Omega_L} \eta(t)\partial_t\eta(t)|w(x,t)|^2 dx\ dt.$$

As $0 \le \eta \le 1$, using Young's inequality, it comes that for any $s > 0$,

$$\int_{\Omega_L} w(x,T/2)^2\ dx \le Cs\int_{Q_L} |w|^2 dx\ dt + \frac{C}{s}\int_{Q_L} |\partial_t w|^2 dx\ dt. \tag{3.1}$$

Then we can conclude replacing w by $e^{s\phi}z$ in (3.1). □

We can state our first main result for a two-by-to linear system which extend precedent results for one equation (see [3] and [4]). We do not follow here the proof of [1, Theorem 1.2] and rather use ideas from [3].

Theorem 3.1 *Let $l > 0$. Let $T > 0$, $L > l$ and $a \in \mathbb{R}^n \setminus \Omega$ satisfying the conditions of Proposition 2.1. We make the following assumption*

$$|\widetilde{u}(., \frac{T}{2})| \geq R \text{ and } |\widetilde{v}(., \frac{T}{2})| \geq R \text{ in } \Omega_L \text{ for some } R > 0. \tag{3.2}$$

Then there exists a sufficiently small number δ_0 such that if $\delta \in (0, \delta_0)$,

$$\|(u-\widetilde{u})(., \frac{T}{2})\|^2_{H^2(\Omega_L)} + \|(v-\widetilde{v})(., \frac{T}{2})\|^2_{H^2(\Omega_L)} + \int_{\gamma_L \times (0,T)} \sum_{k=0}^{2} (|\partial_\nu(\partial_t^k(u-\tilde{u}))|^2 + |\partial_\nu(\partial_t^k(v-\tilde{v}))|^2) d\sigma dt \leq \delta$$

then the following Hölder stability estimate holds

$$\|b-\widetilde{b}\|^2_{L^2(\Omega_l)} + \|c-\widetilde{c}\|^2_{L^2(\Omega_l)} \leq K\delta^\kappa \text{ for all } \delta \in (0, \delta_0). \tag{3.3}$$

Here, $K > 0$ and $\kappa \in (0,1)$ are two constants depending on R, r, L, l, M_0, M_1, M_2, T and a.

Proof. Let χ, η be C^∞ cut-off functions defined by $\chi, \nabla\chi, \Delta\chi \in \Lambda(M_0)$, $0 \leq \chi \leq 1$, $0 \leq \eta \leq 1$,

$$\chi(x) = 0 \text{ if } x \in ((-\infty, -L+\tilde{\epsilon}) \cup (L-\tilde{\epsilon}, +\infty)) \times \omega),$$

$$\chi(x) = 1 \text{ if } x \in (-L+2\tilde{\epsilon}, L-2\tilde{\epsilon}) \times \omega,$$

$$\eta(t) = 0 \text{ if } t \in (0, \tilde{\epsilon}) \cup (T-\tilde{\epsilon}, T),\ \eta(t) = 1 \text{ if } t \in \times(2\tilde{\epsilon}, T-2\tilde{\epsilon}). \tag{3.4}$$

Denote also

$$y = u - \widetilde{u},\ y_0 = \chi\eta y,\ y_1 = \partial_t y_0,\ y_2 = \partial_t y_1,\ z = v - \widetilde{v},\ z_0 = \chi\eta z,\ z_1 = \partial_t z_0 \text{ and } z_2 = \partial_t z_1.$$

Note that (y_0, z_0) satisfies

$$\begin{cases} \partial_t y_0 - \Delta y_0 + a y_0 + b z_0 = \rho_1 := (\widetilde{b} - b)\chi\eta\widetilde{v} + (\partial_t\eta)\chi y - (\Delta\chi)\eta y - 2\nabla\chi \cdot \nabla(\eta y) \text{ in } Q_L, \\ \partial_t z_0 - \Delta z_0 + c y_0 + d z_0 = \rho_2 := (\widetilde{c} - c)\chi\eta\widetilde{u} + (\partial_t\eta)\chi z - (\Delta\chi)\eta z - 2\nabla\chi \cdot \nabla(\eta z) \text{ in } Q_L, \\ y_0 = z_0 = 0 \text{ on } \partial\Omega_L \times (0,T). \end{cases} \tag{3.5}$$

and (y_1, z_1), (y_2, z_2) satisfy

$$\begin{cases} \partial_t y_1 - \Delta y_1 + a y_1 + b z_1 = \partial_t \rho_1 \text{ in } Q_L, \\ \partial_t z_1 - \Delta z_1 + c y_1 + d z_1 = \partial_t \rho_2 \text{ in } Q_L, \\ y_1 = z_1 = 0 \text{ on } \partial\Omega_L \times (0,T) \end{cases} \text{ and } \begin{cases} \partial_t y_2 - \Delta y_2 + a y_2 + b z_2 = \partial_t^2 \rho_1 \text{ in } Q_L, \\ \partial_t z_2 - \Delta z_2 + c y_2 + d z_2 = \partial_t^2 \rho_2 \text{ in } Q_L, \\ y_2 = z_2 = 0 \text{ on } \partial\Omega_L \times (0,T). \end{cases}$$

• First step: Applying (3.5) for $t = \frac{T}{2}$, if we denote

$$J := \int_{\Omega_L} e^{2s\phi(\frac{T}{2})}\chi^2[|c-\widetilde{c}|^2|\widetilde{u}(\frac{T}{2})|^2 + |b-\widetilde{b}|^2|\widetilde{v}(\frac{T}{2})|^2]\ dx$$

then we get

$$J \le Ce^{2sd_2}F_0(\frac{T}{2}) + C\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}(|\partial_t y_0(\frac{T}{2})|^2 + |\partial_t z_0(\frac{T}{2})|^2)\ dx$$

with

$$F_0(T/2) = \|z_0(T/2)\|^2_{H^2(\Omega_L)} + \|z(T/2)\|^2_{H^1(\Omega_L)} + \|y_0(T/2)\|^2_{H^2(\Omega_L)} + \|y(T/2)\|^2_{H^1(\Omega_L)}.$$

Note that

$$F_0(T/2) \le CF(T/2) \text{ with } F(T/2) = \|y(T/2)\|^2_{H^2(\Omega_L)} + \|z(T/2)\|^2_{H^2(\Omega_L)}.$$

Moreover, since $\partial_t y_0 = y_1$, $\partial_t z_0 = z_1$ and $1 \le \phi$, using Lemma 3.1, we obtain

$$J \le Ce^{2sd_2}F(T/2) + Cs\int_{Q_L} e^{2s\phi}\phi^3(|y_1|^2+|z_1|^2)\ dx\ dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}\phi^3(|y_2|^2+|z_2|^2)\ dx\ dt. \quad (3.6)$$

• Second step: Now we evaluate J with the Carleman inequalities (2.5) for y_i and z_i, $i = 1, 2$. Note that all the terms in $\|e^{s\phi}Ay_i\|^2_{L^2(Q_L)}$ or $\|e^{s\phi}Az_i\|^2_{L^2(Q_L)}$ with derivatives of χ or η will be bounded above by Ce^{2sd_1} with C a positive constant. Therefore, for s sufficiently large, there exists a positive constant C such that

$$I(y_i) + I(z_i) \le C\int_{Q_L} e^{2s\phi}\chi^2[|c-\widetilde{c}|^2 + |b-\widetilde{b}|^2]\ dx\ dt + C\int_{Q_L} e^{2s\phi}(|y_i|^2 + |z_i|^2)\ dx\ dt + Ce^{2sd_1}$$
$$+Cs^3e^{2sd_1}(\|y_i\|^2_{H^{2,1}(Q_L)} + \|z_i\|^2_{H^{2,1}(Q_L)}) + Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt.$$

Since $e^{2s\phi} \le e^{2s\phi(T/2)}$, we deduce that

$$I(y_i) + I(z_i) \le C\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2[|c-\widetilde{c}|^2 + |b-\widetilde{b}|^2]\ dx\ dt + Cs^3e^{2sd_1}$$
$$+Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt.$$

Thus

$$s^3\int_{Q_L} e^{2s\phi}\phi^3(|y_i|^2 + |z_i|^2)\ dx\ dt \le C\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2[|c-\widetilde{c}|^2 + |b-\widetilde{b}|^2]\ dx$$
$$+ Cs^3e^{2sd_1} + Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt. \quad (3.7)$$

Therefore, from (3.6) and (3.7), we get for s sufficiently large

$$J \leq Ce^{2sd_2}F(T/2) + \frac{C}{s^2}\left(s^3e^{2sd_1} + \int_{\Omega_L} e^{2s\phi(T/2)}\chi^2[|c-\widetilde{c}|^2| + |b-\widetilde{b}|^2]\,dx\right.$$

$$\left. + s\int_{\gamma_L\times(0,T)} e^{2s\phi}\sum_{i=1}^{2}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\,d\sigma\,dt\right).$$

So we have

$$J \leq Ce^{2sd_2}G(T/2) + Cse^{2sd_1} + \frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2[|c-\widetilde{c}|^2| + |b-\widetilde{b}|^2]\,dx \tag{3.8}$$

with

$$G(T/2) = F(T/2) + \int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu\partial_t^k y|^2 + |\partial_\nu\partial_t^k z|^2)\,d\sigma\,dt.$$

• Third and last step: In this step, we come back to the coefficients $b-\tilde{b}$ and $c-\tilde{c}$. First, from the hypothesis (3.2) we derive from (3.8), for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\widetilde{b}-b|^2 + |\widetilde{c}-c|^2)\,dx \leq Ce^{2sd_2}G(T/2) + Cse^{2sd_1}. \tag{3.9}$$

Moreover, since $e^{2sd_0} \leq e^{2s\phi(T/2)}$ in Ω_l and $\chi = 1$ in Ω_l, we deduce from (3.9) that

$$e^{2sd_0}(\|\widetilde{b}-b\|^2_{L^2(\Omega_l)} + \|\widetilde{c}-c\|^2_{L^2(\Omega_l)}) \leq Ce^{2sd_2}G(T/2) + Cse^{2sd_1}.$$

This last inequality can be rewritten in the following form for s sufficiently large ($s \geq s_2$)

$$\|\widetilde{b}-b\|^2_{L^2(\Omega_l)} + \|\widetilde{c}-c\|^2_{L^2(\Omega_l)} \leq C(e^{2s(d_2-d_0)}G(T/2) + se^{2s(d_1-d_0)}). \tag{3.10}$$

Note that if $G(T/2) = 0$, since (3.10) holds for any $s \geq s_2$ and $d_1 - d_0 < 0$ we get (3.3). Now if $G(T/2) \neq 0$, we recall from (2.4) that $d_1 - d_0 < 0$ and $d_2 - d_0 > 0$ and optimize (3.10) with respect to s. Indeed denote

$$f(s) = e^{2s(d_2-d_0)}G(T/2) + e^{2s(d_1-d_0)} \text{ and } g(s) = e^{2s(d_2-d_0)}G(T/2) + se^{2s(d_1-d_0)}.$$

We have $f(s) \sim g(s)$ at infinity. Moreover the function f has a minimum in

$$s_3 = \frac{1}{2(d_2-d_1)}\ln\left(\frac{d_0-d_1}{(d_2-d_0)G(T/2)}\right) \text{ and } f(s_3) = K'G(T/2)^\kappa$$

with $\kappa = \frac{d_0-d_1}{d_2-d_1}$ and $K' = (\frac{d_0-d_1}{d_2-d_0})^{\frac{d_2-d_0}{d_2-d_1}} + (\frac{d_0-d_1}{d_2-d_0})^{\frac{d_1-d_0}{d_2-d_0}}$. Finally the minimum s_3 is sufficiently large ($s_3 \geq s_2$) if the following condition $G(T/2) \leq \delta_0$, with $\delta_0 = \frac{d_0-d_1}{(d_2-d_0)e^{2s_2(d_2-d_1)}}$, is satisfied. Then we get our result (3.3) and so we complete the proof of Theorem 3.1. □

Remark 1 • Note that the hypothesis (3.2) is quite usual (cf [1, 7] for a parabolic system in a bounded domain) and is removed in [1] by the control theory and in [7] by conditions on $a, \widetilde{b}, \widetilde{c}, d, \widetilde{u}_0, \widetilde{v}_0, h_1, h_2, g_1, g_2$. In some cases, one can also diagonalise the coupling matrix of the coefficients (see [6]) then use a parabolic positivity result (see [9, Theorem 13.5]) for the decoupling system. Of course we could obtain the same result as (3.3) for any coefficient in each equation of (1.1). But if we want to determine the coefficients b and d for example, we only have to assume that $|\widetilde{v}(., \frac{T}{2})| \geq R$ in Ω_L for some $R > 0$, instead of (3.2).
• In fact we can obtain in the right-hand side of (3.3) the term $\int_{\gamma_L\times(0,T)} \sum_{k=1}^{2}(|\partial_\nu(\partial_t^k(u - \tilde{u}))|^2 + |\partial_\nu(\partial_t^k(v-\tilde{v}))|^2)\, d\sigma\, dt$ instead of $\int_{\gamma_L\times(0,T)} \sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u-\tilde{u}))|^2 + |\partial_\nu(\partial_t^k(v-\tilde{v}))|^2)\, d\sigma\, dt$ if we slightly modify d_1 (if we define $d_1 = \sup_{\overline{O_{L,\tilde{\epsilon}}}} \phi$, the inequalities (2.4) still hold and all the terms inside the integrals on γ_L with derivatives of η are therefore bounded above by e^{2sd_1}).

3.2 The second result

Consider now (u, v) (resp. $(\widetilde{u}, \widetilde{v})$) a strong solution of (1.1) associated with $(a,\, b,\, c,\, d,\, u_0,\, v_0,\, g_1,\, g_2,\, h_1,\, h_2)$ (resp. $(\widetilde{a},\, \widetilde{b},\, \widetilde{c},\, \widetilde{d},\, \widetilde{u}_0,\, \widetilde{v}_0,\, g_1,\, g_2,\, h_1,\, h_2)$). Assume that all the coefficients $a, b, c, d, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$ belong to $\Lambda(M_0)$. For our second main result, first we need the following lemma inspired from Klibanov and Timonov ([11]). Recall that χ and η are defined by (3.4).

Lemma 3.2 *There exists a positive constant C such that*

$$\int_{Q_L} e^{2s\phi}\phi\chi^2\eta^2 \left(\int_{T/2}^{t} f(\xi)\, d\xi\right)^2 dx\, dt \leq \frac{C}{s}\left(e^{2sd_1} + \int_{Q_L} e^{2s\phi}\chi^2\eta^2 f^2\, dx\, dt\right),$$

for all $s > 0$ and $f \in L^2(0, T, L^2(\Omega_L)) \cap L^\infty(Q_L)$.

Proof. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_{Q_L} \phi\chi^2\eta^2 e^{2s\phi}\left(\int_{T/2}^{t} f(x,\xi)\, d\xi\right)^2 dx\, dt &\leq \int_{Q_L} \phi\chi^2\eta^2 e^{2s\phi}|t - \frac{T}{2}|\left|\int_{T/2}^{t} f(x,\xi)^2\, d\xi\right| dx\, dt \\ &\leq \int_{\Omega_L}\int_0^{T/2} \phi\chi^2\eta^2 e^{2s\phi}(\frac{T}{2} - t)\left|\int_{T/2}^{t} f(x,\xi)^2 d\xi\right| dxdt \\ &+ \int_{\Omega_L}\int_{T/2}^{T} \phi\chi^2\eta^2 e^{2s\phi}(t - \frac{T}{2})\left(\int_{T/2}^{t} f(x,\xi)^2 d\xi\right) dxdt.\end{aligned} \tag{3.11}$$

Note that

$$\partial_t(e^{2s\phi}) = -4s\lambda(t - \frac{T}{2})\phi e^{2s\phi}.$$

For the second integral of the right hand side of (3.11), since $\eta(T)=0$, by integration by parts we have

$$\int_{\Omega_L}\int_{T/2}^{T}\phi\chi^2\eta^2 e^{2s\phi}(t-\frac{T}{2})\left(\int_{T/2}^{t} f(x,\xi)^2 d\xi\right)\,dx\,dt$$

$$=-\frac{1}{4s\lambda}\int_{\Omega_L}\int_{T/2}^{T}\chi^2\eta^2\partial_t(e^{2s\phi})(\int_{T/2}^{t} f(x,\xi)^2 d\xi)\,dx\,dt$$

$$=-\frac{1}{4s\lambda}\int_{\Omega_L}\left[\chi^2\eta^2 e^{2s\phi}(\int_{T/2}^{t} f(x,\xi)^2 d\xi)\right]_{t=T/2}^{t=T} dx+\frac{1}{4s\lambda}\int_{\Omega_L}\int_{T/2}^{T} e^{2s\phi}\chi^2\eta^2 f^2\,dx\,dt$$

$$+\frac{1}{2s\lambda}\int_{\Omega_L}\int_{T/2}^{T} e^{2s\phi}\chi^2\eta\partial_t\eta(\int_{T/2}^{t} f(x,\xi)^2 d\xi)\,dx\,dt$$

$$=\frac{1}{2s\lambda}\int_{\Omega_L}\int_{T/2}^{T} e^{2s\phi}\chi^2\eta\partial_t\eta(\int_{T/2}^{t} f(x,\xi)^2 d\xi)\,dx\,dt+\frac{1}{4s\lambda}\int_{\Omega_L}\int_{T/2}^{T} e^{2s\phi}\chi^2\eta^2 f^2\,dx\,dt. \quad (3.12)$$

The first integral of (3.12) is bounded above by $\frac{C}{s}e^{2sd_1}$ due to the derivative of η. Therefore

$$\int_{\Omega_L}\int_{T/2}^{T}\phi\chi^2\eta^2 e^{2s\phi}(t-\frac{T}{2})\left(\int_{T/2}^{t} f(x,\xi)^2 d\xi\right)\,dx\,dt\le\frac{C}{s}\left(e^{2sd_1}+\int_{Q_L} e^{2s\phi}\chi^2\eta^2 f^2\,dx\,dt\right).$$

We obtain a similar result for the first integral of (3.11) and this concludes the proof of Lemma 3.2. □

Now we can state our second main result in view to obtain a stability estimate of the four coefficients of (1.1) with nearly the same observations that we obtained in Theorem 3.1 (see the right-hand sides of (3.3) and (3.15)).

Theorem 3.2 *Let $l>0$. Let $T>0$, $L>l$ and $a\in\mathbb{R}^n\setminus\Omega$ satisfying the conditions of Proposition 2.1. We make here the following assumptions*

$$|\widetilde{u}|\ge R \text{ and } |\partial_t(\frac{\widetilde{v}}{\widetilde{u}})|\ge R \text{ in } Q \text{ for some } R>0, \quad (3.13)$$

and

$$|\widetilde{v}|\ge R \text{ and } |\partial_t(\frac{\widetilde{u}}{\widetilde{v}})|\ge R \text{ in } Q \text{ for some } R>0. \quad (3.14)$$

Then there exists a sufficiently small number δ_0 such that if $\delta\in(0,\delta_0)$,

$$\|\sum_{k=0}^{1}\partial_t^k(u-\widetilde{u})(.,\frac{T}{2})\|^2_{H^2(\Omega_L)}+\|\sum_{k=0}^{1}\partial_t^k(v-\widetilde{v})(.,\frac{T}{2})\|^2_{H^2(\Omega_L)}$$

$$+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u-\tilde{u}))|^2+|\partial_\nu(\partial_t^k(v-\tilde{v}))|^2\,d\sigma\,dt\le\delta$$

then the following Hölder stability estimate holds

$$\|a-\widetilde{a}\|^2_{L^2(\Omega_l)}+\|b-\widetilde{b}\|^2_{L^2(\Omega_l)}+\|c-\widetilde{c}\|^2_{L^2(\Omega_l)}+\|d-\widetilde{d}\|^2_{L^2(\Omega_l)}\le K\delta^\kappa \text{ for all } \delta\in(0,\delta_0). \quad (3.15)$$

Here, $K>0$ and $\kappa\in(0,1)$ are two constants depending on R, r, L, l, M_0, M_1, M_2, T and a.

Proof. As in Thereom 3.1 denote $y=u-\widetilde{u}$ and $z=v-\widetilde{v}$. Then (y,z) satisfies

$$\begin{cases}\partial_t y-\Delta y+ay+bz=(\widetilde{a}-a)\widetilde{u}+(\widetilde{b}-b)\widetilde{v} \text{ in } Q,\\ \partial_t z-\Delta z+cy+dz=(\widetilde{c}-c)\widetilde{u}+(\widetilde{d}-d)\widetilde{v} \text{ in } Q,\\ y=z=0 \text{ on } \Sigma.\end{cases}$$

- First step: Let $y_1=\frac{y}{\widetilde{u}}$ and $z_1=\frac{z}{\widetilde{u}}$. Then (y_1,z_1) satisfies

$$\begin{cases}\partial_t y_1-\Delta y_1+ay_1+bz_1=f_1+\widetilde{a}-a+(\widetilde{b}-b)\frac{\widetilde{v}}{\widetilde{u}} \text{ in } Q,\\ \partial_t z_1-\Delta z_1+cy_1+dz_1=f_2+\widetilde{c}-c+(\widetilde{d}-d)\frac{\widetilde{v}}{\widetilde{u}} \text{ in } Q,\\ y_1=z_1=0 \text{ on } \Sigma,\end{cases}$$

with $f_1:=\frac{1}{\widetilde{u}}(-y_1\partial_t\widetilde{u}+y_1\Delta\widetilde{u}+2\nabla y_1\cdot\nabla\widetilde{u})$ and $f_2:=\frac{1}{\widetilde{u}}(-z_1\partial_t\widetilde{u}+z_1\Delta\widetilde{u}+2\nabla z_1\cdot\nabla\widetilde{u})$. Denote now $y_2=\partial_t y_1$, $z_2=\partial_t z_1$, $y_3=\frac{1}{\partial_t(\frac{\widetilde{v}}{\widetilde{u}})}y_2$ and $z_3=\frac{1}{\partial_t(\frac{\widetilde{v}}{\widetilde{u}})}z_2$. Then

$$\begin{cases}\partial_t y_2-\Delta y_2+ay_2+bz_2=\partial_t f_1+(\widetilde{b}-b)\partial_t(\frac{\widetilde{v}}{\widetilde{u}}) \text{ in } Q,\\ \partial_t z_2-\Delta z_2+cy_2+dz_2=\partial_t f_2+(\widetilde{d}-d)\partial_t(\frac{\widetilde{v}}{\widetilde{u}}) \text{ in } Q,\\ y_2=z_2=0 \text{ on } \Sigma,\end{cases}$$

and

$$\begin{cases}\partial_t y_3-\Delta y_3+ay_3+bz_3=f_3+\widetilde{b}-b \text{ in } Q,\\ \partial_t z_3-\Delta z_3+cy_3+dz_3=f_4+\widetilde{d}-d \text{ in } Q,\\ y_3=z_3=0 \text{ on } \Sigma,\end{cases} \quad (3.16)$$

with

$$f_3:=\frac{1}{\partial_t(\frac{\widetilde{v}}{\widetilde{u}})}\left(-y_3\partial_t^2(\frac{\widetilde{v}}{\widetilde{u}})+y_3\Delta(\partial_t(\frac{\widetilde{v}}{\widetilde{u}}))+2\nabla y_3\cdot\nabla(\partial_t(\frac{\widetilde{v}}{\widetilde{u}}))+\partial_t f_1\right)$$

and

$$f_4:=\frac{1}{\partial_t(\frac{\widetilde{v}}{\widetilde{u}})}\left(-z_3\partial_t^2(\frac{\widetilde{v}}{\widetilde{u}})+z_3\Delta(\partial_t(\frac{\widetilde{v}}{\widetilde{u}}))+2\nabla z_3\cdot\nabla(\partial_t(\frac{\widetilde{v}}{\widetilde{u}}))+\partial_t f_2\right).$$

Finally let $y_4=\partial_t y_3$, $z_4=\partial_t z_3$, $y_5=\chi\eta y_4$ and $z_5=\chi\eta z_4$. Then

$$\begin{cases}\partial_t y_5-\Delta y_5+ay_5+bz_5=\chi\eta\partial_t f_3+f_5 \text{ in } Q_L,\\ \partial_t z_5-\Delta z_5+cy_5+dz_5=\chi\eta\partial_t f_4+f_6 \text{ in } Q_L,\end{cases} \quad (3.17)$$

with

$$f_5=(\partial_t\eta)\chi y_4-(\Delta\chi)\eta y_4-2\eta\nabla\chi\cdot\nabla y_4$$

and

$$f_6 = (\partial_t \eta)\chi z_4 - (\Delta\chi)\eta z_4 - 2\eta\nabla\chi\cdot\nabla z_4.$$

Due to the truncation functions, we can apply the Carleman estimates for y_5 and z_5 and now we estimate $I(y_5)+I(z_5)$ with (2.5). We have

$$\begin{aligned} I(y_5)+I(z_5) \le C\int_{Q_L} e^{2s\phi}((Ay_5)^2+(Az_5)^2)dxdt + Cs^3e^{2sd_1}\\ + Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_5|^2+|\partial_\nu z_5|^2)d\sigma dt. \end{aligned} \tag{3.18}$$

As in Thereom 3.1, all the terms in $\int_{Q_L} e^{2s\phi}((Ay_5)^2+(Az_5)^2)\ dx\ dt$ with derivatives of η or χ will be bounded above by Ce^{2sd_1}. So since $\phi \ge 1$

$$\begin{aligned} \int_{Q_L} e^{2s\phi}((Ay_5)^2+(Az_5)^2)\ dx\ dt \le C\int_{Q_L} e^{2s\phi}(y_5^2+z_5^2)\ dx\ dt + Ce^{2sd_1}\\ +C\int_{Q_L} e^{2s\phi}\chi^2\eta^2(|\partial_t f_3|^2+|\partial_t f_4|^2)\ dx\ dt\\ \le C\int_{Q_L} e^{2s\phi}(y_5^2+z_5^2)dxdt + Ce^{2sd_1} + C\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2\sum_{i=1}^{4}(y_i^2+|\nabla y_i|^2+z_i^2+|\nabla z_i|^2)dxdt. \end{aligned} \tag{3.19}$$

Since $\chi\eta y_4 = y_5$ and $\chi\eta z_4 = z_5$, (3.19) implies

$$\begin{aligned} \int_{Q_L} e^{2s\phi}((Ay_5)^2+(Az_5)^2)\ dx\ dt \le C\int_{Q_L} e^{2s\phi}(y_5^2+z_5^2+|\nabla y_5|^2+|\nabla z_5|^2)\ dx\ dt + Ce^{2sd_1}\\ + C\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2\sum_{i=1}^{3}(y_i^2+|\nabla y_i|^2+z_i^2+|\nabla z_i|^2)\ dx\ dt. \end{aligned} \tag{3.20}$$

From (3.18)-(3.20), we get for s sufficiently large

$$\begin{aligned} I(y_5)+I(z_5) \le Cs^3e^{2sd_1} + C\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2\sum_{i=1}^{3}(y_i^2+|\nabla y_i|^2+z_i^2+|\nabla z_i|^2)\ dx\ dt\\ + Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_5|^2+|\partial_\nu z_5|^2)\ d\sigma\ dt. \end{aligned} \tag{3.21}$$

Using now Lemma 3.2 we have

$$\begin{aligned} \int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 y_1^2\ dx\ dt = \int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2\left(\int_{T/2}^{t} \partial_t y_1(\xi)d\xi + y_1(T/2)\right)^2\ dx\ dt\\ \le \frac{C}{s}e^{2sd_1} + \frac{C}{s}\int_{Q_L} e^{2s\phi}\chi^2\eta^2 y_2^2\ dx\ dt + C\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 y_1(T/2)^2\ dx\ dt \end{aligned}$$

$$\leq \frac{C}{s}e^{2sd_1} + \frac{C}{s}\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 y_3^2 \; dx \; dt + C\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 y_1(T/2)^2 \; dx \; dt$$

$$\leq \frac{C}{s}e^{2sd_1}+\frac{C}{s}\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 \left(\int_{T/2}^{t} \partial_t y_3(\xi)d\xi + y_3(T/2)\right)^2 dx \; dt + C\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 y_1(T/2)^2 dx \; dt$$

$$\leq \frac{C}{s}e^{2sd_1}+\frac{C}{s^2}\left(e^{2sd_1} + \int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 y_4^2 \; dx \; dt\right)+C\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2(y_1(T/2)^2+y_3(T/2)^2) \; dx \; dt$$

$$\leq \frac{C}{s}e^{2sd_1} + \frac{C}{s^2}\int_{Q_L} e^{2s\phi}y_5^2 \; dx \; dt + Ce^{2sd_2}\int_{\Omega_L}(y_1(T/2)^2 + y_2(T/2)^2) \; dx. \tag{3.22}$$

Doing the same for $\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 y_i^2 \; dx \; dt$, $\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2 z_i^2 \; dx \; dt$, $\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2|\nabla y_i|^2 \; dx \; dt$ and
$\int_{Q_L} \phi e^{2s\phi}\chi^2\eta^2|\nabla z_i|^2 \; dx \; dt$, for $i = 1, 2, 3$ we get from (3.21)-(3.22) and for s sufficienlty large

$$I(y_5) + I(z_5) \leq Cs^3e^{2sd_1} + Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_5|^2 + |\partial_\nu z_5|^2) \; d\sigma \; dt$$

$$+ Ce^{2sd_2}\int_{\Omega_L}\sum_{i=1}^{2}(y_i(T/2)^2 + z_i(T/2)^2 + |\nabla y_i(T/2)|^2 + |\nabla z_i(T/2)|^2) \; dx. \tag{3.23}$$

Note that (3.23) can be rewritten on the following form

$$I(y_5) + I(z_5) \leq Cs^3e^{2sd_1} + Cse^{2sd_2}\int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu\partial_t^k y|^2 + |\partial_\nu\partial_t^k z|^2) \; d\sigma \; dt$$

$$+Ce^{2sd_2}\int_{\Omega_L}\sum_{k=0}^{1}(\partial_t^k y(T/2)^2 + \partial_t^k z(T/2)^2 + |\nabla\partial_t^k y(T/2)|^2 + |\nabla\partial_t^k z(T/2)|^2) \; dx$$

and so

$$I(y_5) + I(z_5) \leq Cs^3e^{2sd_1} + Cse^{2sd_2}F_1(T/2) \tag{3.24}$$

with

$$F_1(T/2) = \int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu\partial_t^k y|^2+|\partial_\nu\partial_t^k z|^2) \; d\sigma \; dt+\sum_{k=0}^{1}(\|\partial_t^k y(T/2)\|^2_{H^1(\Omega_L)}+\|\partial_t^k z(T/2)\|^2_{H^1(\Omega_L)}).$$

• Second step: Now we evaluate (3.16) at $T/2$. We have

$$\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\widetilde{b} - b|^2 + |\widetilde{d} - d|^2) \; dx \leq C\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\partial_t y_3(T/2)|^2 + |\partial_t z_3(T/2)|^2) \; dx$$
$$+ Ce^{2sd_2}F_2(T/2)$$

with

$$F_2(T/2) = \sum_{i=1}^{2}(\|y_i(T/2)\|^2_{H^2(\Omega_L)} + \|z_i(T/2)\|^2_{H^2(\Omega_L)}).$$

So, since $\eta(T/2)=1$,

$$\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\widetilde{b}-b|^2+|\widetilde{d}-d|^2)\,dx \le C\int_{\Omega_L} e^{2s\phi(T/2)}(|y_5(T/2)|^2+|z_5(T/2)|^2)\,dx+Ce^{2sd_2}F_2(T/2). \tag{3.25}$$

Now let $\psi_1=e^{s\phi}y_5$ and $\psi_2=e^{s\phi}z_5$. Calculate $J_1=\int_{\Omega_L}\int_0^{T/2}\partial_t\psi_1(t)\psi_1(t)\,dx\,dt$ and $J_2=\int_{\Omega_L}\int_0^{T/2}\partial_t\psi_2(t)\psi_2(t)\,dx\,dt$. Since $\eta(0)=0$, we get

$$J_1=\frac{1}{2}\int_{\Omega_L}\psi_1(T/2)^2\,dx=\frac{1}{2}\int_{\Omega_L} e^{2s\phi(T/2)}y_5(T/2)^2\,dx \text{ and } J_2=\frac{1}{2}\int_{\Omega_L} e^{2s\phi(T/2)}z_5(T/2)^2\,dx.$$

Therefore (3.25) becomes

$$\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\widetilde{b}-b|^2+|\widetilde{d}-d|^2)\,dx \le Ce^{2sd_2}F_2(T/2)$$
$$+C\int_{\Omega_L}\int_0^{T/2}\frac{1}{s}\partial_t\psi_1(t)s\psi_1(t)\,dx\,dt+C\int_{\Omega_L}\int_0^{T/2}\frac{1}{s}\partial_t\psi_2(t)s\psi_2(t)\,dx\,dt. \tag{3.26}$$

Using Young inequality, we deduce from (3.26)

$$\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\widetilde{b}-b|^2+|\widetilde{d}-d|^2)\,dx \le \frac{C}{s}(I(y_5)+I(z_5))+Ce^{2sd_2}F_2(T/2). \tag{3.27}$$

From (3.24) and (3.27) we get

$$\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(|\widetilde{b}-b|^2+|\widetilde{d}-d|^2)\,dx \le Cs^2e^{2sd_1}+Ce^{2sd_2}(F_1(T/2)+F_2(T/2)). \tag{3.28}$$

Proceeding as in Theorem 3.1, we obtain from (3.28)

$$\int_{\Omega_l}(|\widetilde{b}-b|^2+|\widetilde{d}-d|^2)\,dx \le Cs^2e^{2s(d_1-d_0)}+Ce^{2s(d_2-d_0)}F_3(T/2) \tag{3.29}$$

with

$$F_3(T/2)=\sum_{k=0}^{1}(\|\partial_t^k y(T/2)\|^2_{H^2(\Omega_L)}+\|\partial_t^k z(T/2)\|^2_{H^2(\Omega_L)})+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu\partial_t^k y|^2+|\partial_\nu\partial_t^k z|^2)\,d\sigma\,dt.$$

Notice that in the first and second steps of this proof, we have only used the hypothesis (3.13).

• Third step: Finally using the hypothesis (3.14), we can proceed exactly as before and obtain

$$\int_{\Omega_l}(|\widetilde{a}-a|^2+|\widetilde{c}-c|^2)\,dx \le Cs^2e^{2s(d_1-d_0)}+Ce^{2s(d_2-d_0)}F_3(T/2). \tag{3.30}$$

From (3.29)-(3.30) we end the proof of Thereom 3.2. □

Remark 2 • First note that our stability results (3.3) and (3.15) are obtained on Ω_l for the left-hand term while the observation data $G(T/2)$ and $F_3(T/2)$ are required on Ω_L for the right-hand term of (3.3), (3.15).

• Second we have used Lemma 3.2 instead of Lemma 3.1 in the proof of Theorem 3.2 in order to avoid a third derivative with respect to t in the observation terms. Indeed, if we no longer used Lemma 3.2 in the proof of Theorem 3.2, we could use a modified version of Lemma 3.1: applying (3.1) with $w = e^{s\phi}\chi\eta z$, we could obtain the following inequality

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}\chi^2|z(T/2)|^2\,dx \le Cs\int_{Q_L} e^{2s\phi}\phi^2\chi^2\eta^2|z|^2\,dx\,dt + \frac{C}{s}e^{2sd_1} + \frac{C}{s}\int_{Q_L} e^{2s\phi}\chi^2\eta^2|\partial_t z|^2\,dx\,dt,$$

for all $z \in H^1(0,T;L^2(\Omega_L))$.

Moreover, if we did so, since we had to give up the end of the first step of the proof of Theorem 3.2, we'd rather follow the ideas of the proof of Theorem 3.1. Therefore, when in the second step we evaluated (3.16) for $t = T/2$, with the above inequality we would have to estimate $\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2|\partial_t y_3(T/2)|^2\,dx$ and $\int_{\Omega_L} e^{2s\phi(T/2)}\chi^2|\partial_t z_3(T/2)|^2\,dx$; thus we could obtain $\int_{Q_L} e^{2s\phi}\chi^2\eta^2|\partial_t y_4|^2\,dx\,dt$ and $\int_{Q_L} e^{2s\phi}\chi^2\eta^2|\partial_t z_4|^2\,dx\,dt$ in the right-hand side of the estimates. Then we would have to apply the Carleman estimates for $\chi\eta y_4, \chi\eta z_4, \chi\eta\partial_t y_4, \chi\eta\partial_t z_4$ and so we would obtain a third derivative in time for the observation terms.

• Third the assumptions (3.13) and (3.14) are equivalent to $|\widetilde{u}| \ge R$, $|\widetilde{v}| \ge R$ and $|\det\left(\begin{pmatrix}\widetilde{u} & \partial_t\widetilde{u}\\ \widetilde{v} & \partial_t\widetilde{v}\end{pmatrix}\right)| \ge R$ with R a positive constant. For example, if $n = 2$ and $\omega = (r_1, r_2)$ with $r_1 > 0$, let $\alpha(x_1)$ be a positive and bounded function in $C^2(\mathbb{R})$ such that $\min_{x_1\in\mathbb{R}}\alpha(x_1) > 2r_2^2$. Then $\widetilde{u}(x,t) = \alpha(x_1)t + x_2$ and $\widetilde{v}(x,t) = tx_2 + 1$ are solutions of the system (1.1) with $g_1 = g_2 = 0$, $\widetilde{a}(x) = \frac{\alpha''(x_1)+\alpha(x_1)x_2}{\alpha(x_1)-x_2^2}$, $\widetilde{b}(x) = \frac{-x_2\alpha''(x_1)-\alpha(x_1)^2}{\alpha(x_1)-x_2^2}$, $\widetilde{c}(x) = \frac{x_2^2}{\alpha(x_1)-x_2^2}$, $\widetilde{d}(x) = \frac{-x_2\alpha(x_1)}{\alpha(x_1)-x_2^2}$, and satisfy the conditions (3.13)-(3.14).

• Finally note that the above results remain valid for the system (1.2) when all the coefficients $a, b, c, d, A_1, A_2, A_3, A_4$ are bounded ($a, b, c, d \in \Lambda(M_0)$ and $A_1, A_2, A_3, A_4 \in (\Lambda(M_0))^n$). We obtain a stability result of at least two coefficients between a, b, c, d with the same observations in the right-hand sides of (3.3) or (3.15). In the next section we study the inverse problem of determining at least one of the coefficient A_1, A_2, A_3, A_4, for example A_3 if we assume that this coefficient has the form $A_3 = \nabla g$.

3.3 The third result

Consider now (u, v) (resp. $(\widetilde{u}, \widetilde{v})$) a strong solution of (1.2) associated with (a, b, c, d, A_1, A_2, A_3, A_4, u_0, v_0, g_1, g_2, h_1, h_2) (resp. (a, $\widetilde{b}$, c, d, A_1, A_2, $\widetilde{A_3}$, A_4, $\widetilde{u_0}$, $\widetilde{v_0}$, g_1, g_2, h_1, h_2)). Assume that all the coefficients a, b, c, d belong to $\Lambda(M_0)$, $A_1, A_2, A_3, A_4, \widetilde{A_3}$ belong to $(\Lambda(M_0))^n \cap (H^1(\Omega))^n$ and that there exist functions $g, \widetilde{g}$ such that

$$A_3 = \nabla g,\ \ \widetilde{A_3} = \nabla\widetilde{g} \text{ in } \Omega. \tag{3.31}$$

The Assumption (3.31) implies conditions on $A_3, \widetilde{A_3}$: if ${}^tA_3 = (c_1, \cdots, c_n)$, it means that for all $i, j = 1, \cdots, n$, $\partial_{x_i} c_j = \partial_{x_j} c_i$, in other words $rot(A_3) = 0$ if $n = 3$.
Now following an idea developed in [10] for Lamé system in bounded domains, also used for example in [4], we obtain the following result

Lemma 3.3 *Assume that the following assumption*

$$|\nabla d \cdot \nabla \widetilde{u}(T/2)| \geq R \text{ in } \Omega_L \text{ for some } R > 0 \tag{3.32}$$

holds. Consider the first order partial differential operator $Pf = \nabla f \cdot \nabla \widetilde{u}(T/2)$. *Then there exist positive constants* $s_4 > 0$ *and* $C > 0$ *such that for all* $s \geq s_4$,

$$s^2 \int_{\Omega_L} e^{2s\phi(T/2)} |f|^2 \, dx \leq C \int_{\Omega_L} e^{2s\phi(T/2)} |Pf|^2 \, dx,$$

for all $f \in H_0^1(\Omega_L)$.

Proof. The proof follows [4]. Let $f \in H_0^1(\Omega_L)$. Denote $w = e^{s\phi(T/2)} f$ and $Qw = e^{s\phi(T/2)} P(e^{-s\phi(T/2)} w)$. So we get $Qw = Pw - sw\nabla\phi(T/2) \cdot \nabla\widetilde{u}(T/2)$. Therefore we have

$$\int_{\Omega_L} |Qw|^2 \, dx \geq s^2 \int_{\Omega_L} w^2 |\nabla\phi(T/2) \cdot \nabla\widetilde{u}(T/2)|^2 \, dx - 2s \int_{\Omega_L} (Pw) w (\nabla\phi(T/2) \cdot \nabla\widetilde{u}(T/2)) \, dx$$

$$\int_{\Omega_L} |Qw|^2 \, dx \geq s^2\lambda^2 \int_{\Omega_L} w^2 (\phi(T/2))^2 |\nabla d \cdot \nabla\widetilde{u}(T/2)|^2 \, dx$$
$$-2s\lambda \int_{\Omega_L} (\nabla w \cdot \nabla\widetilde{u}(T/2)) w \phi(T/2) (\nabla d \cdot \nabla\widetilde{u}(T/2)) \, dx.$$

So

$$\int_{\Omega_L} |Qw|^2 \, dx \geq s^2\lambda^2 \int_{\Omega_L} w^2 (\phi(T/2))^2 |\nabla d \cdot \nabla\widetilde{u}(T/2)|^2 \, dx$$
$$-s\lambda \int_{\Omega_L} \phi(T/2) (\nabla d \cdot \nabla\widetilde{u}(T/2)) (\nabla(w^2) \cdot \nabla\widetilde{u}(T/2)) \, dx.$$

Thus integrating by parts

$$\int_{\Omega_L} |Qw|^2 \, dx \geq s^2\lambda^2 \int_{\Omega_L} w^2 (\phi(T/2))^2 |\nabla d \cdot \nabla\widetilde{u}(T/2)|^2 \, dx$$
$$+s\lambda \int_{\Omega_L} w^2 \nabla \cdot (\phi(T/2)(\nabla d \cdot \nabla\widetilde{u}(T/2)) \nabla\widetilde{u}(T/2)) \, dx$$

and

$$\int_{\Omega_L} e^{2s\phi(T/2)} |Pf|^2 \, dx = \int_{\Omega_L} |Qw|^2 \, dx \geq s^2\lambda^2 \int_{\Omega_L} e^{2s\phi(T/2)} f^2 (\phi(T/2))^2 |\nabla d \cdot \nabla\widetilde{u}(T/2)|^2 \, dx$$
$$+s\lambda \int_{\Omega_L} e^{2s\phi(T/2)} f^2 \nabla \cdot (\phi(T/2)(\nabla d \cdot \nabla\widetilde{u}(T/2)) \nabla\widetilde{u}(T/2)) \, dx.$$

And we can conclude for s sufficiently large. □

The strong positivity assumption (3.32) is frequently involved in inverse problems and is removed in [4] for one equation by the construction of an adapted control. Now we state the third result.

Theorem 3.3 *Let $l > 0$. Let $T > 0$, $L > l$ and $a \in \mathbb{R}^n \setminus \Omega$ satisfying the conditions of Proposition 2.1. Assume that Assumptions (3.31) and (3.32) hold. We also make the following hypothesis*

$$|\widetilde{v}(., \frac{T}{2})| \geq R \text{ in } \Omega_L \text{ for some } R > 0. \tag{3.33}$$

If $g = \widetilde{g}$ and $A_3 = \widetilde{A_3}$ on $\partial\Omega \cap \partial\Omega_L$, then there exists a sufficiently small number δ_0 such that if $\delta \in (0, \delta_0)$,

$$\|\sum_{k=0}^{1} \partial_t^k (u - \widetilde{u})(., \frac{T}{2})\|^2_{H^2(\Omega_L)} + \|\sum_{k=0}^{1} \partial_t^k (v - \widetilde{v})(., \frac{T}{2})\|^2_{H^3(\Omega_L)}$$

$$+ \int_{\gamma_L \times (0,T)} \sum_{k=0}^{2} (|\partial_\nu(\partial_t^k (u - \tilde{u}))|^2 + |\partial_\nu(\partial_t^k (v - \tilde{v}))|^2 \, d\sigma \, dt \leq \delta$$

then the following Hölder stability estimate holds

$$\|b - \widetilde{b}\|^2_{L^2(\Omega_l)} + \|A_3 - \widetilde{A_3}\|^2_{(L^2(\Omega_l))^n} \leq K\delta^\kappa \text{ for all } \delta \in (0, \delta_0). \tag{3.34}$$

Here, $K > 0$ and $\kappa \in (0, 1)$ are two constants depending on R, r, L, l, M_0, M_1, M_2, T and a.

Proof. As in Theorem 3.1 denote

$$y = u - \widetilde{u},\ y_0 = \chi\eta y,\ y_1 = \partial_t y_0,\ y_2 = \partial_t y_1,\ z = v - \widetilde{v},\ z_0 = \chi\eta z,\ z_1 = \partial_t z_0 \text{ and } z_2 = \partial_t z_1.$$

Then (y_0, z_0) satisfies

$$\begin{cases} \partial_t y_0 - \Delta y_0 + a y_0 + b z_0 + A_1 \cdot \nabla y_0 + A_2 \cdot \nabla z_0 = \xi_1 \text{ in } Q_L, \\ \partial_t z_0 - \Delta z_0 + c y_0 + d z_0 + A_3 \cdot \nabla y_0 + A_4 \cdot \nabla z_0 = \xi_2 \text{ in } Q_L, \\ y_0 = z_0 = 0 \text{ on } \partial\Omega_L \times (0, T) \end{cases} \tag{3.35}$$

with

$$\xi_1 := \chi\eta(\widetilde{b} - b)\widetilde{v} + (\partial_t\eta)\chi y - (\Delta\chi)\eta y - 2\nabla\chi \cdot \nabla(\eta y) + \eta y A_1 \cdot \nabla\chi + \eta z A_2 \cdot \nabla\chi$$

and

$$\xi_2 := \chi\eta(\widetilde{A_3} - A_3) \cdot \nabla\widetilde{u} + (\partial_t\eta)\chi z - (\Delta\chi)\eta z - 2\nabla\chi \cdot \nabla(\eta z) + \eta y A_3 \cdot \nabla\chi + \eta z A_4 \cdot \nabla\chi.$$

Then

$$\xi_2 = \eta\nabla(\chi(\widetilde{g} - g)) \cdot \nabla\widetilde{u} - \eta(\widetilde{g} - g)\nabla\chi \cdot \nabla\widetilde{u} + (\partial_t\eta)\chi z - (\Delta\chi)\eta z - 2\nabla\chi \cdot \nabla(\eta z) + \eta y A_3 \cdot \nabla\chi + \eta z A_4 \cdot \nabla\chi.$$

• First step: We evaluate (3.35) for $t = \frac{T}{2}$ and we get

$$\partial_t y_0(T/2) - \Delta y_0(T/2) + a y_0(T/2) + b z_0(T/2) + A_1 \cdot \nabla y_0(T/2) + A_2 \cdot \nabla z_0(T/2)$$

$$= \chi(\widetilde{b} - b)\widetilde{v}(T/2) - (\Delta\chi) y(T/2) - 2\nabla\chi \cdot \nabla y(T/2) + y(T/2) A_1 \cdot \nabla\chi + z(T/2) A_2 \cdot \nabla\chi \quad (3.36)$$

and

$$\partial_t z_0(T/2) - \Delta z_0(T/2) + c y_0(T/2) + d z_0(T/2) + A_3 \cdot \nabla y_0(T/2) + A_4 \cdot \nabla z_0(T/2)$$
$$= P(\chi(\widetilde{g} - g)) - (\widetilde{g} - g)\nabla\chi \cdot \nabla\widetilde{u}(T/2) - (\Delta\chi) z(T/2) - 2\nabla\chi \cdot \nabla z(T/2) + y(T/2) A_3 \cdot \nabla\chi$$
$$+ z(T/2) A_4 \cdot \nabla\chi \quad (3.37)$$

with P the operator defined in Lemma 3.3. From (3.36) we have

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})} \chi^2 |b - \widetilde{b}|^2 |\widetilde{v}(\frac{T}{2})|^2 \, dx \le C \int_{\Omega_L} e^{2s\phi(\frac{T}{2})} |\partial_t y_0(\frac{T}{2})|^2 \, dx$$

$$+ C e^{2sd_2} (\|z_0(T/2)\|^2_{H^1(\Omega_L)} + \|y_0(T/2)\|^2_{H^2(\Omega_L)} + \|y(T/2)\|^2_{H^1(\Omega_L)} + \|z(T/2)\|^2_{L^2(\Omega_L)}).$$

So

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})} \chi^2 |b - \widetilde{b}|^2 |\widetilde{v}(\frac{T}{2})|^2 \, dx \le C e^{2sd_2} F_1(T/2) + C \int_{\Omega_L} e^{2s\phi(\frac{T}{2})} |\partial_t y_0(\frac{T}{2})|^2 \, dx$$

with

$$F_1(T/2) = \|y(T/2)\|^2_{H^2(\Omega_L)} + \|z(T/2)\|^2_{H^1(\Omega_L)}.$$

Then, applying Lemma 3.1 we get

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})} \chi^2 |b - \widetilde{b}|^2 |\widetilde{v}(\frac{T}{2})|^2 \, dx \le C e^{2sd_2} F_1(T/2) + C s \int_{Q_L} e^{2s\phi} \phi^3 |y_1|^2 \, dx \, dt$$
$$+ \frac{C}{s} \int_{Q_L} e^{2s\phi} \phi^3 |y_2|^2 \, dx \, dt. \quad (3.38)$$

Moreover using Lemma 3.3 for (3.37) we have

$$s^2 \int_{\Omega_L} e^{2s\phi(T/2)} \chi^2 (\widetilde{g} - g)^2 \, dx \le C \int_{\Omega_L} e^{2s\phi(T/2)} |P(\chi(\widetilde{g} - g))|^2 \, dx$$

$$\le C e^{2sd_1} + C \int_{\Omega_L} e^{2s\phi(\frac{T}{2})} |\partial_t z_0(\frac{T}{2})|^2 \, dx$$

$$+ C e^{2sd_2} (\|z_0(T/2)\|^2_{H^2(\Omega_L)} + \|y_0(T/2)\|^2_{H^1(\Omega_L)} + \|y(T/2)\|^2_{L^2(\Omega_L)} + \|z(T/2)\|^2_{H^1(\Omega_L)}).$$

Applying again Lemma 3.1 we get

$$s^2 \int_{\Omega_L} e^{2s\phi(T/2)} \chi^2 (\widetilde{g} - g)^2 dx \le C e^{2sd_1} + C e^{2sd_2} F_2(T/2) + C s \int_{Q_L} e^{2s\phi} \phi^3 |z_1|^2 \, dx \, dt$$

$$+\frac{C}{s}\int_{Q_L} e^{2s\phi}\phi^3|z_2|^2dxdt \tag{3.39}$$

with

$$F_2(T/2) = \|y(T/2)\|^2_{H^1(\Omega_L)} + \|z(T/2)\|^2_{H^2(\Omega_L)}.$$

From (3.38)-(3.39) we obtain

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}\chi^2|b-\widetilde{b}|^2|\widetilde{v}(\frac{T}{2})|^2\ dx + \int_{\Omega_L} e^{2s\phi(T/2)}\chi^2(\widetilde{g}-g)^2\ dx$$

$$\leq \frac{C}{s^2}e^{2sd_1} + Ce^{2sd_2}F_3(T/2) + Cs\int_{Q_L} e^{2s\phi}\phi^3(|y_1|^2+|z_1|^2)dxdt + \frac{C}{s}\int_{Q_L} e^{2s\phi}\phi^3(|y_2|^2+|z_2|^2)dxdt \tag{3.40}$$

with

$$F_3(T/2) = \|y(T/2)\|^2_{H^2(\Omega_L)} + \|z(T/2)\|^2_{H^2(\Omega_L)}.$$

Using now Assumption (3.33), we get from (3.40) and for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}\chi^2((b-\widetilde{b})^2 + (\widetilde{g}-g)^2)\ dx \leq \frac{C}{s^2}e^{2sd_1} + Ce^{2sd_2}F_3(T/2)$$

$$+Cs\int_{Q_L} e^{2s\phi}\phi^3(|y_1|^2+|z_1|^2)\ dx\ dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}\phi^3(|y_2|^2+|z_2|^2)\ dx\ dt. \tag{3.41}$$

• Second step: As in Theorem 3.1, now we use the Carleman inequalities (2.5) for y_i and z_i, $i=1,2$. Recall that $\phi \leq \phi(T/2)$ so we get for s sufficiently large

$$I(y_i) + I(z_i) \leq C\int_{\Omega_L} e^{2s\phi(T/2)}(|\nabla(\chi(\widetilde{g}-g))|^2 + \chi^2|b-\widetilde{b}|^2)\ dx + Cs^3e^{2sd_1}$$

$$+Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt.$$

Thus

$$s^3\int_{Q_L} e^{2s\phi}\phi^3(|y_i|^2+|z_i|^2)\ dx\ dt \leq C\int_{\Omega_L} e^{2s\phi(T/2)}(|\nabla(\chi(\widetilde{g}-g))|^2 + \chi^2|b-\widetilde{b}|^2)\ dx$$

$$+Cs^3e^{2sd_1} + Cs\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt. \tag{3.42}$$

Therefore, from (3.41) and (3.42), we get for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}\chi^2((b-\widetilde{b})^2 + (\widetilde{g}-g)^2)\ dx \leq Ce^{2sd_2}F_3(T/2) + Cse^{2sd_1}$$

$$+\frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(T/2)}(|\nabla(\chi(\widetilde{g}-g))|^2 + \chi^2|b-\widetilde{b}|^2)\ dx + \frac{C}{s}\int_{\gamma_L\times(0,T)} e^{2s\phi}\sum_{i=1}^{2}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt.$$

Thus we have for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}\chi^2((b-\widetilde{b})^2+(\widetilde{g}-g)^2)\,dx \le Ce^{2sd_2}F_4(T/2)+Cse^{2sd_1}+\frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(T/2)}|\nabla(\chi(g-\widetilde{g}))|^2|\,dx \tag{3.43}$$

with

$$F_4(T/2) = F_3(T/2) + \int_{\gamma_L\times(0,T)} \sum_{k=0}^{2}(|\partial_\nu\partial_t^k y|^2 + |\partial_\nu\partial_t^k z|^2)\,d\sigma\,dt.$$

• Third step: We apply the same ideas for $\nabla(\chi(\widetilde{g}-g))$. For any integer $1 \le i \le n$, taking the space derivative with respect to x_i in (3.37), we obtain

$$\begin{aligned}
&\partial_t\partial_{x_i}z_0(T/2) - \Delta\partial_{x_i}z_0(T/2) + \partial_{x_i}\left(cy_0(T/2) + dz_0(T/2) + A_3\cdot\nabla y_0(T/2) + A_4\cdot\nabla z_0(T/2)\right)\\
&\quad = P(\partial_{x_i}(\chi(\widetilde{g}-g))) + \nabla(\chi(\widetilde{g}-g))\cdot\nabla(\partial_{x_i}\widetilde{u}(T/2)) - \partial_{x_i}((\widetilde{g}-g)\nabla\chi\cdot\nabla\widetilde{u}(T/2))\\
&\quad - \partial_{x_i}((\Delta\chi)z(T/2) - 2\nabla\chi\cdot\nabla z(T/2) + y(T/2)A_3\cdot\nabla\chi + z(T/2)A_4\cdot\nabla\chi).
\end{aligned} \tag{3.44}$$

We can apply again Lemma 3.3: there exists a positive constant C such that for s sufficiently large,

$$s^2\int_{\Omega_L} e^{2s\phi(T/2)}\partial_{x_i}(\chi(\widetilde{g}-g))^2\,dx \le C\int_{\Omega_L} e^{2s\phi(T/2)}(P(\partial_{x_i}(\chi(\widetilde{g}-g))))^2\,dx.$$

Thus, using (3.44) we obtain

$$\begin{aligned}
s^2\int_{\Omega_L} e^{2s\phi(T/2)}(\partial_{x_i}(\chi(\widetilde{g}-g)))^2\,dx &\le Ce^{2sd_2}F_5(T/2)+Ce^{2sd_1}+C\int_{\Omega_L} e^{2s\phi(T/2)}|\partial_{x_i}z_1(T/2)|^2\,dx\\
&+C\int_{\Omega_L} e^{2s\phi(T/2)}|\nabla(\chi(g-\widetilde{g}))|^2|\,dx
\end{aligned}$$

with $F_5(T/2) = \|z(T/2)\|^2_{H^3(\Omega_L)} + \|y(T/2)\|^2_{H^2(\Omega_L)}$. So using Lemma 3.1 we get

$$\begin{aligned}
s^2\int_{\Omega_L} e^{2s\phi(T/2)}(\partial_{x_i}(\chi(\widetilde{g}-g)))^2\,dx &\le Ce^{2sd_2}F_5(T/2)+Ce^{2sd_1}+C\int_{\Omega_L} e^{2s\phi(T/2)}|\nabla(\chi(g-\widetilde{g}))|^2|\,dx\\
&+Cs\int_{Q_L} e^{2s\phi}(\partial_{x_i}z_1)^2\,dx\,dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}(\partial_{x_i}z_2)^2\,dx\,dt.
\end{aligned} \tag{3.45}$$

Moreover by the Carleman inequality (2.5), we have for $j = 1, 2$,

$$\begin{aligned}
s\int_{Q_L} e^{2s\phi}(z_j^2 + |\nabla z_j|^2)\,dx\,dt &\le C\int_{Q_L} e^{2s\phi}|Az_j|^2\,dx\,dt + Cs^3e^{2sd_1}\|z_j\|^2_{H^{2,1}(Q_L)}\\
&+Cs\int_{\gamma_L\times(0,T)} |\partial_\nu z_j|^2 e^{2s\phi}\,d\sigma\,dt.
\end{aligned}$$

Thus

$$s\int_{Q_L} e^{2s\phi}(z_j^2+|\nabla z_j|^2)\,dx\,dt \le C\int_{Q_L} e^{2s\phi}(y_j^2+|\nabla y_j|^2+z_j^2+|\nabla z_j|^2)\,dx\,dt+\int_{Q_L} e^{2s\phi}|\nabla(\chi(\widetilde{g}-g))|^2\,dx\,dt$$

$$+Cs^3e^{2sd_1}+Cs\int_{\gamma_L\times(0,T)}|\partial_\nu z_j|^2e^{2s\phi}\,d\sigma\,dt. \tag{3.46}$$

By the same way we obtain

$$s\int_{Q_L}e^{2s\phi}(y_j^2+|\nabla y_j|^2)\,dx\,dt\le C\int_{Q_L}e^{2s\phi}(y_j^2+|\nabla y_j|^2+z_j^2+|\nabla z_j|^2)\,dx\,dt+\int_{Q_L}e^{2s\phi}(\chi(\widetilde{b}-b))^2\,dx\,dt$$

$$+Cs^3e^{2sd_1}+Cs\int_{\gamma_L\times(0,T)}|\partial_\nu y_j|^2e^{2s\phi}\,d\sigma\,dt. \tag{3.47}$$

From (3.46) and (3.47) we deduce

$$s\int_{Q_L}e^{2s\phi}(z_j^2+y_j^2+|\nabla z_j|^2+|\nabla y_j|^2)dxdt\le C\int_{Q_L}e^{2s\phi}(y_j^2+|\nabla y_j|^2+z_j^2+|\nabla z_j|^2)dxdt+Cs^3e^{2sd_1}$$

$$+C\int_{Q_L}e^{2s\phi}(|\nabla(\chi(\widetilde{g}-g))|^2+(\chi(\widetilde{b}-b))^2)\,dx\,dt+Cs\int_{\gamma_L\times(0,T)}e^{2s\phi}(|\partial_\nu z_j|^2+|\partial_\nu y_j|^2)\,d\sigma\,dt. \tag{3.48}$$

Since $\phi\le\phi(T/2)$, (3.48) implies for s sufficiently large

$$s\int_{Q_L}e^{2s\phi}(z_j^2+y_j^2+|\nabla z_j|^2+|\nabla y_j|^2)\,dx\,dt\le Cs^3e^{2sd_1}$$

$$+C\int_{\Omega_L}e^{2s\phi(T/2)}(|\nabla(\chi(\widetilde{g}-g))|^2+(\chi(\widetilde{b}-b))^2)\,dx+Cs\int_{\gamma_L\times(0,T)}e^{2s\phi}(|\partial_\nu z_j|^2+|\partial_\nu y_j|^2)\,d\sigma\,dt$$

and so

$$s\int_{Q_L}e^{2s\phi}\sum_{j=1}^{2}(|\nabla z_j|^2+|\nabla y_j|^2)\,dx\,dt\le s\int_{Q_L}e^{2s\phi}\sum_{j=1}^{2}(z_j^2+y_j^2+|\nabla z_j|^2+|\nabla y_j|^2)\,dx\,dt$$

$$\le Cs^3e^{2sd_1}+Cs\int_{\gamma_L\times(0,T)}e^{2s\phi}\sum_{j=1}^{2}(|\partial_\nu z_j|^2+|\partial_\nu y_j|^2)\,d\sigma\,dt$$

$$+C\int_{\Omega_L}e^{2s\phi(T/2)}(|\nabla(\chi(\widetilde{g}-g))|^2+(\chi(\widetilde{b}-b))^2)\,dx. \tag{3.49}$$

Using inequalities (3.45) for $1\le i\le n$ and (3.49), we get

$$s^2\int_{\Omega_L}e^{2s\phi(T/2)}|\nabla(\chi(\widetilde{g}-g))|^2\,dx\le Ce^{2sd_2}F_5(T/2)+C\int_{\Omega_L}e^{2s\phi(T/2)}[|\nabla(\chi(g-\widetilde{g}))|^2|+|\chi(b-\widetilde{b})|^2]\,dx$$

$$+Cs^3e^{2sd_1}+Cs\int_{\gamma_L\times(0,T)}e^{2s\phi}\sum_{j=1}^{2}(|\partial_\nu z_j|^2+|\partial_\nu y_j|^2)\,d\sigma\,dt.$$

Therefore for s sufficienlty large

$$s^2\int_{\Omega_L}e^{2s\phi(T/2)}|\nabla(\chi(\widetilde{g}-g))|^2\,dx\le Ce^{2sd_2}F_5(T/2)+C\int_{\Omega_L}e^{2s\phi(T/2)}(\chi(b-\widetilde{b}))^2\,dx$$

$$+ Cs^3 e^{2sd_1} + Cs \int_{\gamma_L \times (0,T)} e^{2s\phi} \sum_{j=1}^{2} (|\partial_\nu z_j|^2 + |\partial_\nu y_j|^2) \, d\sigma \, dt. \tag{3.50}$$

• Fourth step: Now we gather (3.43) and (3.50) and we get for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(T/2)} \chi^2 (|\widetilde{b} - b|^2 + |\widetilde{g} - g|^2 + |\nabla(\chi(\widetilde{g} - g))|^2) \, dx \leq C e^{2sd_2} F_6(T/2) + C s e^{2sd_1}, \tag{3.51}$$

with $F_6(T/2) = F_4(T/2) + F_5(T/2)$. Moreover, since $e^{2sd_0} \leq e^{2s\phi(T/2)}$ in Ω_l and $\chi = 1$ in Ω_l, we deduce that

$$\|\widetilde{b} - b\|^2_{L^2(\Omega_l)} + \|\widetilde{g} - g\|^2_{H^1(\Omega_l)} \leq C(e^{2s(d_2 - d_0)} F_6(T/2) + s e^{2s(d_1 - d_0)}).$$

This concludes the proof of Theorem 3.3. □

Remark 3 In Theorem 3.3 we have presented the case of determining the coefficients b and A_3. Of course we could obtain similar results for at least two coefficients between $a, b, c, d, A_1, A_2, A_3, A_4$. If we want to determine A_1 and A_3, we only have to assume that Assumption (3.32) holds intead of (3.32)-(3.33). If we want to estimate the coefficients A_2 and A_3, we still have to assume the hypothesis (3.32) satisfied but in this case, we should also assume that the following hypothesis

$$|\nabla d \cdot \nabla \widetilde{v}(T/2)| \geq R \text{ in } \Omega_L \text{ for some } R > 0$$

holds. Note also that the last item of Remark 1 still holds for (3.34). To conclude, if we would like to determine more than two coefficients, we could procede with the same method used in Theorem 3.2.

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On the Square-Triangular Numbers and Balancing-Numbers

ABSTRACT. In 1770, Euler looked for positive integers n and m such that $n(n+1)/2 = m^2$. Integer solutions for this equation produce what he called square-triangular numbers. In this paper, we present a new explicit formula for this kind of numbers and establish a link with balancing numbers.

KEY WORDS. Triangular number, Square number, Square-triangular number, Balancing number

1 Introduction

A triangular number counts objects arranged in an equilateral triangle. The first five triangular numbers are 1, 3, 6, 10, 15, as shown in Figure 1. Let T_n denote the n^{th} triangular number, then T_n is equal to the sum of the n natural numbers from 1 to n, i.e.,

$$T_n = 1 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

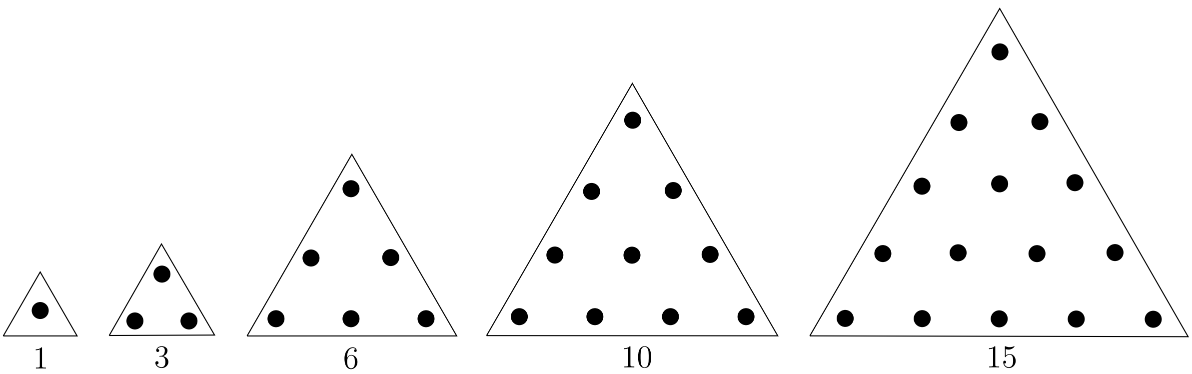

Figure 1: The first five triangular numbers

Similar considerations lead to square numbers which can be thought of as the numbers of objects that can be arranged in the shape of a square. The first five square numbers are 1, 4, 9, 16, 25, as shown in Figure 2. Let S_n denote the n^{th} square number, then we have

$$S_n = n^2.$$

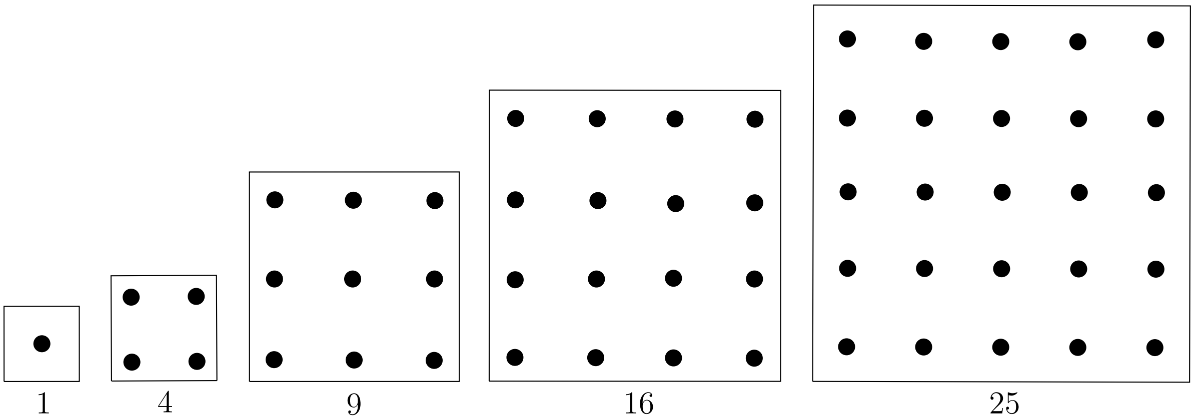

Figure 2: The first five square numbers

A square-triangular number is a number which is both a triangular and square number. The firsts non-trivial square-triangular number is 36, see Figure 3. A square-triangular number is a positive integer solution of the diophantine equation:

$$\frac{n(n+1)}{2} = m^2. \tag{1}$$

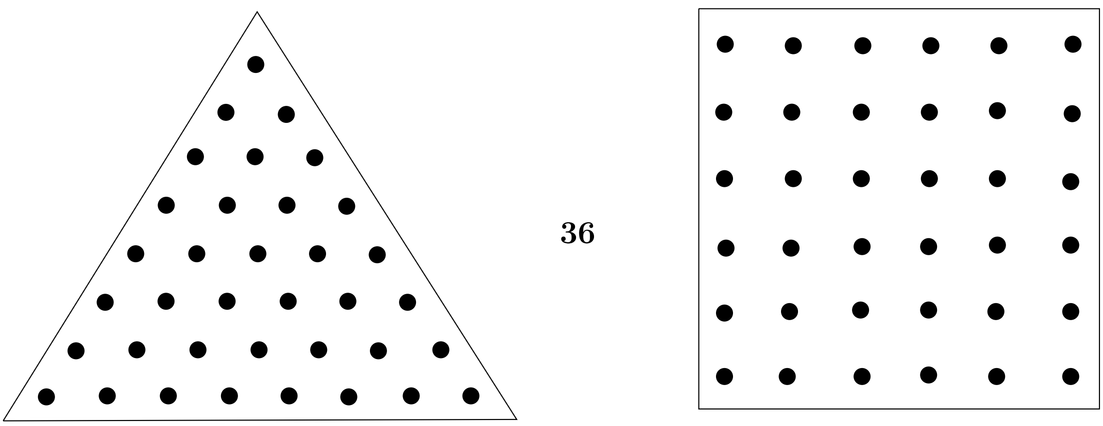

Figure 3: The first non-trivial square-triangular number

2 Main Results

Lemma 1 *(n, m) is a solution of Equation (1) if, and only if*

$$n = \sum_{i=0}^{k-1} \binom{2k}{2i+2} 2^i \quad \textit{and} \quad m = \sum_{i=-1}^{k-2} \binom{2k}{2i+3} 2^i, \ \textit{for } k \in \mathbb{N}^*.$$

Proof. From Equation (1), we have

$$n^2 + n - 2m^2 = 0. \tag{2}$$

Equation (2) can be rewritten as follows:

$$(2n+1)^2 - 2(2m)^2 = 1. \tag{3}$$

Letting $x = 2n + 1$ and $y = 2m$, Equations (3) becomes the Pell equation:

$$x^2 - 2y^2 = 1. \tag{4}$$

It is well known, that the form $x^2 - 2y^2$ is irreducible over the field $\mathbb{Q}$ of rational numbers, but in the extension field $\mathbb{Q}(\sqrt{2})$ it can be factored as a product of linear factors $(x + y\sqrt{2})(x - y\sqrt{2})$. Using the norm concept for the extension field $\mathbb{Q}(\sqrt{2})$, Equation (4) can be written in the form:

$$N\left(x + y\sqrt{2}\right) = 1. \tag{5}$$

It is easily checked that the set of all numbers of the form $x+y\sqrt{2}$, where x and y are integers, form a ring, which is denoted $\mathbb{Z}[\sqrt{2}]$. The subset of units of $\mathbb{Z}[\sqrt{2}]$, which we denote $\mathcal{U}$ forms a group. It is easy to show that $\alpha \in \mathcal{U}$ if and only if $N(\alpha) = \pm 1$ [2]. Applying Dirichlet's Theorem, we can show that $\mathcal{U} = \{\pm \left(1 + \sqrt{2}\right)^k, \ k \in \mathbb{Z}\}$.

Since

$$N\left(\left(1 + \sqrt{2}\right)^k\right) = \left(N\left(1 + \sqrt{2}\right)\right)^k = (-1)^k, \tag{6}$$

we obtain

$$N(\alpha) = 1 \Leftrightarrow \alpha = \left(1 + \sqrt{2}\right)^{2k}, \ k \in \mathbb{Z}. \tag{7}$$

Thus, all integral solutions of Equation (4) are given by:

$$\begin{aligned} x + \sqrt{2}y &= \left(1 + \sqrt{2}\right)^{2k} \\ &= \sum_{i=0}^{2k} \binom{2k}{i} 2^{i/2} \\ &= \left(\sum_{i=0}^{k} \binom{2k}{2i} 2^i\right) + \sqrt{2}\left(\sum_{i=0}^{k-1} \binom{2k}{2i+1} 2^i\right). \end{aligned} \tag{8}$$

We get, after identification

$$2n+1 = x = \sum_{i=0}^{k} \binom{2k}{2i} 2^i,$$

and

$$2m = y = \sum_{i=0}^{k-1} \binom{2k}{2i+1} 2^i.$$

Equivalently, we have

$$n = \sum_{i=0}^{k-1} \binom{2k}{2i+2} 2^i,$$

and

$$m = \sum_{i=-1}^{k-2} \binom{2k}{2i+3} 2^i.$$

This completes the proof. □

We have thus proved, via Lemma 2, the following theorem.

Theorem 2 *Let ST_n denotes the n^{th} square-triangular number. Then*

$$ST_n = S_m = T_k,$$

where

$$m = \sum_{i=-1}^{n-2} \binom{2n}{2i+3} 2^i \quad \textit{and} \quad k = \sum_{i=0}^{n-1} \binom{2n}{2i+2} 2^i.$$

3 A Link Between Square-Triangular Numbers and Balancing Numbers

Behera and Panda [1] introduced balancing numbers $m \in \mathbb{Z}^+$ as solutions of the equation:

$$1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)\,. \tag{9}$$

Theorem 3 *Let B_n be the n^{th} balancing number. Then*

$$ST_n = B_n^2.$$

Proof. By making the substitution $m+r=n$, with $n \geq m+1$, Equation (9) becomes

$$1+2+\cdots+(m-1) = (m+1)+(m+2)+\cdots+n. \tag{10}$$

Therefore

$$
\begin{aligned}
m \text{ is a balancing number} &\iff 1+2+\cdots+(m-1) = (1+2+\cdots+n)-(1+2+\cdots+m) \\
&\iff \frac{m(m-1)}{2} = \frac{n(n+1)}{2} - \frac{m(m+1)}{2} \\
&\iff \frac{m(m-1)}{2} + \frac{m(m+1)}{2} = \frac{n(n+1)}{2} \\
&\iff m^2 = \frac{n(n+1)}{2} \\
&\iff m^2 \text{ is a square-triangular number}
\end{aligned}
$$

This completes the proof. □

Table 1 bellow summarizes the first ten square-triangular numbers with there associated triangular and balancing numbers, based on Theorem 2 and Theorem 3.

n	$N=\sum_{i=0}^{n-1}\binom{2n}{2i+2}2^i$	$T_N=\frac{N(N+1)}{2}$	$B_n=\sum_{i=-1}^{n-2}\binom{2n}{2i+3}2^i$	$ST_n=B_n^2$
1	1	1	1	1
2	8	36	6	36
3	49	1225	35	1225
4	288	41616	204	41616
5	1681	1413721	1189	1413721
6	9800	48024900	6930	48024900
7	57121	1631432881	40391	1631432881
8	332928	55420693056	235416	55420693056
9	1940449	1882672131025	1372105	1882672131025
10	11309768	63955431761796	7997214	63955431761796

Table 1: The first ten square-triangular numbers

4 Recurrence Relations for Square-Triangular Numbers

Theorem 4 *The sequence of square-triangular numbers* $(ST_n)_n$ *satisfies the recurrence relation:*

$$ST_n = 34ST_{n-1} - ST_{n-2} + 2, \ for \ n \geq 3,$$

with $ST_1 = 1$ *and* $ST_2 = 36$.

Proof. It is well known that the sequence of balancing numbers satisfies the following recurrence relations [1]:

$$B_{n+1} = 6B_n - B_{n-1}, \tag{11}$$

and

$$B_n^2 - B_{n+1}B_{n-1} = 1\cdot \tag{12}$$

Hence

$$\begin{aligned} B_n^2 &= (6B_{n-1} - B_{n-2})^2 \\ &= 36B_{n-1}^2 - 12B_{n-1}B_{n-2} + B_{n-2}^2\cdot \end{aligned}$$

From Equation (11), we get

$$\begin{aligned} B_n^2 &= 36B_{n-1}^2 - 12\left(\frac{B_n + B_{n-2}}{6}\right)B_{n-2} + B_{n-2}^2 \\ &= 36B_{n-1}^2 - 2B_nB_{n-2} - B_{n-2}^2 \\ &= 34B_{n-1}^2 - 2\left(B_nB_{n-2} - B_{n-1}^2\right) - B_{n-2}^2\cdot \end{aligned}$$

From Equation (12), we get

$$B_n^2 = 34B_{n-1}^2 - B_{n-2}^2 + 2\cdot$$

This completes the proof according to Theorem 3. □

5 Generating Function for Square-Triangular Numbers

In this section, we present the generating function based on some relations on balancing numbers.

Theorem 5 *The generating function of ST_n is:*

$$f(x) = \frac{x(1+x)}{(1-x)(x^2 - 34x + 1)}\cdot$$

Proof. Let $f(x) = \sum_{n\geq 1} ST_n\, x^n$. Then

$$34xf(x) = \sum_{n\geq 2} 34ST_{n-1}\, x^n,$$

and

$$x^2 f(x) = \sum_{n\geq 3} ST_{n-2}\; x^n \cdot$$

Therefore

$$\begin{aligned} 34xf(x) - x^2 f(x) &= 34x^2 + \sum_{n\geq 3} (34ST_{n-1} - ST_{n-2})\; x^n \\ &= 34x^2 + \sum_{n\geq 3} (34ST_{n-1} - ST_{n-2} + 2)\; x^n - 2\sum_{n\geq 3} x^n \cdot \end{aligned}$$

By Theorem 6, we have

$$\begin{aligned} 34xf(x) - x^2 f(x) &= 34x^2 + \sum_{n\geq 3} ST_n\; x^n - 2\left(\frac{1}{1-x} - 1 - x - x^2\right) \\ &= 34x^2 + \left(f(x) - x - 36x^2\right) - 2\left(\frac{1}{1-x} - 1 - x - x^2\right) \\ &= f(x) - \frac{x(1+x)}{1-x} \cdot \end{aligned}$$

Hence

$$(1 - 34x + x^2) f(x) = \frac{x(1+x)}{1-x} \cdot$$

This completes the proof. □

By using the generating function we can have the following equivalent explicit formula for the sequence of square-triangular numbers $(ST_n)_n$ that may be convenient to include.

Theorem 6 *For $n \geq 1$, we have*

$$ST_n = \frac{\left(17 + 12\sqrt{2}\right)^n + \left(17 - 12\sqrt{2}\right)^n - 2}{32} \cdot$$

Proof. From expanding the generating function of ST_n in partial fractions, we obtain

$$f\,(x) = \frac{1}{16\,(x-1)} + \frac{12\sqrt{2} - 17}{32\left(12\sqrt{2} - 17 + x\right)} + \frac{12\sqrt{2} + 17}{32\left(12\sqrt{2} + 17 - x\right)} \cdot$$

Therefore

$$\begin{aligned} f\,(x) &= -\frac{1}{16}\sum_{n\geq 0} x^n + \frac{1}{32}\sum_{n\geq 0} \frac{(-x)^n}{\left(-17 + 12\sqrt{2}\right)^n} + \frac{1}{32}\sum_{n\geq 0} \frac{x^n}{\left(17 + 12\sqrt{2}\right)^n} \\ &= -\frac{1}{16}\sum_{n\geq 0} x^n + \frac{1}{32}\sum_{n\geq 0} \left(17 + 12\sqrt{2}\right)^n x^n + \frac{1}{32}\sum_{n\geq 0} \left(17 - 12\sqrt{2}\right)^n x^n \cdot \end{aligned}$$

Then

$$ST_n = -\frac{1}{16} + \frac{1}{32}\left(17 + 12\sqrt{2}\right)^n + \frac{1}{32}\left(17 - 12\sqrt{2}\right)^n \cdot$$

Hence, the result follows. □

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