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Valeri B. Alekseev

On the number of functions in some maximal closed classes of partial k-valued logic

We investigate the number of functions that depend on n fixed variables in some closed classes of partial k-valued logic. The asymptotic for the algorithm of this number when $n \to \infty$ is established for maximal closed classes of functions preserving binary relations.

A function $f(x_1,\dots,x_n)$ is said to be a function of partial k-valued logic iff the values of all its variables are taken from the set $E_k = \{0,1,\dots,k-1\}$ and the value of the function itself is taken from the set $E_k \cup \{*\}$. The set of all such functions is denoted by P_k^*. The superposition

$$g(x_1,\dots x_n) = f(h_1(x_1,\dots,x_n), \dots, h_m(x_1,\dots,x_n))$$

in P_k^* is as usual with the following additional rule: if $h_i(a_1,\dots,a_n) = *$ for some i then $g(a_1,\dots,a_n) = *$.

In studying the completeness theory of partial k-valued logic maximal closed classes (those that are not contained in any other closed class except P_k^*) are of great importance. The set of all maximal closed classes in P_k^* is described in [1,2]. All but one of them are classes of functions preserving some relation on E_k. By $R(a_1,\dots,a_m)$ (or $\neg R(a_1,\dots,a_m)$) we denote that the string $(a_1,\dots,a_m)$ of elements from E_k satisfies (does not satisfy) the relation R on E_k. For an m-ary relation R on E_k and for any natural number n we define the relation R^n on the set of all strings of length n with elements from E_k as follows:

$$R^n(\bar{a}^1,\dots,\bar{a}^m) \iff \forall\ i\ \ R(a_i^1,\dots,a_i^m) \tag{1}$$

where $\bar{a}^j = (a_1^j,\dots,a_n^j)$.

A function $f(x_1,\dots x_n) \in P_k^*$ is said to preserve the m-ary relation R on E_k iff for any m strings $\bar{a}^1,\dots,\bar{a}^m$ of length n the following implication holds:

$$R^n(\bar{a}^1,\dots,\bar{a}^m) \Longrightarrow [(\exists\ j\ (f(\bar{a}^j) = *)) \vee R(f(\bar{a}^1),\dots,f(\bar{a}^m))]. \tag{2}$$

We denote by $U^*(R)$ the set of all functions from P_k^* preserving a relation R. If $Q = U^*(R)$, then by $Q^{(n)}$ we denote the set of all functions from Q that depend on n fixed variables $x_1,\dots,x_n$.
If $Q = U^*(R)$ and R is an unary relation on E_k and $R(a)$ is true for exactly p elements from E_k then

$$|Q^{(n)}| = (p+1)^{p^n}(k+1)^{k^n-p^n}.$$

If $Q = U^*(R)$ and R is an m-ary relation, where $m \geqslant 2$, then the problem of finding a formula for $|Q^{(n)}|$ is very hard. First we prove the following common proposition.

<u>Theorem 1:</u> If $Q = U^*(R)$ where R is an arbitrary relation on E_k then $\log_2|Q^{(n)}| \asymp k^n$ when $n \to \infty$.

<u>Proof:</u> Obviously $|Q^{(n)}| \leqslant (k+1)^{k^n}$ so $\log_2|Q^{(n)}| \leqslant k^n\log_2(k+1)$. Consider all functions $f(x_1,\dots,x_n)$ such that $f(a_1,\dots,a_n)$ is equal to a_1 or $*$ for any string $(a_1,\dots,a_n)$. From (1) and (2) we have that any such function belongs to $U^*(R)$. So $|Q^{(n)}| \geqslant 2^{k^n}$ and $\log_2|Q^{(n)}| \geqslant k^n$. Hence $\log_2|Q^{(n)}| \asymp k^n$. ■

Consider now in more detail maximal closed classes $U^*(R)$ in P_k^* where R is a binary relation on E_k. Any binary relation R on E_k can be represented by a directed graph G_R with the set E_k of vertices and the following set of arcs: $X = \{(a_1,a_2)|\ R(a_1,a_2)\}$. If a binary relation R is symmetric, then it is convenient to replace any pair of opposite arcs by one undirected edge and to consider the graph G_R as undirected. It is known [1,2] that $U^*(R)$ is a maximal closed class in P_k^* iff one of the following properties holds:

a) G_R is a nonempty directed bipartite graph (it means that all arcs go from A to B where $A \cup B = E_k$, $A \cap B = \emptyset$);

b) G_R is a directed graph with loops for all vertices which has at least one arc not being a loop and which has no cycles except loops;

c) G_R is an undirected graph with loops for all vertices which has at least one edge not being a loop and which has at least one pair of vertices which are not connected by an edge;

d) G_R is a nonempty undirected bipartite graph.

We prove that the following theorems hold.

<u>Theorem 2:</u> Let $Q = U^*(R)$ where G_R satisfies a). Then $\log_2|Q^{(n)}| \sim k^n \log_2(k+1)$ when $n \to \infty$.

<u>Theorem 3:</u> Let $Q = U^*(R)$ where G_R satisfies b). Then $\log_2|Q^{(n)}| \sim k^n$ when $n \to \infty$.

<u>Theorem 4:</u> Let $Q = U^*(R)$ where G_R satisfies c), and let d be the size of a maximum clique (complete subgraph) in G_R. Then $\log_2|Q^{(n)}| \sim k^n \log_2(d+1)$ when $n \to \infty$.

<u>Theorem 5:</u> Let $Q = U^*(R)$ where G_R satisfies d). Then the following holds:

i) if the bipartite graph G_R is a perfect matching (i.e., the set of its edges covers each vertex exactly once) then $|Q^{(n)}| = (3k+1)^{(1/2)k^n}$;

ii) if G_R has no perfect matching as a subgraph then $\log_2|Q^{(n)}| \sim k^n \log_2(k+1)$;

iii) if G_R has a perfect matching as a subgraph and has at least one additional edge then construct a new graph G by adding a vertex $*$ to G_R and connecting it by an edge with all other vertices of G_R. Let $d = \max\{|A|\cdot|B|\}$ where the maximum is taken over all pairs $A \subseteq E_k \cup \{*\}$, $B \subseteq E_k \cup \{*\}$ such that any pair of vertices $a \in A$, $b \in B$ is adjacent in G. Then $\log_2|Q^{(n)}| \approx (k/2)^n \log_2 d$.

<u>Proof of Theorem 2:</u> Let all arcs go from A to B where $A \cup B = E_k$, $A \cap B = \emptyset$. Let $\bar{a} = (a_1,\ldots,a_n)$, $\bar{b} = (b_1,\ldots,b_n)$ and $R^n(\bar{a},\bar{b})$. Then according to (1) $\forall\, i\ R(a_i,b_i)$, i.e., (a_i,b_i) is an arc of G_R. It means that $\bar{a} \in A^n$ where $A^n = \{(c_1,\ldots,c_n) \mid \forall\, i\ (c_i \in A)\}$ and $\bar{b} \in B^n$ where $B^n = \{(d_1,\ldots,d_n) \mid \forall\, i\ (d_i \in B)\}$. Consider all functions $f(x_1,\ldots,x_n)$ such that $f(\bar{a}) = *$ for all $\bar{a} \in A^n$ and for all $\bar{a} \in B^n$. It is easy to see that all such functions satisfy (2). So

$$\log_2|Q^{(n)}| \geq \log_2((k+1)^{k^n-|A|^n-|B|^n}) \gtrsim k^n \log_2(k+1)$$

because $|A| < k$, $|B| < k$. The upper bound is obvious. ■

Theorem 3 is a particular case of the following theorem.

<u>Theorem 6:</u> If $Q = U^*(R)$ where G_R is a directed graph with loops for all vertices which has at least one arc that is not a loop and has no pair of opposite arcs then $\log_2|Q^{(n)}| \sim k^n$ when $n \to \infty$.

<u>Proof:</u> Let $R(a,b)$ and $a \neq b$. Consider the following subrelation R_1 of the relation R: $\forall x\, R_1(x,x)$, $R_1(a,b)$ and $\neg R_1(x,y)$ in the other cases. Then R_1 is a relation of a partial order. Hence R_1^n also is a relation of a partial order on the set E_k^n of all strings of length n, and it can be proved that E_k^n can be devided into chains (linearly ordered sets) $L_1, L_2, \ldots, L_r$ in such a way that $r \leq Ck^n/\sqrt{n}$ where C is a constant that does not depend on n (see, for example, Lemma 15, Theorem 3 and eq. (1) in [3]). It is easy to see that the size l_i of L_i satisfies the inequality $l_i \leq n+1$.

Let $\bar{a}, \bar{b}, \bar{c}$ lie on a chain L_i and $R_1^n(\bar{a},\bar{b})$, $R_1^n(\bar{b},\bar{c})$ hold and $p, q \in E_k$, $p \neq q$. Because R_1 is a subrelation of R then R_1^n is a subrelation of R^n. Thus $R^n(\bar{a},\bar{b})$ and $R^n(\bar{b},\bar{c})$. So if $f \in U^*(R)$ then it can not be $f(\bar{a}) = p$, $f(\bar{b}) = q$, $f(\bar{c}) = p$ because then it would be $R(p,q)$ and $R(q,p)$ but in G_R there are not opposite arcs. Hence if $f \in U^*(R)$ then all the same values of f on the chain L_i stand in succession (if we do not consider $*$). So for defining a function $f \in U^*(R)$ on a chain L_i it is enough to indicate all strings $\bar{a}$ such that $f(\bar{a}) = *$ and for each $p \in E_k$ to indicate the first string $\bar{a}$ on L_i (if it exists) such that $f(\bar{a}) = p$. Thus

$$|Q^{(n)}| \leq \prod_{i=1}^{r} 2^{l_i}(l_i+1)^k \leq 2^{k^n(n+2)Ck^n/\sqrt{n}}$$

and $\log_2|Q^{(n)}| \lesssim k^n$ when $n \to \infty$.

Because all functions with the only values $*$ and 0 belong to $U^*(R)$ for R from Theorem 6 we have $\log_2|Q^{(n)}| \gtrsim k^n$. So $\log_2|Q^{(n)}| \sim k^n$ when $n \to \infty$. ■

Proof of Theorem 4: Let $A \subseteq E_k$, $|A| = d$ and let A be a clique in G_R. It means that $R(a_i, a_j)$ for any $a_i \in A$ and $a_j \in A$. Consider all functions that take on only values from $A \cup \{*\}$. All of them satisfy (2) and so belong to $U^*(R)$. So

$$\log_2|Q^{(n)}| \geq \log_2(d+1)^{k^n} = k^n\log_2(d+1). \qquad (3)$$

Let $a,b \in E_k$, $a \neq b$ and $R(a,b)$. Without loss of generality suppose that $a = 0$, $b = 1$. Let $D = (F_1,\ldots,F_{k-1})$ where $F_1 \cup F_2 \cup \ldots \cup F_{k-1} = \{1,\ldots,n\}$ and $F_i \cap F_j = \emptyset$ for $i \neq j$ ($F_i = \emptyset$ is allowed). Let $H(D) = \{(a_1,\ldots,a_n)|\ (i \in F_1 \rightarrow (a_i = 0 \vee a_i = 1)) \wedge ((i \in F_j \wedge j \geq 2) \Rightarrow a_i = j)\}$.

It is easy to see that $R^n(\bar{a},\bar{b})$ for any two strings $\bar{a},\bar{b}$ from the same $H(D)$. So if $f(x_1,\ldots,x_n) \in U^*(R)$ then all its values on $H(D)$ (except $*$) have to form a complete subgraph in G_R. Hence

$$|Q^{(n)}| \leq \prod_D \sum_A (|A| + 1)^{|H(D)|}$$

where the product is taken over all $(k-1)^n$ ordered strings D and the sum is taken over all cliques in G_R. Because $|A| \leq d$ then

$$|Q^{(n)}| \leq \prod_D [(d + 1)^{|H(D)|}C] = (d + 1)^{k^n}C^{(k-1)^n}$$

where C is the number of cliques in G_R. Thus $\log_2|Q^{(n)}| \leq k^n\log_2(d+1)$. According to (3) we have $\log_2|Q^{(n)}| \sim k^n\log_2(d+1)$. ■

<u>Proof of Theorem 5, case i):</u> Let G_R have only edges (a_i,b_i), $i = 1,\ldots,k/2$. In this case all k^n strings from E_k^n fall into pairs $(\bar{a},\bar{b})$ such that $R^n(\bar{a},\bar{b})$ if and only if $\bar{a},\bar{b}$ belong to the same pair. A function $f(x_1,\ldots,x_n) \in U^*(R)$ can be defined on any pair by $3k+1$ ways $((a_i,b_i),(b_i,a_i),(a_i,*),(*,a_i),(b_i,*),(*,b_i),(*,*))$ independently of other pairs.
So $|Q^{(n)}| = (3k+1)^{(1/2)k^n}$. ■

For any binary relation R on $E_k = \{0,1,\ldots,k-1\}$ we can consider the bipartite directed graph H_R with the set of vertices $V \cup W$ where $V = \{v_o, v_1, \ldots, v_{k-1}\}$ and $W = \{w_o, w_1, \ldots, w_{k-1}\}$ and with the set of arcs $\{(v_i,w_j)|\ R(i,j)\}$. Let G_R be an undirected bipartite graph with parts of vertices A and B. Then the sets V and W in H_R are divided into parts $V = A' \cup B'$, $W = A'' \cup B''$ such that A', A'' correspond to A and B',B'' correspond to B and all arcs in H_R go from A'

to B" or from B' to A". If H_R has a directed perfect matching then this matching consists of two parts: one over A' and B" and another over B' and A". Any of these two parts produces a perfect undirected matching in G_R. Thus <u>case ii) of Theorem 5</u> is a corollary of the following theorem.

<u>Theorem 7:</u> Let $Q = U^*(R)$ where R is any binary relation. Then one of the following propositions must hold:
$\log_2|Q^{(n)}| \sim k^n \log_2(k+1)$ or in H_R there is a directed perfect matching.

<u>Proof:</u> Let β be any constant such that $1/2 < \beta < 1$ and $\tau = n^\beta$. Let $n_i(\bar{a})$ indicate how many times i occurs in the string $\bar{a}$ $(0 \leq i \leq k-1)$. Consider the following subset of E_k^n: $N = \{\bar{a} = (a_1,\ldots,a_n) \mid \forall\, i\ ((n/k)-\tau \leq n_i(\bar{a}) \leq (n/k)+\tau)\}$. It is known that $N \sim k^n$ when $n \to \infty$ (see, for example, [3], p. 12). Suppose first that there exists a number n_o such that for any $n \geq n_o$ there is no pair $\bar{a},\bar{b} \in N$ for which $R^n(\bar{a},\bar{b})$. Then consider all functions $f(x_1,\ldots x_n)$ from P_k^* such that $f(\bar{a}) = *$ for all $\bar{a} \notin N$. It is easy to see that for $n \geq n_o$ all such functions satisfy (2) and so all of them belong to $U^*(R)$. So in this case $|Q^{(n)}| \geq (k+1)^{|N|}$ and $\log_2|Q^{(n)}| \geq |N|\log_2(k+1)$ for $n \geq n_o$. Because $|N| \sim k^n$ we have $\log_2|Q^{(n)}| \gtrsim k^n\log_2(k+1)$ when $n \to \infty$ and therefore $\log_2|Q^{(n)}| \sim k^n\log_2(k+1)$.

Now consider the opposite case. It means that there exists an infinite sequence $n_1, n_2, \ldots$ with $n_i \to \infty$ such that for any n_i the following holds: there exist two strings $\bar{a},\bar{b} \in N$ for which $R^{n_i}(\bar{a},\bar{b})$. Consider any $n = n_i$ and some $\bar{a},\bar{b} \in N$ for which $R^n(\bar{a},\bar{b})$. Let $g(v_i,w_j)$ be equal to the number of indices s such that $(a_s,b_s) = (i,j)$. Note that if there is no arc (v_i,w_j) in H_R then $\neg R(i,j)$ and then $g(v_i,w_j) = 0$ because $R^n(\bar{a},\bar{b})$. Because $\bar{a},\bar{b} \in N$ we have

$$\frac{n}{k} - \tau \leq \sum_{j \in E_k} g(v_i,w_j) \leq \frac{n}{k} + \tau \quad \text{for all } i \in E_k \text{ and}$$

$$\frac{n}{k} - \tau \leq \sum_{i \in E_k} g(v_i,w_j) \leq \frac{n}{k} + \tau \quad \text{for all } j \in E_k.$$

Add to H_R two new vertices v and w and arcs (v,v_i), (w_i,w) for all $i \in E_k$. Set

$$g(v,v_i) = \sum_{j \in E_k} g(v_i,w_j), \qquad g(w_j,w) = \sum_{i \in E_k} g(v_i,w_j).$$

Then g is a (stationary) flow from v to w ([5]). Define the capacity of arcs in the following way: $c(v,v_i) = 1$, $c(w_j,w) = 1$ for all $i,j \in E_k$, $c(v_i,w_j) = +\infty$ for all arcs (v_i,w_j) of H_R. Set $h = g/(\frac{n}{k} + \tau)$ for all arcs. Then h is a (stationary) flow from v to w satisfying the capacity restrictions $h \leqslant c$.
The value of this flow is $t \geqslant (k(\frac{n}{k} - \tau))/(\frac{n}{k} + \tau)$. The last value tends to k when $n \to \infty$, because $\tau = n^{\beta}$ and $\beta < 1$. Since such flow can be constructed for any n_i and $n_i \to \infty$ the maximum value of a flow in the constructed network is not less than k. But a flow of value greater than k can not go out of v. So the maximum flow in this network is of value k and because all capacities are integers there exists an integer flow of value k in this network ([5]). This flow must consist of k flows along the k paths from v to w of length 3 which have no common points except v and w. The second arcs of these paths produce a directed perfect matching in H_R. ■

Concerning part iii) of Theorem 5 the lower bound is rather simple but the proof of the upper bound is very complicated and will be published elsewhere. This proof uses an idea analogous to one used in [3,4] for obtaining the asymptotic for the logarithm of the number of functions on n variables in some maximal closed classes in the set of all completely defined functions of k-valued logic P_k.

References:

1. Luo Zhukai: Maximal closed sets in the set of partial many-valued logic functions (Chinese). Acta Math. Sinica 27, 795-800 (1984)

2. Luo Zhukai: The completeness theory of partial many-valued logic functions. (Chinese) Acta Math. Sinica 27, 676-683 (1984)

3. Алексеев, В.Б.: О числе монотонных к-значных функций. Проблемы Кибернет. 28, 5-24 (1974)

4. Алексеев, В.Б.: О числе функций в классах, задаваемых центральными предикатами. Мат. Заметки. 37, 880-886 (1985)

5. Ford, L.R.,Jr., and Fulkerson, D.R.: Flows in networks. Princeton 1962

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Author:

Dr. Valeri B. Alekseev
Moscow State University
Department of Computational Mathematics and Cybernetics
Leningori
SU-119 899 Moscow
USSR

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Tomáš Dvořák; Ivan Havel;
Jean-Marie Laborde; Petr Liebl

Generalized hypercubes and graph embedding with dilation

The main notion investigated here is that of a graph embedding. The graphs considered are finite, undirected, without loops and multiple edges. As usual, $V(G)$ is the vertex set and $E(G)$ the edge set of the graph G, $d_G(x,y)$ denotes the distance of the vertices x,y and $\Delta(G)$ the maximum degree of vertices in G. Define as usual the d-th power H^d of the graph H by

$$V(H^d) = V(H),$$

$$E(H^d) = \{(u,v) \mid u \neq v,\ d_H(u,v) \leq d\}.$$

Let G, H be graphs. A 1-1 mapping f of $V(G)$ into $V(H)$ we call an embedding of G into H. Its expansion is the ratio $|V(H)|/|V(G)|$. The dilation of f is defined by

$$\mathrm{dil}(f) = \max_{(u,v)\in E(G)} d_H(f(u),f(v)).$$

Both are well-known quantitative characteristics of graph embeddings.

Consider an embedding f of G into H such that $\mathrm{dil}(f) = 1$. It maps adjacent vertices of G into adjacent vertices of H, such that there is a subgraph H_1 of H isomorphic to G. If such an embedding f exists we write $G \subseteq H$. If there is an embedding of G into H with dilation $\leq d$ we write $G \overset{d}{\subseteq} H$. Clearly,

$$G \overset{d}{\subseteq} H \Longleftrightarrow G \subseteq H^d.$$

The study of embeddings and their characteristics has practical motivation in computer science. They can be viewed as high level descriptions of efficient methods to simulate an algorithm designed for one type of architecture on a different network structure and/or techniques to distribute data/program variables to achieve optimum use of all available processors. Then the

expansion estimates the economy of an embedding and the dilation measures the delay in the data flow.

We shall be concerned with the special case of the host graph being a hypercube Q_n. $G \overset{d}{\subseteq} Q_n$ trivially implies $n \geq \log_2|V(G)|$. Given G, $Q_{\lceil \log_2|V(G)| \rceil}$ is called its optimal hypercube and $Q_{\lceil \log_2|V(G)| \rceil+1}$ its next-to-optimal hypercube. An embedding of G into its optimal hypercube has expansion less than 2.

Consider the following definition:

Let $S \subseteq \{1,\dots,n\}$. By $Q_n(S)$ we denote the graph with

$$V(Q_n(S)) = V(Q_n),$$

$$E(Q_n(S)) = \{(u,v) \mid d_{Q_n}(u,v) \in S\}.$$

$Q_n(S)$ so defined is a special case of Lovasz's cube-like graphs, cf. [2].
$Q_n(\{1,\dots,d\})$ is the power Q_n^d. Several further special cases of $Q_n(S)$ have already be studied; e.g., $Q_{2k-1}(\{1,2k-1\})$ is the halfcube $\frac{1}{2}Q_{2k}$ (cf. [11]) which arises from Q_{2k} by identifying each vertex with its opposite and $Q_{2k}(\{1,2k\})$ is the extended odd graph E_{k+1} (cf. [11,13]). $Q_n(S)$ is not connected for $S = \{n\}$ and for S not containing odd numbers; otherwise it is connected.

$Q_n(S)$ is regular of degree $\Sigma_{i \in S}\binom{n}{i}$. Comparing graphs $Q_n(S)$ of the same degree, one gets the following facts on isomorphisms.

<u>Assertion 1:</u> Assume $M \subseteq \{1,\dots,\lfloor \frac{n}{2} \rfloor\}$ and put $S_1 = \{2k \mid k \in M\}$, $S_2 = \{2k \mid k \in M,\ 2k \leq n-1\} \cup \{2k-1 \mid k \in M\}$; assume $S_1 \neq \{n\}$. Then $Q_n(S_1)$ consists of two components, each of which is isomorphic to $Q_{n-1}(S_2)$.

<u>Assertion 2:</u> Let n be odd, assume M, S_1, S_2 as in Assertion 1. Put $S_3 = S_1 \cup \{n\}$. Then $Q_n(S_3)$ is isomorphic to the cartesian product $K_2 \times Q_{n-1}(S_2)$.

<u>Assertion 3:</u> Let n be even, $S_1 \subseteq \{1,\dots,n\}$, $S_2 = \{k \mid k \in S_1,\ k \text{ even}\} \cup \{n-k \mid k \in S_1,\ k \text{ odd}\}$. Then $Q_n(S_1)$ and $Q_n(S_2)$ are isomorphic.

It seems interesting to study some well-known graph-theoretical characteristics of the graphs $Q_n(S)$. Obviously, if S contains only odd numbers it is bipartite, hence the chromatic number $\chi(Q_n(\{1,n\}))$ is 2 for n odd; it was proved in [13] that it is 4 for n even. From results in [1] there follow values for the clique number ω of Q_n^d:

Corollary 4:

$$\omega(Q_n^{2m}) = \sum_{k=o}^{m} \binom{n}{k} \quad \text{for } 2m < n,$$

$$\omega(Q_n^{2m+1}) = 2\sum_{k=o}^{m} \binom{n-1}{k} \quad \text{for } 2m+1 < n.$$

Using basic notions of coding theory ([10]), it can be seen that any (n,k)-code is an independent set in Q_n^d for any $d < k$. If the code is linear the classes of the factor group represent a partition of $V(Q_n^d)$ into independent sets and hence a colouring of Q_n^d. Using this, one may get

Corollary 5:

$$\chi(Q_n^2) = 2^m \quad \text{for } n = 2^m-k,\ k \in \{1,2,3,4\},$$

$$\chi(Q_n^3) = 2^m \quad \text{for } n = 2^m-k,\ k \in \{0,1,2,3\}.$$

(See also [9]).
From what we know, we are able to give the table for $\chi(Q_n^d)$ (Figure 1). Here the symbol ? denotes a value we do not know.

d\n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2		4	4	8	8	8	8	?	?	?	?	16	16	16	16	?
3			8	8	16	16	16	16	?	?	?	?	32	32	32	32
4				16	16	32	64	64	?	?	?	?	?	?	?	?
5					32	32	64	128	128	?	?	?	?	?	?	?
6						64	64	128	256	512	?	?	?	?	?	?

Figure 1

Turn now to the problem of embedding trees into hypercubes. Various results support the following conjecture from [4]:

Conjecture 1: Any balanced tree T with $\Delta(T) \leq 3$ and 2^n vertices can be embedded into Q_n with dilation 1.

Here balanced means the considering of T as a bipartite graph, both parts have an equal number of vertices. The condition $\Delta(T) \leq 3$ is essential; in [6] for any $n \geq 4$ a balanced tree of 2^n vertices is constructed with $\Delta(T) = 4$ (and only one vertex of degree 4) such that $T \not\subseteq Q_n$.

The assertion of Conjecture 1 has been proved for some classes of trees [4, 7, 5]. It has also been verified for $n \leq 4$ [8]. Trees T with $\Delta(T) \leq 3$ are often called binary trees, with the allusion to a directed tree with indegree 1 and outdegree ≤ 2. Let us reformulate Conjecture 1 in an equivalent form:

<u>Conjecture 1':</u> If both parts of the bipartition of a binary tree T have $\leq 2^{n-1}$ vertices then $T \subseteq Q_n$.

In [14] two related conjectures are stated:

<u>Conjecture 2:</u> Any binary tree can be embedded into its next-to-optimal hypercube with dilation 1.

<u>Conjecture 3:</u> Any binary tree can be embedded into its optimal hypercube with dilation 2.

To show that Conjecture 2 is weaker then Conjecture 1, given a binary tree, take two copies of it, find a vertex with degree ≤ 2 and join it by a new edge with its counterpart in the other copy; the obtained tree is balanced and Conjecture 1' may be applied.
Also Conjecture 3 is weaker than Conjecture 1. We can however state a stronger version of Conjecture 3 and prove that it is still weaker than Conjecture 1.

<u>Conjecture 4:</u> Any tree T with $\Delta(T) \leq 5$ can be embedded into its optimal hypercube with dilation 2.

<u>Lemma:</u> Let G and H be two graphs, G connected. Assume there is an embedding f of G into H with $dil(f) \leq k$ such that $d_H(f(u),f(v))$ is even for every $(u,v) \in E(G)$.
Then $H \subseteq Q_{n+1} \Longrightarrow G \overset{k}{\subseteq} Q_n$.

<u>Proof:</u> Let g be an embedding of H into Q_{n+1} (with dilation 1). Let p be the projection of Q_{n+1} onto Q_n which omits one fixed coordinate. Now consider the mapping

$h = p \circ g \circ f: V(G) \to V(Q_n)$. From the condition of even distances it follows that h is 1-1, i.e., it is an embedding.
It is easy to see that $dil(h) \leq k$. ■

To show that the Conjecture 4 follows from Conjecture 1, take the tree G, choose any vertex for its root and orient G from the root. Construct from G the binary tree H by adding new ("black") to the old ("white") vertices according to the scheme in Figure 2.

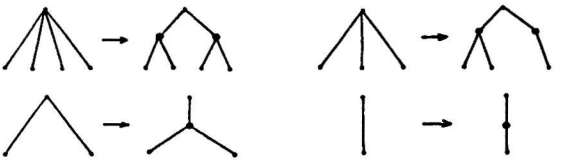

Figure 2

The number of the black vertices is smaller than that of the white ones, so apply Conjecture 1' to H. Now use the Lemma for $k = 2$.

Using the Lemma and the result ([6]) that the complete binary tree can be embedded into its next-to-optimal hypercube one can prove

<u>Corollary 7:</u> Any complete 2^{2m}-ary tree can be embedded into its optimal hypercube with dilation $2m$.

From the Lemma and the result ([5]) that any generalized one-legged caterpillar (a binary tree with all the vertices of degree 3 lying on the same path) with legs of even length can be embedded into its optimal hypercube with dilation 1 it follows

<u>Corollary 8:</u> Any generalized one-legged caterpillar can be embedded into its optimal hypercube with dilation 2.

$\Delta(T) \leq 5$ in Conjecture 4 is essential. This follows from the existence, for any $n \geq 5$, $n \neq 6$, of a tree T_n with $\Delta(T_n) = 6$, $|V(T_n)| = 2^n$ such that $T_n \not\subseteq Q_n^2$. These trees are constructed in the following way.

First the case n odd. Consider the trees S_n for $n \geq 5$, arising from S_5 on Figure 3 by rooting at each end vertex of S_5 a complete 4-ary tree of $(n-5)/2$ levels.

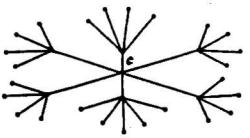

Figure 3

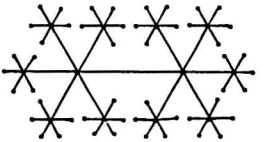

Figure 4

S_n has $2^n + \frac{4}{3}(2^{n-5} - 1)$ vertices and the distance of its vertices from the center c is at most $(n-1)/2$. Construct now T_n by removing from S_n $\frac{4}{3}(2^{n-5} - 1)$ end vertices (and the corresponding edges) so that T_n has 2^n vertices. In Q_n^2 opposite vertices have distance $(n+1)/2$; hence no embedding f of dilation 1 can map any vertex of T_n into the vertex of Q_n^2 opposite to the vertex $f(c)$. Notice that T_n has only two (adjacent) vertices of degree 6.
For n even, consider the trees S_n given as double complete 5-ary trees of $(n-2)/2$ levels (Figure 4 shows S_6). S_n has now $\frac{1}{2}(5^{n/2} - 1)$ vertices which is more than 2^n for $n \geq 8$. Construct T_n, for $n \geq 8$, by removing end vertices from S_n so that T_n has 2^n vertices. A slightly refined argument shows that again no embedding of dilation 1 can map any vertex of T_n into a certain vertex of Q_n^2.

Let us conclude with three problems:

<u>Problem 1:</u> The values known so far for $\chi(Q_n(S))$ are all powers of 2. Is this so in general ?

<u>Problem 2:</u> In particular, we know only that $13 \leq \chi(Q_8^2) \leq 16$. Is it equal to 16 ?

Problem 3: It is well known that the complete binary tree can be embedded into its optimal hypercube with dilation 2. From Corollary 7 this also holds for the complete 4-ary tree. Does it hold for the complete 3-ary tree ? (This would be a consequence of Conjecture 4.)

References:

1. Bezrukov, S.: Characterization of maximal subsets of the Hamming space having given diameter (Russian). Problemy Peredači Informacii 23, 106-109 (1987)

2. Harary, F.: Four difficult unsolved problems in graph theory. In: Recent advances in graph theory. Proc. Second Czechoslovak Sympos., Prague, 1974. pp. 249-256, Prague 1975

3. Havel, I.: Embedding certain trees into cubes. In: Recent advances in graph theory. Proc. Second Czechoslovak Sympos., Prague, 1974. pp. 257-261, Prague 1975

4. Havel, I.: On hamiltonian circuits and spanning trees of hypercubes. Časopis Pěst. Mat. 1, 135-152 (1984)

5. Havel, I., Laborde, J.-M., and Liebl, P.: Caterpillar-like trees spanning hypercubes. Časopis Pěst. Mat. (to appear)

6. Havel, I., and Liebl, P.: Embedding the dichotomic tree into the cube. (Czech with English summary). Časopis Pěst. Mat. 97, 201-205 (1972)

7. Havel, I., and Liebl, P.: One-legged caterpillars span hypercubes. J. Graph Theory 10, 69-76 (1986)

8. Laborde, J.-M.: Le plongement des arbres a an plus 16 sommets dans l'hypercube. Rapport de Recherche 7551, LSD IMAG, Grenoble

9. Linial, N., Meshulam, R., and Tarsi, M.: Matroidal bijections between graphs. J.Combin.Theory Ser. B 45, 31-44 (1988)

3*

10. Sloane, N.J.A., and MacWilliams, F.J.: The theory of error correcting codes. Amsterdam 1978

11. Mulder, H.M.: The interval function of a graph. Amsterdam 1980

12. Nebesky, L.: On cubes and dichotomic trees. Časopis Pěst. Mat. 99, 164-167 (1974)

13. Sokolova, M.: The chromatic number of extended odd graphs is four. Časopis Pěst. Mat. 112, 308-311 (1987)

14. Wagner, A.S.: Embedding trees in the hypercube. Techn. Report 204/87, University of Toronto 1987

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Authors:

Prof. Dr. Tomáš Dvořák
Prof. Dr. Ivan Havel
Prof. Dr. Petr Liebl
ČSAV, Matematický ústav
Žitna 25
CS-115 67 Praha
ČSSR

Prof. Dr. Jean-Marie Laborde
Lab. Struct. Discr. et Didactique CNRS Grenoble
BP68
F-38402 Saint-Martin-d'Hères
France

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Hans Jürgen Prömel; Bernd Voigt

Ramsey theory for the Cantor space

1. Introduction

There exist many roads to Ramsey Theory and there certainly exist several ways to recapitulate its essence. For our present purposes it is most fitting to say that Ramsey Theory deals with structural generalizations of the ordinary pigeonhole principle. We examine Ramsey's Theorem from this point of view. Let us partition the set $\left[\begin{smallmatrix}\omega\\2\end{smallmatrix}\right]$ of two-element subsets of $\omega = \{0,1,2,\dots\}$ into two parts, say, $\left[\begin{smallmatrix}\omega\\2\end{smallmatrix}\right] = A \cup B$. The ordinary pigeonhole principle tells us that one of the sets A or B must be infinite. But Ramsey's Theorem tells us that there must exist an infinite set $X \subseteq \omega$ such that all of its two-element subsets either belong to A or to B, i.e., $\left[\begin{smallmatrix}X\\2\end{smallmatrix}\right] \subseteq A$ or $\left[\begin{smallmatrix}X\\2\end{smallmatrix}\right] \subseteq B$. The later statement exhibits the improvement and the additional comprehension one may gain from Ramsey type theorems most clearly. Not just infinitely many two-element sets but all two-element subsets of an infinite set X belong to either A or B.

In this paper we present some principles and ideas of a branch of Ramsey Theory which is known as Topological Ramsey Theory. As the guiding lines merely evolved during the last few years we are not giving an historical account but rather discuss results which are concerned with the particular structure 2^ω of infinite 0-1-sequences.

Via characteristic functions 2^ω is interpreted as the power set of ω. From a topological point of view the space 2^ω is Cantor's Discontinuum.

2. Intervals in 2^ω

We tacitly interpret elements $X = (X(0),X(1),\dots) \in 2^\omega$ as subsets of ω. Thus the elementary set theoretic operations like union, intersection and difference as well as the relation "$\subseteq$" are well defined.

2.1. Consider the following question: Is it true that for every partition $2^\omega = A \cup B$ there exist sets $X \subseteq Y$ in 2^ω such that $Y \setminus X$ is infinite and such that the interval $[X,Y] = \{Y^* \in 2^\omega \mid X \subseteq Y^* \subseteq Y\}$ belongs to either A or B ?

2.2. The first answer to this question, using the axiom of choice, is simply "no". Observe that for every choice of X and Y such that $Y \setminus X$ is an infinite set the interval $[X,Y]$ is a nonempty perfect subset of 2^ω, i.e., $[X,Y]$ is a nonempty closed set without isolated vertices. Hereby 2^ω bears the Tychonoff product topology inherited from the discrete space $2 = \{0,1\}$. For $g \in \{0,1\}^n$ let $T(g) = \{X \in 2^\omega \mid g(i) = X(i)$ for all $i < n\}$ be the Tychonoff cone generated by g. The set of all Tychonoff cones forms a basis for the topology on 2^ω. It is a complete separable metric space, i.e., a Polish space, where the Baire metric is defined by $d(X,Y) = 1/(k+1)$ if $X(i) = Y(i)$ for all $i < k$ but $X(k) \neq Y(k)$. A Bernstein set $Z \subseteq 2^\omega$ is any set with the property that every nonempty perfect subset P intersects Z as well as the complement $2^\omega \setminus Z$.

Bernstein [1], cf., also Kuratowski [5, p. 514], showed that Bernstein sets exist. However, the construction of such Bernstein sets involves the axiom of choice. This can be seen, e.g., from the fact that Bernstein sets are not Lebesgue measurable, nor do they have the property of Baire (cf., Oxtoby [12]), together with the result of Shelah [18] that the consistency of ZF implies the consistency of ZF plus the axiom of dependent choices plus the assertion "every set in 2^ω has the property of Baire".

2.3. The second answer to the question addressed above is "yes". The guiding idea is to exclude partitions which are defined nonconstructively, that is, by means of the axiom of choice. Moran and Strauss [9] investigated Baire partitions, resp., measurable partitions of 2^ω.

2.4. To appreciate the constructive approach of Moran and Strauss we introduce the following notation:

<u>Notation:</u> Let $[2]_{one}{}^{(\omega}_{\omega)}$ consists of all words (mappings) $F: \omega \to \{0,1\} \cup \{\lambda_i \mid i < \omega\}$ such that for each $j < \omega$ there exists exactly one $i < \omega$ with $F(i) = \lambda_j$. For sets $X = (X(0), X(1), \ldots) \in 2^\omega$ and parameter words $F \in [2]_{one}{}^{(\omega}_{\omega)}$

the insertion of X into F, viz., $F\cdot X$, is defined by

$$(F\cdot X)(i) = \begin{cases} F(i) & \text{if } F(i) \in \{0,1\}, \\ X(j) & \text{if } F(i) = \lambda_j. \end{cases}$$

Notice that for every $F \in [2]_{one}(^{\omega}_{\omega})$ the set $F\cdot 2^{\omega}$ is of the form $F\cdot 2^{\omega} = [F\cdot(\bar{0}), F\cdot(\bar{1})]$, where $(\bar{0}) = (0,0,\ldots)$ and $(\bar{1}) = (1,1,\ldots)$. Conversely, every interval $[X,Y]$ with $Y\setminus X$ an infinite set may be represented by such a parameter word $F \in [2]_{one}(^{\omega}_{\omega})$.

2.5. Recall that a set $M \subseteq 2^{\omega}$ is meager if it belongs to the complement of a countable intersection of dense open sets D_n, i.e., $M \subseteq 2^{\omega}\setminus\bigcap_{n<\omega} D_n$. A set $B \subseteq 2^{\omega}$ is a Baire set if it can be written as a symmetric difference of an open set O and a meager set M, i.e., $B = (O\setminus M) \cup (M\setminus O)$.

Using a typical Baire category construction (viz., a procedure which is commonly used to prove the Baire category theorem) Moran and Strauss showed that meager sets in 2^{ω} are Ramsey null:

<u>Lemma:</u> Let $O \subseteq 2^{\omega}$ be a nonempty open set and let $M \subseteq 2^{\omega}$ be meager. Then there exists an $F \in [2]_{one}(^{\omega}_{\omega})$ such that $F\cdot 2^{\omega} \subseteq O\setminus M$.

<u>Proof:</u> Let $D_n \subseteq 2^{\omega}$ be dense open sets such that $M \subseteq 2^{\omega}\setminus \bigcap_{n\in\omega} D_n$.

As O is a nonempty open set there exists a $f_o \in \{0,1\}^{<\omega}$ such that the Tychonoff cone $T(f_o)$ is contained in O. By induction let $f_k \in (\{0,1\} \cup \{\lambda_o,\ldots,\lambda_{k-1}\})^{<\omega}$ be such that

$$T(f_k\cdot g) \subseteq O \cap \bigcap_{i<k} D_i$$

for every $g \in \{0,1\}^k$ and such that $|f_k^{-1}(\lambda_i)| = 1$ for every $i < k$. The insertion $f_k\cdot g$ is defined as before.

Let $(g_x g)_{\xi<q}$ be an enumeration of $\{0,1\}^{k+1}$, thus $q = 2^{k+1}$. By induction on $l \leq q$ there exist $h_o,\ldots,h_{l-1} \in \{0,1\}^{<\omega}$ such that

$$T(((f_k \otimes \lambda_k)\cdot g_{\xi}) \otimes h_o \otimes \ldots \otimes h_{l-1}) \subseteq O \cap \bigcap_{i\leq k} D_i$$

for all $\xi < l$. Here $\otimes$ denotes the concatenation of sequences. Notice that for $l = 0$ the assertion holds vacuously. We show how to construct h_l. Consider the Tychonoff cone

$$T = T(((f_k \otimes \lambda_k)\cdot g_l) \otimes h_o \otimes \ldots \otimes h_{l-1}).$$

As D_k is a dense open set the intersection $T \cap D_k$ still contains a Tychonoff cone. Thus there exists a $h_l \in \{0,1\}^{<\omega}$ such that

$$T(((f_k \otimes \lambda_k)\cdot g_l) \otimes h_o \otimes \ldots \otimes h_l) \subseteq O \cap \bigcap_{i<k} D_i.$$

Finally, $f_{k+1} = f_k \otimes \lambda_k \otimes h_o \otimes \ldots \otimes h_{q-1}$ satisfies $T(f_{k+1}\cdot g) \subseteq O \cap \bigcap_{i<k} D_i$ for every $g \in \{0,1\}^{k+1}$.

Define $F \in [2]_{one}(^{\omega}_{\omega})$ as the limit of the f_k, i.e., $F(i) = f_i(i)$. This is a proper definition as f_k is an initial segment of f_{k+1}. By construction then $F\cdot X \in O\backslash M$ for every $X \in 2^{\omega}$. Consider $X]\![k]$, the restriction of X to $\{0,\ldots,k-1\}$. Then $f_k\cdot X]\![k]$ is an initial segment of $F\cdot X$. Thus

$$F\cdot X \in T(f_k\cdot X]\![k]) \subseteq O \cap \bigcap_{i<k} Di.$$

Hence $F\cdot X \in O \cap \bigcap_{k<\omega} D_k \subseteq O\backslash M$. ■

<u>Corollary:</u> For every partition $2^{\omega} = \bigcup_{n<\omega} A_n$ into Baire sets A_n there exists an $F \in [2]_{one}(^{\omega}_{\omega})$ and there exists some $n < \omega$ such that $F\cdot 2^{\omega} \subseteq A_n$.

<u>Proof:</u> Every A_n may be written as $A_n = (O_n\backslash M_n) \cup (M_n\backslash O_n)$, where O_n is open and M_n is meager. As certainly 2^{ω} itself is not meager there exists $n < \omega$ such that O_n is nonempty. By the Lemma there exists an interval $F \in [2]_{one}(^{\omega}_{\omega})$ such that $F\cdot 2^{\omega} \subseteq O_n\backslash M_n \subseteq A_n$. ■

2.6. Knowing the duality between the concept of category and measure one is tempted to ask about partitions into measurable sets. By a measure μ on 2^{ω} we understand the completion of the product measure coming from any discrete distribution $\tilde{\mu}$ on $\{0,1\}$. Thus $\mu T(f(0),\ldots,f(n-1)) = \prod_{i<n} \tilde{\mu}(f(i))$ for any $f \in \{0,1\}^n$. By Caratheodory's Theorem this extends uniquely to a

measure which is defined on Borel sets and, finally, we add all subsets of sets of measure zero and consider the such generated σ-algebra of measurable sets.

Moran and Strauss [9] investigated the particular case where $\tilde{\mu}$ is the uniform distribution. Here we present a slight modification of the original argument which works in the general setting. The crucial lemma is:

<u>Lemma:</u> Let $C \subseteq 2^{\omega}$ be a set of positive measure. Then there exists a finite sequence $g = (g(0),\dots,(g(n-1)) \in \{0,1\}^{<\omega}$ and there exists a closed set $D \subseteq 2^{\omega}$, again of positive measure, such that $g \otimes \{0,1\} \otimes D \subseteq C$, where " $\otimes$ " still refers to the concatenation of sequences.

<u>Proof:</u> Without loss of generality we can assume that C is closed (otherwise there exists $C' \subseteq C$, C' closed and $\mu(C') > \mu(C)/2$). Let $\varepsilon > 0$ be such that $\varepsilon \cdot \sum_{x<2} \tilde{\mu}(x)^{-1} < 1$.

Let $W \subseteq 2^{\omega}$ be an open set such that $C \subseteq W$ and $\mu(W) < \mu(C)\cdot(1+\varepsilon)$. W can be written as an union of Tychonoff cones. As C is compact there exist finitely many Tychonoff cones $T(g_i)$, $i = 0,\dots,m-1$, such that $C \subseteq \bigcup_{i<m} T(g_i)$. As we may assume that $T(g_i) \cap T(g_j) = \emptyset$ for $i \neq j$ and thus $\mu(\bigcup_{i<m} T(g_i)) = \sum_{i<m} \mu(T(g_i)) < \mu(C)\cdot(1+\varepsilon)$, there exists $i < m$ such that $\mu(T(g_i)) < \mu(C \cap T(g_i))\cdot(1+\varepsilon)$.

Let $C' \subseteq 2^{\omega}$ be such that $C \cap T(g_i) = g_i \otimes C'$. Then C' is measurable and $\mu(C') = (\mu(C \cap T(g_i)))/(\mu(T(g_i))) > 1/(1+\varepsilon) > 1-\varepsilon$. Thus there exists a closed set $D \subseteq 2^{\omega}$ with $\mu(D) > 1 - \varepsilon \cdot \sum_{x<2} \mu(x)^{-1} > 0$ such that $\{0,1\} \otimes D \subseteq C'$.
Altogether we obtain $D \subseteq 2^{\omega}$ with $\mu(D) > 0$ and $g = g_i$ such that $g \otimes \{0,1\} \otimes D \subseteq C$, as desired. ■

<u>Corollary 1:</u> For every set $C \subseteq 2^{\omega}$ of positive measure there exists an $F \in [2]_{one}\binom{\omega}{\omega}$ such that $F\cdot 2^{\omega} \subseteq C$. ■

So this Corollary actually says that sets of measure zero are Ramsey null. Observe that the dual statement is obtained by replacing the term "positive measure" by "Baire, but not meager".

Corollary 2: For every partition $2^\omega = \bigcup_{n<\omega} A_n$ into measurable sets there exists an $F \in [2]_{one}{}^{(\omega)}_{\omega}$ and there exists a positive integer $n < \omega$ such that $F \cdot 2^\omega \subseteq A_n$. ■

2.7. In the previous section we considered partitions of 2^ω with countably many equivalence classes. Next we are much more generous. We would like to know what are typical patterns with respect to unbounded partitions. Basically there exist two approaches.

2.8. Recall that we still should avoid partitions which are defined using the existence of Bernstein sets in one way or the other. One possibility to rule this out is to consider Baire mappings into metric spaces, resp., measurable mappings into separable metric spaces.

Let Y be a metric space. A mapping $\Delta: 2^\omega \to Y$ is a Baire mapping, resp., a measurable mapping if $\Delta^{-1}(O)$ is a Baire set, resp., a measurable set for every open set $O \subseteq Y$. Baire mappings into metric separable spaces, resp., measurable mappings into metric spaces are very close to continuous mappings. This is made precise by the following well-known result:

Theorem: Let $\Delta: 2^\omega \to Y$ be a baire mapping, resp., a measurable mapping, where Y is a metric space, resp., a separable metric space. Then there exists a meager set M, resp., for every $\varepsilon > 0$ there exists a measurable set M of measure $\mu(M) < \varepsilon$ such that $\Delta]2^\omega \backslash M$ is a continuous mapping.

We sketch proofs for the case that Y is separable. For the general case compare Emeryk, Frankiewicz and Kulpa [4] or Lefmann and Voigt [8].

Proof: Let $\Delta: 2^\omega \to Y$ be a Baire mapping. Let $(O_n)_{n<\omega}$ be a basis of Y. Such an enumeration exists as Y is a separable metric space. As Δ is a Baire mapping each of the preimages $\Delta^{-1}(O_n) \subseteq 2^\omega$ is a Baire set, so it may be written as $\Delta^{-1}(O_n) = B_n \backslash M_n \cup M_n \backslash B_n$ for an open set $B_n \subseteq 2^\omega$ and a meager set $M_n \subseteq 2^\omega$. Let $M = \bigcup_{n<\omega} M_n$. We claim that $\Delta]2^\omega \backslash M$ is continu-

ous. Let $\bigcup_{n\in I} O_n \subseteq Y$ be an open set in Y. Then $(\Delta\rceil 2^\omega\backslash M)^{-1}(\bigcup_{n\in I} O_n)$ $= (\bigcup_{n\in I}(B_n\backslash M_n \cup M_n\backslash B_n))\backslash M = \bigcup_{n\in I} B_n\backslash M$, so this is an open set in $2^\omega\backslash M$. Next let $\Delta\colon 2^\omega \to Y$ be a measurable mapping. Let $(O_n)_{n<\omega}$ be a basis of Y. As Δ is measurable each of the preimages $\Delta^{-1}(O_n) \subseteq 2^\omega$ is measurable. Let $B_n \subseteq 2^\omega$ be open sets such that $\mu(B_n\backslash\Delta^{-1}(O_n)) < \varepsilon\cdot 2^{-n}$. Let $M = \bigcup_{n<\omega} B_n\backslash\Delta^{-1}(O_n)$. Then $\mu(M) < \varepsilon$. We claim that $\Delta\rceil 2^\omega\backslash M$ is continuous. As $(\Delta\rceil 2^\omega\backslash M)^{-1}(O_n)\backslash M = B_n\backslash M$ is an open set in $2^\omega\backslash M$ the claim follows. ■

<u>Corollary:</u> Let $\Delta\colon 2^\omega \to Y$ be a Baire mapping, resp., a measurable mapping, where Y is a metric space, resp., a separable metric space. Then there exists an interval $F \in [2]_{one}{\binom{\omega}{\omega}}$ such that $\Delta\rceil F\cdot 2^\omega$ is a continuous mapping.

<u>Proof:</u> Combine Theorem 2.8. with Lemma 2.5. and Corollary 2.6.1. resp. ■

2.9. In view of the foregoing Corollary it suffices to restrict attention to continuous mappings $\Delta\colon 2^\omega \to Y$, where Y is a metric space. The following Lemma is from Lefman and Voigt [8].

<u>Lemma:</u> Let $\Delta_i\colon 2^\omega \to Y$, $i \in \{0,1\}$, where Y is a metric space, be two continuous mappings. Then there exists a $F \in [2]_{one}{\binom{\omega}{\omega}}$ such that either $\Delta_0(F\cdot 2^\omega) \cap \Delta_1(F\cdot 2^\omega) = \emptyset$, i.e., the two mappings have disjoint images on $F\cdot 2^\omega$, or such that $\Delta_0(F\cdot X) = \Delta_1(F\cdot X)$ for all $X \in 2^\omega$, i.e., the two mappings agree on $F\cdot 2^\omega$.

<u>Proof:</u> Assume that there exist some $X \in 2^\omega$ with $\Delta_0(X) \neq \Delta_1(X)$. As $\Delta_0(X)$ and $\Delta_1(X)$ are distinct elements of a metric space they may be separated by disjoint open sets. Hence, as Δ_0 and Δ_1 are continuous, there exists a positive integer n such that

$$\{\Delta_0(Y) \mid Y \in 2^\omega \text{ and } Y(i) = X(i) \text{ for all } i<n\}$$
$$\cap \{\Delta_1(Y) \mid Y \in 2^\omega \text{ and } Y(i) = X(i) \text{ for all } i<n\} = \emptyset.$$

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Then any $F \in [2]_{one}\binom{\omega}{\omega}$ such that $F(i) = X(i)$ for all $i<n$ has the property that $\Delta_0(F\cdot 2^\omega) \cap \Delta_1(F\cdot 2^\omega) = \emptyset$. ■

We may use this Lemma to give a short proof of the canonizing theorem for continuous mappings $\Delta: 2^\omega \to Y$, where Y is a metric space. Originally the result is due to some unpublished work of Silver's from about 1960. The proof given by Prömel, Simpson and Voigt [17] adapts ideas from recursion theoretic investigations of Lachlan [6] and is not straightforward.

<u>Theorem:</u> For every continuous mapping $\Delta: 2^\omega \to Y$, where Y is a metric space, there exists an interval $F \in [2]_{one}\binom{\omega}{\omega}$ such that $\Delta\rceil F\cdot 2^\omega$ either is a constant or a one-to-one mapping.

<u>Proof:</u> By induction, using the Lemma, one constructs an $F \in [2]_{one}\binom{\omega}{\omega}$ such that for every $n < \omega$ and every $g \in 2^n$ either

(i) $\{\Delta(F\cdot(g\otimes(0)\otimes X))\mid X \in 2^\omega\} \cap \{\Delta(F\cdot(g\otimes(1)\otimes X))\mid X \in 2^\omega\} = \emptyset$

or

(ii) $\Delta(F\cdot(g \otimes (0) \otimes X)) = \Delta(F\cdot(g \otimes (1) \otimes X))$ for all $X \in 2^\omega$.

If there exist some $g \in 2^{<\omega}$ such that property (i) holds for all sequences in $g \otimes 2^{<\omega}$ then for $G = g \otimes (\lambda_0, \lambda_1, \ldots)$ the restriction $\Delta\rceil F\cdot G\cdot 2^\omega$ is a one-to-one mapping. Otherwise there exists an infinite ascending chain of g's which all have property (ii). In other words, there exists a $G' \in \{0,1\}^\omega$ and there exists a strictly ascending sequence $(n_i)_{i<\omega}$ such that each $G'\rceil[n_i]$ has property (ii).
Consider $G \in [2]_{one}\binom{\omega}{\omega}$ which is defined by

$G(n^i) = \lambda_i$ and

$G(k) = G'(k)$ otherwise.

A simple transitivity argument shows that if for some h and $h \otimes (0) \otimes h'$ property (ii) holds, where $h, h' \in 2^{<\omega}$, then property (ii) also holds with respect to $h \otimes (1) \otimes h'$. This argument actually implies that for every $X \in 2^\omega$ and every $i < \omega$ also the restriction $(G\cdot X)\rceil[n_i]$ has property (ii).

Let X and X' be elements of 2^ω. By induction on n it follows that

$\Delta(F\cdot G(X(0),\ldots,X(n-1)) \otimes Y) = \Delta(F\cdot G(X'(0),\ldots,X'(n-1)) \otimes Y)$ for all $Y \in 2^\omega$. Hence, by continuity, $\Delta(F\cdot G\cdot X) = \Delta(F\cdot G\cdot X')$. ■

<u>Corollary:</u> For every Baire mapping $\Delta: 2^\omega \to Y$, resp., measurable mapping $\Delta: 2^\omega \to Y$, where Y is a metric space, resp., a separable metric space, there exists an interval $F \in [2]_{one}(^\omega_\omega)$ such that $\Delta\restriction F\cdot 2^\omega$ either is a constant or a one-to-one mapping.

<u>Proof:</u> Combine Theorem 2.9. and Corollary 2.8. ■

<u>Remark:</u> It is not clear to what extent the requirement that Y should be separable (if $\Delta: 2^\omega \to Y$ is a measurable mapping) may be replaced by weaker conditions or, probably, dismissed at all.

2.10. So far we considered partitions (equivalence relations) on 2^ω which can be represented by Baire mappings (resp., measurable mappings) into metric spaces. Under these conditions, there always exists an interval such that the partition restricted to that interval either is an one-block equivalence relation or the identity. This property is called selectivity. One may wonder whether such a conclusion is still valid for more general partitions. A partition (equivalence relation) $\equiv$ on 2^ω may be viewed as given by a set of pairs $(X,Y) \in 2^\omega\times 2^\omega$ such that $X \equiv Y$. Thus $\equiv$ is a subset of $2^\omega\times 2^\omega$. Let us call $\equiv$ a Baire partition if $\equiv \subset 2^\omega\times 2^\omega$ is a Baire set with respect to the product topology. The following example shows that Theorem 2.8 does not extend to Baire partitions in general.
Let $X \equiv Y$ if and only if the symmetric difference $(X\setminus Y) \cup (Y\setminus X)$ is finite. This even is a Borel partition, i.e., the corresponding set $\{(X,Y) \in 2^\omega\times 2^\omega \mid (X\setminus Y) \cup (Y\setminus X)$ finite$\}$ is a Borel set. But for no interval $F \in [2]_{one}(^\omega_\omega)$ is the restriction to $F\cdot 2^\omega$ an one-block partition nor the identity.

This suggests two questions: 1. What are typical (i.e., hereditary) patterns of Baire partitions on 2^ω with respect to intervals ?
Even for Borel partitions the situation has not been investigated.
2. What are appropriate substructures of 2^ω which still admit a selectivity result ?
The later question is treated in the next section.

3. Sublattices in 2^ω

In this section we discuss sublattices of 2^ω. A subset $L \subseteq 2^\omega$ is a sublattice if it is closed under the operations union and intersection. L is a 2^ω-sublattice if, additionally, it is isomorphic to 2^ω.

3.1. Let $[2]\binom{\omega}{\omega}$ consist of all words (mappings) $F: \to \{0,1\} \cup \{\lambda_i \mid i < \omega\}$ such that $\Delta^{-1}(\lambda_j) \neq \emptyset$ for every $j < \omega$. The insertion $F \cdot X$ is defined as in 2.4.

For $F \in [2]\binom{\omega}{\omega}$ the set $F \cdot 2^\omega$ is a 2^ω-sublattice and, conversely, every 2^ω-sublattice may be represented in this way. To obtain a rigid representation we require that the first occurences of the parameters λ_i in F are ordered ascendingly, i.e., $\min F^{-1}(\lambda_i) < \min F^{-1}(\lambda_j)$ for all $i < j < \omega$. But this is just a technical remark.

3.2. Analogously we define parameter words to represent 2^k-sublattices. By a 2^k-lattice we understand the lattice of subsets of a k-element set.

Let $[2]\binom{\omega}{k}$ consist of all words (mappings) $g: \omega \to \{0,1\} \cup \{\lambda_0, \lambda_1, \ldots, \lambda_{k-1}\}$ such that $g^{-1}(\lambda_j) \neq \emptyset$ for every $j < k$ and such that $\min g^{-1}(\lambda_i) < \min g^{-1}(\lambda_j)$ for all $i < j < k$ (the later requirement assures rigid representations). For $F \in [2]\binom{\omega}{\omega}$ and $g \in [2]\binom{\omega}{k}$ the insertion (composition) $F \cdot g \in [2]\binom{\omega}{k}$ again is defined as in 2.4., analogously we have $g \cdot (a_0, \ldots, a_{k-1}) \in 2^\omega$ for $(a_0, \ldots, a_{k-1}) \in \{0,1\}^k$.
For $g \in [2]\binom{\omega}{k}$ the set $g \cdot \{0,1\}^k \subseteq 2^\omega$ is a 2^k-sublattice and, conversely, every 2^k-sublattice may be represented in this way. Notice that $[2]\binom{\omega}{0} = 2^\omega$.

3.3. Now we may consider the following question, which is a straightforward extension of 2.1.: Is it true that for every partition $[2]\binom{\omega}{k} = A \cup B$ there exists an $F \in [2]\binom{\omega}{\omega}$ such that $F \cdot [2]\binom{\omega}{k}$ belongs to either A or B ?

3.4. As in 2.2., the first answer to this question is "no". Observe that $[2]\binom{\omega}{k}$ is an open subset of the space $(\{0,1\} \cup \{\lambda_0, \ldots, \lambda_{k-1}\})^\omega$, where the later space bears the Tychonoff product topology inherited from the discrete space $\{0,1\} \cup$

$\{\lambda_0,\ldots,\lambda_{k-1}\}$. For $F \in [2]\binom{\omega}{\omega}$ the set $F\cdot[2]\binom{\omega}{k}$ is a countable union of perfect sets and thus we may again use Bernstein sets to construct counter-examples.

3.5. Generalizing a previous result of Carlson and Simpson [3] it has been shown in Prömel and Voigt [14] that Baire sets in $[2]\binom{\omega}{k}$ are Ramsey. In particular, meager sets are Ramsey null:

<u>Theorem:</u> For every partition $[2]\binom{\omega}{k} = A_0 \cup \ldots \cup A_{r-1}$ into Baire sets there exists $j < r$ and there exists an $F \in [2]\binom{\omega}{\omega}$ such that $F\cdot[2]\binom{\omega}{k} \subseteq A_j$. For every meager set $M \subseteq [2]\binom{\omega}{k}$ there exists an $F \in [2]\binom{\omega}{\omega}$ such that $M \cap F\cdot\binom{\omega}{k} = \emptyset$.

We do not prove this result here. Notice that for $k = 0$ the Moran-Strauss results 2.5. apply.

3.6. It has been conjectured that the dual result of 3.5. for measurable partitions is also valid. However, this is not the case, as shown by the result of Prömel and Voigt [16]. The following Lemma says that for $k > 0$ there exist sets $N \subseteq [2]\binom{\omega}{k}$ of Lebesgue measure zero which are completely Ramsey:

<u>Lemma:</u> Let $k > 0$ and let μ be the complete measure on $[2]\binom{\omega}{k}$ which is generated from equal distribution on $\{0,1\} \cup \{\lambda_0,\ldots,\lambda_{k-1}\}$. Then there exists a set $N \subseteq [2]\binom{\omega}{k}$ of measure zero such that for every $F \in [2]\binom{\omega}{\omega}$ there exists some $G \in [2]\binom{\omega}{\omega}$ with $F\cdot G\cdot[2]\binom{\omega}{k} \subseteq N$.

<u>Proof:</u> For $j < \omega$ let $N_j = \{f \in [2]\binom{\omega}{k} \mid$ there exists an $m < \omega$ such that $f(m) = \lambda_{k-1}$ but $f(m+i) \neq \lambda_{k-1}$ for all $0<i\leq m+j\}$. Then N_j is an open set of measure $\mu(N_j) \leq c\cdot((2+k-1)/(2+k))^j$, where $c > 0$ does not depend on j. Hence $N = \bigcap_{j<\omega} N_j$ is a measurable set with $\mu(N) = 0$ which has the desired properties.

■

Recall from 2.6. that the situation is slightly different for $k = 0$.

Corollary: Let $k > 0$. There exists a partition $[2]\binom{\omega}{k} = A \cup B$ into measurable sets such that for every $F \in [2]\binom{\omega}{\omega}$ there exist $g,h \in [2]\binom{\omega}{k}$ with $F\cdot g \in A$ and $F\cdot h \in B$.

Proof: Let N be as in the Lemma and let $Z \subseteq [2]\binom{\omega}{k}$ be a Bernstein set. Then $A = N \cap Z$ and $B = [2]\binom{\omega}{k}\setminus A$ have the desired properties. Notice that A is measurable, although the Bernstein set Z is not, as N has measure zero. ∎

3.7. In a sense, the counter-example of 3.6. reflects inherent properties of the structure of sublattices. If ascending sublattices are considered, a positive result may be obtained (Prömel and Voigt [16]). Mycielski [10, 11] and Blass [2] offer a somewhat different approach, partitioning k-element subsets and asking for monochromatic perfect sets. Mycielski shows that meager sets as well as sets of measure zero are Ramsey null. Lefmann [7] investigates the canonical partition behaviour.

3.8. Now we come back to a question mentioned in 2.10. We consider Baire partitions $\equiv$ on 2^ω. The following result has been conjectured in Prömel, Simpson and Voigt [17], where also the particular case of restricted Baire partitions has been treated.

Theorem: For every Baire partition $\equiv$ on 2^ω there exists an $F \in [2]\binom{\omega}{\omega}$ such that either $F\cdot X \equiv F\cdot Y$ for all $X,Y \in 2^\omega$ or $F\cdot X \not\equiv F\cdot Y$ for all $X \neq Y$ in 2^ω.

Proof: Observe that the mapping $S\colon [2]\binom{\omega}{2} \to 2^\omega\times 2^\omega\times 2^\omega\times 2^\omega$, where $S(f) = (f\cdot(0,0), f\cdot(1,1), f\cdot(0,1), f\cdot(1,0))$, is a continuous open mapping. Hence the mapping $\Delta\colon [2]\binom{\omega}{2} \to \{0,1,2\}$ which is defined by

$$\Delta(f) = \begin{cases} 0 & \text{if } f\cdot(0,0) \equiv f\cdot(1,1), \\ 1 & \text{if } f\cdot(0,0) \not\equiv f\cdot(1,1) \text{ and } f\cdot(0,1) \equiv f\cdot(1,0), \\ 2 & \text{if } f\cdot(0,0) \not\equiv f\cdot(1,1) \text{ and } f\cdot(0,1) \not\equiv f\cdot(1,0), \end{cases}$$

is a Baire mapping, i.e., $\Delta^{-1}(i)$ is a Baire set for $i = 0,1,2$. By Theorem 3.5. we may assume that Δ is a constant mapping. If Δ is constant in either 0 or 2 we are done. So it remains to show that Δ cannot be constant in 1.

Consider any 2^4-sublattice, say, $\{0,1\}^4 \otimes (\bar{0})$. Then

$(0,0,0,1) \otimes (\bar{0}) \equiv (0,0,1,0) \otimes (\bar{0}) \equiv (1,1,0,1) \otimes (\bar{0})$, as $(0,0,0,1)$ and $(0,0,1,0)$, resp., $(0,0,1,0)$ and $(1,1,0,1)$ are the $(0,1)$ and $(1,0)$ vertices of a 2^2-sublattice. But $(0,0,0,1)$ and $(1,1,0,1)$ are the $(0,0)$ and $(1,1)$ vertices of a 2^2-sublattice, contradicting that Δ is a constant in color 1. ■

3.9. We do not know whether Theorem 3.8. admits a dualization. In spite of the negative result 3.6. we have the following:

Conjecture: For every measurable partition $\equiv$ on 2^ω there exists an $F \in [2]\binom{\omega}{\omega}$ such that either $F \cdot X = F \cdot Y$ for all $X, Y \in 2^\omega$ or such that $F \cdot X \not\equiv F \cdot Y$ for all $X \not\equiv Y$ in 2^ω.

3.10. With respect to Baire partitions canonical patterns may be determined for all finite lattices. Using an approach similar to the one in 3.8. the results of Prömel and Voigt [13] carry over.

4. Concluding Remarks

Definitions 3.1. and 3.2. may be extended straightforwardly to defining parameter words over finite sets A (here we considered $A = \{0,1\}$). This is related to the concept of Graham-Rothschild parameter words (cf. Prömel and Voigt [15]). With the only exception of Theorem 2.9. (cf. Lefman and Voigt [8]) all results may be extended to general finite sets A. However, in Theorem 3.8. a few additional hereditary patterns occur.

References:

1. Bernstein, F.: Zur Theorie der trignometrischen Reihen. Leipz. Ber. 60, 325-338 (1908)

2. Blass, A.: A partition theorem for perfect sets. Proc. Amer. Math. Soc. 82, 271-277 (1981)

3. Carlson, T.J., and Simpson, S.G.: A dual form of Ramsey's theorem. Adv. in Math. 53, 265-290 (1984)

4. Emeryk, A., Frankiewicz, R., and Kulpa, W.: On functions having the Baire property. Bull. Polish Acad. Sci. Math. 27, 489-491 (1979)

5. Kuratowski, K.: Topology, Vol. I. New York 1966

6. Lachlan, A.H.: Solution to a problem of Spector. Canad. J. Math. 23, 247-256 (1971)

7. Lefman, H.: Canonical partition behaviour of Cantor spaces. Proceedings of the Conference "Irregularities and partitions", Hungary 1986 (to appear)

8. Lefman, H., and Voigt, B.: Partition theorems for A^{ω}. Preprint

9. Moran, G., and Strauss, D.: Countable partitions of product spaces. Matematika 27, 213-224 (1980)

10. Mycielski, J.: Independent sets in topological algebra. Fund. Math. 55, 139-147 (1964)

11. Mycielski, J.: Algebraic independence and measure. Fund. Math. 61, 165-169 (1967)

12. Oxtoby, J.C.: Measure and Category, 2nd edition. New York 1980

13. Prömel, H.J., and Voigt, B.: Canonical partition theorems for finite distributive lattices. Rend. Circ. Mat. Palermo. (2) Suppl. 2, 223-237 (1982)

14. Prömel, H.J., and Voigt, B.: Baire sets of k-parameter words are Ramsey. Trans. Amer. Math. Soc. 291, 189-201 (1985)

15. Prömel, H.J., and Voigt, B.: Graham-Rothschild parameter sets. WP85403 Institut für Ökonometrie und Operations Research, Universität Bonn 1985. To appear in: Nešetřil, J., and Rödl, V. (Eds.): The Mathematics of Ramsey Theory.

16. Prömel, H.J., and Voigt, B.: Graham-Rothschild parameter words and measurable partitions. WP88524 Institut für Ökonometrie und Operations Research, Universität Bonn 1988

17. Prömel, H.J., Simpson, S.G., and Voigt, B.: A dual form of Erdös-Rado's canonization theorem. J. Combin. Theory Ser. A. 42, 159-178 (1986)

18. Shelah, S.: Can you take Solovay's inaccessible away ? Israel J. Math. 48, 1-47 (1984)

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Authors:

Prof. Dr. Hans Jürgen Prömel
Institut für Ökonometrie und Operations Research
Abteilung Operations Research
Rheinische Friedrich-Wilhelms-Universität Bonn
Nassestraße 2
D-5300 Bonn 1

Prof. Dr. Bernd Voigt
Fakultät für Mathematik
Universität Bielefeld
Universitätsstraße
D-4800 Bielefeld

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Henry Martyn Mulder

Triple convexities for graphs

1. Introduction

This paper surveys a line of research in metric graph theory that has proven to be quite successful with respect to results and theory as well as applications. There are many interconnections with algebra, geometry and abstract convexity theory. The applications concern location problems (finding vertices simultaneously close to a family of vertices), dynamic search in graphs (meant for optimal self-adjusting algorithms for information retrieval), and models for measuring dissimilarities (like in mediaeval manuscripts).

An abstract convexity on a set V is a family if subsets $\mathcal{F}$ of V, with Φ, V $\in \mathcal{F}$, such that $\mathcal{F}$ is closed under taking arbitrary intersections and taking unions of chains (subfamilies linearly ordered by inclusion). The sets in $\mathcal{F}$ are the <u>convex</u> sets, and the <u>convex closure</u> of a subset S of V is the intersection of all convex sets containing S. A <u>triple convexity</u> is a convexity with conditions on the convex closures of the pairs of any triple u,v,w of elements of V.

In abstract convexity theory the main line of research is the study of parameters like the Helly, Carathéodory and Radon numbers. In a way, a convexity can also be seen as measuring 'nearness' and 'betweennes', and precisely this feature is what interests us here.

Several convexities can be defined on a graph. Below we will encounter two of these, viz. the geodesic convexity and the Δ-convexity. For a survey of convexities on graphs, see [19]. A subgraph H of a graph G is <u>geodesically convex,</u> for short <u>convex,</u> if with any two vertices u,v of H all (u,v)-geodesics (i.e. shortest (u,v)-paths) are contained in H. This convexity will feature in five different but closely related triple convexities, each giving an interesting class of graphs. In all these graphs the triple convexity makes it possible, for any triple of vertices, to find vertices that are 'simultaneously closest' to all vertices of the triple. From this starting-point a rich structure theory has been developed, interconnec-

tions with algebraic and geometric systems have been established, and necessary results have been derived for the applications mentioned above.
The theory of the 'median-like' triple convexities, which we describe in sections 3 to 6, is sandwiched between two applications: location theory in section 2 and dynamic searching in graphs in section 7.
For all definitions, results, proofs and applications we have included the necessary references. Some more are given to indicate the scope of the literature, but the list of references is far from exhaustive. For more relevant references one is referred to those given below.

2. A location problem

Let G be a graph where facilities are placed at a number of vertices. Somewhere at a vertex we want to locate a service station from which the facilities have to be served. Serving a facility at a vertex v from a station at vertex u entails a cost of $d(u,v)$, where $d(u,v)$ is the distance in G between u and v. If more facilities are placed at the same vertex, then the cost of serving them is a multiple of the distance. Now the problem is where to locate the service station to minimize the costs of serving all facilities on the graph.
Otherwise formulated this location problem reads as follows.
Let $G = (V,E)$ be a graph with distance function d. Let $F = (v_1,\ldots,v_p)$ be a multiset of vertices from V. A <u>median vertex</u> of the set F is a vertex x that minimizes

$$D(x,F) = \sum_{i=1}^{p} d(x,v_i).$$

Any median vertex of F is a solution for the above location problem. The set M_F consisting of all median vertices of F is the <u>median-set</u> of F. When F is just the vertex-set V of G, then M_V is precisely the median-set of G in the usual sense, cf. [31]. Not much can be said about median-sets in arbitrary graphs. So even less can be said about median-sets of arbitrary multisets of vertices. There are two trivial cases. The median-set of a single vertex consists, of course, of the vertex itself. The median-set of a pair u and v of vertices is precisely the <u>interval</u> $I(u,v)$ between u and v defined by

$$I(u,v) = \{w \mid w \text{ lies on a shortest } (u,v)\text{-path}\}.$$

The interval function I of a graph is an indispensable notion in metric graph theory, see [25] for an extensive study.
Clearly, if the intersection $M = \bigcap_{i<j} I(v_i, v_j)$ of all intervals between pairs of vertices in $F = (v_1, \ldots, v_p)$ is non-empty, then M is the median-set of F.
In trees median-sets are always paths, where a single vertex is considered to be a path of length 0, cf. [34]. This result is a consequence of the theory discussed in the next section. There we consider graphs in which every triple has a unique median vertex. Trees have this property. It is proven in [5] that in such graphs every odd-numbered multiset has a unique median vertex, which can be found in an efficient way, and every even-numbered multiset has an interval as median-set. Note that in a tree an interval is always a path.
We will call a subset U of the vertex-set V of a graph G <u>convex</u> if the subgraph induced by U is convex in G.

3. Median graphs

In this section we consider a class of graphs on which our location problem has a nice solution: the median graphs. These were introduced independently by several authors, see [1, 27, 23, 21].
A connected graph G is a <u>median graph</u> if, for every triple u,v,w of vertices, there exists a unique vertex x, called the <u>median</u> of u,v,w, such that x lies simultaneously on geodesics joining u and v, v and w, and u and w. Figure 1 gives the characteristic diagramm. The vertex x is the median vertex of the triple u,v,w in the sense of the previous section. Trees and hypercubes are notable examples of median graphs.

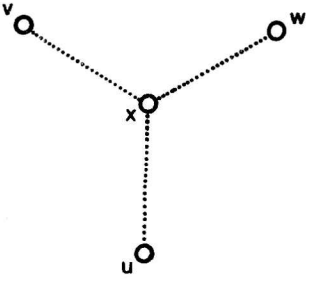

Figure 1

It is easily seen that median graphs are bipartite. It turns out that these graphs are precisely the connected graphs in which every triple of vertices admits a unique median vertex in the sense of the previous section.
Median graphs possess a rich structure theory. They have many nice characterizations and some interesting applications, and they can be interpreted as algebraic systems in several ways. The structural aspect is the focus of this section. In the next sections, when considering generalizations of median graphs, we will come across some other aspects.
In describing the structure of median graphs we follow [23], see also [25] for an extensive study.
Choose an arbitrary edge uv joining u and v in a median graph G. We view G from the edge uv, see Figure 2 for an example.

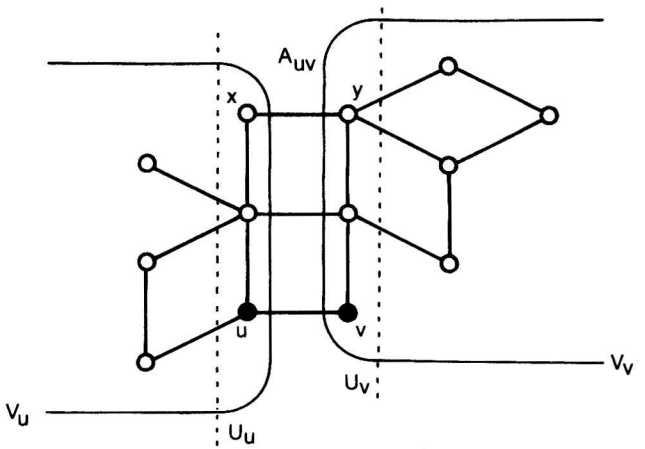

Figure 2

Let V_u be the set of vertices nearer to u than to v, and let V_v be the set of vertices nearer to v than to u. Then these two sets partition the vertex-set of G. The edges between V_u and V_v form a matching A_{uv} containing the edge uv. Let U_u consists of the ends of A_{uv} in V_u and U_v of those in V_v, see Figure 2. It turns out that, if we choose any edge xy of A_{uv} with x in U_u and y in U_v, then we obtain the same sets of vertices and the same matching, i.e. $U_x = U_u$, $V_x = V_v$, $U_y = U_v$, $V_y = V_v$, and $A_{xy} = A_{uv}$. So we can define an equivalen-

ce relation $\approx$ on the edges by $uv \approx xy$ if $A_{uv} = A_{xy}$. This equivalence relation yields a partition of the edges in such matchings: the canonical edge-colouring of the median graph.

Another essential feature is that the sets U_u and U_v (and hence also V_u and V_v) are convex sets in G.
Finally, the matching A_{uv} produces a natural bijection along its edges, which turns out to be an isomorphism between the convex subgraphs of G induced by U_u and U_v. This allows us to contract the edges of A_{uv}, by which we obtain a smaller median graph G', see Figure 3. Another way of stating this fact is that the subgraph induced by $U_u \cup U_v$ is the Cartesian product of K_2 and the subgraph induced by U_u.

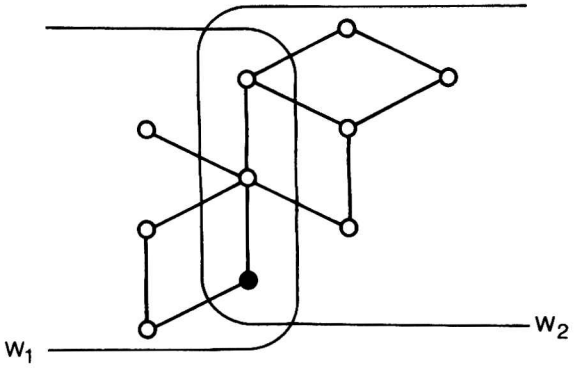

Figure 3

Contracting the other colour classes successively, we arrive at the one-vertex graph K_1.
We can reverse this procedure: let the median graph G' be covered by two convex sets W_1 and W_2 with non-empty intersection U such that there are no edges between $W_1 \setminus W_2$ and $W_2 \setminus W_1$, see Figure 3. Now we 'pull' these two sets apart, concurrently doubling the set U and joining corresponding vertices by an edge. Thus we get from G' in Figure 3 the graph G in Figure 2. Loosely speaking, the main theorem of [23] reads as follows.

<u>Theorem A:</u> Any (finite) median graph can be obtained from the one-vertex graph by repeated "expansions along convex sets".

Note that we can obtain all trees from K_1 by this expansion procedure, where the two covering sets W_1 and W_2 always have only one vertex in common. To get the hypercubes from K_1 both sets W_1 and W_2 are to contain all vertices in each step.
With this theorem in hand we can derive many results by induction on the number of vertices or the number of colour classes in the canonical edge colouring. We list some consequences.

Corollary A1: A connected graph G is a median graph if and only if, for every edge uv, the associated sets U_u and U_v are convex.

Corollary A2: A connected graph G is a median graph if and only if G is an induced subgraph of a hypercube Q such that with any triple of vertices of G their median in Q is also in G.

Corollary A3: Glueing together two median graphs along a common convex subgraph produces a median graph.

Corollary A4: Any median graph can be obtained from a collection of hypercubes by successively glueing these together along subhypercubes.

Corollary A5: The Cartesian product of two median grahs is again a median graph.

As can be seen from Corollaries A2 and A4, hypercubes play an important role in median graph theory. They are precisely the regular median graphs, see [24]. Any edge-preserving map f of a median graph into itself fixes some hypercube Q, i.e. f(Q) = Q, cf. [12]. Finally, a graph is a median graph if and only if it is a retract of a hypercube, see [2]. Loosely speaking, a retract G of a graph H is an isometric (i.e. distance-preserving) subgraph of H such that there is an edge-preserving mapping of H onto G of which the restriction to G is the identity mapping.
There are several algebraic structures related to median graphs. For instance, in a median graph G = (V,E) one can associate with each triple of vertices its median. Thus one obtains a ternary algebra on V, for which several sets of axioms have been given, cf. [30, 1, 27, 25]. Another way to view G as an algebraic structure is as follows: choose an arbitrary vertex a

of G and order V with $\leqslant_a$, where

$$u \leqslant_a v \text{ if and only if } u \in I(a,v).$$

Loosely speaking, this poset is the lower part of a distributive lattice: a meet-semilattice such that with any triple u,v,w, for which the joins $u \vee v$, $u \vee w$ and $v \vee w$ exist, the join $u \vee v \vee w$ also exists, see [30, 1, 25]. Especially this last interpretation of median graphs is useful. Often one can derive results on median graphs (or median semilattices) by using a mix of graph-theoretic and order-theoretic arguments, by which proofs are shortened considerably.
We close this section with another interesting application of median graphs (a third one will be discussed in the last section). When one wants to construct a genealogical tree for the copying of medieaval manuscripts, one 'measures' the dissimilarities between the texts. In [14] Buneman develops a method to recover a tree from measures of dissimilarities satisfying certain compatibilities. So one gets a genealogical tree for the copying of manuscripts. But in history it occurred quite often that a new manuscript was produced using two different older, and possibly partial, manuscripts. Thus one needs a model having cycles. Barthelemy has shown in [13] that median graphs provide 'optimal' models in this case. The main result of [13] generalizes a result on the relation between median graphs and Helly hypergraphs in [26].

4. Modular graphs

What happens when we drop the requirement of unicity of the median of a triple in the previous section? The graphs we get are called <u>modular graphs</u>: conected graphs such that, for each triple of vertices u,v,w the intersection $I(u,v) \cap I(v,w) \cap I(w,u)$ is non-empty.
In this case almost (?) all is lost of the structure theory and the equivalence with algebraic systems we had for median graphs. So far not much work has been done on modular graphs. They play a role in the study of Condorcet vertices: a vertex in a graph such that there is no vertex closer to a strict majority of 'users' located at vertices; see [3], which also contains more references on this topic.
Here we consider only one aspect. First note that again it is easily seen that modular graphs are bipartite. In the defining

condition above we required that, for each triple, the intersection of the three intervals is non-empty. This turns out to be superfluous.

<u>Theorem B:</u> A connected graph G is modular if and only if G is triangle-free and, for each triple of vertices u,v,w with

$$1 \leq d(v,w) \leq 2, \text{ and } d(u,v) = k = d(u,w)$$

there exists a common neighbour x of v and w with $d(u,x) = k-1$.

The 'only if' part follows by definition.
For the 'if' part, note that, because G is triangle-free, we won't find vertices u,v,w with v and w adjacent and $d(u,v) = d(u,w)$. Hence the characteristic diagram for modular graphs is that in Figure 4.

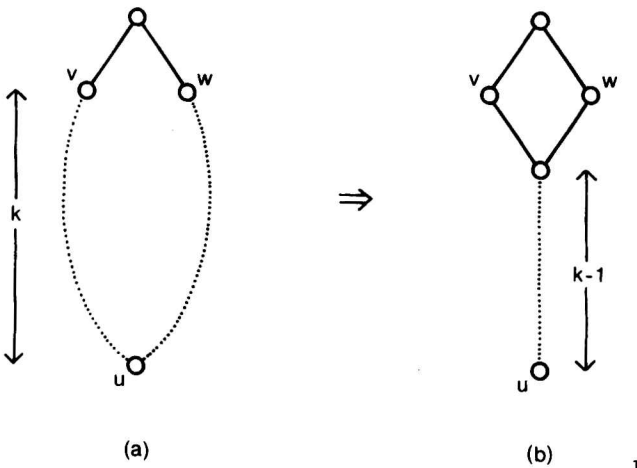

Figure 4

We can read Figure 4 as follows. In (a), going from v to w and seen from u, we see an edge going upwards followed by a downward edge. Then we can 'flipflop' this to a downward edge followed by an upward edge as in (b). Now to prove the 'if' part, we take three vertices u,v,w and choose a geodesic between v and w. We go along this geodesic from v to w. Seen from u the edges go upwards or downwards. By flipflop-

ping, we can transform this geodesic into one consisting of a sequence of downward edges followed by a sequence of upward edges. The bottom vertex x in this last geodesic, nearest to u, is a vertex in $I(u,v) \cap I(v,w) \cap I(w,u)$.
If we require unicity of the vertex x in Theorem B, then we obtain one of the characterizations of median graphs in [25].

5. Pseudo-median and pseudo-modular graphs

Modular graphs generalize median graphs, but they are still bipartite. What can we do in the non-bipartite case? Then necessarily there are triples for which the intersection of the three associated intervals is empty. The first possible deviation from bipartiteness is the case where there exists a triangle "in between" the vertices u,v,w, as in Figure 5.

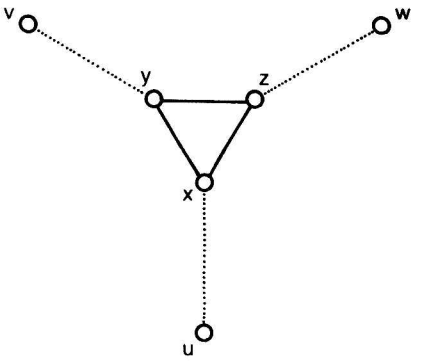

Figure 5

Here x,y lie on a (u,v)-geodesic, y and z lie on a (v,w)-geodesic, and z,x lie on a (w,u)-geodesic. Such a triangle is called a <u>pseudo-median</u> of u,v,w. Then a connected graph G is a <u>pseudo-median graph</u> if every triple of vertices has a unique median or a unique pseudo-median, see [9]. Figures 1 and 5 are the characteristic diagrams for these graphs.
It turns out that these graphs also possess a rich structure theory. What is lost in comparison with the median graphs is that the class of pseudo-median graphs is not closed under Cartesian multiplication, although the Cartesian product of a median graph and a pseudo-median graph is still a pseudo-median graph. Moreover, there is no reasonable algebraic interpretation available, but on the other hand we have a geometric one:

pseudo-median graphs are join spaces; for the relevant definitions and proofs see [10].
As for the structure of pseudo-median graphs, they can be built up from a specific collection of graphs by glueing these together successively, see [9]. The building stones are hypercubes, wheels, complete graphs minus some matching, and snakes (maximal outerplanar graphs such that each face has an edge in common with the outer face - that is, a snake of triangles), and moreover the Cartesian product of a hypercube with any of these graphs. We have to be careful along what kind of subgraphs we glue them together: just convex is not enough. The subgraph has to be Δ-convex, i.e. convex and such that, whenever the subgraph contains an edge of a triangle, then it contains the whole triangle, see [9]. Note that, whenever a Δ-convex subgraph contains an edge of a wheel, then it contains the whole wheel. The same holds for a snake, and for a complete graph minus a matching (except for the 4-cycle). Hence we may glue together such graphs only along a vertex.
Again this structural characterization yields the possibility of using induction on the number of vertices. Quite a number of results for median graphs thus have an analog for pseudo-median graphs. Just to mention some: the regular pseudo-median graphs are the Cartesian products of a hypercube and a hyperoctahedron (i.e. complete graphs minus a perfect matching), and any edge-preserving map of a pseudo-median graph into itself fixes such a regular subgraph. Also similar results hold for the location problem on pseudo-median graphs. The reader is referred to [8].
As in the case of modular graphs we can drop unicity of medians and pseudo-medians, and the graphs we get are, of course, called <u>pseudo-modular-graphs</u>, see [7]. These graphs turn out to be a common generalization of several classes of graphs that at first sight seem to be quite different: the median graphs, the pseudo-median graphs, the chordal bipartite graphs (sensu [20]), the 3-sun-free chordal graphs (sensu [15]), the absolute retracts of reflexive graphs (cf. [28, 29]), and the distance-hereditary graphs (sensu [22], see also [6]). For the hereditary case, where each isometric subgraph is also pseudo-modular, nice characterizations involving forbidden isometric subgraphs are available, see [7].
The flipflop technique we had in the modular case is also available here. Besides Figure 4, there are two other flipflops possible. If we go from v to w seen from u, then we can flipflop a 'horizontal' edge followed by a downward edge into a

downward edge followed by a horizontal edge. Here a horizontal edge has both its ends at the same distance from u. Furthermore, two consecutive horizontal edges can be replaced by a downward edge and an upward edge. Thus we can transform any geodesic from v to w into one that is involved in a pseudo-median triangle, see [7].

6. Quasi-median graphs

Dropping bipartiteness, as we did in the last section to generalize median graphs, does not necessarily lead to graphs without algebraic interpretation. In [25] the quasi-median graphs were introduced. They were independently introduced in [21]. The characteristic diagram is given in Figure 6.

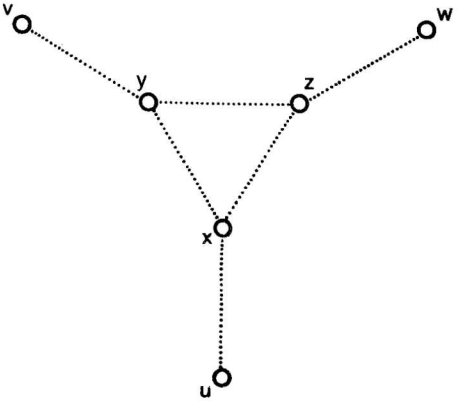

Figure 6

The triple x,y,z is a quasi-median for u,v,w if x and y lie on a (u,v)-geodesic, y and z lie on a (v,w)-geodesic, and z and x lie on a (w,u)-geodesic, and moreover

$$d(x,y) = d(y,z) = d(z,x),$$

and d(x,y) is minimal under these conditions.
Then a quasi-median graph is a graph in which any triple of vertices has a unique quasi-median, and which satisfies two extra conditions, viz. K_4 minus an edge does not occur, and any isometric 6-cycle has the 3-cube as convex closure (i.e. the smallest convex subgraph containing the 6-cycle). These two extra conditions are necessary to obtain a class of graphs that neatly generalizes median graphs.

The vertex x in the quasi-median x,y,z of the triple u,v,w is the unique vertex at largest distance from u and lying simultaneously on a (u,v)-geodesic and a (u,w)-geodesic. A similar property holds for y and z.
Hypercubes play a special role with respect to median graphs. For quasi-median graphs this role is reserved for the Hamming graphs, i.e. the Cartesian products of complete graphs. For example, quasi-median graphs are isometric "quasi-median preserving" subgraphs of Hamming graphs, see [25], they are the retracts of Hamming graphs, see [32, 17], and the regular ones are precisely the Hamming graphs, see [25]. Any quasi-median graph can be obtained from a collection of Hamming graphs by successively glueing them together along sub-Hamming graphs, see [21]. This is also a corollary of the structural characterization in [25]. A new proof of this result is given in [11].

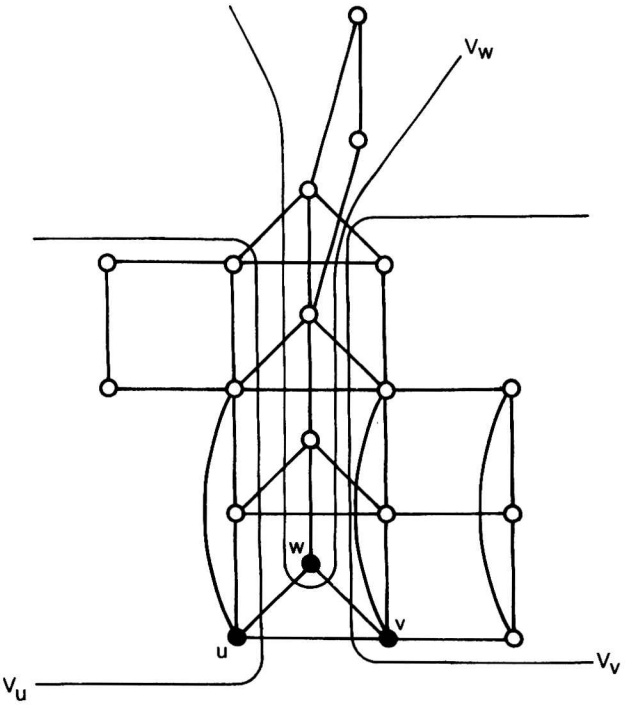

Figure 7

The structure of quasi-median graphs is quite similar to that of median graphs, see [25]. Of course we have to allow some modifications. Instead of choosing just an edge, we have to choose a maximal clique containing the edge. Now we partition the vertex-set into parts consisting of vertices nearer to some vertex of the clique than to all other vertices of the clique. For any two parts we have the same type of structure as in the median case: there is a matching between the two parts that induces a graph isomorphism between the subgraphs connected by the matching. These subgraphs are now Δ-convex. So in a sense we have a 'matching of cliques' between all the parts, and the subgraph induced by these cliques is a Cartesian product of a complete graph (of the size of the clique) and a Δ-convex subgraph induced by the ends of the matching cliques in one part, see Figure 7.
Again we have a contradiction procedure and, conversely, an expansion procedure, which we will not decribe explicitly but will be clear from Figures 7 and 8.

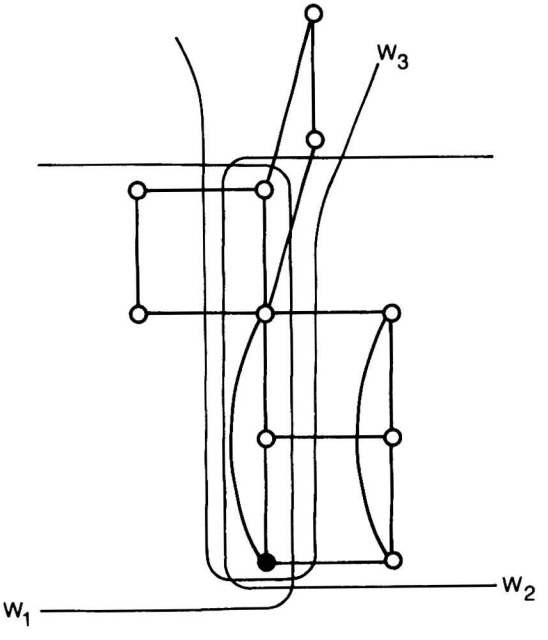

Figure 8

For several other characterizations involving the interval function see [25]. Here we mention just one other fact, which can be deduced from the properties described above: the convex closure as well as the Δ-convex closure of the quasi-median x,y,z of a triple of vertices is the Cartesian product of k copies of K_3, where $k = d(x,y)$.
One can associate a ternary algebra with a quasi-median graph in the following way: let x,y,z be the quasi-median of u,v,w, then we assign x to (u,v,w), and y to (v,w,u), and z to (w,u,v). In [25, 21, 18, 11] several equivalent axiom systems for quasi-median graphs are given. One can recover the quasi-median graph from these algebraic systems.
Quasi-median graphs generally are not bipartite, so they cannot be the Hasse diagram of some poset. But it seems reasonable that there exists a partial order and an equivalence relation on the vertices that are interconnected and constitute an analog for the median semi-lattices in section 3. So far this is an open problem.

7. Dynamic search in graphs

In [16, 17] Chung, Graham and Saks have developed an interesting theory for dynamic searching in graphs. Apparently the authors were not aware of the fact that quasi-median graphs had already been studied in [21, 25], cf. [33]. Here we give two results involving median and quasi-median graphs.
Applications of the theory are with respect to optimal self-adjusting search algorithms, but for the sake of simplicity we restrict ourselves to an imaginary application.
We have a graph $G = (V,E)$, where an information is located at the vertices. Somewhere on a vertex u a librarian has his desk. When he receives a quest for a piece of information located at v, for short a quest for v, then obtaining this information costs him $d(u,v)$. Moving his desk to a vertex w costs him $d(u,w)$. So if all quests are for the same vertex v, then to minimize his costs the librarian better moves his desk to v. In general the librarian gets an arbitrary succession of quests. If he waits until he has k quests, then it may be sensible to move his desk to a new vertex 'closer' to the k quests vertices, before actually retrieving the information at the quested vertices. Now the question is whether the librarian has an optimal strategy for moving his desk if, at any time, he has foreknowledge of the next k quests. This, of course,

depends on the structure of the graph.
In [16] it is shown that, if the librarian at any time has foreknowledge of the next 2 quests, then the graphs on which he has an optimal strategy for moving his desk are precisely the median graphs. In [17] this result is generalized to foreknowledge of k quests at any time. Then the optimal graphs are precisely the quasi-median graphs.
Median graphs and quasi-median graphs were introduced independently by several authors without any application in mind. A rich structure was developed and related algebraic systems were studied. Now it turns out that these graphs, and related graphs such as the pseudo-median graphs, have interesting applications. Thus this line of research in metric graph theory has proven to be successful, and there are still many interesting problems left.

References:

1. Avann, S.P.: Metric ternary distributive semi-lattices. Proc. Amer. Math. Soc. 12, 407-414 (1961)

2. Bandelt, H.-J.: Retracts of hypercubes. J. Graph Theory 8, 501-510 (1984)

3. Bandelt, H.-J.: Networks with Condorcet solutions. European J. Oper. Res. 20, 314-326 (1985)

4. Bandelt, J.-H.: Hereditary modular graphs. Combinatorica 8, 149-157 (1988)

5. Bandelt, H.-J., and Barthelemy, J.P.: Medians in median graphs. Discrete Appl. Math. 8, 131-142 (1984)

6. Bandelt, H.-J., and Mulder, H.M.: Distance-hereditary graphs. J. Combin. Theory Ser. B 41, 182-208 (1986)

7. Bandelt, H.-J., and Mulder, H.M.: Pseudo-modular graphs. Discrete Math. 62, 245-260 (1986)

8. Bandelt, H.-J., and Mulder, H.M.: Regular pseudo-median graphs. J. Graph Theory 12, 533-549 (1988)

9. Bandelt, H.-J., and Mulder, H.M.: Pseudo-median graphs: decomposition via amalgamation and Cartesian multiplication. (submitted)

10. Bandelt, H.-J., and Mulder, H.M.: Pseudo-median graphs are join spaces. Preprint

11. Bandelt, H.-J., Mulder, H.M., and Wilkeit, E.: Quasi-median graphs and algebras. Preprint

12. Bandelt, H.-J., and van de Vel, M.: A fixed cube theorem for median graphs. Discrete Math. 67, 129-137 (1987)

13. Barthelemy, J.P.: From copair hypergraphs to median graphs with latent vertices. Discrete Math. (to appear)

14. Buneman, P.: The recovery of trees from measures of dissimilarities. In: Hodson, F.R., Kendall, D.G., and Tautu, P., (Eds.): Mathematics in Archeological and Historical Sciences. Edinburgh 1971

15. Chang, G.J., and Nemhauser, G.L.: The k-domination and k-stability problems on sun-free chordal graphs. SIAM J. Algebraic Discrete Methods 5, 332-345 (1984)

16. Chung F.K., Graham, R.L., and Saks, M.E.: Dynamic search in graphs. In: Johnson, D.S., Nishizeki, T., Nozaki, A., and Wilf, H., (Eds.): Discrete algorithms and complexity. pp. 351-387, Orlando 1987

17. Chung, F.K., Graham, R.L., and Saks, M.E.: A dynamic location problem for graphs. Preprint

18. Draskovicova, H.: Modular median graphs. Math. Slovaca 32, 267-281 (1982)

19. Duchet, P.: Convexity in combinatorial structures. Proceedings of the 14th winter school on abstract analysis, Srni 1986. Rend, Circ. Mat. Palermo (2) Suppl. 14, 261-293 (1987)

20. Golumbic, M.C., and Goss, C.F.: Perfect elimination and chordal bipartite graphs. J. Graph Theory 2, 155-163 (1978)

21. Isbell, J.: Median algebra. Trans. Amer. Math. Soc. 206, 319-362 (1980)

22. Howorka, E.: A characterization of distance-hereditary graphs. Quart. J. Math. Oxford Ser. (2) 28, 417-420 (1977)

23. Mulder, H.M.: The structure of median graphs. Discrete Math. 24, 197-204 (1978)

24. Mulder, H.M.: n-cubes and median graphs. J. Graph Theory 4, 107-110 (1980)

25. Mulder, H.M.: The interval function of a graph. Amsterdam 1980

26. Mulder, H.M., and Schrijver, A.: Median graphs and Helly hypergraphs. Discrete Math. 25, 41-50 (1979)

27. Nebesky, L.: Median graphs. Comment. Math. Univ. Carolin. 12, 317-325 (1971)

28. Nowakowsky, R., and Rival, I.: The smallest graph variety containing all paths. Discrete Math. 42, 223-234 (1983)

29. Quilliot, A.: Homomorphismes, points fixes, rétractions et jeux de pursuite dans les graphes, les ensembles ordonnés et les espaces métriques. Thèse, Paris 1983

30. Sholander, M.: Trees, lattices, order, and betweenness. Proc. Amer. Math. Soc. 3, 369-381 (1952)

31. Slater, P.J.: Medians of arbitrary graphs. J. Graph Theory 4, 389-392 (1980)

32. Wilkeit, E.: Isometrische Untergraphen von Hamming-Graphen. Dissertation, Oldenburg 1986

33. Winkler, P.M.: The metric structure of graphs: theory and applications. In: Whitehead, C., (Ed.): Surveys in combinatorics 1987. London Math. Soc. Lecture Note Ser. 123, 197-221 (1987)

34. Zelinka, B.: Medians and peripherians of trees. Arch. Math. (Brno) 4, 87-95 (1968)

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Author:

Dr. Henry Martyn Mulder
Econometrisch Institut
Erasmus Universiteit Rotterdam
Postbus 1738
NL-3000 DR Rotterdam
Netherlands

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Gustav Burosch; Ulrike Franke; Susanne Röhl

Über Ordnungen von Binärworten

1. In dieser Arbeit definieren wir endliche Ordnungen auf Binärworten mit Hilfe der Teilwortrelation. Bisher ist eine recht geringe Anzahl Standardordnungen bekannt und gut untersucht ([1]). Wir sehen in den hier behandelten Ordnungen ein weiteres potentielles Beispiel. Wir behandeln eine induktive Konstruktion des Hasse-Graphen dieser Ordnungen, die sehr an das Vorgehen beim n-dimensionalen Einheitswürfel erinnert. Andererseits sind die betrachteten Ordnungen wesentlich komplizierter. Wir beweisen die Unimodalität der Niveauzahlen und zeigen, daß es für unsere Ordnungen P_n, $n \geq 2$, genau zwei Automorphismen gibt.

In [2] und [3] wurden bis auf geringfügige Unterschiede die gleichen Ordnungen betrachtet und Schälbarkeitseigenschaften nachgewiesen.

2. Sei A das Alphabet $\{0,1\}$. Ein Binärwort $\underline{a}$ der Länge $|\underline{a}| = n$ ist eine Folge $a_1a_2\ldots a_n$, $a_i \in A$. Das Leerwort ist zugelassen und wird mit e bezeichnet. Es hat die Länge 0.
Wir setzen $\bar{0} = 1$, $\bar{1} = 0$, und für $\underline{a} := a_1a_2\ldots a_n$ heißt $\bar{\underline{a}} = \bar{a}_1\bar{a}_2\ldots\bar{a}_n$ das duale Wort zu $\underline{a}$. Das Wort $\underline{a}$ heißt normal ([2]), falls $a_i \neq a_{i+1}$, $i = 1,\ldots,n-1$, ist. Ist $\underline{a}$ normal und $a_1 = 0$ oder $a_1 = 1$, so heißt $\underline{a}$ 0-normal bzw. 1-normal. Mögen $\underline{u}_n$ und $\underline{v}_n$ stets das 1- bzw. 0-normale Wort der Länge n bezeichnen, $n \geq 1$.
Für gegebene Worte $\underline{a}$ und $\underline{b}$ heißt $\underline{a}$ ein Teilwort von $\underline{b}$, falls $\underline{b} = a_1a_2\ldots a_n$ und $\underline{a} = a_{i_1}a_{i_2}\ldots a_{i_t}$, $1 \leq i_1 < i_2 < \ldots < i_t \leq n$, ist. In diesem Fall schreiben wir $\underline{a} \leq \underline{b}$ oder $\underline{b} \geq \underline{a}$. Ist gleichzeitig $\underline{a} \leq \underline{b}$ und $\underline{a} \neq \underline{b}$, so schreiben wir $\underline{a} < \underline{b}$. Mögen P_n und Q_n die Ordnungen aller Teilworte des 1-normalen bzw. 0-normalen Wortes der Länge n bezeichnen. Offenbar sind P_n und Q_n isomorph. Die Beschränkung auf normale Worte ist dadurch motiviert, daß man für jedes Wort $\underline{a}$ die Ordnung sämtlicher Teilworte von $\underline{a}$ ordnungsisomorph in die Ordnung P_{2n}, $n := |\underline{a}|$, einbetten kann.
Wie üblich veranschaulichen wir endliche Ordnungen durch ihre Hasse-Diagramme und schreiben $\underline{a} \lessdot \underline{b}$, falls $\underline{a} < \underline{b}$ ist und es

kein Wort $\underline{c}$ mit $\underline{a} < \underline{c} < \underline{b}$ gibt. Dann heißt $\underline{a}$ ein Vorgängen von $\underline{b}$, $\underline{b}$ ein Nachfolger von $\underline{a}$, und $\underline{a}$ und $\underline{b}$ werden als benachbart bezeichnet. $V(\underline{a})$ und $N(\underline{a})$ seien die Mengen aller Vorgänger bzw. Nachfolger von $\underline{a}$. Die Worte $\underline{u}_n$ und e sind Eins bzw. Null in P_n, und durch $r(\underline{a}) := |\underline{a}|$ ist eine Rangfunktion gegeben. In P_n bildet die Menge der Elemente vom Rang $k \geqslant 0$ das k-te Niveau $N_k(P_n)$, und $|N_k(P_n)|$ heißt die k-te Whitney-Zahl von P_n. Analog ist $N_k(Q_n)$ definiert. Wir vereinbaren die Bezeichnungen

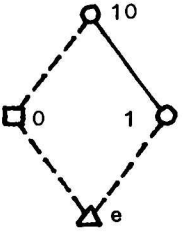
$t_{n,k} := |N_k(P_n)|$ und $t_n := \Sigma_{k=o}^{n} t_{n,k} = |P_n|$. Für ein Element $\underline{a}$ aus P_n möge $d_{P_n}(\underline{a})$, $d^+_{P_n}(\underline{a})$ und $d^-_{P_n}(\underline{a})$ die Anzahl der Nachbarn, die Anzahl der Nachfolger bzw. die Anzahl der Vorgänger von $\underline{a}$ in p_n bedeuten. Häufig lassen wir den Index P_n weg. Die Zahl $d_{P_n}(\underline{a})$ heißt der Grad von $\underline{a}$ in P_n.

Zur rekursiven Beschreibung von P_n vereinbaren wir, daß die Ordnungen $P_{n,o}$ und $P_{n,1}$ aus P_n entstehen, indem jedes Element $a_1a_2\ldots a_n$ durch $0a_1a_2\ldots a_n$ bzw. $1a_1a_2\ldots a_n$ ersetzt wird. Natürlich sind P_n, $P_{n,o}$ und $P_{n,1}$ untereinander isomorph. Analog seien $Q_{n,o}$ und $Q_{n,1}$ aus Q_n gewonnen. Für die Elemente $\underline{x}$ aus $P_{n,i}$ und $Q_{n,i}$, $i = 0,1$, gilt $r(\underline{x}) = |\underline{x}| - 1$.

Seien H_n und H'_n die Hasse-Graphen von P_n bzw. Q_n und $H_{n,o}$, $H_{n,1}$, $H'_{n,o}$, $H'_{n,1}$ diejenigen von $P_{n,o}$, $P_{n,1}$, $Q_{n,o}$ bzw. $Q_{n,1}$. In den Abbildungen 1 - 4 ist jeweils die Bildung von P_n aus $P_{n-2,o}$, $Q_{n-1,1}$ sowie e veranschaulicht, wobei die Elemente durch ein Kästchen, einen Kreis bzw. ein Dreieck dargestellt wurden.

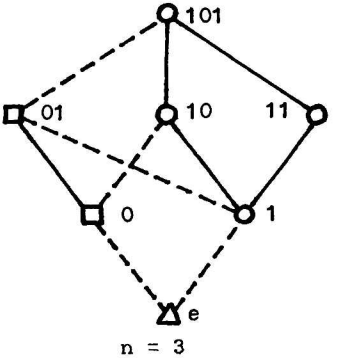
n = 1

Abb. 1



n = 2

Abb. 2



n = 3

Abb. 3

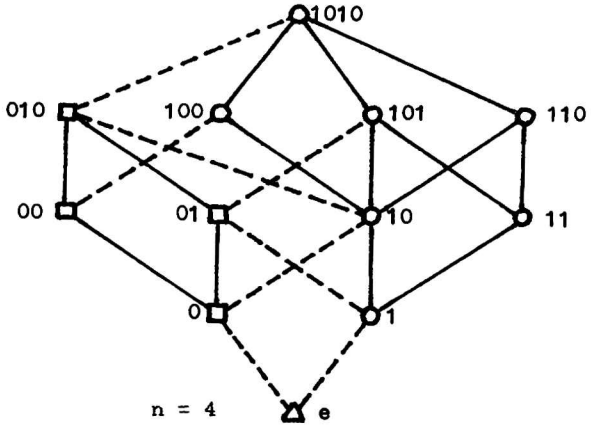


Abb. 4

Die Folge $x_o, x_1, \ldots, x_n$ von reellen Zahlen heißt unimodal (vgl. [1]), falls es einen Index q gibt, so daß

$$x_o \leq x_1 \leq \ldots \leq x_{q-1} \leq x_q \geq x_{q+1} \geq \ldots \geq x_{n-1} \geq x_n$$

gilt. Die Folge heißt logarithmisch konkav, falls

$$x_k^2 \geq x_{k-1}x_{k+1}, \quad k = 1, \ldots, n-1, \tag{1}$$

gilt. Bekanntlich folgt aus der logarithmischen Konkavität die Unimodalität (vgl. [1]).
Wir beweisen folgende Aussagen:

<u>Satz 1:</u> Die Folge $\{t_{n,k}\}$ der Niveauzahlen der Ordnung P_n ist logarithmisch konkav.

<u>Satz 2:</u> Die Ordnung P_n, $n \geq 2$, hat neben der identischen Abbildung genau einen Automorphismus f. Er ist durch $f(0\underline{u}_{n-2}) = \underline{u}_{n-1}$ charakterisiert.

Die Sätze 1 und 2 werden in den Abschnitten 4. und 6. bewiesen. In den Abschnitten 3. und 5. werden Hilfsaussagen bereitgestellt.

3. Die folgenden Aussagen sind leicht einzusehen (vgl. Abbildungen 1-4).

<u>Lemma 1:</u> Für $n \geq 2$ gilt

$$N_k(P_n) = N_{k-1}(Q_{n-1,1}) \cup N_{k-1}(P_{n-2,o}), \quad 1 \leq k \leq n-1, \tag{2}$$

$$N_k(Q_n) = N_{k-1}(P_{n-1,o}) \cup N_{k-1}(Q_{n-2,1}), \quad 1 \leq k \leq n-1, \tag{3}$$

$$N_n(P_n) = \{\underline{u}_n\}, \quad N_n(Q_n) = \{\underline{v}_n\}, \tag{4}$$

$$N_o(P_n) = N_o(Q_n) = e. \tag{5}$$

<u>Lemma 2:</u> Es gilt

a) Jedes Element $O\underline{b} \in N_k(P_n)$, $1 \leq k \leq n-1$, hat als Nachbarn in $N_{k+1}(P_n)$ neben seinen Nachbarn in $N_k(P_{n-2,o})$ noch genau das Element $1O\underline{b}$. Jedes Element $1\underline{b} \in N_k(P_n)$, $1 \leq k \leq n-1$, hat als Nachbarn in $N_{k+1}(P_n)$ seine Nachbarn in $N_k(Q_{n-1,1})$ und darüber hinaus für $1\underline{b} \in P_{n-2}$ noch genau das Element $O1\underline{b}$ und für $1\underline{b} \notin P_{n-2}$ kein weiteres Element als Nachfolger.

b) Jedes Element $Ox\underline{b} \in N_k(P_n)$, $2 \leq k \leq n-1$, $x = 0,1$, hat als Nachbarn in $N_{k-1}(P_n)$ neben seinen Nachbarn in $N_{k-2}(P_{n-2,o})$ für $x = 1$ noch genau das Element $1\underline{b}$ und für $x = 0$ kein weiteres Element als Vorgänger. Jedes $1x\underline{b} \in N_k(P_n)$, $2 \leq k \leq n-1$, hat als Nachbarn in $N_{k-1}(P_n)$ neben seinen Nachbarn in $N_{k-2}(Q_{n-1,1})$ für $x = 0$ noch genau das Element $O\underline{b}$ und für $x = 1$ kein weiteres Element als Vorgänger.

c) Die Aussagen a) und b) bleiben richtig, falls in ihnen durchgängig $P, Q, 0, 1$ durch $Q, P, 1$ bzw. 0 ersetzt werden.

(Die Aussagen a) und b) decken sich.)

Die Lemmata 1 und 2 sind leicht zu beweisen. Wir verzichten hier auf die Darstellung der Beweise.

Aus Lemma 1 ergibt sich sofort die nächste Aussage:

<u>Lemma 3:</u> Es gilt

$$t_{n,o} = 1 \quad \text{für} \quad n \geq 0, \tag{6}$$

$$t_{n,n} = 1 \quad \text{für} \quad n \geq 0, \tag{7}$$

$$t_{n,k} = t_{n-2,k-1} + t_{n-1,k-1} \quad \text{für} \quad n \geq 2,\ 1 \leq k \leq n-1. \tag{8}$$

4. <u>Beweis von Satz 1:</u> Wir vereinbaren $t_{n,q} := 0$ für $q < 0$ oder $q > n$ und beweisen zunächst durch Induktion über $n \geq 1$ die folgenden Ungleichungen:

$$t_{n,k}^2 \geq t_{n,k-1} \cdot t_{n,k+1}, \qquad k = 0,\ldots,n, \tag{9}$$

$$t_{n-1,k-1} \cdot t_{n,k-1} \geq t_{n,k} \cdot t_{n-1,k-2}, \qquad k = 0,\ldots,n, \tag{10}$$

$$t_{n-1,k-1} \cdot t_{n,k} \geq t_{n,k-1} \cdot t_{n-1,k}, \qquad k = 0,\ldots,n, \tag{11}$$

$$t_{n-1,k-1} \cdot t_{n,k-1} \geq t_{n,k-2} \cdot t_{n-1,k}, \qquad k = 0,\ldots,n. \tag{12}$$

Für $n = 1,2$ und beliebiges k sind diese vier Ungleichungen richtig. Sei nun $n \geq 2$. Die Ungleichung

$$t_{n+1,k}^2 \geq t_{n+1,k-1} \cdot t_{n+1,k+1} \tag{9'}$$

folgt unter Beachtung von (8) aus (9), (10) und (12). Die Ungleichung

$$t_{n,k-1} \cdot t_{n+1,k-1} \geq t_{n+1,k} \cdot t_{n,k-2} \tag{10'}$$

ist wegen (8) gleichwertig zu

$$t_{n,k-1} \cdot t_{n-1,k-2} \geq t_{n-1,k-1} \cdot t_{n,k-2},$$

und diese ist wegen (11) mit $k-1$ statt k erfüllt.
Die Ungleichung

$$t_{n,k-1} \cdot t_{n+1,k} \geq t_{n+1,k-1} \cdot t_{n,k} \tag{11'}$$

ergibt sich wie folgt: Zunächst zeigt man unter Beachtung von (8), daß sich

$$t_{n+1,k-1} \cdot t_{n-1,k-1} \leq t_{n-1,k-2} \cdot t_{n+1,k} \tag{13}$$

aus (11) mit $k-1$ statt k ergibt. Multipliziert man (13) mit $t_{n,k}$ und (10) mit $t_{n+1,k}$, so erhält man

$$\begin{aligned} t_{n+1,k-1} \cdot t_{n-1,k-1} \cdot t_{n,k} &\leq t_{n-1,k-2} \cdot t_{n+1,k} \cdot t_{n,k} \\ &\leq t_{n-1,k-1} \cdot t_{n,k-1} \cdot t_{n+1,k}. \end{aligned}$$

Daraus schließt man für $t_{n-1,k-1} \neq 0$ auf (11').
Nun ist $t_{n-1,k-1} = 0$ genau dann, wenn $k \leq 0$ oder $k \geq n+1$ ist. In diesen beiden Fällen ist (11') aber gewiß richtig. Schließlich bleibt

$$t_{n,k-1} \cdot t_{n+1,k-1} \geq t_{n+1,k-2} \cdot t_{n,k} \qquad (12')$$

zu zeigen. Von den beiden Ungleichungen

$$t_{n+1,k-2} \cdot t_{n,k-1} \leq t_{n,k-2} \cdot t_{n+1,k-1},$$
$$t_{n,k-2} \cdot t_{n,k} \leq t_{n,k-1} \cdot t_{n,k-1}$$

ergibt sich die erste aus (11') mit $k-1$ statt k, die zweite folgt aus (9) für $k-1$ an Stelle von k. Aus beiden zusammen resultiert für $t_{n,k-1} \neq 0$ tatsächlich (12').
Nun ist $t_{n,k-1} = 0$ gleichbedeutend mit $k \leq 0$ oder $k \geq n+2$. In diesen Fällen ist (12') auch richtig.

Damit ist der Induktionsbeweis für (9)-(12) abgeschlossen.
Aus dem Vergleich von (1) und (9) schließt man auf die logarithmische Konkavität der Folge $\{t_{n,k}\}$. ■

5. In diesem Abschnitt formulieren wir einige Aussagen, die im Beweis von Satz 2 verwendet werden.

<u>Lemma 4:</u> Sei $n \geq 1$. Die Elemente $\underline{u}_n \in P_n$ und $\underline{v}_n \in Q_n$ haben in P_n bzw. Q_n genau n Vorgänger.

<u>Beweis:</u> Aus (7) und (8) folgt sofort $t_{n,n-1} = n$. ■

<u>Lemma 5:</u> Sei $n \geq 2$ und $0 \leq k \leq n-2$. Für jedes Element $\underline{a}$ von $N_k(P_n)$ ist

$$d^+_{P_n}(\underline{a}) \geq 2. \qquad (14)$$

<u>Beweis:</u> Die Aussage ist richtig für $n = 2$. Sei ihre Richtigkeit für alle $n' \leq n-1$ und alle k' mit $0 \leq k' \leq n'-2$ vorausgesetzt, und sei $\underline{a} \in N_k(P_n)$. Für $k = n-2$ ist $\underline{a}$ Vorgänger gewisser Worte $\underline{b}$, $\underline{c}$ aus $N_{n-1}(P_n)$, und nach Lemma 4 ist $\underline{b} \neq \underline{c}$, also (14) erfüllt. Für $k \leq n-3$ gilt $\underline{a} \in N_{k-1}(P_{n-2,o})$ oder $\underline{a} \in N_{k-1}(Q_{n-1,1})$, und (14) ist nach Induktionsvoraussetzung richtig. ■

Aus Lemma 2 folgt, daß in H_n jedes Element aus $H_{n-2,o}$ und $H_{n-1,1}$ höchstens zwei Nachbarn in $H_{n-1,1}$ bzw. $H_{n-2,o}$ hat. Diesen Fakt verwenden wir mehrfach beim Beweis der nächsten Aussagen.

Lemma 6: Für $n \geq 3$ gilt

a) $d_{P_n}(0\underline{u}_{n-2}) = d_{P_n}(\underline{u}_{n-1}) = n$,

b) $d_{P_n}(\underline{u}_{n-2}) = 2n-2$,

c) $\forall\ \underline{a} \in P_n \quad \underline{a} \neq \underline{u}_{n-2} \Longrightarrow d_{P_n}(\underline{a}) < 2n-2$,

d) $\forall\ \underline{a} \in N_{n-1}(P_n) \quad \underline{a} \neq (0\underline{u}_{n-2}),\ \underline{u}_{n-1} \Longrightarrow d_{P_n}(\underline{a}) < n$.

Beweis: Die Aussage a) ergibt sich aus Lemma 2 b) und Lemma 4. Das Wort 1 hat in P_3 den Grad $d_{P_3}(1) = 4 = 2\cdot 3-2$, und dieser Maximalgrad wird in P_3 von keinem anderen Wort angenommen. Sei $n \geq 4$ und die Richtigkeit der Aussagen b) und c) für $n' \leq n-1$ angenommen. Dann hat jedes Element aus $P_{n-2,o}$ oder aus $Q_{n-1,1}$ in P_n höchstens den Grad $2(n-2)-2+2 = 2n-4$ bzw. $2(n-1)-2+2 = 2n-2$. In P_{n-1} gibt es genau ein Element vom Maximalgrad $2n-4$, das Wort $\underline{u}_{n-3}$. Also hat in Q_{n-1} genau das Wort $\underline{v}_{n-3}$ und in $Q_{n-1,1}$ genau das Wort $(1\underline{v}_{n-3}) = \underline{u}_{n-2}$ den Maximalgrad $2n-4$. Zusammen mit Lemma 2 liefert das die Aussagen b) und c).
Die Aussage d) ist für $n = 3$ richtig. Für $n \geq 4$ gilt für jedes $\underline{a} \in N_{n-1}(P_n) \cap Q_{n-1,1}$ mit $\underline{a} \neq (1\overline{0\underline{u}_{n-3}}) = (11\underline{v}_{n-3})$ und $\underline{a} \neq (1\overline{\underline{u}_{n-2}}) = (1\underline{v}_{n-2}) = \underline{u}_{n-1}$ nach Induktionsvoraussetzung $d_{Q_{n-1,1}}(\underline{a}) \leq n-2$, und da jedes $\underline{a} \in Q_{n-1,1}$ in $P_{n-2,o}$ höchstens einen Vorgänger und keinen Nachfolger hat, ist $d_{P_n}(\underline{a}) \leq (n-2)+1 = n-1$. Schließlich hat $(11\underline{v}_{n-3})$ nach Lemma 2 in $P_{n-2,o}$ weder einen Vorgänger noch einen Nachfolger. Damit ist $d_{P_n}(11\underline{v}_{n-3}) \leq n-2$, und insgesamt ist die Aussage d) bewiesen. ■

Lemma 7: Sei $n \geq 5$. Es gilt:

$$\forall\ \underline{x} \in P_n(\exists\ \underline{y} \in V(\underline{v}_{n-1}),\ \underline{y} \notin \{(0\underline{v}_{n-3}),\ \underline{u}_{n-2},\ \underline{v}_{n-2}\},$$
$$\underline{y} = (0\underline{x}) \Longleftrightarrow \underline{x} \in V(\underline{u}_{n-2}),\ \underline{x} \notin \{\underline{u}_{n-3},\ \underline{v}_{n-3}\}).$$

Beweis: Die folgenden Aussagen (15) und (16) sind leicht nachprüfbar, und aus ihnen ergibt sich die Behauptung:

$\underline{y} \in V(\underline{v}_{n-1})$, $\underline{y} \notin \{(0\underline{v}_{n-3}), \underline{u}_{n-2}, \underline{v}_{n-2}\}$,

$\Longleftrightarrow \exists\, s,t \geq 0\ \exists\, c \in \{0,1\}$ $\underline{y} = (\underline{z}cc\underline{w})$, mit $\underline{z} = (\underline{u}_s)$, $s+t = n-5$, c ist vom letzten Buchstaben von $\underline{z}$ verschieden, und es ist (15)

$$\underline{w} = \begin{cases} \underline{u}_t & \text{für } c = 0, \\ \underline{v}_t & \text{für } c = 1; \end{cases}$$

$\underline{x} \in V(\underline{u}_{n-2})$, $\underline{x} \notin \{\underline{u}_{n-3}, \underline{v}_{n-3}\}$

$\Longleftrightarrow \exists\, s,t \geq 0\ \exists\, c \in \{0,1\}$ $\underline{x} = (\underline{u}_s cc\underline{w})$, $s+t = n-5$, für $s > 0$ ist c vom letzten Buchstaben von $\underline{u}_s$ verschieden, und es gilt (16)

$$\underline{w} = \begin{cases} \underline{u}_t & \text{für } c = 0, \\ \underline{v}_t & \text{für } c = 1. \end{cases}$$

■

<u>Lemma 8:</u> Sei $n \geq 4$. Es gilt:

$\underline{x}' \in V(\underline{u}_{n-1} \subseteq P_n$, $\underline{x}' \neq \underline{u}_{n-2}$, $\underline{x}' \neq \underline{v}_{n-2} \Longrightarrow \underline{x}'$ hat in $V(\underline{u}_n)\setminus\{\underline{v}_{n-1}, \underline{u}_{n-1}\}$ genau einen Nachfolger $\underline{x}''$, und die Abbildung $h : \underline{x}' \rightarrow \underline{x}''$ bildet $V(\underline{u}_{n-1}\setminus\{\underline{u}_{n-2}, \underline{v}_{n-2}\}$ injektiv in $V(\underline{u}_n)\setminus\{\underline{u}_{n-1}, \underline{v}_{n-1}\}$ ab.

<u>Beweis:</u> Für $\underline{x}' \in V(\underline{u}_{n-1}) \subseteq P_n$, $\underline{x}' \notin \{\underline{u}_{n-2}, \underline{v}_{n-2}\}$ verwende man die Darstellung (16) mit $n-1$ an Stelle von $n-2$. Jeder Nachfolger $\underline{x}''$ von $\underline{x}'$ entsteht durch Hinzufügen eines Buchstaben d zu $\underline{x}'$. Da für $\underline{x}'' \in V(\underline{u}_n)\setminus\{\underline{u}_{n-1}, \underline{v}_{n-1}\}$ wiederum eine zu (16) analoge Darstellung $\underline{x}'' = (\underline{u}_s, cc\underline{w}_t,)$ gilt, darf d weder vor noch zwischen die Buchstaben von $\underline{x}'$ gesetzt werden, muß also als letzter Buchstabe von $\underline{x}''$ hinter $\underline{x}'$ angefügt werden. Jedes solche $\underline{x}''$ ist auch tatsächlich in $V(\underline{u}_n)\setminus\{\underline{u}_{n-1}, \underline{v}_{n-1}\}$. Damit ist bei gegebenem $\underline{x}'$ ein Nachfolger $\underline{x}''$ eindeutig bestimmt und h offenbar injektiv. ■

<u>Lemma 9:</u> Sei $n \geq 3$, $1 \leq k \leq n-2$. Es gilt:

$\forall\, \underline{x} \in N_k(P_n)\ \forall\, \underline{y} \in N_k(P_n)\ \underline{x} \neq \underline{y} \Longrightarrow N(\underline{x}) \neq N(\underline{y})$.

<u>Beweis:</u> Die Aussage ist richtig für $n = 3$. Sei $n \geq 4$ und die Aussage für $n-1$ als richtig vorausgesetzt. Falls $\underline{x}$, $\underline{y}$ aus $P_{n-2,o}$ sind, so haben sie nach Lemma 2 verschiedene Nachfolger in $Q_{n-1,1}$. Sind $\underline{x}$, $\underline{y}$ aus $Q_{n-1,1}$, so haben sie nach Induk-

tionsvoraussetzung verschiedene Nachfolgermengen in $Q_{n-1,1}$, und damit gilt auch $N(\underline{x}) \neq N(\underline{y})$. Für $\underline{x} \in P_{n-2,o}$ und $\underline{y} \in Q_{n-1,1}$ hat $\underline{x}$ nach Lemma 2 höchstens einen und $\underline{y}$ nach Lemma 5 mindestens zwei Nachfolger in $Q_{n-1,1}$. ■

6. Beweis von Satz 2: Sei f eine isomorphe Abbildung von P_n auf sich. Dann ist f rangtreu und daher $f(N_k(P_n)) = N_k(P_n)$, $k = 0,1,\ldots,n$. Außerdem müssen für $f(\underline{x}) = \underline{y}$ die Elemente $\underline{x}$ und $\underline{y}$ die gleiche Anzahl von Vorgängern und die gleiche Anzahl von Nachfolgern haben. Da nach Lemma 6 d) in $N_{n-1}(P_n)$ nur die Elemente $\underline{v}_{n-1}$ und $\underline{u}_{n-1}$ den Maximalgrad n in dieser Schicht besitzen, gibt es nur die Möglichkeiten $f(\underline{v}_{n-1}) = \underline{v}_{n-1}$ und $f(\underline{v}_{n-1}) = \underline{u}_{n-1}$. Wir werden zeigen, daß der Automorphismus f durch $f(\underline{v}_{n-1})$ bereits eindeutig bestimmt ist.
Jedenfalls gilt $f(\underline{u}_{n-2}) = \underline{u}_{n-2}$ nach Lemma 6 b,c).
Wir verwenden Induktion über n. Für $n = 2,3,4$ gibt es genau die beiden behaupteten Automorphismen. Sei $n \geq 5$ und die Behauptung für alle $n' < n$ als richtig angenommen.

Fall 1: $f(\underline{v}_{n-1}) = \underline{v}_{n-1}$.

Dann ist $f(\underline{u}_{n-1}) = \underline{u}_{n-1}$. Ferner geht bei f die Menge $N_{n-2}(P_n) \cap P_{n-2,o}$ in sich über, denn ihre Elemente sind außer dem Fixpunkt $\underline{u}_{n-2}$ die einzigen Vorgänger des Fixpunktes $\underline{v}_{n-1}$. Dann geht jede Menge $N_k(P_n) \cap P_{n-2,o}$, $k = n-3, n-4,\ldots,1$, bei f in sich über, da jedes ihrer Elemente nach Lemma 4 in $N_{k+1}(P_n) \cap P_{n-2,o}$ mindestens zwei Nachfolger besitzt, während jedes Element aus $N_k(P_n) \cap Q_{n-1,1}$ in $N_{k+1}(P_n) \cap P_{n-2,o}$ höchstens einen Nachfolger hat. Folglich erzeugt f auf $P_{n-2,o} \subset P_n$ bzw. $Q_{n-1,1} \subset P_n$ einen Isomorphismus f_1 bzw. f_2. Dabei ist f_1 einer der beiden nach Induktionsannahme auf $P_{n-2,o}$ möglichen Isomorphismen, und genau für $f_1(OO\underline{u}_{n-4}) = (OO\underline{u}_{n-4})$ ist f_1 die identische Abbildung. Für $f_1(OO\underline{u}_{n-4}) \neq (OO\underline{u}_{n-4})$ muß $f_1(OO\underline{u}_{n-4}) = (O\underline{u}_{n-3})$ sein. Letzteres ist unmöglich, da $\underline{u}_{n-1}$ bei f Fixpunkt ist und den Nachbarn $\underline{v}_{n-2}$, aber nicht den Nachbarn $(OO\underline{u}_{n-4})$ hat. Folglich ist f_1 die identische Abbildung.
Aus der Kenntnis von f_1 folgt $f_2(N_k(P_n) \cap Q_{n-1,1}) = N_k(P_n) \cap Q_{n-1,1}$, $k = 1,2,\ldots,n$. Nach Induktionsvoraussetzung ist f_2 genau dann nicht die identische Abbildung, falls

$f_2(\underline{u}_{n-1}) = (1\underline{u}_{n-2})$ ist. Letzteres ist unmöglich, da nach Lemma 2 zwar $\underline{u}_{n-1}$, aber nicht $(1\underline{u}_{n-2})$ einen Vorgänger in $P_{n-2,o}$ hat und $f(P_{n-2,o}) = P_{n-2,o}$ ist.
Also ist auch f_2 und damit f die identische Abbildung.

Fall 2: $f(\underline{v}_{n-1}) = \underline{u}_{n-1}$. (17)

Dann ist $f(\underline{u}_{n-1}) = \underline{v}_{n-1}$. Da $\underline{u}_{n-2}$ ein Fixpunkt ist, erzeugt f auf $P_{n-2} \subseteq P_n$ einen Isomorphismus f'. Für f' gibt es nur die Möglichkeiten $f'(\underline{v}_{n-3}) \in \{\underline{v}_{n-3}, \underline{u}_{n-3}\}$, und f' ist nach Induktionsvoraussetzung durch $f'(\underline{v}_{n-3})$ eindeutig festgelegt. In jedem Fall ist $\underline{v}_{n-2}$ ein Fixpunkt von f, denn $\underline{u}_{n-2}$ ist ein solcher, und $\underline{u}_{n-2}$ sowie $\underline{v}_{n-2}$ sind die einzigen gemeinsamen Vorgänger von $\underline{u}_{n-1}$ und $\underline{v}_{n-1}$.

Fall 2.1: $f'(\underline{v}_{n-3}) = \underline{v}_{n-3}$. (18)

Aus der Darstellung (16) ergibt sich leicht

$$N(\underline{u}_{n-3}) = V(\underline{u}_{n-1}). \tag{19}$$

Analog gilt

$$N(\underline{v}_{n-3}) = V(\underline{v}_{n-1}). \tag{20}$$

Aus (17) bis (20) ergibt sich

$$V(\underline{u}_{n-1}) = f(V(\underline{v}_{n-1})) = f(N(\underline{v}_{n-3})) = N(\underline{v}_{n-3}) = V(\underline{v}_{n-1}).$$

Das ist für $n \geqslant 4$ ein Widerspruch, da $V(\underline{v}_{n-1}) \cap V(\underline{u}_{n-1}) = \{\underline{u}_{n-2}, \underline{v}_{n-2}\}$ und $|V(\underline{u}_{n-1})| = |V(\underline{v}_{n-1})| = n-1 \geqslant 3$ ist.
Folglich kann Fall 2.1 nicht eintreten.

Fall 2.2: $f'(\underline{v}_{n-3}) = \underline{u}_{n-3}$. (21)

Wir wollen zunächst zeigen, daß f auf $V(\underline{v}_{n-1})$ eindeutig bestimmt ist. Wir wissen bereits, daß $\underline{v}_{n-2}$ und $\underline{u}_{n-2}$ Fixpunkte sind. Nach Lemma 2 und Lemma 7 liefert die Zuordnung

$$\underline{y} \in V(\underline{v}_{n-1}) \setminus \{\underline{u}_{n-2}, \underline{v}_{n-2}, (O\underline{v}_{n-3})\} \xrightarrow{g} \underline{x} \in V(\underline{u}_{n-2}) \setminus \{\underline{u}_{n-3}, \underline{v}_{n-3}\}$$

eine Bijektion, bei der jedes Element $\underline{y}$ auf einen Vorgänger $\underline{x} \lessdot \underline{y}$ abgebildet wird. Ferner ist wegen (21) nach Induktionsvoraussetzung f' eindeutig bestimmt und daher insbesondere für jedes Element x von $V(\underline{u}_{n-2}) \setminus \{\underline{u}_{n-3}, \underline{v}_{n-3}\}$ das Element $\underline{x}' := f(\underline{x}) = f'(\underline{x})$ in dieser Menge eindeutig festgelegt. Nach

Lemma 8 (mit $n-2$ statt $n-1$) hat $\underline{x}'$ in $V(\underline{u}_{n-1})\setminus\{\underline{u}_{n-2},\underline{v}_{n-2}\}$ einen eindeutig bestimmten Nachfolger $h(\underline{x}') =: \underline{x}''$, und h ist eine Bijektion. Folglich ist $f(\underline{y}) = \underline{x}''$. Durch Abzählen erhält man

$$V(\underline{u}_{n-1})\setminus(\{\underline{u}_{n-2},\underline{v}_{n-2}\} \cup g\cdot h(V(\underline{v}_{n-1})\setminus\{\underline{u}_{n-2},\underline{v}_{n-2},(0\underline{v}_{n-3})\}) = \{\underline{z}\},$$

und wegen $f(V(\underline{v}_{n-1})) = V(\underline{u}_{n-1})$ ist $f(0\underline{v}_{n-3}) = \underline{z}$. Daher ist f auf $V(\underline{v}_{n-1})$ eindeutig bestimmt. (Siehe hierzu auch die unten folgende Anmerkung.)

Wir wollen nun zeigen, daß f auf $N_{n-1}(P_n)$ eindeutig bestimmt ist.

Sei $Y := N_{n-1}(P_n)\setminus\{\underline{v}_{n-1},\underline{u}_{n-1},(1\underline{u}_{n-2})\}$. Nach Lemma 2 liefert die Vorgängerbildung eine injektive Abbildung $k : Y \longrightarrow k(Y)$, $k(Y) = V(\underline{v}_{n-1})\setminus\{\underline{u}_{n-2},\underline{v}_{n-2}\}$. Für jedes $\underline{y}' = k(\underline{y})$, $\underline{y} \in Y$ ist $f(\underline{y}') \in V(\underline{u}_{n-1})\setminus\{\underline{u}_{n-2},\underline{v}_{n-2}\}$ eindeutig festgelegt und hat nach Lemma 8 genau einen Nachfolger $\underline{y}'' \in V(\underline{u}_n)\setminus\{\underline{u}_{n-1},\underline{v}_{n-1}\}$, wobei die Abbildung $f(\underline{y}') \xrightarrow{j} \underline{y}''$ injektiv ist. Offenbar muß $f(\underline{y}) = \underline{y}''$ sein. Die Abbildung $k\cdot f\cdot j$ stimmt auf ihrem Definitionsbereich mit f überein und bildet Y in $N_{n-1}(P_n)\setminus\{\underline{u}_{n-1},\underline{v}_{n-1}\}$ ab, wobei genau ein Element $\underline{q} \in N_{n-1}(P_n)\setminus\{\underline{u}_{n-1},\underline{v}_{n-1}\}$ nicht Bild ist. Dann muß $f(1\underline{u}_{n-2}) = \underline{q}$ sein, und f ist auf $N_{n-1}(P_n)$ eindeutig bestimmt.
Nun erkennt man leicht schrittweise für $k = n-2, n-3, \ldots, 1$, daß f auf $N_k(P_n)$ und damit auf ganz P_n eindeutig bestimmt ist. Sei dazu f auf $N_{k+1}(P_n)$ bereits als eindeutig erkannt und $\underline{x} \in N_k(P_n)$. Dann muß $N(f(\underline{x})) = f(N(\underline{x}))$ sein, und durch $f(N(\underline{x}))$ ist $f(\underline{x})$ nach Lemma 9 als Abbildung $f : P_n \longrightarrow P_n$ bestimmt. Es gibt also höchstens zwei Automorphismen auf P_n.

Es bleibt zu zeigen, daß es außer dem identischen Automorphismus tatsächlich noch einen weiteren Automorphismus gibt. Dazu vereinbaren wir die Bezeichnung $(a_1a_2\ldots a_t)^{-1} := (a_ta_{t-1}\ldots a_2a_1)$ und setzen für ein beliebiges Wort $\underline{a} \in P_n$

$$f(\underline{a}) := \begin{cases} \underline{a}^{-1} & \text{für ungerades } n, \\ \overline{\underline{a}^{-1}} & \text{für gerades } n. \end{cases} \tag{22}$$

Wegen $\underline{u}_{2q+1}^{-1} = \underline{u}_{2q+1}$ und $\overline{\underline{u}_{2q}^{-1}} = \overline{\underline{v}_{2q}} = \underline{u}_{2q}$ ist durch (22) eine Bijektion $f : P_n \to P_n$ gegeben. Sie ist offenbar ein Ordnungsisomorphismus. ■

Anmerkung: Diesen Teil des Beweises könnte man durch folgende Überlegung abkürzen: Sei l derjenige Isomorphismus von P_{n-1} auf Q_{n-1}, der jedes Wort auf das duale Wort abbildet, und sei f_2 der durch $f_2(\underline{u}_{n-2}) \to \underline{v}_{n-2}$ nach Induktionsvoraussetzung eindeutig bestimmte Isomorphismus von P_{n-1}. Dann muß mit $f'' := f_{|Q_{n-1}}$ gelten: $l \cdot f'' = f_2$.
Daher ist sogar f'' als $l^{-1} \cdot f_2$ eindeutig bestimmt.

8. Wir kennen für die Ordnung P_n eine Reihe weiterer Struktureigenschaften, aber auf die folgenden Fragen wissen wir bisher keine Antwort.

Problem 1:

Man bestimme $|N_k(P_n)|$.

Man erkennt unschwer, daß $|N_k(P_n)| = 2^k$ für $k = 0,1,\ldots,\lfloor \frac{n}{2} \rfloor$ ist. Natürlich ist $|N_{n-1}(P_n)| = n$.

Problem 2:

Man bestimme k mit $|N_k(P_n)| \geq |N_i(P_n)|$, $i = 0,1,\ldots,n$.

Literatur:

1. Aigner, M.: Kombinatorik I, II. Berlin 1975, 1976

2. Björner, A., and Wachs, M.: On lexicographically shallable posets. Trans. Amer. Math. Soc. 277, 323-341 (1983)

3. Farmer, F.D.: Cellular homology for posets. Math. Japon. 23 607-613 (1979)

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Autoren:

Prof. Dr. Gustav Burosch
stud. math. Ulrike Franke
stud. math. Susanne Röhl
Wilhelm-Pieck-Universität Rostock
Sektion Mathematik
Universitätsplatz 1
Rostock
DDR-2500

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Janos Demetrovics; Lajos Ronyai

Generating sets for Crowns and Fences [1)]

1. Introduction

First we introduce some terminology. Let A be a nonempty finite set. An n-ary operation over A is a function from A^n to A. $P_n(A)$ denotes the set of all n-ary operations over A and we put $P(A) = \bigcup_{n\geq 0} P_n(A)$. A set of operations $C \subseteq P(A)$ is a clone over A if it contains the projections and is closed under arbitrary superpositions (cf. Jablonskii [10], Pöschel, Kalužnin [13], Szendrei [16]). The set of all clones over A is denoted by $L(A)$. $L(A)$ is a partially ordered set with respect to inclusion. It is not difficult to see that $L(A)$ is in fact a complete lattice. For a subset $H \subseteq P(A)$ let $[H]$ denote the smallest clone containing H. This is the clone generated by H. A clone C is finitely generated if $C = [H]$ for some finite subset H of C. The dual atoms of $L(A)$ are the maximal clones over A.

A nonempty subset $\varrho \subseteq A^n$ is an n-ary relation over A. An operation $f \in P_k(A)$ preserves ϱ or ϱ is an invariant relation of f if ϱ is a subalgebra of the algebra $\langle A, f\rangle^n$. For a relation ϱ over A we define

$$\mathrm{Pol}(\varrho) = \{f \in P(A);\ f \text{ preserves } \varrho\}.$$

It is easy to see that $\mathrm{Pol}(\varrho)$ is a clone.
In this paper we study clones of the form $\mathrm{Pol}(\varrho)$ where ϱ is a partial ordering on the finite set P. As the ordering will always be clear from the context, we shall write $\mathrm{Pol}(P)$ instead of $\mathrm{Pol}(\varrho)$. $\mathrm{Pol}(P)$ is the set of all order preserving operations over P. We shall also use the terms monotone operation or isotone operation as well. It is known that $\mathrm{Pol}(P)$ is a maximal clone if and only if P has elements 0 and 1 such that $0 < x < 1$ holds for every $x \in P$.

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The Slupecki relation η_A over A with $\mathrm{card}(A) = k$ is defined as

$$\eta_A = \{(a_1, a_2, \ldots, a_k) \in A^k;\ \mathrm{card}(\{a_1, a_2, \ldots, a_k\}) < k\}.$$

$S_A = \mathrm{Pol}(\eta_A)$ is the Slupecki clone over A. An operation $f\colon A^n \to A$ is essentially unary if there exists an integer i, $1 \leq i \leq n$, and a function $f' \in P_1(A)$ such that for every $a = (a_1, a_2, \ldots, a_n) \in A^n$ we have $f(a) = f'(a_i)$. It is known (cf. Pöschel, Kalužnin [13]) that S_A consists of all non-surjective operations and all essentially unary operations over A. Moreover, S_A is a maximal clone.
An operation $f\colon A^n \to A$ is a near unanimity function if $n > 2$ and the following identities are valid for all $a, b \in A$

$$f(b, a, a, \ldots, a) = f(a, b, a, \ldots, a) = \ldots = f(a, a, a, \ldots, b) = a.$$

A result of Baker and Pixley [1] implies that a clone containing a near unanimity function is finitely generated. Let B be a nonempty subset of A. A function $m\colon A^3 \to B$ is called a B-majority operation over A if

$$m(a, a, b) = m(a, b, a) = m(b, a, a) = a \quad \text{holds for every} \quad a, b \in B.$$

In the special case $B = A$, m is simply called a majority operation. Majority operations are the simplest and perhaps the most important examples of near unanimity functions. Here we shall use the following slight variant of the interpolation technique introduced by Baker and Pixley [1, Corollary 5.1, esp.], see also Szendrei [16, Corollary 1.26] and Demetrovics, Hannak, Ronyai [4, Section 2] for an application in the context of monotone maximal clones.

<u>Lemma 1.1:</u> Let C be a clone over the finite set A and let B be a nonempty subset of A. Let C_B denote the subclone consisting of the operations $f \in C$ such that $\mathrm{Im}(f) \subseteq B$. Suppose that there exists a B-majority operation $m \in C_B$. Then C_B is finitely generated. Suppose further that we have a set $H \subseteq C$ such that for every $g \in C_B$ and $x, y \in \mathrm{dom}(g)$ there exists an $f \in [H]$ such that $g(x) = f(x)$ and $g(y) = f(y)$.
Then $[H \cup \{m\}] \supseteq C_B$.

In fact, Baker and Pixley formulated this method for arbitrary near unanimity functions.

Lau [11] showed that every monotone maximal clone over a base set having at most 7 elements is finitely generated. In a previous paper by Demetrovics, Hannak, Ronyai [3,4,5] the generating properties of maximal clones of the form Pol(P) were studied, where P is a partial order. A sufficient condition for the existence of a near unanimity operation $m \in Pol(P)$ was given. This result allowed us to prove that Pol(P) is finitely generated for a large class of posets. Rosenberg [15] studied the existence of near unanimity functions for arbitrary partially ordered sets. Rosenberg and Füredi [8] addressed the existence of monotone majority operations. They obtained a beautiful existence theorem for tree-like orders. Tardos [17] provided the first example of a monotone maximal clone which is not finitely generated. It is not known whether this clone has a basis (i.e., a minimal generating set). Our aim here is to extend these investigations to some other important partially ordered sets.

For a positive integer n F_n denotes the ordered set (Figure 1) defined on the set $\{0,1,\dots,n\}$ with the following inequalities:

$$0 < 1,\ 1 > 2,\ 2 < 3,\ \dots,\ n-1 < n, \quad \text{if } n \text{ is odd}$$

or

$$0 < 1,\ 1 > 2,\ 2 < 3,\ \dots,\ n-1 > n, \quad \text{if } n \text{ is even.}$$

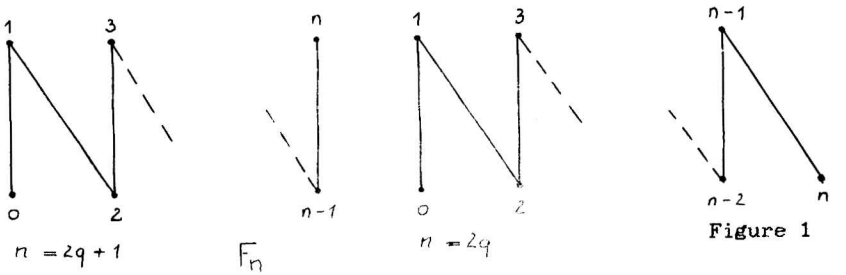

Figure 1

Let G_n denote the dual of F_n. An ordered set isomorphic to either F_n or G_n is a fence.

The image of 0 is the starting point and the image of n is the endpoint of the fence. The length of a fence is defined by

$$l(F_n) = l(G_n) = n.$$

For an integer $n > 0$ let H_{2n} denote the partial order (Figure 2) on the set $\{0,1,\dots,2n-1\}$ defined by the following inequalities:

$$0 < 1,\ 1 > 2,\ 2 < 3,\ \dots,\ 2n-2 < 2n-1,\ \ 2n-1 > 0.$$

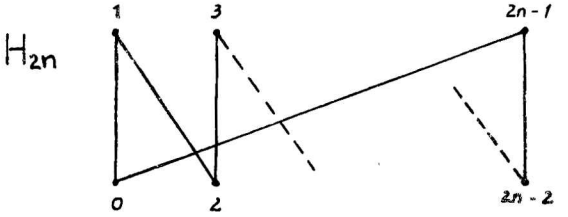

Figure 2

An ordered set isomorphic to H_{2n} for an integer n is a crown.

Crowns and fences arise frequently in the theory of finite ordered sets. Crowns play an important role in studies of symmetries of ordered sets (G. Birkhoff [2], Frucht [7]) and of the fixed point problem (Rival [14]). Fences arise in the study of connectivity of ordered sets, Hashimoto [9]. Both structures are considered by Duffus and Rival [6] in their theory of order varieties. Algebraic properties of both structures are considered in Rosenberg [15] and Rosenberg, Füredi [8].
In this paper we study the clones of all order preserving operations of crowns and fences. By presenting explicit generating sets, we show that these clones are finitely generated. Fences admit a uniquely determined order preserving majority operation. This enables us to exhibit a finite generating set for Pol(F) where F is a fence. In Section 3 we prove first that every crown H, Pol(H) is contained in the Slupecki clone S_H (here H denotes also the ground set of the ordered set H). From this result we deduce two important known results on crowns. Next using our knowledge on fences, we show that Pol(H) is finitely generated. As in the case of fences, we show that the binary order preserving operations generate the clone.

Here we give the main statements only. The details will appear elsewhere.

2. Fences

Following Nevermann and Rival [12], for elements x,y in an ordered set P, the updistance $D_P(x,y)$ from x to y is the least positive integer n (if it exists) such that there exists an order preserving map from F_n to P which maps 0 to x and n to y. The down distance $d_P(x,y)$ from x to y is defined dually (using G_n instead of F_n).
The following statement will be used as the first step in interpolation arguments.

Lemma 2.1: Let F,G be fences, $x,y \in F^n$ and $g: F^n \to G$ be a monotone map. Then there exists an integer i, $1 \leq i \leq n$, and a monotone map $f: F \to G$ such that $g(x) = f(x_i)$ and $g(y) = f(y_i)$.

Now we construct some useful binary order preserving maps. For integers i,j let $\max(i,j)$ ($\min(i,j)$) denote the maximal (minimal) element of the set $\{i,j\}$ with respect to the usual ordering of integers. Let M_n and dually m_n denote the restriction of max (min, resp.) to the set $\{0,1,\dots,n\}$. The following statement is straightforward. The easy proof is left to the reader.

Lemma 2.2: $M_n, m_n \in Pol(F_n)$ and $M_n, m_n \in Pol(G_n)$, i.e., they are order preserving operations over F_n as well as over G_n.

Theorem 2.3: Let F be a fence. Then there exists exactly one order preserving majority function $m: F^3 \to F$. Moreover, m can be obtained as a superposition of binary order preserving operations.

Remark: The existence and uniqueness of m was obtained by Rosenberg and Füredi [8]. The last statement seems to be new.
The main result of this section can be stated as follows.

Theorem 2.4: Let F be a fence. Then the clone $C = Pol(F)$ is finitely generated. In fact, $Pol(F) = [C_2]$.

3. Crowns

Here we investigate the clones of the form $Pol(F)$ where F is a crown. Our approach is based on Theorem 3.3, which states that the surjective order preserving operations on F are essentially unary We need the following

Lemma 3.1: Let i be an arbitrary element of the crown $H = H_{2n}$. Then

(a) There exists exactly one $j \in H$ such that

$$\max\{D_H(i,j),\ d_H(i,j)\} = n + 1.$$

For these elements $i, j \in H$ we have

$$\min\{D_H(i,j),\ d_H(i,j)\} = n.$$

Moreover, for any element $j \neq k \in H$ we have $D_H(i,k) < n + 1$ and $d_H(i,k) < n + 1$.

(b) For every proper subset $K \subset H$ there exists an $i \in H$ such that $D_H(i,k) < n + 1$, and $d_H(i,k) < n + 1$ holds for every $k \in K$.

The following well-known lemma gives a convenient way to construct invariant relations for clones of the form $Pol(Q)$, where Q is a partially ordered set. For a proof we refer to Pöschel, Kalužnin [13].

Lemma 3.2: Let P, Q be posets and $c_1, c_2, \ldots, c_k \in P$. Then

$$\{(f(c_1), f(c_2), \ldots, f(c_k));\ f\colon P \to Q \text{ is a monotone map}\}$$

is an invariant relation of every element of $Pol(Q)$.

Now we can prove the following

Theorem 3.3: Let F be a crown. Then $Pol(F) \subset S_F$, where S_F is the Slupecki clone on the set F.

Theorem 3.3 can be interpreted as a strengthening of two important results on crowns. The first of these says that we cannot apply directly the interpolation technique of Baker and Pixley to crowns.

Corolllary 3.4 (Rosenberg [15, Proposition 3.10]): Crowns admit no order preserving near unanimity functions.

The next result was obtained by Duffus and Rival [6, Theorem 6.7]. A collection $(P_i;\ i \in I)$ is a representation of the poset P if every P_i is a retract of P and P is a retract of $\Pi_{i \in I} P_i$. P is irreducible if for every representation $(P_i;\ i \in I)$ of P there exists an $i \in I$ such that P is a retract of P_i. We notice that the irreducibility is an "order preserving" analogue of the notion of subdirect irreducibility from universal algebra.

Corollary 3.5: Every crown is irreducible.

Now we return to the clones of the form $Pol(H_{2n})$. We have seen, that a function $f \in Pol(H_{2n})$ is either essentially unary, or $Im(f)$ is a proper subset of H_n. We shall look more closely at the latter case.

Lemma 3.6: Let $f \in Pol(H_{2n})$ and suppose that $Im(f)$ is a proper subset of H_n. Then there exists a fence $F \subset H_{2n}$ such that $Im(f) \subseteq F$ and the length of F is at most n. Every such fence is a retract of H_{2n} and any two points $x, y \in H_{2n}$ are contained in a proper retract.

Next we prove an equivalent of Lemma 2.1 for crowns.

Lemma 3.7: Let H be a crown $x, y \in H^n$ and $g: H^n \to H$ be a monotone map. Then there exists an integer i, $1 \leq i \leq n$, and a monotone map $f: H \to H$ such that $g(x) = f(x_i)$ and $g(y) = f(y_i)$.

Our aim is to show that the binary order preserving operations constitute a generating set of $Pol(H)$. Theorem 3.3 says that it suffices to give generators for the non-surjective operations. Let F be a maximal proper retract of H_{2n} and $r: H_{2n} \to F$ is the corresponding order preserving retraction. By Lemma 3.6, F is a fence of length n.

Let $m \in Pol(F)$ be the unique isotone majority operation. We define the ternary operation $m_F: H_{2n}^3 \to F$ as

$$m_F(x_1,x_2,x_3) = m(r(x_1),r(x_2),r(x_3)) \text{ for every } (x_1,x_2,x_3) \in H_{2n}^3.$$

Clearly $m_F \in Pol(H_{2n})$ and m_F is an F-majority operation. Also, using the fact that $m \in [Pol(F)_2]$ we obtain that $m_F \in [Pol(H_{2n})_2]$. We can prove the following

<u>Theorem 3.8:</u> The clone $C = Pol(H_{2n})$ is generated by the set of functions C_2.

<u>References:</u>

1. Baker, K.A., and Pixley, A.F.: Polynomial interpolation and the Chinese Remainder Theorem for algebraic systems. Math. Z. 143, 165-174 (1975)

2. Birkhoff, G.: Sobre los grupos de automorfismos. Rev. Un. Mat. Argentina 11, 155-157 (1946)

3. Demetrovics, J., Hannak, L., and Ronyai, L.: Near unanimity functions of partial orderings. Proceedings of the 14th International Symposium on Multiple-Valued Logics, Manitoba 1984, 52-56

4. Demetrovics, J., Hannak, L., and Ronyai, L.: On algebraic properties of monotone clones. Order 3, 219-225 (1986)

5. Demetrovics, J., Hannak, L., and Ronyai, L.: On monotone clones. Tanulmanyok-MTA Szamitastech. Automat. Kutato. Int. Budapest 202, 39-62 (1987)

6. Duffus, D., and Rival, I.: A structure theory of ordered sets. Discrete Math. 35, 53-118 (1981)

7. Frucht, R.: On the construction of partially ordered sets with a given group of automorphisms. Amer. J. Math. 72, 195-199 (1950)

8. Füredi, Z., and Rosenberg, I.G.: Orders admitting an isotone majority operation. Preprint C.R.M. Université de Montréal, 1985

9. Hashimoto, J.: On the product decomposition of partially ordered sets. Math. Japon. 1, 120-123 (1948)

10. Jablonskii, S.V.: Functional constructions in a k-valued logic (Russian). Trudy Mat. Inst. Steklov. 51, 5-142 (1958)

11. Lau, D.: Bestimmung der Ordnung maximaler Klassen von Funktionen der k-wertigen Logik. Z. Math. Logik Grundlag. Math. 24, 79-96 (1978)

12. Nevermann, P., and Rival, I.: Holes in ordered sets. Technical Report, The University of Calgary 1984

13. Pöschel, R., and Kalužnin, L.A.: Funktionen und Relationenalgebren. Berlin 1979

14. Rival, I.: A fixed point theorem for finite partially ordered sets. J. Combin. Theory Ser. A 21, 309-318 (1976)

15. Rosenberg, I.G.: Near unanimity orders. Preprint C.R.M. Université de Motréal 1985

16. Szendrei, A.: Clones in universal algebra. Les Presses de l'Université de Montréal 1986

17. Tardos, G.: A not finitely generated maximal clone of monotone operations. Order 3, 211-218 (1986)

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Authors:

Dr. Janos Demetrovics
Dr. Lajos Ronyai
Computer and Automation Institute
Hungarian Academy of Sciences
H-1502 Budapest, P.O.B. 63
Hungary

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Dragoš M. Cvetkovic

Some results on graph angles

0. Introduction

Characterizing a graph up to an isomorphism by the graph spectrum is possible only in trivial or exceptional cases. To find all graphs having the given spectrum is a difficult task and generally we do not know any better procedure than to construct all graphs with the required number of vertices and edges, to calculate spectra of all these graphs and to select those we need.

Adding to the eigenvalues some invariants of the corresponding eigenspaces we come to a better position if we want to tell something about the graph structure. In particular, some attention has been paid to the so-called graph angles.

Let G be a graph on n vertices. Let $\mu_1, \mu_2, \ldots, \mu_m$ $(\mu_1 > \mu_2 > \ldots > \mu_m)$ be the distinct eigenvalues of (the adjacency matrix A of) G with the corresponding eigenspaces $S_1, S_2, \ldots, S_m$. Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of R_n. The quantities $\alpha_{ij} = \cos\beta_{ij}$ $(i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$, where β_{ij} is the angle between S_i and e_j, are called angles of G. The matrix $\mathfrak{A} = \|\alpha_{ij}\|_{m,n}$ is called the angle matrix of G. We assume that columns of $\mathfrak{A}$ are ordered lexicographically so that $\mathfrak{A}$ is a graph invariant. The rows of $\mathfrak{A}$ are associated with the eigenvalues and are called eigenvalue angle sequences while the columns of $\mathfrak{A}$ are associated with the vertices and are called vertex angle sequences.

Let $P(\lambda) = \det(\lambda I - A)$ be the characteristic polynomial of G and let $P_i(\lambda)$, $i = 1,2,\ldots,n$, be the characteristic polynomials of vertex deleted subgraphs $G-i$ of G. We have (c.f. [6])

$$P^j(\lambda) = P(\lambda) \sum_{i=1}^{m} \frac{\alpha_{ij}^2}{\lambda-\mu_i} \qquad (j = 1,2,\ldots,n). \tag{1}$$

Let N_k^i be the number of closed walks of length k starting and terminating at vertex i. For the generating function

$H_i(t) = \sum_{k=0}^{+\infty} N_k^i t^k$ we have

$$H_i(t) = P_i(1/t)/tP(1/t). \tag{2}$$

Some early references concerning graph angles are given in expository papers [4, 6]. For a general discussion on graph angles see also [9]. Recent papers in which graph angles are used are [7, 5, 8, 2, 16].

Angles helped in studying the largest eigenvalue of unicyclic graphs [7]. [5] will be cited latter concerning construction of trees. Angles appear also in the graph reconstruction (from vertex deleted subgraphs) [8]. The change in the largest eigenvalue when adding a pendant edge has been studied through angles and perturbations in [2, 16].

Graph angles have been used in [1] to prove that isomorphism testing for graph with bounded multiplicities of eigenvalues has a polynomial complexity.

A result from [10] can be reformulated as follows. Let b be the number of non-zero entries in the i-th column of the angle matrix. Then the eccentricity of vertex i is bounded from above by b-1.

1. Trees

It was shown in [5] that for trees angles, added to the eigenvalues, "almost" suffice for the graph structure reconstruction. Although we have nonisomorphic trees with the same eigenvalues and angles, even although this happens "almost" always, we are still in the position easily to construct all trees with given eigenvalues and angles. Treating the vertices from leaves towards the centre we can "calculate" the remaining neighbour of each particular vertex by the use of a reconstruction lemma given below.

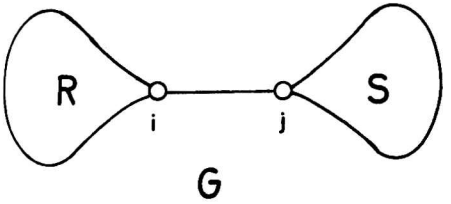

Figure 1

Suppose there is a bridge i-j in the graph G (see Figure 1). Suppose further that we completely know the structure of the subgraph R including the fact that the bridge starts at vertex i. Now the problem is how to cross the bridge, i.e. to identify vertex j. We have ([5]) the following

<u>Reconstruction lemma:</u> Given the subgraph R and the vertex i (see Figure 1), the vertex j is among those vertices k for which

$$P_k(\lambda) = \frac{P_R(\lambda)}{(P_{R-i}(\lambda))^2}\,(P_R(\lambda)P_i(\lambda) - P_{R-i}(\lambda)P(\lambda)). \tag{3}$$

We can now strengthen the result from [5] on constructing trees with given eigenvalues and angles. We suppose that instead of eigenvalues and angles we are given only the characteristic polynomials $P_i(\lambda)$, $(i = 1,2,\ldots,n)$ of vertex deleted subgraphs G-i of a graph G. We shall show that we still can construct all trees G.

Since $P'(\lambda) = \sum_{i=1}^{n} P_i(\lambda)$ we can readily calculate the characteristic polynomial $P(\lambda) = \sum_{k=o}^{n} a_k\lambda^{n-k}$ except for the constant term a_n.

If we determine a_n anyhow, we can calculate eigenvalues and angles of G and proceed with the construction of G in the same way as in [5].

The problem of a unique reconstruction of the characteristic polynomial $P(\lambda)$ of a graph G from the collection of characteristic polynomials $P_i(\lambda)$ $(i = 1,2,\ldots,n)$ of vertex deleted subgraphs G-i of G has been treated in [14]. If we know an eigenvalue of G the constant term a_n is uniquely determined. In particular any multiple root of any of the polynomials $P_i(\lambda)$ would cause (by interlacing theorem) the existence of the same eigenvalue in the spectrum of G. It was shown in [14] that for many trees a_n is uniquely reconstructable. The only trees for which this reconstruction is perhaps not unique are trees having an 1-factor.

For our purposes it is sufficient to note ([14]) that in the last case for any tree satisfying the requirement we have $a_n = (-1)^{n/2}$. We can show that if a_n is not unique then the other graphs are bipartite disconnected graphs, at least one

component being a tree; further all components have an even number of vertices, zero is not an eigenvalue and all components have simple eigenvalues.

It is clear now how to construct all trees G having the given collection of characteristic polynomials $P_i(\lambda)$ $(i = 1,2,\ldots,n)$ of vertex deleted subgraphs. We try to reconstruct $P(\lambda)$ and to establish that the graph considered is a tree and proceed with the construction as in [5]. In the case discussed above we suppose that G is a tree, determine $P(\lambda)$ and again perform the tree construction. If there is no tree G, this will be manifested during the construction of G.

Let us denote the described algorithm for constructing trees as Algorithm T.

There is a well known result by P.S. Kelly [15] saying that a tree is uniquely reconstructable from the collection of its vertex deleted subgraphs. The above considerations show that we 'almost' can weaken the hypothesis of the Kelly theorem by replacing vertex deleted subgraphs by their characteristic polynomials. The reconstruction is now not unique but we do have a good procedure to determine all trees satisfying the requirements given. Hence, we can formulate the following theorem.

Theorem 1: Let the characteristic polynomials $P_i(\lambda)$ $(i = 1,2,\ldots,n)$ of vertex deleted subgraphs $G-i$ of a graph G be given. Using Algorithm T we can construct all trees G or establish that they do not exist.

2. Tree-like cubic graphs

It was noted in [5] that the Reconstruction lemma can be also applied to unicyclic graphs. More generally, we can reconstruct in a similar way all graphs consisting of a central part, for which we know that it is characterised by eigenvalues (and angles), and trees attached to the central part.

We shall show further possibilities of the application of the Reconstruction lemma and develop other reconstruction tools based on angles.

From eigenvalues and the angle sequence of a particular vertex we can calculate the number N_k of closed walks of any length k starting and terminating at vertex i ([5]).

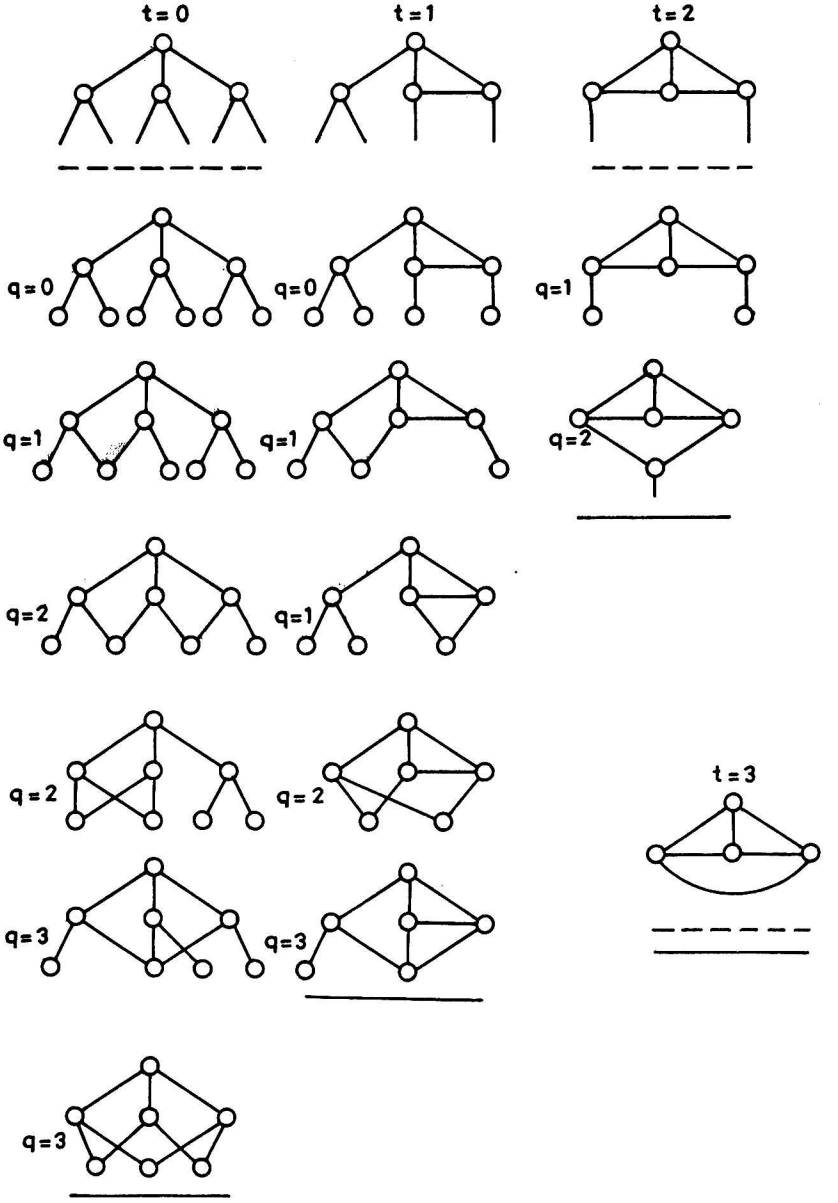

Figure 2

N_2 is the degree of i and N_3 is twice the number t_i of triangles passing through i. Further, t_i is equal to the number of edges in the subgraph induced by the neighbours of i. Numbers N_k (k = 4,5,...) contain some information on the further neighbourhood of vertex i but generally it is impossible to conclude very much about the actual graph structure. In regular graphs of degree r we have

$$q_i = (N_4^i - r^2)/2, \tag{4}$$

where q_i is the number of quadrangles passing through i. However, the number of pentagons passing through i is not uniquely determined in this way even in regular case. On the other hand, the number of shortest circuits passing through i can be calculated in regular case. We omit details.

After having considered trees we turn to cubic graphs.

Because of low degree in cubic graphs the neighbourhood of a vertex can be described fairly well. The subgraph induced by the three neighbours of a vertex is uniquely determined by t_i. Using q_i and N_5^i we can say something about the second neighbours of vertex i and in some cases even more. Figure 2 displays the possibilities.

The ordered pair (t_i, q_i) is called the <u>type</u> of vertex i. As Figure 2 illustrates, the type of a vertex sometimes uniquely determines the types of its neighbourhood or essentially limits possible types of neighbours. We shall restrict ourselves to a special class of cubic graphs.

A connected cubic graph is called <u>tree-like</u> if it consists of blocks displayed on Figure 3.

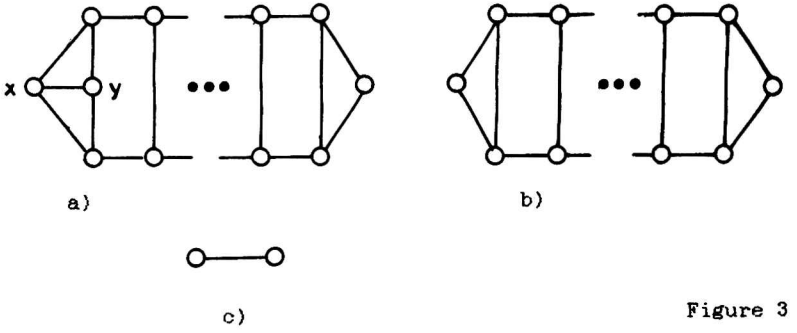

Figure 3

Blocks of the type a) on Figure 3 are called <u>terminal</u> blocks while those of type b) or c) are <u>internal</u> blocks. Vertices x and y (see Figure 3) in a terminal block are called <u>terminal</u> vertices. Blocks of type a) and b) are called <u>non-trivial</u> while those of type c) are <u>trivial</u>. Substituting non-trivial blocks by trivial ones in a tree-like cubic graph results in a tree called <u>skeleton tree</u>. If the skeleton tree of a tree-like cubic graph is a path, the graph is called a <u>path-like</u> cubic graph.

<u>Theorem 2:</u> A path-like cubic graph is characterized up to an isomorphism by eigenvalues and angles.

<u>Proof:</u> Given eigenvalues and angles of a graph we can first establish whether or not it is a cubic connected graph. If yes, we can, for any vertex x, determine the numbers N_k ($k = 1,2,\ldots,$) of closed walks of length k starting and terminating at x. We can recognize a terminal vertex by $N_3 = 4$. Now we can gradually reconstruct the whole graph starting from terminal vertex (see Figure 4).

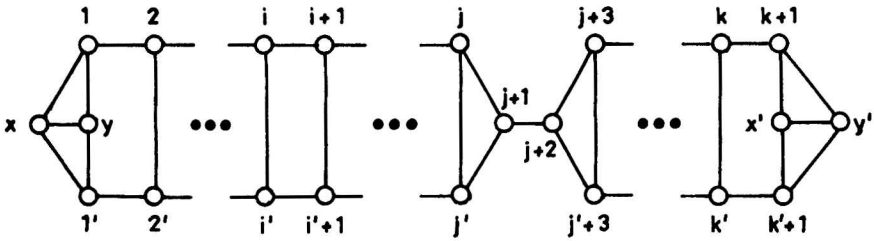

Figure 4

Suppose we have established the graph structure between vertices x and i, i'. Note that the vertices 1, 1 are at distance 1 from x. The value of N_{2i+2} tells us whether i+1 and i'+1 are identic or distinct. In the first case N_{2i+2} is by 2 greater than in the second one. Both values can easily be determined since all closed walks of length 2i+2 with exception of only a few pass through the subgraph already reconstructed. Looking at N_{2i+3} we can conclude in a similar way whether i+1 and i'+1 are adjacent or not.

If $i+1$ and $i'+1$ were identic we would have a situation like with vertices j, j', $j+1$, also in Figure 4. Vertex $j+1$ is an end vertex of a block and we consider vertices $j+1$, $j+2$, $j+3$, $j'+3$. Looking at N_{2j+7} we can decide whether $j+3$ and $j'+3$ are adjacent or not. Finally, we come to vertices $k+1$, $k'+1$ which are not adjacent while only two vertices x',y' remain. Their adjacencies must be as given in Figure 4 and the whole graph is reconstructed. Hence, we can uniquely construct a path-like cubic graph with given eigenvalues and angles or establish that it does not exist. This completes the proof. ■

Note that we have used in the proof only the angle sequence of the vertex x and that we have not identified graph vertices in the sense which angle sequence belongs to which vertex.

Now we combine Theorem 1 and 2 in order to study tree-like cubic graphs. First we need some more definitions.

Consider a tree-like cubic graph G. Vertices in which three blocks meet are called <u>centres</u>. A maximal connected subgraph of G not containing any centre as a cut point is called a <u>branch</u>. Branches of G contain one or two centres. Branches with one centre are called <u>terminal</u> while those with two centres <u>internal</u> branches. A terminal branch contains a terminal block (including two terminal vertices) and a centre which is called the <u>end</u> of the branch. In internal branches <u>ends</u> are the two centres of G belonging to the branch. A branch consisting of a single edge is called <u>trivial</u>. <u>Non-trivial</u> branches contain non-trivial blocks.

A <u>limb</u> of a tree-like cubic graph is recursively defined:

i) a terminal branch is a limb,
ii) two limbs meeting at a point form a limb;
iii) limbs are obtained by a finite number of applications of i), ii).

Using the procedure from the proof of Theorem 2 we can reconstruct all terminal branches. Now we have the problem of identifying ends of these branches i.e. to determine the angle sequences of endpoints.

Suppose that the vertex i is the end of the last non-trivial block in the branch and j is the end of the branch. Since the edge i-j is a bridge of G, if we know the angle sequence of i we can calculate the angle sequence of j by the Reconstruc-

tion lemma.

We can restrict the set of possible candidates for vertex i. First, we know the type of vertex i (type (1,0) on Figure 1). Second, the polynomial $P_i(\lambda)$ should contain as a factor the characteristic polynomial of the part of the branch containing vertices reconstructed before vertex i. Third, again applying the Reconstruction lemma, thus calculating $P_j(\lambda)$, there must be at least one vertex to which $P_j(\lambda)$ is associated.

If there is no vertex satisfying these three requirements, the graph is not tree-like. If there is more than one such vertex we have to examine all possibilities in further reconstruction. Hence, in general case, for each terminal branch we have a set of candidates for the endpoint of the branch.

If the graph is tree-like we must have two terminal branches with the common end-point. This end-point is a centre. By the Reconstruction lemma we find the third neighbour of it. The type of this neighbour determines whether it is the end of a nontrivial block or another centre.

1. In the first case a non-trivial branch has just started and we can reconstruct it like in Theorem 2 using angle sequence of the centre considered. Thus we are further reconstructing a limb of G consisting of the two terminal branches, the branch that we just consider and perhaps further branches.

2. In the second case we check whether some other branches have their ends at the new centre. If yes, we can continue the construction like in 1., while in the opposite case we leave it and consider other branches (or limbs).

Gradually we can reconstruct the whole graph. Then we check whether it has the given eigenvalues and angles.

Let us denote the algorithm described as <u>Algorithm C</u>. We can formulate the following theorem.

<u>Theorem 3:</u> Let eigenvalues and angles of a graph G be given. Using Algorithm C we can construct all tree-like cubic graphs with given eigenvalues and angles or establish that they do not exist.

3. Some computational results and implications

Trying to learn how eigenvalues and angles describe graphs other than trees we did some computations.

Many pairs of cospectral graphs are distinguishable by angles. This applies, for example, to the unique cospectral pairs of graphs on 5 vertices, of connected graphs on 6 vertices, of trees on 8 vertices, and to all 3 cospectral pairs of cubic graphs on (up to) 14 vertices. Further, we have treated connected graphs on 7 vertices (29 cospectral pairs and 1 triplet), trees up to 11 vertices (besides a pair on 8 vertices there are 5 pairs on 9 vertices, 4 pairs on 10 vertices and 27 pairs and 1 triplet on 11 vertices), unicyclic graphs up to 9 vertices (5 pairs on 8 vertices and 10 pairs and 3 triplets on 9 vertices) and bicyclic graphs on 8 vertices (11 cospectral pairs) without finding cospectral graphs with the same angles. We treated also a remarkable set of 10 cospectral graphs on 9 vertices with 16 edges found by C. Godsil and B. McKay [11]. They all have mutually different angles. The same holds for their complements which also are cospectral.

There are two pairs of cospectral regular graphs of degree 4 on 10 vertices ([11], one pair is displayed on Figure 5).

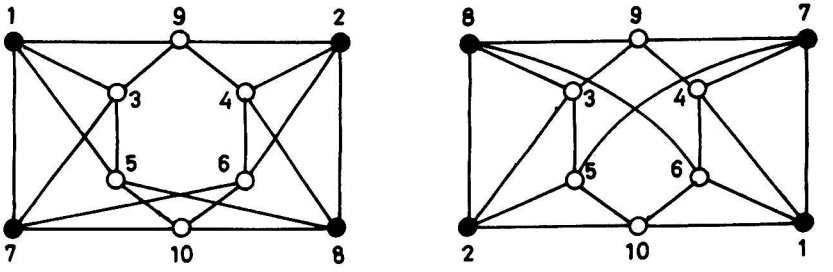

Figure 5

Their complements form two pairs of cospectral regular graphs of degree 5. No pairs of cospectral regular graphs on less than 10 vertices exist. Graphs from these pairs have the same angles. No examples of cospectral graphs with the same angles on less than 10 vertices are known. Graphs from these pairs are switching equivalent (with respect to black vertices in Figure 5) and this

is why they have the same angles. Generally, regular switching equivalent graphs of the same degree are cospectral and have the same angles ([4]). In addition, in strongly regular graphs all vertices have the same vertex angle sequences which are determined by eigenvalues. Hence, in strongly regular graphs angles do not carry any additional information on graph structure.

Because of the low degree cubic graphs cannot be switching equivalent. Therefore it seems to be reasonable to look to what extent the structure of cubic graphs is reflected in eigenvalues and angles.

First, it is not true that all cubic graphs are determined up to an isomorphism by eigenvalues and angles.

In [3] two non-isomorphic cospectral cubic graphs on 20 vertices are given. They have the eigenvalues 3, 2, 1, -1, -2, -3 with multiplicities 1, 4, 5, 5, 4, 1, respectively. One of these graphs is the Desargues graph and the other has been constructed in [3]. Computations show that both graphs have the same angles.

The Desargues graph is transitive while its cospectral mate has 12 vertices of eccentricity 5 and 8 vertices of eccentricity 4. However both are walk regular. (A graph is called <u>walk regular</u> if the generating function (2) is independent of the vertex i). It has been shown in [13] that a graph is walk regular if and only if all vertices have the same angle sequences (c.f. [9]). Now, inspired by the above example and having in mind the fact, that the sum of the squares of angles with a given eigenvalue is equal to the dimension of the eigenspace, we can formulate the following theorem.

<u>Theorem 4:</u> Let G be a walk regular graph. Then the angles of G can be calculated from eigenvalues of G.

<u>Corollary:</u> Cospectral walk regular graphs have the same angles.

The existence of cospectral cubic graphs with the same angles does not mean to much having in view the fact that in [3] these graphs have been constructed from the eigenvalues only. However, further investigations are necessary to conclude to what extent cubic graphs are characterized by eigenvalues and angles.

In order to see how cospectral graphs with the same angles can be distinguished we analyse graphs from Figure 5.

Vertices labelled by the same number in the two graphs have

equal angle sequences. With respect to angle sequences there are three types of vertices: angle type α: 1,2,3,4, angle type β: 5,6,7,8, and angle type γ: 9,10 (we omit numerical values).

As a contrast to cubic graphs, because of the higher degree the subgraph induced by neighbours of a particular vertex is not uniquely determined by eigenvalues and angles. For example, this subgraph in G_1 is isomorphic to $K_{1,3}$ for α type vertices, to $2K_2$ for β type vertices and again to $2K_2$ for γ type vertices while in G_2 it is isomorphic to P_4, $P_3 \cup K_1$, and $2K_2$ for α, β, γ type vertices respectively.

In addition, all α types vertice (in both G_1 and G_2) have neighbour types α, β, β, γ while all β type vertices have neighbour types α, α, β, γ. However, neighbour types of vertices 9 and 10 are α, α, α, α and β, β, β, β in G_1 while both vertices have the same neighbour types α, α, β, β in G_2. Hence, these graphs can be distinguished by angle types of neighbours of vertices (in particular, of vertices 9 and 10).

Let G be a graph on n vertices. Let λ be an eigenvalue of G of multiplicity k. Let vectors $(x_{i1}, x_{i2}, \ldots, x_{in})$ $(i = 1,2,\ldots,k)$ form an orthonormal basis of the eigenspace belonging to λ. Let j be a vertex of G of degree d and let $j_1, j_2, \ldots, j_d$ be its neighbours. Then

$$\lambda x_{ij} = \sum_{s=1}^{d} x_{ij_s} \quad (i = 1,2,\ldots,k). \tag{5}$$

Summing the squares of both sides of (5) we get

$$\lambda^2\alpha_j^2 = \sum_{s=1}^{d} \alpha_{j_s}^2 + 2\sum_{i=1}^{k} \sum_{\substack{s,t=1 \\ s<t}}^{d} x_{ij_s}x_{ij_t},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ is the eigenvalue angle sequence belonging to λ.

Clearly, the double sum does not depend on the choice of the basis of the eigenspace and it is an invariant of the vertex j. Let us denote it by B_j.

We have

$$\sum_{s=1}^{d} \alpha_{j_s}^2 = \lambda^2\alpha_j^2 - 2B_j. \tag{6}$$

Note that vertices 9 and 10 have different values B_9 and B_{10} for graphs G_1 and G_2 in Figure 5.

Now we change the notation introducing another subscript. Let α_{ij} be the angle corresponding to the eigenvalue μ_i and to the vertex j having neighbours as above. The quantities B_{ij} defined by

$$\sum_{s=1}^{d} \alpha_{ij_s}^2 = \mu_i^2 \alpha_{ij}^2 - 2B_{ij} \quad (i = 1,2,\ldots,m), \tag{7}$$

can be put into the matrix $\mathfrak{B} = \|B_{ij}\|_{m,n}$. The vertices of G can be ordered lexicographically first by angle sequences and second, by columns of $\mathfrak{B}$. In this case $\mathfrak{B}$ is also a graph invariant.

The graphs in Figure 5 show that $\mathfrak{B}$ is an invariant generally independent of eigenvalues and angles. These graphs can uniquely be reconstructed from the spectrum, angles and the matrix $\mathfrak{B}$.

In walk regular graphs the matrix $\mathfrak{B}$ does not help much since its entries B_{ij} can be calculated from eigenvalues and angles. If the degree is r and $\alpha_{ij} = a_i$ $(i = 1,2,\ldots,n)$, (7) is reduced to

$$B_{ij} = (\mu_i^2 - r)a_i^2/2, \ (j = 1,2,\ldots,n). \tag{8}$$

It can easily be shown that the sum

$$\sum_{i=1}^{m} B_{ij}\mu_i^k \qquad (k = 1,2,\ldots)$$

is equal to the number of walks of length k between different neighbours of vertex j.

An efficient reconstruction of cubic graphs from the spectrum and matrices $\mathfrak{A}$ and $\mathfrak{B}$ seems to be quite possible.

In addition to tools from Section 2, quantities B_{ij} would help very much in determining the neighbours of a particular vertex.

It seems reasonable to implement an <u>artificial intelligence</u> computer program (an expert system) which would construct all cubic graphs having a given spectrum and matrices $\mathfrak{A}$ and $\mathfrak{B}$. The reconstruction consists of many simple steps of a limited number of types. It would be difficult for a human being to take care of all little details appearing in due course while the computer, provided by a sufficiently intelligent program, could cope with these difficulties. After some experiments, perhaps a general theorem could be stated and proved.

References:

1. Babai, L., Grigoriev, D.Yu., and Mount, D.M.: Isomorphism graphs with bounded eigenvalue multiplicity. In: Proc. 14th annual ACM symposium on theory of computing, San Francisco 1982. pp. 310-324, New York 1982

2. Bell, F.K., and Rowlinson, P.: The change in the index of a graph resulting from the attachment of a pendant edge. Proc. Roy. Soc. Edinburgh Sect. A 108, 67-74 (1988)

3. Bussemaker, F.C., Cvetkovic, D.: There are exactly 13 connected, cubic, integral graphs. Univ. Beograd. Publ. Electrotehn. Fak. Ser. Mat. Fiz. No. 544-576, 43-48 (1976)

4. Cvetkovic, D.: Graph invariants based on the eigenvectors of the adjacency matrix. Presented at the Fifth Yugoslav Seminar on Graph Theory, Beograd 1984. Proc. Eighth Yugoslav Seminar on Graph Theory (to appaer)

5. Cvetkovic, D.: Constructing trees with given eigenvalues and angles. Linear Algebra Appl. 105, 1-8 (1988)

6. Cvetkovic, D., and Doob, M.: Developments in the theory of graph spectra. Linear and Multilinear Algebra, 18, 153-181 (1985)

7. Cvetkovic, D., and Rowlinson, P.: Spectra of unicyclic graphs. Graphs Combin. 3, 7-23 (1987)

8. Cvetkovic, D., and Rowlinson, P.: Seeking counter-examples to the reconstruction conjecture for graphs - a research note. Proc. Eighth Yugoslav Seminar on Graph Theory (to appear)

9. Cvetkovic, D., and Rowlinson, P.: Further properties of graph angles. Scientia (Valparaiso) (to appear)

10. Godsil, C.D.: Matching and walks in graphs. J. Graph Theory. 5, 85-297 (1981)

11. Godsil, C. and McKay, B.: Some computational results on the spectra of graphs. In: Casse, L.R.A., and Wallis, W.D. (Eds.): Combinatorial mathematics IV. Proc. 4th Austral. Conf., Adelaide 1975. Lecture Notes in Math. 560, 73-92, Berlin 1976

12. Godsil, C., Holton, D.A., and McKay, B.: The spectrum of a graph. In: Little, C.H.C. (Ed.): Combinational mathematics V. Proc. 5th Aust. Conf., Melbourne 1976. In: Lecture Notes in Math. 622, 91-117, Berlin 1977

13. Godsil, C.D., and McKay, B.D.: Feasibility conditions for the existence of walk-regular graphs. Linear Algebra Appl. 30, 51-61 (1980)

14. Gutman, I., and Cvetkovic, D.: The reconstruction problem for characteristic polynomials of graphs. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 498-541, 45-48 (1975)

15. Kelly, P.J.: A congruence theorem for trees. Pacific J. Math. 7, 961-968 (1957)

16. Rowlinson, P.: On angles and perturbation of graphs. Bull. London Math. Soc. 20, 193- 297 (1988)

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Author:

Prof. Dr. Dragoš M. Cvetkovic
University of Belgrade
Faculty of Electrical Engineering
POB 816
YU-11001 Belgrade
Yugoslavia

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Dietmar Cieslik

Knotengrade kürzester Bäume in endlich-dimensionalen Banachräumen

A_d bezeichne den d-dimensionalen affinen Raum. In ihm seien die Punkte mit den Ortsvektoren bez. des Ursprungs o des A_d identifiziert.

$B \subseteq A_d$ heißt Eichkörper (des A_d), sofern B eine kompakte, konvexe und bez. o zentralsymmetrische Menge ist. Dann definiert $\|x\| := \inf \{t \mid x/t \in B\}$ eine Norm des A_d und es gilt $B = \{x \in A_d \mid \|x\| \leq 1\}$.

$M_d(B)$ bezeichne den A_d mit dem Eichkörper B und der davon abgeleiteten Norm. Er stellt einen d-dimensionalen Banachraum dar, der i. allg. als Banach-Minkowski-Raum bezeichnet wird. In ihm ist die Strecke $\underline{xy} = \{tx+(1-t)y \mid 0 \leq t \leq 1\}$ stets eine Kürzeste.

Sei N eine endliche Menge in $M_d(B)$. Dann heißt ein in den $M_d(B)$ eingebetteter Baum $T = (V,E)$ Steiner-Minimal-Tree (SMT), falls $V \supseteq N$ und $l(T) := \sum_{vv' \in E} \|v-v'\| = \min$ gilt.

Für eine gegebene natürliche Zahl k heißt er k-SMT, falls $V \supseteq N$, $\operatorname{card} V \leq \operatorname{card} N + k$ und $l(T) = \min$ gilt. Für $k = 0$ erhält man in einem 0-SMT das sogenannte Minimalgerüst für N. Es bezeichne $\deg_T(v)$ den Grad eines Knotens v in einem Baum.

<u>Bemerkung 1:</u> Für jede endliche Menge N in $M_d(B)$ existiert ein SMT bzw. k-SMT $T = (V,E)$ für N, wobei gilt:

a) $\deg_T(v) \geq 1$ für alle v aus V,

b) $\deg_T(v) \geq 3$ für alle v aus V-N,

c) $\operatorname{card}(V-N) \leq k$ für k-SMT und
$\operatorname{card}(V-N) \leq \operatorname{card} N - 2$ für SMT.

Obere Schranken für Knotengrade kürzester Bäume findet man nun wie folgt:

$\{B_1,\ldots,B_l\}$ heißt Newton-Familie für den Eichkörper B des A_d, falls

$B_i = B + x_i$ für gewisses $x_i \in A_d$, $i = 1,\ldots,l$,

$\operatorname{int} B_i \cap \operatorname{int} B_j = \emptyset$ für $i,j = 1,\ldots,l$ und $i \neq j$,
$B_i \cap B \neq \emptyset$ für $i = 1,\ldots,l$,
$\operatorname{int} B_i \cap \operatorname{int} B = \emptyset$ für $i = 1,\ldots,l$ gilt.

Dann heißt

$$z_d(B) = \operatorname{Max}\{l \mid \{B_1,\ldots,B_l\} \text{ Newton-Familie für } B \subseteq A_d\}$$

Newton-Zahl für B in A_d.
Vermöge der Homogenität der Norm und ihrer Dreiecksungleichung beweist man leicht (vgl. [3]):

<u>Lemma:</u> Sei B Eichkörper des A_d. Dann gilt
$z_d(B) = \operatorname{Max}\{\operatorname{card} W \mid W \subseteq \operatorname{bd} B,\ \|w-w'\| \geq 1$ für w,w' aus W und $w \neq w'\}$.

Damit kann nun bewiesen werden:

<u>Satz 1:</u> Sei N eine endliche Menge in $M_d(B)$ und $T = (V,E)$ ein SMT oder k-SMT für N. Dann gilt $\deg_T(v) \leq z_d(B)$ für alle v aus V.

<u>Beweis:</u> Seien v ein Knoten in T und $v_1,\ldots,v_l$ seine Nachbarn. Da Normen translationsinvariant sind, kann $v = o$ angenommen werden. Wäre für $i,j \in \{1,\ldots,l\}$ $\|v_i-v_j\| < \|v-v_j\|$, so könnte man den Baum T durch Entfernung der Kante $\underline{vv_i}$ und Hinzunahme von $\underline{v_iv_j}$ im Widerspruch zur Minimalität von T verkürzen. Somit definieren wir folgende Bedingung, die für die Minimalität notwendig ist:

$\{v_1,\ldots,v_l\}$ heißt zulässig bez. $v = o$, sofern
$v_i \notin B(v_j) := \{x \in A_d \mid \|x-v_j\| < \|v_j\|\}$ für $i \neq j$ gilt.

Definieren wir w_i durch $w_i = \underline{ov_i} \cap \{x \mid \|x\| = m\}$ für $i = 1,\ldots,l$, mit $m = \operatorname{Min}\{\|v_j\| \mid j = 1,\ldots,l\}$. Dann ist auch $\{w_1,\ldots,w_l\}$ bez. o zulässig. Dies kann man wie folgt einsehen:
Sei $v_i = t_iw_i$ mit $t_i \geq 1$. Dann gilt

(i) $B(w_i) \subseteq B(v_i)$ für $i = 1,\ldots,l$,

denn für x aus $B(w_i)$ folgt $\|x-v_i\| = \|x-t_iw_i\| =$
$\|x-w_i+w_i-t_iw_i\| \leq \|x-w_i\| + (t_i-1)\|w_i\| < t_i\|w_i\| = \|v_i\|$.

(ii) Ist $\|v_i\| = \|v_j\|$ und $\{v_i,v_j\}$ zulässig bez. o, so ist dies auch $\{w_i,w_j\}$ für $\|w_i\| = \|w_j\|$,

denn die Normgleichheit zieht $t_i = t_j =: t$ nach sich, womit

weiter geschlossen werden kann: Wäre $w_j \in B(w_i)$, so wäre auch $t\|v_i-v_j\| < t\|v_i\|$, also $\|v_i-v_j\| < \|v_i\|$ im Widerspruch zur Zulässigkeit von $\{v_i, v_j\}$.
Indem wir (i) und (ii) geeignet nutzen, kann man bei v' mit $\|v'\| = \mathrm{Max}\ \{\|v_j\| \mid j = 1,\dots,l\}$ beginnend sukzessiv die w_i's wie oben angegeben bilden. Dabei bleibt in jedem Schritt die Zulässigkeit erhalten.

Aufgrund der Homogenität der Norm und (ii) genügt es nun, $\mathrm{Min}\ \{\|v_j\| \mid j = 1,\dots,l\} = 1$ anzunehmen, d.h., wir betrachten zulässige Mengen auf dem Rand bd B des Eichkörpers. Die maximale Mächtigkeit solcher Mengen ist eine obere Schranke für $\deg_T(v)$.
Der Beweis ist nun durch die Anwendung des Lemmas beendet. ■

<u>Satz 2:</u> Für eine gegebene natürliche Zahl k ist die Aussage von Satz 1 für k-SMT nicht zu verbessern.

<u>Beweis:</u> Sei zunächst $k = 0$.
Nach obigem Lemma existiert eine Menge $N_o = \{v_1,\dots,v_z\} \subseteq \mathrm{bd}\ B$ mit $\|v_i-v_j\| \geq 1$ für $i,j = 1,\dots,z$, $i \neq j$ und $z = z_d(B)$. Es ist leicht zu sehen, daß für $N = N_o \cup \{o\}$ der Baum $T = (N, \{\underline{ov_i} \mid i = 1,\dots,z\})$ ein 0-SMT ist. In ihm gilt $\deg_T(o) = z_d(B)$.

Sei nun $k > 0$.
Auf einer Geraden durch o seien Punkte $x_1=o, x_2,\dots,x_l$ in dieser Reihenfolge mit $\|x_i-x_{i+1}\| = 4$ $(i = 1,\dots,l-1)$ lokalisiert. Nun definieren wir, ausgehend von obiger Menge N_o, eine endliche Menge N wie folgt:

$$N = \bigcup_{i=1}^{l} (N_o+x_i) \cup \{x_1,\dots,x_l\}.$$

Offenbar gilt dann $\mathrm{card}\ N = (z+1)\cdot l$. Es ist leicht zu sehen, daß ein 0-SMT für N existiert, in dem die l Knoten $x_1,\dots,x_l$ den Grad $z_d(B)$ haben. Ist nun l nur hinreichend groß, d.h. also $\mathrm{card}\ N$ hinreichend groß, so kann nicht bei all diesen Knoten ein Grad kleiner als $z_d(B)$ auftreten. ■

<u>Bemerkung 2:</u> Aus dem ersten Teil des letzten Beweises erhält man eine weitere Charakterisierung der Newton-Zahlen:

$$z_d(B) = \max_{\substack{N \subseteq M_d(B) \\ \text{endliche Menge}}} \ \max_{\substack{T=(V,E) \\ \text{0-SMT für } N}} \ \max_{v \in V} \deg_T(v).$$

Bemerkung 3: Sei B ein Eichkörper des A_d. Dann gilt:

a) $z_d(B) \leq 3^d - 1$ (Hadwiger [4], die Schranke wird bei Parallelepipeden angenommen);

b) $z_d(B) \geq d^2 + d$ (Grünbaum [3], die Schranke wird bei Vektorkörpern von Simplexen angenommen);

c) $z_d(B) \leq 2(2^d - 1)$, sofern B streng konvex ist (Minkowski [1]).

d) Die Newton-Zahlen in euklidischen Räumen sind nur für die Dimensionen $d = 1,2,3,8$ und 24 genau bekannt (sie haben die Werte 2,6,12,240 und 196560). Für $d \leq 23$ existieren gute Abschätzungen (Seidel [5], Odlycko, Sloane, [6]).

Durch die bisher angeführten Sätze und Bemerkungen ist die Bestimmung oberer Grenzen für Knotengrade von k-SMT im wesentlichen abgeschlossen. Für SMT sieht dies etwas anders aus.

Bemerkung 4: Die naheliegende Vermutung, daß in SMT die maximalen Knotengrade stets echt kleiner sind als in den k-SMT, ist i. allg. nicht richtig, wie man in der Ebene erkennt, deren Eichkörper ein affin-reguläres Sechseck ist. (Man vgl. hierzu [2].)

Vermutung: Für jeden Knoten v eines SMT einer endlichen Menge in einem d-dimensionalen Banach-Minkowski-Raum gilt

$$\deg_T(v) \leq 2(2^d - 1).$$

Die Schranke wird genau in den Räumen angenommen, in denen der Eichkörper B sich wie folgt darstellen läßt:

$$B = \operatorname{conv}(I \cup -I) \text{ mit } I = \{(t_1,\ldots,t_d) \mid 0 \leq t_i \leq 1,\ i = 1,\ldots,d\}.$$

Diese Vermutung ist für $d = 1$ trivialerweise richtig und für $d = 2$ in [2] bewiesen.

Bemerkung 5: In Banach-Minkowski-Ebenen sind die maximalen Knotengrade in Abhängigkeit vom Eichkörper B wie folgt angebbar:

$$\text{max. Knotengrad in einem k-SMT} \leq \begin{cases} 8 & \text{für Parallelogramm } B, \\ 6 & \text{sonst,} \end{cases}$$

$$\text{max. Knotengrad in einem SMT} \leq \begin{cases} 6 & \text{für affin-reguläre Sechsecke,} \\ 5 & \text{sonst.} \end{cases}$$

(Man vgl. hierzu Satz 1 und [3] bzw. [2].)

Literatur:

1. Cassels, J.W.: An Introduction to the Geometry of Numbers. Berlin 1959

2. Cieslik, D.: Das Steinerproblem für Bäume minimaler Länge in der Banach-Minkowski-Ebene. Preprint-Reihe Math. 15, Ernst-Moritz-Arndt Univ. Greifswald 1986

3. Grünbaum, B.: On a conjecture of H. Hadwiger. Pacific J. Math. 11, 215-219 (1961)

4. Hadwiger, H.: Über Treffanzahl bei translationsgleichen Eichkörpern. Arch. Math. 8, 212-213 (1957)

5. Seidel, J.J.: Harmonics and Combinatorics. In: Askey, R.A., Koornwinder, T.H., and Schempp, W. (Eds.): Special functions: group theoretical aspects and applications. pp. 287-303, Dordrecht, Boston, Mass. 1984

6. Odlyzko, A., and Sloane, N.J.A.: New bounds on the number of unit spheres that can touch a unit sphere in n dimensions. J. Combin. Theory Ser. A, 26, 210-214 (1979)

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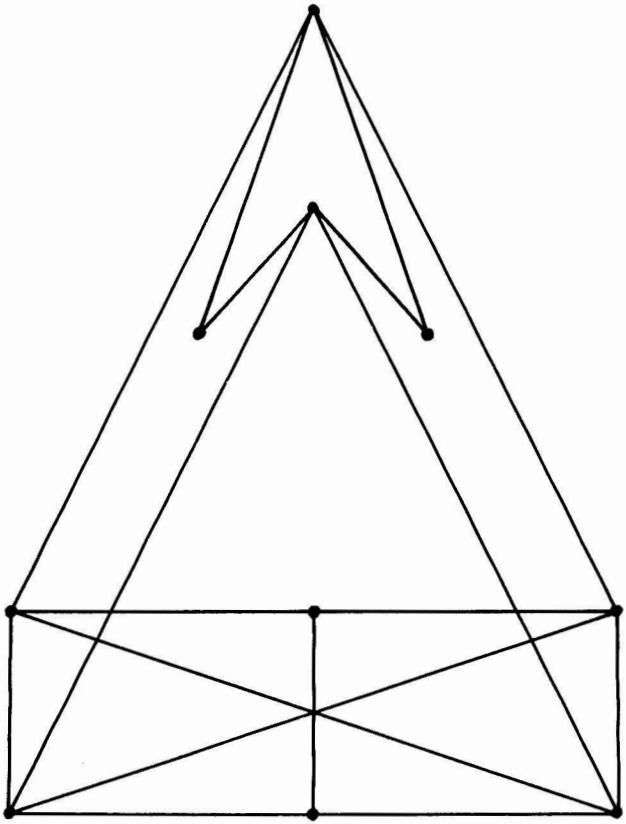
Verfasser:

Dr. Dietmar Cieslik
Ernst-Moritz-Arndt-Universität Greifswald
Sektion Mathematik
Jahnstraße 15a
Greifswald
DDR-2200



The W u s t r o w graph

(Created by Dr. Roger Labahn)

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