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Jiří Novák

Coverings, packings and representations

The present paper contains an example of covering of k-tuples by 2k-tuples, a method of representing of k-tuples by triples, further constructions of minimum packings of pairs with quadruples and of pairs with quintuples for some sizes of the basic set. Finally it deals with cyclic representations of triples by pairs.

1. Coverings

Definition 1.1: Let $n > g > k > 0$ be positive integers, let $E = \{1,2,...,n\}$ be an n-set. A covering of all k-subsets (k-tuples) of E by g-subsets (g-tuples) of E is a system $C(n,k,g)$ of g-tuples such that every k-tuple is contained in at least one g-tuple of the system. If the number of g-tuples is as minimal as possible at given parameters n, k, g, we have minimum coverings. If every k-tuple occurs exactly once in the system $C(n,k,g)$ then such a system is called tactical and denoted by $T(k,g,n)$.

Theorem 1.1 ([23]): The following estimates hold:

a) $|C(n,k,g)| \geq \binom{n}{k}/\binom{g}{k}$,

b) $|C(n,k,g)| \geq \lceil n/g \lceil (n-1)/(g-1) \lceil \ldots$
$\ldots \lceil (n-k+2)/(g-k+2) \lceil (n-k+1)/(g-k+1) \rceil \rceil \ldots \rceil \rceil \rceil$,

where $\lceil x \rceil$ denotes the least integer $\geq x$, while $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

Tactical coverings $T(k,g,n)$ are together minimum coverings $C(n,k,g)$. Moreover the existence of $T(k,g,n)$ implies a construction of minimum coverings $C(n-1,k,g)$, $C(n,k-1,g-1)$ ([23]). Minimum coverings $C(n,2,3)$ and $C(n,2,4)$ have been constructed in [5] and in [11, 12], respectively. Other papers dealing with coverings are [13] and [9].

Construction 1.1: Let n be an even integer. A method of covering of k-tuples by 2k-tuples is the following: Decompose the n-set E into two disjoint classes $E_1 = \{1, 2, \ldots, n/2\}$, $E_2 = \{n/2 + 1, n/2 + 2, \ldots, n\}$ and form systems $\binom{E_1}{k}$ and $\binom{E_2}{k}$, where $\binom{E_i}{k}$ denotes the system of all k-subsets of E_i. Let $\{i_1, i_2, \ldots, i_k\}$ be any k-tuple of the class E_1. Then the system of all 2k-tuples having the form $\{i_1, i_2, \ldots, i_k, n/2 + i_1, n/2 + i_2, \ldots, n/2 + i_k\}$ is a covering $C(n,k,2k)$.

Proof: If the k-tuple K to be covered belongs to $\binom{E_1}{k}$ or $\binom{E_2}{k}$ then it is covered according to the construction of 2k-tuples. Assume the k-tuple is formed by elements of E_1 and t elements of E_2, where $s + t = k$. Thus
$K = \{i_1, i_2, \ldots, i_s, n/2 + j_1, n/2 + j_2, \ldots, n/2 + j_t\}$. Consider the sequence $Q = \{i_1, i_2, \ldots i_s, j_1, j_2, \ldots, j_t\}$. If elements of Q are pairwise different then there is the 2k-tuplo
$\{i_1, \ldots, i_s, j_1, \ldots, j_t, n/2 + i_1, \ldots, n/2 + i_s, n/2 + j_1, \ldots, n/2 + j_t\}$ of $C(n,k,2k\}$ containing K.

If any element j_m, $m = 1, \ldots, t$ is equal to some element i_k, $k = 1, \ldots, s$, then replace j_m in Q in such a way that all elements in Q are pairwise different. We shall obtain a k-tuple $K' \in \binom{E_1}{k}$. By adding $n/2$ to the elements of K' we shall obtain a k-tuple $K'' \in \binom{E_2}{k}$. Then $K' \cup K''$ is a 2k-tuple containing the original k-tuple K, as we can easily see. Thus the k-tuple K is covered.

Example 1.1: Covering of triples by sixtuples, if $n = 10$:

$E = \{1, 2, \ldots, 10\}$, $E_1 = \{1, 2, 3, 4, 5\}$, $E_2 = \{6, 7, 8, 9, 10\}$;

$\binom{E_1}{3} = \{1\ 2\ 3,\ 1\ 2\ 4,\ 1\ 2\ 5,\ 1\ 3\ 4,\ 1\ 3\ 5,\ 1\ 4\ 5,\ 2\ 3\ 4,\ 2\ 3\ 5,$
$2\ 4\ 5,\ 3\ 4\ 5\},$

$\binom{E_2}{3} = \{6\ 7\ 8,\ 6\ 7\ 9,\ 6\ 7\ 10,\ 6\ 8\ 9,\ 6\ 8\ 10,\ 6\ 9\ 10,\ 7\ 8\ 9,$
$7\ 8\ 10,\ 7\ 9\ 10,\ 8\ 9\ 10\}$

The wanted covering is formed by the sixtuples arised from the corresponding triples in both systems, i.e., 1 2 3 6 7 8, 1 2 4 6 7 9, etc.

2. Representations

Definition 2.1: Let $n > k > s > 0$ be positive integers, let $E = \{1,2,\dots,n\}$ be an n-set. A representation of k-subsets by s-subsets is a system $R(n,k,s)$ of s-subsets of E such that any k-subset of E contains at least one s-subset of E contained in the system $R(n,k,s)$.

A representation is said to be minimum if it contains a minimal number of s-subsets for given parameters n,k,s.

P. Turán gave a construction of $R(n,k,s)$ if $\lfloor (k-1)/(s-1) \rfloor > 1$. The set E is decomposed into $r = \lfloor (k-1)/(s-1) \rfloor$ classes $E_1,\dots,E_r$ such that their sizes differ at most by one. Then the systems $\binom{E_i}{s}$, $i = 1,\dots,r$, together form a representation $R(n,k,s)$. Such representations are called Turán's representations. P. Turán proved in [24], that this construction leads to a minimum representation $R(n,k,s)$ for $s = 2$.
If $s > 2$ then this construction need not lead to minimum representations what the case of $R(2k-1,k,3)$ shows, where k is an odd number ≥ 5 ([20]). In this case there exist representations having less triples than Turán's representations.
For $k = 4$ and $s = 3$ there is a well known construction connected with a famous Turán's conjecture about a minimum $R(n,4,3)$. We generalize this construction for the case $s = 3$ and arbitrary k.

Construction 2.1: Let $E = \bigcup_{i=1}^{k-1} E_i$, $E_i \cap E_j = \emptyset$, $||E_i| - |E_j|| \leq 1$. Then $R(n,k,3) = A \cup B$, where $A = \bigcup_{i=1}^{k-1} \binom{E_i}{3}$, $B = \bigcup_{i=1}^{k-1} \binom{E_i}{2} E_{i+1}$, $(E_k = E_1)$.

The notation $\binom{E_i}{2} E_{i+1}$ means the set of all triples having two elements from E_i and one element from E_{i+1}.

Example 2.1: Construct $R(15,6,3)$ by the described method:

$E = \{1,2,\dots,15\}$, $E_1 = \{1,2,3\}$, $E_2 = \{4,5,6\}$, ...,
..., $E_5 = \{13,14,15\}$;
$A = \{1\,2\,3,\ 4\,5\,6,\ \dots,\ 13\,14\,15\}$,
$\binom{E_1}{2} E_2 = \{1\,2\,4,\ 1\,3\,4,\ 2\,3\,4,\ 1\,2\,5,\ 1\,3\,5,\ 2\,3\,5,\ 1\,2\,6,\ 1\,3\,6,\ 2\,3\,6\}$,

$B = \binom{E_1}{2}E_2 \cup \binom{E_2}{2}E_3 \cup \binom{E_3}{2}E_4 \cup \binom{E_4}{2}E_5 \cup \binom{E_5}{2}E_1$,

$R(15,6,3) = A \cup B$.

Proof of the Construction 2.1: Any k-subset $K \subset E$ meets some class E_i either in at least three elements or in at most two elements. The first possibility implies that we must include all systems $\binom{E_i}{3}$, $i = 1,\dots,k-1$, into the representation $R(n,k,3)$. If the other possibility arises, then we shall show that two elements of K will belong to some class E_i and one element of K will belong to the class E_{i+1}. Therefore we must include the triples $\binom{E_i}{2}E_{i+1}$ into the representation. Let j denote the number of such classes E_i that each of them contains two elements of K. Evidently the relation $1 \leq j \leq \lfloor k/2 \rfloor$ holds. Determine the number of classes containing no element of K. The number of classes containing exactly one element of K is equal to $k-2j$. Thus the number of classes containing no element of K is equal to $k-1-(j+k-2j) = j-1$.

Assume that any class containing two elements of K is followed by a class E_{i+1} containing no element of K. Thus we have j classes containing no element of K. This contradicts the previous result. Therefore there is at least one class E_i with two elements of K followed by a class E_{i+1} containing one element of K at least. This motivates our construction.

Theorem 2.1: Denote $f(n,k,s)$ the number of s-tuples in a minimum representation $R(n,k,s)$. Then the following relation holds:

$$f(n,k,s) \geq (n/(n-s))\cdot f(n-1,k,s).$$

The proof is given in [10].

Remark: We can easily see that for $q \doteq |R(n,k,s)|$ the estimate $q \geq \binom{n}{k}/\binom{n-s}{k-s}$ holds.

Definition 2.2: Let π be the cyclic permutation of the set E, i.e. $\pi(i) = i+1$, $i = 1,2,\dots,n$, with numbers taken modulo n. A representation $R(n,k,s)$ will be called cyclic (notation $R_c(n,k,s)$), if it admits the automorphism π.

Theorem 2.2: Turán's representations are cyclic iff n is divisible by $r = \lfloor (k-1)/(s-1) \rfloor$.

Proof: If r divides n then $|E_i| = |E_j|$, $i,j = 1,2,\ldots,r$. We put into the class E_i all numbers $e \equiv i (\bmod r)$, $i = 1,2,\ldots,r$, $e \in E$. The permutation π transforms any s-subset of E_i into an s-subset of E_{i+1}, where $E_{r+1} = E_1$.

If r does not divide n then there exist two classes E_i, E_j with different sizes. This implies that the multiplicity of any element of E_i in $\binom{E_i}{s}$ differs from the multiplicity of any element of E_j in $\binom{E_j}{s}$.

This result contradicts the obvious fact that in any cyclic representation all elements of E have the same multiplicity.

From Theorem 2.2 it follows, if $k = 3$, Turán's representations $R(n,3,2)$ are cyclic iff n is even. The construction of minimum $R_c(n,3,2)$, if n is an odd number, remains an open problem. This problem can be transformed into another which we shall describe.

Problem 2.1: Let $n = 2m + 1$. Construct a system T of triples $\{x,y,z\}$ from the elements $1, 2, \ldots, m$ in the following manner:

$\{x,y,z\} \in T$ iff these relations hold:

a) if $x + y \leq m$, then $z = x + y$,

b) if $x + y > m$, then $z = n - (x + y)$.

It may be $x = y$ or $x = y = z$.

A minimum representation M of the triples of T by elements of $E' = \{1,2,\ldots,m\}$ is to be constructed, or equivalently, a maximum independent set I of elements for the system T is to be found.

(An independent set I of elements contains no triple of the system T.)

Example 2.2: Let $n = 15$, i.e. $m = 7$, $E' = \{1,2,\ldots,7\}$;

$T = \{1\ 1\ 2,\ 1\ 2\ 3,\ 1\ 3\ 4,\ 1\ 4\ 5,\ 1\ 5\ 6,\ 1\ 6\ 7,\ 1\ 7\ 7,\ 2\ 2\ 4,$
$2\ 3\ 5,\ 2\ 4\ 6,\ 2\ 5\ 7,\ 2\ 6\ 7,\ 3\ 3\ 6,\ 3\ 4\ 7,\ 3\ 5\ 7,\ 3\ 6\ 6,$
$4\ 4\ 7,\ 4\ 5\ 6,\ 5\ 5\ 5\}$,

$M = \{1,4,5,6\}$, $I = E' - M = \{2,3,7\}$.

We see that the triple $2\ 3\ 7$ does not exist in T, therefore the set I is independent.

Our problem has been settled for $n < 30$. If $n \neq 15, 25$ then $|M| = \lfloor n/3 \rfloor$, otherwise $|M| = \lfloor n/3 \rfloor - 1$.

3. Packings of p-tuples with k-tuples

Definition 3.1: Let $n > k > p > 0$ be positive integers, let $E = \{1,2,\dots,n\}$ be an n-set. A packing of p-tuples with k-tuples is a system $P(n,k,p)$ of k-subsets (blocks) $K \subset E$ such that a) every p-subset of E is contained in at most one k-subset of the system $P(n,k,p)$, b) no further k-subset can be added to the system along with a), i.e. the system P is maximal.
If the number of k-subsets is maximum (minimum) for the given parameters n, k, p, we have maximum (minimum) packings.

Obviously tactical systems $T(k,g,n)$ defined in section 1 are together maximum packings $P(n,g,k)$, alternatively known as exact packings.

Theorem 3.1: Denote $q = |P(n,k,p)|$. Then the following estimates hold:

a) $q \leqslant \lfloor (n-d)/(k-d) \rfloor \binom{n}{d} / \binom{k}{d}$, where $d = p-1$ ([14]),

b) $q \leqslant \lfloor \frac{n}{k} \lfloor \frac{n-1}{k-1} \lfloor \dots \lfloor \frac{n-p+2}{k-p+2} \lfloor \frac{n-p+1}{k-p+1} \rfloor \rfloor \dots \rfloor \rfloor \rfloor$ ([22]),

c) $q \geqslant \lceil \binom{n}{k} / \sum_{i=o}^{k-p} \binom{k}{p+i} \binom{n-k}{k-p-i} \rceil$.

Proof: Let $K = \{b_1,\dots,b_k\} \in P(n,k,p)$. The packing $P(n,k,p)$ cannot contain such k-subsets of E different from K which comprise p elements of K at least. Thus we must omit from $\binom{E}{k}$ some k-subsets whose number (including the k-subset K) is equal to $\sum_{i=o}^{k-p} \binom{k}{p+i} \binom{n-k}{k-p-i}$.
Therefore the number of blocks in $P(n,k,p)$ cannot be less than the right side of relation b).

Theorem 3.2: If we omit all blocks with the element n from a tactical system $T(p,k,n)$, then the remaining blocks form a maximum packing $P(n-1,k,p)$ ([22]).

Theorem 3.3: Necessary and sufficient conditions for exact packings $P(n,3,2)$, $P(n,4,3)$, $P(n,4,2)$, $P(n,5,2)$ to exist are $n \equiv 1$ or $3 \pmod 6$, $n \equiv 2$ or $4 \pmod 6$, $n \equiv 1$ or $4 \pmod{12}$, $n \equiv 1$ or $5 \pmod{20}$, respectively ([21, 8, 7]).

Maximum packings $P(n,3,2)$ and $P(n,4,2)$ for any n have been constructed in [22] and [2], respectively.
Maximum packings $P(n,4,3)$ for $n \equiv 4$ or $3 \pmod 6$ follow from Theorems 3.2 and 3.3, the maximum packing $P(n,4,3)$ for $n \equiv 0 \pmod 6$ has been constructed in [6]. Some constructions of maximum packings $P(n,4,2)$ have been found independently in [1].

The problem of construction of minimum packings $P(n,k,p)$ was investigated only in a few cases. A construction of minimum $P(n,3,2)$ has been performed in [19] using Turán's Theorem [24] and the construction of maximum packings $P(n,3,2)$. The construction of minimum packings $P(n,k,p)$ is closely connected with Turán's hypergraph problem.
Any packing $P(n,k,2)$ has the property that for every $K \in \binom{E}{k}$ there is a block B in $P(n,k,2)$ such that $K \cap B \geq 2$. Therefore the block B of $P(n,k,2)$ comprise pairs representing all k-tuples of E. The number of pairs in blocks of $P(n,k,2)$ must be at least equal to the number of pairs in a minimum representation $R(n,k,2)$. Thus both following theorems hold:

Theorem 3.4: $|P(n,k,2)| \geq |\text{minimum } R(n,k,2)|/\binom{k}{2}$.

Theorem 3.5: Suppose $k > 2$ and n divisible by $k-1$. Decompose the set E into $k-1$ classes E_i, $|E_i| = n_1 = n/(k-1)$, $i = 1,2,\ldots,k-1$.
Assume the existence of a system $T(2,k,n_1)$ defined in section 1. Then there exists a minimum packing $P(n,k,2)$ formed by the systems $T(2,k,n_1)$ on the classes E_i, $i = 1,2,\ldots,k-1$ ([18]).

Using Theorems 3.4, 3.5 and methods contained in [19] the following results can be obtained. Detailed proofs will be given elsewhere.

Theorem 3.6: Let $n = 36t + 6$. Decompose the set E into three disjoint classes E_1, E_2, E_3, where $|E_1| = |E_2| = 12t + 1$, $E_3 = 12t + 4$, then the minimum packing $P(36t + 6,4,2)$ is formed by three systems $T(2,4,n_i)$ on the sets E_i, $i = 1,2,3$, where $n_1 = n_2 = 12t + 1$, $n_3 = 12t + 4$.

Remark: An analogous theorem holds also for $n = 36t + 9$.

Theorem 3.7: Let $n = 80t + 8$. Decompose the set E into disjoint classes E_i, $i = 1,\ldots,4$, where $n_1 = |E_1| = |E_2| = |E_3| = 20t + 1$, $n_4 = |E_4| = 20t + 5$. Then a minimum packing $P(80t + 8)$ is formed by three systems $T(2,5,n_1)$ on the classes E_i, $i = 1,2,3$, and by one system $T(2,5,n_4)$ on the class E_4.

Remark: Analogous theorems hold for $n = 80t + 12$ and for $n = 80t + 16$.

Remark: The references [3, 4, 15, 16, 17] not introduced in the paper concern the section 3.

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Hansjoachim Walther

Imbedding of graphs into a special family of infinite graphs

All graphs considered in this paper will be undirected and without loops or multiple edges. The set of vertices of a graph G will be denoted by $V(G)$ and the set of edges by $E(G)$.

For an arbitrary subset ϑ of the set N of all positive integers we define a graph $G = G(\vartheta)$ in the following way. $V(G) = \Gamma$ will be the set of all integers and an edge $e = (i,j)$ with $i,j \in \Gamma$ belongs to $E(G)$ if there is a $d \in \vartheta$ with $|i-j| = d$. Obviously, the infinite graph $G(\vartheta)$ is regular of degree $|\vartheta| \cdot 2$ for a finite set ϑ.

We call U an <u>induced subgraph</u> of a graph G if $V(U) \subseteq V(G)$ and if $(X,Y) \in E(U)$ holds for $X,Y \in V(U)$ and $(X,Y) \in E(G)$.

Two graphs H and U are said to be <u>isomorphic</u> if there is a bijection φ: $V(H) \longrightarrow V(U)$,
$E(H) \longrightarrow E(U)$
such that $\varphi(X,Y) = (\varphi(X),\varphi(Y))$.

<u>Theorem 1:</u> Let H be any finite graph. Then there exists a set $\vartheta = \{d_1, d_2, \ldots, d_r\} \subseteq N$ such that $G(\vartheta)$ contains an induced subgraph U isomorphic to H.

<u>Proof:</u> Let $V(H) = \{X_1, X_2, \ldots, X_n\}$. We choose $\vartheta = \{(2^j - 2^i):$ $1 \leq i < j \leq n \wedge (X_i, X_j) \in E(H)\}$ and look for the subgraph U of $G(\vartheta)$ spanned by $V(U) = \{2^j: 1 \leq j \leq n\}$. We have
$E(U) = \{(2^i, 2^j): 1 \leq i < j \leq n \wedge 2^j - 2^i \in \vartheta\}$ and that means that $E(U) = \{(2^i, 2^j): 1 \leq i < j \leq n \wedge (X_i, X_j) \in E(H)\}$.

Since the $\binom{n}{2}$ differences $2^j - 2^i$ are pairwise distinct we have found an isomorphism as asserted in Theorem 1.

A surjection f: $V(G) \longrightarrow \{f_1, f_2, \ldots, f_k\}$ is called to be a <u>feasible colouration</u> of G with k colours if $f(X) \neq f(Y)$ for any edge $(X,Y) \in E(G)$. The <u>chromatic number</u> $\chi(G)$ of G is the smallest number k such that there exists a feasible colouration of G with k colours.

In the following, we will search for the chromatic number $\chi(\vartheta) = \chi(G(\vartheta))$ for ϑ being a set of positive integers.

Theorem 2: For $|\vartheta| < \infty$ it holds $\chi(\vartheta) \leq |\vartheta| + 1$.

Proof: Let $\vartheta = \{d_1, d_2, \ldots, d_r\}$ with $d_1 < d_2 < \ldots < d_r$. We will colour the vertices of $G(\vartheta)$ inductively with $k \leq |\vartheta| + 1 = r + 1$ colours feasibly.
At first, we colour the non-negative integers:

$$f(0) := f(1) := \ldots := f(d_1-1) := f_1.$$

Provided that the vertices of $L = \{0,1,2,\ldots,l-1\}$ with $l \geq d_1$ are already coloured feasibly by $\leq r + 1$ colours $f_1, f_2, \ldots, f_k$. For colouring the vertex l we take in consideration that l has at most r neighbours $i_1, i_2, \ldots, i_s$ in L. That means that there is at least one colour $f_t \in \{f_1, f_2, \ldots, f_{r+1}\}$ with $f(i_j) \neq f_t$, $j = 1,2,\ldots,s$. If we put $f(l) := f_t$ we obtain a feasible colouration of $L \cup \{l\}$ with at most $r + 1$ colours.
For colouring the negative integers we can proceed in an analogous way. This completes the proof of Theorem 2.

Let ϑ_e be the set of all even integers of ϑ and ϑ_o be the set of all odd integers of ϑ, respectively.

Theorem 3: For any $\vartheta \subseteq N$ it holds $\chi(\vartheta) \leq 2(|\vartheta_e| + 1)$ provided that ϑ_e is finite.

Proof: Because of Theorem 2 we have $\chi(\vartheta_e) \leq |\vartheta_e| + 1$. Let f be a feasible colouration of $G(\vartheta_e)$ with $k \leq |\vartheta_e| + 1$ colours $f_1, f_2, \ldots, f_k$. For finding a feasible colouration f' of $G(\vartheta)$ we add k further colours $g_1, g_2, \ldots, g_k$ and colour the vertices of $G(\vartheta)$ in the following way:

If $Y \in V(G)$ is an even integer and $f(Y) = f_i$ then $f'(Y) = f_i$.
If $Y \in V(G)$ is an odd integer and $f(Y) = f_i$ then $f'(Y) = g_i$.

As any $d \in \vartheta_o$ induces edges only between an odd and an even vertex of $G(\vartheta)$ we have found a feasible colouration f' of

$G(\vartheta)$ with at most $2(|\vartheta_e| + 1)$ colours and Theorem 3 is proved.

The upper bound $2(|\vartheta_e| + 1)$ for the chromatic number $\chi(\vartheta)$ proves to be sharp. For this, let us consider the set $\vartheta = \{1,2,\dots,2m+1\}$. We have $|\vartheta_e| = m$. As $G(\vartheta)$ contains a complete graph K_{2m+2}, for example the graph induced by the $2m+2$ vertices $0,1,2,\dots,2m+1$, we have $\chi(\vartheta) \geqslant 2m+2$, implying

$$2m+2 \leqslant \chi(\vartheta) \leqslant 2(|\vartheta_e| + 1) = 2(m+1) = 2m+2, \text{ that means}$$

$$\chi(\vartheta) = 2(|\vartheta_e| + 1).$$

A graph H is said to be <u>bipartite</u> if there is a partition $V(H) = V_1 \cup V_2$ $(V_1 \cap V_2 = \emptyset)$ of the vertex set $V(H)$ such that for any edge $e \in E(H)$ the two vertices incident to e belong to different parts. For a given finite graph H we define the <u>unbipartiteness</u> $\gamma(H)$ as the smallest integer k such that there is a set $\vartheta \subseteq N$ with $|\vartheta_e| = k$ and $G(\vartheta)$ contains an induced subgraph U isomorphic to H.

<u>Theorem 4:</u> Let H be a finite graph. Then H is bipartite if and only if $\gamma(H) = 0$.

<u>Proof:</u>
1. Let H be bipartite. We have to find a $\vartheta \subseteq N$ without even integers such that $G(\vartheta)$ has an induced subgraph isomorphic to H. Let be $V(H) = V_1 \cup V_2$ with $V_1 = \{X_1,X_2,\dots,X_r\}$ and $V_2 = \{Z_1,Z_2,\dots,Z_s\}$ the partition of $V(H)$.

Let $Y_i = 2i,\ i = 1,2,\dots,r,$
$W_j = 2jr + 1,\ j = 1,2,\dots,s,$
$\vartheta := \{d\colon d = W_j - Y_i \wedge (X_i,Z_j) \in E(H)\}.$

Obviously, we have $\vartheta_e = \emptyset$.
Analogously to the proof of Theorem 1 one can prove that H is isomorphic to the subgraph U induced by the vertex set $\{Y_i\colon i = 1,2,\dots,r\} \cup \{W_j\colon j = 1,2,\dots,s\}$.

2. Let H be a graph with $\gamma(H) = 0$. That means, there is a $\vartheta \subseteq N$ with $\vartheta_e = \emptyset$ and H is isomorphic to the suitably choosen induced subgraph U of $G(\vartheta)$. It suffices to show that $G(\vartheta)$ is bipartite.
Supposing $G(\vartheta)$ not to be bipartite. Then there exists a cycle

$C = (Y_1, Y_2, \ldots, Y_{2m}, Y_{2m+1}, Y_{2m+2} = Y_1)$ in $G(\vartheta)$ with an odd number $2m+1$ of edges (Y_i, Y_{i+1}), $i = 1,2,\ldots,2m+1$. Because of the construction of $G = G(\vartheta)$ we have the property
$(Y_2-Y_1) + (Y_3-Y_2) + \ldots + (Y_{2m+2}-Y_{2m+1}) = 0$.
As $\vartheta_e = \emptyset$ for any edge $(X,Y) \in E(G)$ we have $|Y-X| \equiv 1(\mathrm{mod}\ 2)$. That implies that the sum of an odd number of odd integers equals zero, but that is a contradiction, and Theorem 4 is thus proved.

<u>Theorem 5:</u> For the complete graph K_n of n vertices it holds

$$\frac{n-2}{2} \leq \gamma(K_n) \leq \frac{n-1}{2} .$$

The assertion of Theorem 5 is equivalent to

$$\gamma(K_{2m}) = m-1 \quad \text{and} \quad \gamma(K_{2m+1}) = m.$$

<u>Proof:</u> For $\vartheta = \{1,2,\ldots,n-1\}$ we have

$$|\vartheta_e| = \begin{cases} m-1 & \text{if } n = 2m, \\ m & \text{if } n = 2m+1, \end{cases}$$

and $G(\vartheta)$ contains a subgraph isomorphic to K_n.
Therefore, we have

$$\gamma(K_{2m}) \leq m-1 \quad \text{and} \quad \gamma(K_{2m+1}) \leq m.$$

Let $n = 2m$ and suppose that $\gamma(K_{2m}) < m-1$.
That means, there is a set $\vartheta \subseteq N$ with $|\vartheta_e| < m-1$ and $G(\vartheta)$ contains an induced subgraph isomorphic to K_{2m}.
That implies that $\chi(\vartheta) \geq n = 2m$ and together with Theorem 3 we get the contradiction

$$2m = n \leq \chi(\vartheta) \leq 2(|\vartheta_e| + 1) < 2(m-1+1) = 2m.$$

In an analogous way one can deal with the case $n = 2m+1$. This completes the proof of the Theorem 5.

The motivation for making these investigatins was given in [1]. You can find some more results in this field in [2].

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Heinz-Jürgen Voss

Groups and triangulations of oriented surfaces

Many authors dealt with groups and graphs. Here we shall assign to a group triangulations of some oriented (closed) surfaces in the sense of H. Sachs [2]. This assignment was intensively studied by W. Voss [4].

The present paper presents a survey on old and new results. An extended version with all the proofs is in preparation [5].

1. Concepts

Let $\mathfrak{G}$ be a finite group. A triple (x,y,z) of elements $x,y,z \in \mathfrak{G}$ is said to be regular if $x \neq y \neq z \neq x$ and $xyz = e$. With $xyz = e$ also $yzx = e$ and $zxy = e$. If (x,y,z) is a regular triple then (x,y,z') with $z' = (yx)^{-1}$ is regular, too.
To each regular triple (x,y,z) an oriented triangle $D(x,y,z)$ with arcs (x,y), (y,z), (z,x) and vertices x, y, z is assigned (see Fig. 1) so that $D(x,y,z) \equiv D(x',y',z')$ iff $(x',y',z') \in \{ (x,y,z), (y,z,x), (z,x,y) \}$; otherwise $D(x,y,z)$, $D(x',y',z')$ are disjoint. In the set Σ of all so obtained triangles an arc occurs at most once. If (x,y) is in some triangle of Σ then (y,x) is in a triangle of Σ, too.

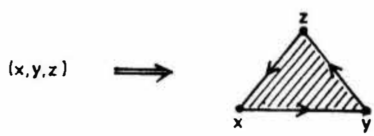

Fig. 1

In the sense of combinatorial topology opposite directed arcs (x,y), (y,x) are identified so that an edge $[x,y]$ is obtained (see Fig. 2). Identifying step by step all opposite directed arcs, triangulations of oriented surfaces are obtained.
Thus to each group $\mathfrak{G}$ a set of topological invariants of these triangulations is assigned.

We ask: Let P be a property of these invariants. Which property of the groups corresponds to the property P of these invariants and vice versa ?

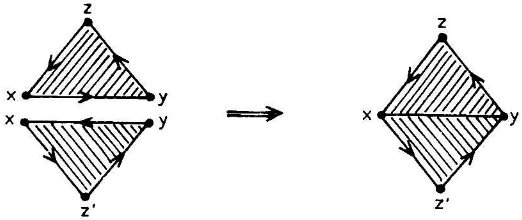

Fig. 2

If the elements x, y of the regular triple (x,y,z) commute then $xyz = yxz = e$ and the triangulation T is formed by the two triangles $D(x,y,z)$ and $D(y,x,z)$ only. Hence T is the sphere with a 3-cycle on it which triangulates that sphere.
Consequently, in the following we shall only consider <u>non-commutative</u> elements of $\mathfrak{G}$. Really, two non-commutative elements x, y form with $(xy)^{-1}$ a regular triple. Thus to each pair of non-commutative elements $x,y \in \mathfrak{G}$ a so called group-triangulation $T(x,y)$ corresponds, where to each vertex a group element is assigned.
The octahedron $T($ (12)(34), (1234) $)$ with (12)(34) $\cdot$ (1234) $\neq$ (1234) $\cdot$ (12)(34) is presented as an example (see Fig. 3).

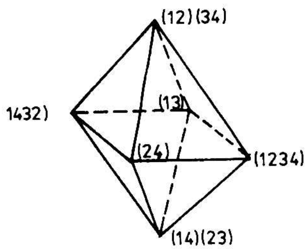

Fig. 3

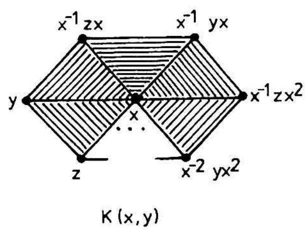

Fig. 4

In $T(x,y)$ any two different triangles are either disjoint or have precisely one vertex in common or have precisely one edge and its endvertices in common. Such triangulations are said to be simple.

2. Symmetries of group-triangulations

The basic disc $K(x,y)$ of $T(x,y)$ with centre x through y conists of the edge $[x,y]$ and all triangles of $T(x,y)$ incident with this vertex x according to Fig. 4 (note that an element of $\mathcal{G}$ can be on $T(x,y)$ more than once).
The inner automorphism f_x of $\mathcal{G}$ with $f_x(g) = x^{-1}gx$ for each $g \in \mathcal{G}$ induces a face preserving automorphism on $T(x,y)$ so that $K(x,y)$ is clockwise rotated over two sectors, i.e. $x \to x$, $y \to x^{-1}yx$, $z \to x^{-1}zx$, $x^{-1}yx \to x^{-2}yx^2$, etc.
Consequently, the triangulation $T(x,y)$ has many symmetries.

3. Further properties of group-triangulations

The clockwise rotation of the basic disc $K(x,y)$ over two sectors maps the vertex z onto the vertex $z' = x^{-1}zx$ (see Fig. 2). Hence z and z' have the same valency. The high symmetry properties of $T(x,y)$ imply that each triangle D can be mapped onto $D(x,y,z)$ or $D(y,x,z')$ by a series of clockwise rotations over two sectors of some basic discs ($D(x,y,z) \cup D(y,x,z')$ is in Fig. 2). Since this results in an automorphism of $T(x,y)$ the triangles D and $D(x,y,z)$ have the same valencies.
Consequently, on $T(x,y)$ all vertices have one of the three valencies v_1, v_2, v_3 (two or all of these vertices can be equal). Thus to each triangulation $T(x,y)$ a valency triple (v_1, v_2, v_3) is assigned.
Using Euler's formula, W. Voss [4] has proved:

<u>Theorem 3.1:</u> Let T be a group-triangulation of an orientable surface of genus P and (v_1, v_2, v_3) the triple of valencies of the vertices of T. Then

$$\left(\frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} - \frac{1}{2}\right) f = 2 - 2P,$$

where f denotes the number of triangles of T. ■

This formula and further arguments imply:

Theorem 3.2: Let T, P and (v_1, v_2, v_3) be as in Theorem 3.1. If $P = 0$ then all possible triples for (v_1, v_2, v_3) are

(3,3,3), (5,5,5), (3,6,6), (3,8,8), (3,10,10),
(4,6,6), (5,6,6), (4,6,8), (4,6,10).

If $P = 1$ then all possible triples for (v_1, v_2, v_3) are

(4,8,8), (4,6,12), (3,12,12), (6,6,6). ■

If $P = 0$ then to each triple (v_1, v_2, v_3) of Theorem 3.2 a group-triangulation of the sphere exists (see [5]).
If $P = 1$ then to each of the four possible triples an infinite series of pairwise nonisomorphic group-triangulations of the torus exists (see [5] and [3]).

4. Simple triangulations of high symmetry

In simple triangulations T the triangles incident with one vertex X form a basic disc $K(X)$. By d_X a clockwise rotation of $K(X)$ over two sectors is denoted. If d_X can be extended to an automorphism a_X of the whole triangulation T then we say that "a_X exists". It holds ([5]):

Theorem 4.1: If for an edge $[X,Y]$ of a simple triangulation T the automorphisms a_X and a_Y exist then for each vertex V of T the automorphism a_V exists. ■

5. Simple triangulations which are group-triangulations

In 2 we stated that the group-triangulations have the following property

(P): For each vertex V the automorphism a_V exists.

Next we ask: which simple triangulations T with (P) are group-triangulations, i.e. when to each vertex of T can be assigned an element of some group $\mathfrak{G}$ so that T with this assignment is a group-triangulation of $\mathfrak{G}$?
Let the triple $(T, \mathfrak{A}(T), \tau)$ denote a simple triangulation T with (P), the automorphism group $\mathfrak{A}(T)$ of T and the mapping τ of the vertex set of T into $\mathfrak{A}(T)$ which assigns to each vertex V of the automorphism $\tau(V) := a_V$ (a_V is defined in 4).
The triple $(T, \mathfrak{A}(T), \tau)$ is a group-triangulation if it satisfies some additional condition.

$D(X,Y,Z)$ denotes a triangle of T with vertices X, Y, Z, where X, Y, Z are met in this cyclic order by an anti-clockwise walk on its boundary.

Theorem 5.1: Let T be a simple triangulation of an orientable surface with (P). Then

(i) for each triangle $D(X,Y,Z)$ of T the triple (a_X, a_Y, a_Z) of automorphisms is regular.

(ii) If there is no pair of edges $[P,Q]$, $[P',Q']$ such that $a_P = a_{P'}$ and $a_Q = a_{Q'}$ then $(T, \mathfrak{A}(T), \tau)$ is a group-triangulation. ■

In [3] I investigated extensions of Theorem 5.1 (i) to 2-cell-decompositions of orientable surfaces, where each cell is bounded by a k-cycle.

6. Triangulations with group-assignments

We shall extend our investigations to triangulations with group-assignments possibly having no high symmetries. The symbol $(T, \mathfrak{G}, \varphi)$ means a simple triangulation T of an orientable surface, a group $\mathfrak{G}$ and a mapping φ of the vertex set of T into the group $\mathfrak{G}$. We only consider $(T, \mathfrak{G}, \varphi)$ with the additional property that for each triangle $D(X,Y,Z)$ of T the triple $(\varphi(X), \varphi(Y), \varphi(Z))$ is regular.
$(T, \mathfrak{G}, \varphi)$ and $(T', \mathfrak{G}, \varphi')$ are said to be isomorphic iff there exist 1-1-mappings of the vertices, edges and faces of T' onto the vertices, edges and faces of T, respectively, such that for each vertex X' of T' holds: X' and its image X in T are labelled by the same group element, i.e. $\varphi'(X') = \varphi(X)$.
Let Q be a 2-cell-decomposition of the sphere and L one of its faces. Let all faces different from L be triangles. Deleting the face L we obtain a polygon-triangulation P from Q, where the polygon $\bar{P}$ of P is the bounding cycle of L. Thus P is the triangulation of a bounded orientable surface, namely, of a disc with boundary $\bar{P}$.
The symbol $(P, \mathfrak{G}, \varphi)$ is defined similar as $(T, \mathfrak{G}, \varphi)$. Let P_1, P_2, ..., P_m be m pairwise disjoint copies of $(P, \mathfrak{G}, \varphi_P)$. Let α, β, γ be incidence-preserving mappings of the vertices, edges and triangles of $P_1 \cup P_2 \cup \ldots \cup P_m$ onto the vertices,

edges and triangles of $(T, \mathfrak{G}, \varphi_T)$, respectively, such that γ is a 1-1-mapping and for each vertex X of $P_1 \cup \ldots \cup P_m$ it holds: X and its image $\alpha(X)$ in T are labelled by the same group element. Then $(T, \mathfrak{G}, \varphi_T)$ is said "covered by m copies of $(P, \mathfrak{G}, \varphi_P)$". Thus $(T, \mathfrak{G}, \varphi_T)$ can be covered by m copies of $(P, \mathfrak{G}, \varphi_P)$ so that each triangle is covered precisely once so that vertices which coincide have the same group element.
If the endvertices of an edge e are labelled by $p, q \in \mathfrak{G}$, then e is said to be an p,q-edge.
In Theorem 5.1 (ii) nothing could be said about some $(T, \mathfrak{A}(T), \tau)$ having two or more distinct edges with the same group elements. For such triangulations I have proved ([5]):

<u>Theorem 6.1:</u> Let m denote the maximum number of edges of $(T, \mathfrak{G}, \varphi)$ to which the same elements of $\mathfrak{G}$ are assigned. Then

(i) the triangulation $(T, \mathfrak{G}, \varphi_T)$ can be covered by m copies of a polygon-triangulation $(P, \mathfrak{G}, \varphi_P)$, where $(P, \mathfrak{G}, \varphi_P)$ has the property: if $\bar{P}$ contains a p,q-edge then $\bar{P}$ has <u>precisely two</u> p,q-edges.
(ii) By identifying each pair of p,q-edges of $\bar{P}$ a triangulation $(T', \mathfrak{G}, \varphi_{T'})$ is obtained being isomorphic with $T(x,y)$. ■

In the case of Theorem 6.1 we briefly say that "$(T, \mathfrak{G}, \varphi_T)$ is covered by m copies of $T(x,y)$" and "$T(x,y)$ is the quotient triangulation of $(T, \mathfrak{G}, \varphi_T)$".
An x,y-edge of $(T, \mathfrak{G}, \varphi_T)$ has multiplicity m iff $(T, \mathfrak{G}, \varphi_T)$ contains precisely m different x,y-edges.
From Theorem 6.1 we derive

<u>Theorem 6.2:</u> All edges of $(T, \mathfrak{G}, \varphi)$ have the same multiplicity. ■

Next we apply Theorem 6.1 to quotient groups.

<u>Theorem 6.3:</u> Let $\mathfrak{G}$ be a group, $\mathfrak{N}$ a normal subgroup of $\mathfrak{G}$ and $\mathfrak{G}/\mathfrak{N}$ the quotient group. Let (x,y,z) be a regular triple of non-commutative elements of $\mathfrak{G}$ with $x\mathfrak{N} \neq y\mathfrak{N} \neq z\mathfrak{N} \neq x\mathfrak{N}$, i.e. $(x\mathfrak{N}, y\mathfrak{N}, z\mathfrak{N})$ is regular too.
Let $(T, \mathfrak{G}/\mathfrak{N}, \varepsilon)$ be obtained from $T(x,y)$ by replacing each element $g \in \mathfrak{G}$ by $g\mathfrak{N}$ on $T(x,y)$.
Then $(T, \mathfrak{G}/\mathfrak{N}, \varepsilon)$ is covered by m copies of $T(x\mathfrak{N}, y\mathfrak{N})$. ■

In a later paper I shall explicitly study the concept of quotient triangulation and compare it with the divisor in [1].

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Norbert Grünwald

Optimale Kantennumerierung des $K_{n,m}$

Bezeichnen wir mit $K_{n,m}$ den vollständigen paaren Graphen mit den Knotenmengen X, $|X| = n$, und Y, $|Y| = m$, $m \geq n$, und der Kantenmenge $E = X \times Y$. Eine eineindeutige Abbildung f der Kanten aus E auf die Menge der natürlichen Zahlen $N = \{1, 2, \ldots, nm\}$ nennen wir eine Kantennumerierung des $K_{n,m}$. Für jeden Knoten $v \in X \cup Y$ bezeichnen wir mit $E(v)$ die Menge aller Kanten aus E, welche mit v inzidieren. Dann sei

$$D_f(v) = \max_{e,e' \in E(v)} |f(e) - f(e')| + 1 \quad \text{und}$$

$$W_f = \max_{v \in X \cup Y} D_f(v).$$

In dieser Arbeit beschäftigen wir uns mit Kantennumerierungen, welche W_f minimieren und bestimmen den Wert

$$W(n,m) = \min_f W_f.$$

Die Arbeit ist somit eine Verallgemeinerung des Theorems 1 aus [3], wo diese Fragestellung für den $K_{n,n}$ gelöst wurde.

Die Knoten der Menge X bzw. Y seien von 1 bis n bzw. von 1 bis m durchnumeriert. Wir ordnen jeder Kantennumerierung f eineindeutig eine $n \times m$-Matrix A_f wie folgt zu:

Die Nummer der Kante $(x,y) \in E = X \times Y$ wird in das Feld $a_{x,y}$ der Matrix A_f eingetragen, $1 \leq x \leq n$, $1 \leq y \leq m$.

So ist $D_f(r)$ für einen Knoten $r \in x$ bzw. $D_f(s)$ für $s \in Y$ um 1 größer als die maximale Differenz zweier Zahlen der r-ten Zeile bzw. s-ten Spalte.

<u>Theorem 1.:</u>

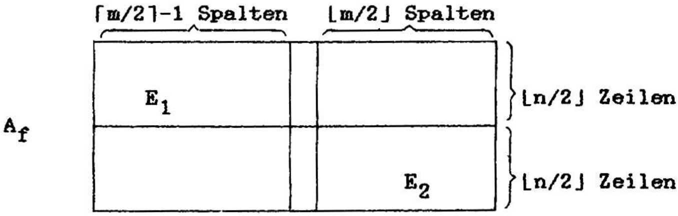
$$W(n,m) = R = \begin{cases} \frac{n(m+1)}{2} & \text{für gerades } n, \\ \frac{(n+1)m}{2} & \text{für ungerades } n. \end{cases}$$

Beweis: 1. Wir wollen zuerst $W(n,m) \geq R$ zeigen.

Nach [3] Theorem 1 gilt $W(n,n) = \binom{n+1}{2}$. Wir betrachten nun noch die Fälle $m > n$.

Sei f eine beliebige Kantennumerierung des $K_{n,m}$ und E_1 die Menge der Kanten mit den Nummern $1, 2, \ldots, \lfloor n/2 \rfloor(\lceil m/2 \rceil - 1)$ und E_2 die Menge der Kanten mit den Nummern $nm - (\lfloor n/2 \rfloor \lfloor m/2 \rfloor) + 1, \ldots, nm$.

1.1: n sei gerade. Kommt in der Matrix A_f keine Kante aus E_1 mit einer Kante aus E_2 in einer Zeile oder Spalte vor, so kann maximal eine Spalte in A_f kein Element aus E_1 und E_2 enthalten (s. Abb. 1).



Die Nummern der Kanten aus E_1 und E_2 sind o.B.d.A. wie oben angeordnet. Abb. 1

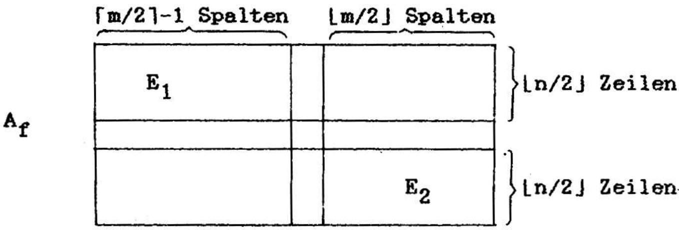
Folglich muß die Kante mit der Nummer $\lfloor n/2 \rfloor(\lceil m/2 \rceil - 1) + 1$ mit einer Kante einer Nummer größer gleich $nm - \lfloor n/2 \rfloor \lfloor m/2 \rfloor$ in einer Spalte oder Zeile liegen oder die Kante mit der Nummer $nm - \lfloor n/2 \rfloor \lfloor m/2 \rfloor$ mit einer Kante einer Nummer kleiner gleich $\lfloor n/2 \rfloor(\lceil m/2 \rceil - 1) + 1$ in einer Spalte oder Zeile liegen.
Somit gilt

$$W(n,m) \geq nm - \lfloor n/2 \rfloor \lfloor m/2 \rfloor - (\lfloor n/2 \rfloor(\lceil m/2 \rceil - 1) + 1) + 1$$

$$= \frac{n(m+1)}{2} .$$

1.2: n sei ungerade. Kommt in der Matrix A_f keine Kante aus E_1 mit einer Kante aus E_2 in einer Zeile oder Spalte vor, so kann genau eine Zeile keine Kanten aus E_1 und E_2 enthalten (s. 1.2.1), oder in jeder Zeile kommen Kanten aus E_1 oder E_2 vor (s. 1.2.2).

1.2.1: Dann enthält maximal eine Spalte aus A_f keine Kanten aus E_1 und E_2 (s. Abb. 2).



Die Nummern der Kanten aus E_1 und E_2 sind o.B.d.A wie oben angeordnet. Abb. 2

Folglich müssen die Kanten mit den Nummern $nm - \lfloor n/2 \rfloor \lfloor m/2 \rfloor$ und $\lfloor n/2 \rfloor(\lceil m/2 \rceil - 1) + \lceil n/2 \rceil$ oder die Kanten mit den Nummern $nm - \lfloor n/2 \rfloor \lfloor m/2 \rfloor - \lfloor n/2 \rfloor$ und $\lfloor n/2 \rfloor(\lceil m/2 \rceil - 1) + 1$ in einer gemeinsamen Zeile liegen. Somit gilt

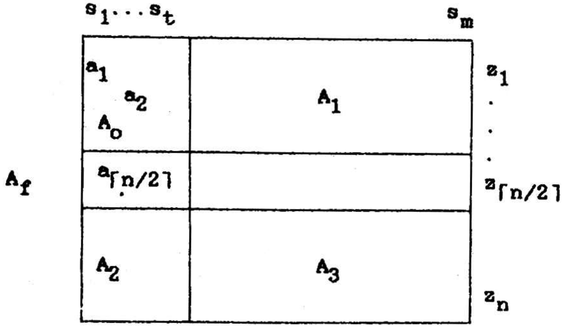
$$W(n,m) \geq nm - \lfloor n/2 \rfloor \lfloor m/2 \rfloor - \lfloor n/2 \rfloor(\lceil m/2 \rceil - 1) - \lceil n/2 \rceil + 1$$

$$= \frac{(n+1)m}{2} .$$

1.2.2: Durch Umnumerierung der Zeilen kann man erreichen, daß die Nummern der Kanten aus E_1 in den ersten $k \geq \lfloor n/2 \rfloor$ Zeilen liegen (in jeder Zeile mindestens eine Nummer einer Kante aus E_1) und die Nummern der Kanten aus E_2 nur in den restlichen Zeilen vorkommen.

Angenommen, es gibt eine Kantennumerierung f mit $W_f(n,m) < \frac{(n+1)m}{2}$. Seien die ersten $\lceil n/2 \rceil$ Zeilen nun so geordnet, daß für die minimalen Elemente $a_i \in E_1$ der Zeilen i, $1 \leq i \leq \lceil n/2 \rceil$, $a_1 < a_2 < \ldots < a_{\lceil n/2 \rceil}$ gilt.

Wir ordnen nun die Spalten mit Kanten aus E_1 so um, daß die ersten t Spalten jeweils mindestens eine Nummer $a_j < a_{\lceil n/2 \rceil}$, $1 \leq j \leq t$, $t \in \{1,2,\ldots,m\}$, enthalten und für jede Nummer a einer Kante aus E_1 aus den restlichen Spalten $a > a_{\lceil n/2 \rceil}$ gilt (s. Fig 3).



Unterteilung der Matrix A_f in die Teilmatrizen A_o, A_1, A_2, A_3 und $z_{\lceil n/2 \rceil}$. Abb. 3

Dann darf in den Teilmatrizen A_o, A_1, A_2 und der Zeile $z_{\lceil n/2 \rceil}$ keine Nummer $a > a_{\lceil n/2 \rceil} + \frac{(n+1)m}{2} - 1$ stehen.
Nun stehen aber bei der Kantennumerierung f in den Teilmatrizen A_o und A_3 zusammen genau $\frac{(n-1)m}{2}$ Nummern, d.h. genau die Nummern $1, 2, \ldots, a_{\lceil n/2 \rceil} - 1, a_{\lceil n/2 \rceil} + \frac{(n+1)m}{2}, \ldots, nm$.
Damit müssen in A_o nur die Nummern $1, 2, \ldots, a_{\lceil n/2 \rceil} - 1$ und in A_3 die Nummern $a_{\lceil n/2 \rceil} + \frac{(n+1)m}{2}, \ldots, nm$ stehen.

Die Nummer $a_{\lceil n/2 \rceil} + \frac{(n+1)m}{2} - 1$ muß folglich in A_1, A_2 oder $z_{\lceil n/2 \rceil}$ stehen, woraus der Widerspruch $W_f(n,m) \geq \frac{(n+1)m}{2}$ folgt.

2. Wir zeigen nun $W(n,m) \leq R$.

2.1: n sei gerade. Wir verteilen die Nummern $1, 2, \ldots, nm$ in der Matrix A_f wie folgt:

$$a_{i,j} = \begin{cases} (j-1)n/2 + i & \text{für } 1 \leq i \leq n/2,\ 1 \leq j \leq m, \\ nm-((m-j)n/2)-(n-i) & \text{für } n/2 < i \leq n,\ 1 \leq j \leq m. \end{cases}$$

Für $n = 4$ und $m = 9$ ist die Numerierung in Abb. 4 angegeben.

	1	2	3	4	5	6	7	8	9	Y
1	1	3	5	7	9	11	13	15	17	
2	2	4	6	8	10	12	14	16	18	
3	19	21	23	25	27	29	31	33	35	
4	20	22	24	26	28	30	32	34	36	

X

Die Matrix A_f für eine optimale Kantennumerierung des $K_{4,9}$. Abb. 4

Für einen Knoten $r \in X$ gilt dann

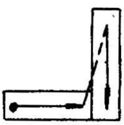
$$D_f(r) = \frac{(m-1)n}{2} + 1 \leq \frac{(m+1)n}{2}.$$

Für einen Knoten $s \in Y$ gilt dann

$$D_f(s) = nm - (m-j)n/2 - (j-i)n/2 = \frac{(m+1)n}{2}.$$

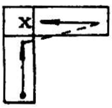
2.2: n sei ungerade. Wir verteilen die Nummern $1, 2,, \ldots, nm$ in der Matrix A_f wie folgt:

$$a_{i,j} = \begin{cases} (j-i)\lfloor n/2 \rfloor + i & \text{für } 1 \leq j \leq \lfloor m/2 \rceil - 1,\ 1 \leq i \leq \lfloor n/2 \rfloor, \\ nm - (m-j)\lfloor n/2 \rfloor - (n-i) & \text{für } \lceil m/2 \rceil < j \leq n,\ \lceil n/2 \rceil < i \leq n, \\ nm - \lfloor n/2 \rfloor \lceil m/2 \rceil - \lfloor n/2 \rfloor - \lfloor m/2 \rfloor - (\lceil m/2 \rceil - 1 - j)\lfloor n/2 \rfloor - (n-i) & \text{für } 1 \leq j < \lceil m/2 \rceil,\ \lceil n/2 \rceil < i \leq n, \\ \lfloor n/2 \rfloor \lfloor m/2 \rfloor + \lceil m/2 \rceil + \lfloor n/2 \rfloor + (\lceil m/2 \rceil + 1 - j)\lfloor n/2 \rfloor + i & \text{für } \lceil m/2 \rceil < j \leq m,\ 1 \leq i \leq \lfloor n/2 \rfloor, \end{cases}$$



$s_{\lceil m/2 \rceil}$

$z_{\lceil n/2 \rceil}$ von $\lfloor n/2 \rfloor \lfloor m/2 \rfloor + 1$ bis $\lfloor n/2 \rfloor \lfloor m/2 \rfloor + \lceil m/2 \rceil + \lfloor n/2 \rfloor$



von $nm - \lfloor n/2 \rfloor \lfloor m/2 \rfloor$ bis $nm - \lfloor n/2 \rfloor \lfloor m/2 \rfloor - \lfloor n/2 \rfloor - \lfloor m/2 \rfloor$.

Für $n = 5$ und $m = 11$ ist die Numerierung in Abb. 5 angegeben.

	1	2	3	4	5	6	7	8	9	10	11	Y
1	1	3	5	7	9	16	19	21	23	25	27	
2	2	4	6	8	10	17	20	22	24	26	28	
3	11	12	13	14	15	16	39	40	41	42	43	
4	29	31	33	35	37	44	46	48	50	52	54	
5	30	32	34	36	38	45	47	49	51	53	55	
X												

Die Matrix A_f für eine optimale Kantennumerierung des $K_{5,11}$. Abb. 5

Für einen Knoten $r \in X$ gilt dann

$$D_f(r) = \begin{cases} \frac{(n+1)m}{2} - \lfloor m/2 \rfloor - \lfloor n/2 \rfloor + 1 & \text{für } 1 \leq r \leq \lfloor n/2 \rfloor, \\ \frac{(n+1)m}{2} & \text{für } r = \lceil n/2 \rceil, \\ \frac{(n+1)m}{2} - \lfloor m/2 \rfloor - \lfloor n/2 \rfloor & \text{für } \lceil n/2 \rceil + 1 \leq r \leq n. \end{cases}$$

Für einen Knoten $s \in Y$ gilt dann

$$D_f(s) = \begin{cases} \frac{(n+1)m}{2} - \lfloor m/2 \rfloor + \lfloor n/2 \rfloor & \text{für } 1 \leq s \leq \lceil m/2 \rceil, \\ \frac{(n+1)m}{2} - \lceil m/2 \rceil + \lfloor n/2 \rfloor & \text{für } \lceil m/2 \rceil < s \leq m. \end{cases}$$

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What graphs have bounded tree-width?

One approach in the treatment of NP-complete problems consists in developing efficient algorithms for special graph classes like trees, k-outerplanar graphs, series-parallel graphs, bandwidth-limited graphs, interval graphs and others (see [10] for a survey). Recently some attempts are made to unify these results concerning several problems by defining general graph classes for which a lot of problems are polynomial-time solvable (see [4, 6, 19]). The most general class for which efficient algorithms are known is the class of graphs of bounded tree-width. Seese first defined by logic means the wide class of EMS-properties decidable for graphs of bounded tree-width in linear time ([15, 2]), the author described a dynamic programming algorithm for most of them ([14]). In the present paper we demonstrate that graphs of many frequently considered classes are graphs with bounded tree-width (compare also [5]). The limited space forces us to leave out some proofs here.

Definition 1: A tree-decomposition of width w for a finite undirected graph G is a pair $(T,\mathcal{X})$ consisting of a tree T and a family of subsets of the vertex set V(G) $\mathcal{X} = \{X_t\colon t \in V(T),\ X_t \subseteq V(G)\}$ with:

(1) $\bigcup_{t\in V(T)} X_t = V(G)$.

(2) For every edge e of G there exists a $t \in V(T)$ such that e has both ends in X_t.

(3) For every vertex v of G the subgraph T_v of the tree T induced by the node set $\{t\colon v \in X_t\}$ is connected.

(4) For all t it holds $|X_t| \leq w+1$.

Definition 2: The complete graph with k vertices K_k is a k-tree. Another graph G is a k-tree iff it contains a simplicial vertex v of degree k such that $G\setminus\{v\}$ is a k-tree. Every subgraph of a k-tree with the same vertex set is a partial k-tree.

Theorem 1: A graph is a partial k-tree iff its tree-width is not greater than k.

First we prove some useful simple properties of tree-decompositions. Obviously, the tree-width of a graph cannot be less than that of its subgraph and a tree-decomposition of G can be obtained from tree-decompositions for its connected components.

Lemma 1: The tree-width of a graph G is equal to the maximum tree-width of its 2-connected components (blocks).

Proof: Let $(T_1,\mathcal{X}_1)$, ..., $(T_l,\mathcal{X}_l)$ be tree-decompositions of the blocks of G. Consider the tree T representing the connection structure of the blocks of G. ($V(T) = \{v_{i,j}\} \cup \{u_i\}$ where u_i corresponds to the block G_i and $v_{i,j}$ is the articulation point belonging to G_i and G_j and $E(T) = \{(v_{i,j},u_i)\} \cup \{(v_{i,j},u_j)\}$.) In this tree each node u_i replace by the tree T_i connecting T_i with every $v_{i,j}$ in T by exactly one edge (starting from a node t in T_i such that $v_{i,j} \in X_t$). Furthermore we set $X_t := X_{i,t}$ for $t \in V(T_i)$ and $X_t := \{v_{i,j}\}$ for $t = v_{i,j}$. The pair $(T,\{X_t\})$ is a tree-decomposition of the needed width of G. ■

Lemma 2: Let $(T,\mathcal{X})$ be a tree-decomposition of graph G. Then there exists for every clique $K \subseteq G$ at least one node $t \in V(T)$ in the tree with $V(K) \subseteq X_t$.

Proof: For cliques of size one and two this is true according to Definition 1. So let $|V(K)| = k > 2$, $v \in V(K)$, and $K' = K\setminus\{v\}$. Then (by induction) there is a node t' in the tree for a clique K' with $V(K') \subseteq X_{t'}$. If now $v \notin X_{t'}$, then in T there is an unique branch T_v containing all nodes with $v \in X_s$. Let t be the first node on the path from t' into T_v such that $v \in X_t$. Then it holds $V(K) \subseteq X_t$, because of (2) (every edge $(v,w) \in E(K)$ must be realized within the branch T_v) and (3). ■

Lemma 3: A graph with tree-width w is at most w-connected.

Proof: If G has more than $w+1$ vertices then the tree of each decomposition has at least one edge (s,t). Without loss of generality we may assume $|X_t \cap X_s| = w$. On the other hand, the vertex set $X_t \cap X_s$ is a seperator of G. ■

Lemma 4: For the number of edges m of a graph G with n vertices and tree-width w it holds $m \leq wn - w(w+1)/2$.

Proof: G is a partial graph of a w-tree. For w-trees we have indeed $m = w(w-1)/2 + (n-w)w = wn - w(w+1)/2$. ■

Obviously all nontrivial trees have tree-width one. All simple cycles have tree-width 2 (tree-decompositions of the plane cycle correspond to its planar triangulations) and the complete graph K_n has tree-width $n-1$. The nxn-grid has tree-width n, thus being an example for planar graphs with arbitrary large tree-width. Outerplanar graphs have tree-width at most 2. For graphs obtained from an edge by series and parallel compositions (for a formal definition see e.g. [17] or [18]) it holds

Theorem 2: A graph is a series-parallel-graph iff its tree-width is at most 2.

Definition 3: A connected graph without vertices of degree 2 is a Halin graph iff it has such a plane embedding that after deleting the edges of the outer cycle it remains a tree ([16, 19]).

Theorem 3: All Halin graphs have tree-width 3.

Halin graphs are 2-outerplanar graphs, k-outerplanar graphs are defined recursively generalizing the outerplanar graphs ([3]).

Theorem 4 ([3]): The tree-width of a k-outerplanar graph is at most $3k-1$.

Nevertheless there are 2-trees which are k-outerplanar for arbitrary large k. Yet another class is the class of k-almost trees (each block of G differs from a tree by at most k edges [8]), for which the difference between k and the tree-width can be also arbitrary large, but it holds:

Theorem 5 [5]: The tree-width of a k-almost tree is at most $k+1$.

Chordal graphs, interval graphs (and other subclasses) and circular arc graphs are intersection graphs ([11]).

Definiton 4: A graph G is an intersection graph of a graph H iff there is a 1-1-correspondence between the vertices of G and the connected subgraphs of H such that $(u,v) \in E(G)$ iff $V(H_u) \cap V(H_v) \neq \emptyset$.

Theorem 6: Let G be an intersection graph of the graph H and $\omega(G)$ its clique number (i.e. the maximum clique size). Then for the tree-widths of both graphs it holds $w(G) \leq w(H) * \omega(G) - 1$.

Proof: Let $(T,\mathcal{X})$ be a tree-decomposition of width w for H. In the connected graph G for any $a \in V(H)$ consider the vertex set $C_a := \{ v \in V(G)$ with $a \in V(H_v)\}$. Clearly, $|C_a| \leq \omega(G)$ for all $a \in V(H)$. Now choose a root r in T and delete one vertex from each set of $\mathcal{X}$ according to the following rules:

- Start at the root r, consider the father s before its son t, proceed all nodes of the tree.
- If $a_s \in X_t$, then set $a_t := a_s$.
  Otherwise take any a_t from $X_s \cap X_t$, in the root case from X_r.
- Set $X'_t := X_t \setminus \{a_t\}$.
- Add a new leaf r' to T with edge (r',r) and $X_{r'} = \{a_r\}$.

Now the pair $(T', \{Y_t\})$ is a tree-decomposition of the graph G if we define $Y_t := \bigcup_{a \in X'_t} C_a$. Moreover $|Y_t| \leq \omega(G) * w(H,T)$. ■

Consider as an example the class of circular arc graphs (i.e. the intersection graphs of paths in a simple cycle).

Theorem 7: Let G be a circular arc graph and $\omega(G)$ its clique number. Then its tree-width satisfies $w(G) \leq 2\omega(G) - 1$.

Definition 5: A graph is a chordal graph iff it does not contain a chordless cycle of length 4 or more.
It is a k-chordal graph iff its clique number is $k+1$.

It is known that a graph is chordal iff it is the intersection graph of subtrees in a tree ([9]), hence we can apply Theorem 5 with $w(H) = 1$. On the other hand, the tree-width of a graph is never less than its clique size minus 1 (Lemma 3). So we have

Theorem 8: A graph has tree-width w or less iff it is a partial graph of a w-chordal graph.

This fact is interesting from point of view of algorithm design since the class of chordal graphs and its subclasses are well investigated. There are a lot of problems for which polynomial-time algorithms exist and others which remain NP-complete if restricted to chordal graphs ([10, 9]). Now the reasons for such a different behaviour become more clear: The intractability of some problems is caused by the existence of arbitrary large cliques in the input graphs. Furthermore, some polynomial-time algorithms can be modified possibly in order to apply them not only to chordal graphs but also to their partial graphs.

Interval graphs ([9]) are special chordal graphs - the intersection graphs of subgraphs of a simple path. Hence, the tree-width of an interval graph is equal to its clique number minus 1. But for interval graphs there is always a tree-decomposition $(T,\mathcal{X})$ such that T is a simple path. This is called path-decomposition and the minimum width of path-decompositions for a graph G is its path-width $p(G)$. Obviously, for all graphs it holds $w(G) \leq p(G)$. In the case of interval graphs equality holds, the same is true for simple cycles, grids and complete graphs. But there are trees of arbitrary large path-width. The question, how to characterize path-like graphs (with $w(G) = p(G)$) is open. Path-width has close relations to other problems and practical applications (e.g. gate matrix layout in VLSI-design). We proved that it is never greater than the bandwidth and the cutwidth of a graph.

Definition 6: The minimum clique number of an interval graph with G as its partial graph is called interval thickness $\Theta(G)$ of G.

Kirousis and Papadimitriou [12] proved that the interval thickness of a graph is always equal to its node search number.

Theorem 9: For all graphs G it holds $\Theta(G) = p(G) + 1$.

The comparison of our results with the survey of Johnson [10] shows that really most graph classes for which polynomial-time algorithms are known for intractable problems are subsets of the class of graphs with bounded tree-width. (There remains one open question: What is to say about the tree-width of comparability graphs?) Consequently, the linear time algorithm [14] for EMS-properties is an improvement of all these results and moreover makes available efficient solutions for new problems. Most of the above considered graph classes have simple and efficient membership test algorithms ([9]). Thus for graphs of these classes a bound for their tree-width can be found in low-degree polynomial (often linear) time, in contrast to the general case where the problem to determine the tree-width of a graph is NP-complete. So fare only a $O(n^{w+2})$-algorithm is available to decide the problem "Is tree-width of G less or equal to w?" for fixed w (see [1, 13]). But for the considered special graph classes the general approach of [2] and [14] may well give practical algorithms thorouhly.

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Distance graphs on finite graph sets

1. Introduction

The problem we are dealing with is the following: how to define distances between graphs determining their structural similarity in a certain sense. A. Sanfeliu, K.S. Fu [4] and M.A. Eshera, K.S. Fu [1] used the minimal number of given modifications, transforming one graph into another one, to determine their distance. An approach basing on graph grammers was presented by D. Gernert [2]. L.G. Shapiro [5] defined a graph distance by the maximal number of coinciding elements of the adjacency matrices after a suitable rearrangement of rows and columns. The minimal vertex number of induced subgraphs contained in exactly one of two graphs determines a graph distance introduced by F. Sobik [6]. Our approach is based on the Zelinka-distance, which is defined by maximal common induced subgraphs (see B. Zelinka [7]). This graph metric is modified by using only subgraphs and supergraphs fulfiling certain properties. We introduce the so-called distance graph and formulate the problem of isometry. The main results of this paper concern this problem with respect to distance graphs defined on several finite graph sets and corresponding graph metrics.

2. Graph distances

We consider finite undirected graphs without loops and multiple edges. The notion graph is used in the sense of isomorphy classes. We define several distances between graphs by means of common subgraphs and common supergraphs fulfilling certain conditions. Let $n(G)$ be the number of vertices of the graph $G = (V,E)$, whose vertex and edge sets are V and E, respectively. A graph $G' = (V',E')$ is called induced subgraph of a graph $G = (V,E)$ (in terms $G' \leqslant G$), if there are a subset $U \subseteq V$ of the vertex set V and a bijection $f : U \longrightarrow V'$ so that $\{u,v\} \in E$ if and only if $\{fu,fv\} \in E'$ for all $u,v \in U$. If $G' \leqslant G$ and both graphs are not isomorphic to each other we write $G' < G$ and call G' a proper induced subgraph of G.

Given three graphs H, G, and G' the graph H is called a common subgraph (or supergraph) of G and G' if $H \leq G$ and $H \leq G'$ (or $G \leq H$ and $G' \leq H$) hold.
Let us consider graph sets M containing common subgraphs and common supergraphs of any two graphs $G, G' \in M$. We define

$$n_M(G,G') = \max \{n(H) \mid H \leq G,G',\ H \in M\} \quad \text{and} \tag{1}$$

$$n'_M(G,G') = \min \{n(H') \mid G,G' \leq H',\ H' \in M\} \tag{2}$$

the maximal vertex number $n_M(G,G')$ of common subgraphs and the minimal vertex number $n'_M(G,G')$ of common supergraphs for any two graphs $G, G' \in M$. The functions d_M and d'_M called graph distances are defined as follows

$$d_M(G,G') = \max \{n(G),\ n(G')\} - n_M(G,G'), \tag{3}$$

$$d'_M(G,G') = n'_M(G,G') - \min \{n(G),\ n(G')\}. \tag{4}$$

Next, we ask the question whether the functions d_M and d'_M are metrics for given graph sets M. The following conditions are sufficient for the graph distanes d_M and d'_M, respectively, to be metrics:

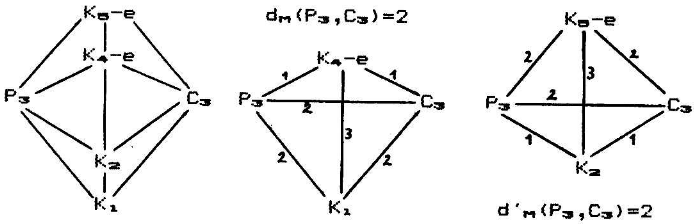
If $H' \in M$ is a common supergraph of $G, G' \in M$, then there is a common subgraph $H \in M$ of G, G' with (5)
$n(H) \geq n(G) + n(G') - n(H')$.

If $H \in M$ is a common subgraph of $G, G' \in M$, then there is a common supergraph $H' \in M$ of G, G' with (6)
$n(H') \leq n(G) + n(G') - n(H)$.

<u>Theorem 1:</u> The functions d_M and d'_M are metrics on the set M if the conditions (5) or (6) hold, respectively.

For a proof we have only to show that the triangle inequality holds. It is easy to see that the conditions stated above are sufficient for our graph distances to be metrics. On the other hand, the following example shows that these conditions are not necessary.
Let $M = \{K_1, P_3, C_3, K_4-e\}$ (see Figure 1a).
If we take $G = P_3$, $G' = C_3$, and $H' = K_4-e$ we see that

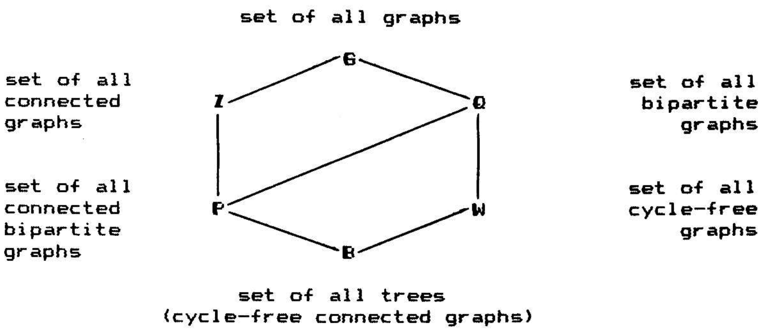
condition (5) fails. Nevertheless, the function d_M is a metric on the set M (see Figure 1b).
Considering $M' = \{K_2, P_3, C_3, K_5-e\}$ (see Figure 1a) the graphs $G = P_3$, $G' = C_3$, and $H = K_2$ miss condition (6) although d'_M is a metric on M (see Figure 1c).



a) subgraph relation b) the metric d_M c) the metric d'_M,

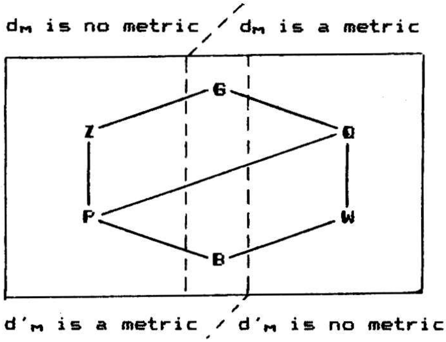
The conditions (5) and (6) are not necessary. Figure 1

Now we consider the graph sets given in Figure 2 which shows their inclusion scheme.



Inclusion scheme of six graph sets. Figure 2

With the help of Theorem 1 we show which distances d_M and d'_M are metrics. If the sufficient condition fails then in our cases there are counterexamples for the triangle inequality (see Figure 3).



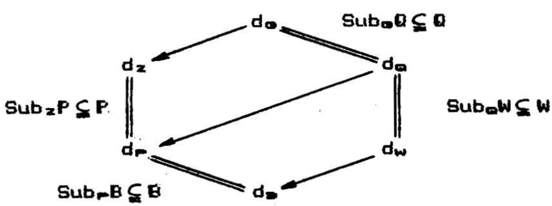
Graph metrics and graph distances. **Figure 3**

We have $d_G = d'_G$ (= d_o - Zelinka-distance). Furthermore, the distances d_B and d'_B are identical, but different from the Zelinka-distance restricted to the set **B**. This can be shown by the graphs $K_{1,6}$ and P_7.
Our next aim is to study relations between graph distances defined on different graph sets. Of course graph distances can only be compared on their common domain of definition.

Let Sub M be the set of all induced subgraphs of graphs contained in the set M. Given two graph sets M and M' with $M \subseteq M'$ we define

$$\mathrm{Sub}_{M'}M = M' \cap \mathrm{Sub}\ M. \tag{7}$$

Evidently, $M \subseteq M'$ implies $d_M \geq d_{M'}$ and $d'_M \geq d'_{M'}$ on M. Furthermore, the distances d_M and $d_{M'}$, $M \subseteq M'$, are identical on M if $\mathrm{Sub}_{M'}M \subseteq M$ holds.



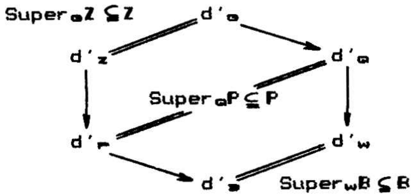
Relations between the graph distances d_M $d \longrightarrow d'$ **Figure 4**
denotes $d \leq d'$ but not $d = d'$ ($d < d'$).

With respect to the graph distances d'_M we define

$$\text{Super } M = \{ H' \mid \exists\, G,G' \in M \ [G,G' \leq H' \text{ and } \exists\, H \ (G,G' < H < H')] \} \tag{8}$$

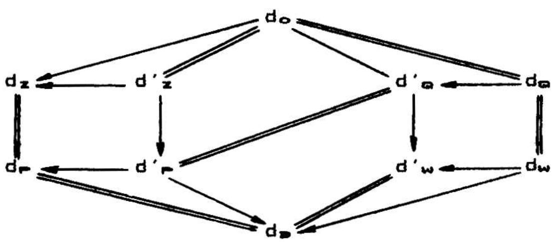
$$\text{Super}_{M'} M = M' \cap \text{Super } M. \tag{9}$$

The functions d'_M and $d'_{M'}$ are identical on M if $\text{Super}_{M'} M \subseteq M$ holds.



Relations between the graph distances d'_M. Figure 5

The following figure summarizes the relations between the graph metrics d_M and d'_M.



Relations between the graph distances d_M and d'_M. Figure 6

3. Distance graphs

Let (M,d) be a finite metric space and d an integer-valued metric. The corresponding distance graph $D(M,d) = (M,E,d)$ is defined on the vertex set M, so that any two vertices $u,v, \in M$ are adjacent in D (in terms $\{u,v\} \in E$) if and only if $d(u,v) = 1$ holds. Graph-theoretic terms applied to the distance graph $D = (M,E,d)$ are used in the sense of the graph (M,E). All distance graphs D to be considered here are connected. Let $\delta_D(u,v)$ be the distance between any two vertices u,v of D which is defined by the length of a shortest path connecting them.

The problem we are dealing with is the question whether the metrics d and δ_D are identical on the set M. With the help of the triangle inequality for the metric d the relation $d(u,v) \leq \delta_D(u,v)$, $u,v \in M$ is proved. The distance graph $D = D(M,d)$ is called isometric if d and δ_D are identical on M. The so called degree of non-isometry $\deg D$ is defined as follows

$$\deg D = \max \{\delta_D(u,v) - d(u,v) \mid u,v \in M\}. \tag{10}$$

Our aim is to decide whether the distance graphs corresponding to the graph sets and metrics introduced in Chapter 2 are isometric and if not to calculate or estimate their degree of non-isometry.

Let $\mathbb{G}_n$ be the set of all graphs of vertex number n. The corresponding distance graph G_n^o with respect to the Zelinka-distance is isometric in all cases (see B. Zelinka [7]) .

Now let $\mathbb{Z}_n$ be the set of all connected graphs of vertex number n. The distance graph $Z_n^o = D(\mathbb{Z}_n, d_o)$ is isometric if and only if n is less than ten. Furthermore, it can be proved, that the degree of non-isometry of Z_n^o is equal to one in all other cases.

<u>Theorem 2:</u> $\deg Z_n^o = \begin{cases} 0 & \text{if } n < 10, \\ 1 & \text{else.} \end{cases}$

We do not give the proof here (see F. Kaden [3]) because some cases have to be considered.

Another interesting case appears when we consider the set $\mathbb{P}_n$ of all connected bipartite graphs of vertex number n and the graph metric d_P' defined by minimal bipartite supergraphs. The corresponding distance graph $P'_n = D(\mathbb{P}_n, d'_P)$ fulfils an equation similar to Theorem 2.

<u>Theorem 3:</u> $\deg P'_n = \begin{cases} 0 & \text{if } n < 11, \\ 1 & \text{else.} \end{cases}$

A lot of cases have to be considered to prove the isometry in the cases $n < 11$. On the other hand, it is easy to see, that the degree of non-isometry of P'_n is less then or equal to two. It requires a little bit more time to show that degree two never occurs.

In the case of the set $\mathbb{B}_n$ containing all trees of vertex number n we have to consider two distance graphs denoted by $B_n^o = D(\mathbb{B}_n, d_o)$ and $B_n = D(\mathbb{B}_n, d_B)$. The last one is isometric for all vertex numbers n (B. Zelinka unpublished). On the other hand, the degree of non-isometry of the first one tends to infinity with the vertex number n. In all other cases corresponding to the Zelinka-distance we only give an estimation of the degree of non-isometry (see the next table).

graph set	distance-graph	estimation
$\mathbb{B}_n$	$B_n^o = D(\mathbb{B}_n, d_o)$	$\lim \deg B_n^o/n = 1$
$\mathbb{W}_n$	$W_n^o = D(\mathbb{W}_n, d_o)$	$C \lfloor n^{1/3} \rfloor \leq \deg W_n^o \leq C'n$
$\mathbb{Q}_n$	$Q_n^o = D(\mathbb{Q}_n, d_o)$	$C'' \lfloor n^{1/3} \rfloor \leq \deg Q_n^o \leq C'n$
$\mathbb{P}_n$	$P_n^o = D(\mathbb{P}_n, d_o)$	$C'' \lfloor n^{1/3} \rfloor \leq \deg P_n^o \leq C'n$

Towards the problem of isometry. Table

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Maximal graphs with bounded maximum degree: structure, asymptotic enumeration, randomness

0. Introduction

A graph G is called a d-graph if it is not a spanning subgraph of any graph of maximum degree d. A vertex of degree less than d is called unsaturated. In this paper we examine the structure of d-graphs and enumerate them asymptotically with respect to the number of unsaturated vertices. We also discuss the question of Erdős about the limit distribution of the number of unsaturated vertices at the end of the graph process in which edges are added one by one, equiprobably, and so that the maximum degree does not exceed d.

1. Random graph models

By a random graph (a random graph process) we mean a probabilistic space whose elements are graphs (sequences of graphs) on the vertex set $V = \{1,2,\dots,n\}$. Most natural are equiprobable models: the Erdős-Rényi model $K_{n,N}$ consisting of all graphs with N edges and Bollobás' process Π_n formed by all sequences $(G_0,G_1,\dots,G_{\binom{n}{2}})$, where G_i has i edges and is contained in G_{i+1}, $i = 0,\dots,\binom{n}{2}-1$. Both can be thought of as results of random experiments: $K_{n,N}$ - drawing N out of $\binom{n}{2}$ pairs of vertices, Π_n - adding edges one by one to the empty graph. (See [4] for an extensive account on random graphs.)

In many applications in chemistry and physics these models are not satisfactory due to the lack of degree restrictions quite natural in the world of molecules. In the simplest case the condition that no vertex has degree larger than d is required. Then the appropriate model could be the equiprobable space of all graphs on V with N edges and maximum degree at most d - an analogue of $K_{n,N}$. However, in this paper we do not deal with this model. More appealing is an analogue of Π_n which we denote by $\Pi_{n,d}$ and define as follows.

Let $[V]^2$ be the set of 2-element subsets of V, $\Delta(G)$ and $E(G)$ stand for the maximum degree and edge set of a graph G, respectively, and $G \cup x = (V, E(G) \cup \{x\})$, where $x \in [V]^2 - E(G)$.
The space $\Pi_{n,d}$ consists of all sequences $(G_o, G_1, \ldots, G_m)$ satisfying

(i) $|E(G_i)| = i$, $i = 0, \ldots, m$,

(ii) $\emptyset = E(G_o) \subset E(G_1) \subset \ldots \subset E(G_m)$,

(iii) $\Delta(G_i) \leqslant d$, $i = 0, \ldots, m$,

(iv) $A(G_m) = \emptyset$,

where $A(G) = \{ x \in [V]^2 - E(G) : \Delta(G \cup x) \leqslant d \}$.
The probability assigned to $(G_o, \ldots, G_m)$ equals to $\prod_{i=o}^{m-1} a_i^{-1}$, where $a_i = |A(G_i)|$.
More intuitively, one keeps adding edges one by one, each time choosing equiprobably a pair of vertices which are not yet joined and both have degree less than d.
Note that the length of the process varies, since it terminates when for the first time $A(G_i) = \emptyset$. At first glance we see two differences between Π_n and $\Pi_{n,d}$. The space $\Pi_{n,d}$ is not equiprobable and the last term, G_m, is not unique. Indeed, G_m ranges over all graphs G with $\Delta(G) = d$ and $\Delta(G \cup x) > d$ for all $x \in [V]^2 - E(G)$. Such graphs are called <u>d-graphs</u> here. This notion was introduced (under different name) by Kennedy and Quintas [5]. Some probabilistic spaces of d-graphs where also investigated by Balińska and Quintas [1].

In Section 2 we examine some structural properties of d-graphs, whereas Section 3 contains an asymptotic enumeration of n-vertex d-graphs (as $n \to \infty$) with respect to the number of vertices of degree less than d. Such vertices are called <u>unsaturated</u> here. It is an open problem posed by Erdős (a personal communication) to determine the limit distribution (as $n \to \infty$) of the number of unsaturated vertices at the end of the random process $\Pi_{n,d}$.

In Section 4 we present a simple procedure to find the distribution for $d = 2$ and any fixed n. For larger d, a computer simulation is all one is able to do at the moment. A simple procedure of generating a random process $\Pi_{n,d}$ goes as follows.

Procedure 1:

Set $E = \emptyset$, $c_1 = \ldots = c_n = n-1$, $d_1 = \ldots = d_n = 0$.
(*) Pick R = a random integer from $\{1, 2, \ldots, c_1 + \ldots + c_n\}$.
Set i for the smallest integer satisfying
$$R \leq c_1 + \ldots + c_i.$$
If $i = 1$ set $j = 0$ otherwise set j for the largest integer satisfying
$$R > c_1 + \ldots + c_j.$$
If $j = 0$ set $r = R$ otherwise set $r = R - (c_1 + \ldots + c_j)$.
Set $l = 0$.
For $k = 1, \ldots, n$, $k \neq i$ if $\{i,k\} \notin E$ and $d_k < d$ then $l = l+1$ until $l = r$.
Set $E = E \cup \{i,k\}$, $d_i = d_i+1$, $d_k = d_k+1$.
For $t = i,k$ if $d_t < d$ then $c_t = c_t - 1$ otherwise set $c_t = 0$ and for $s = 1, \ldots, n$, $s \neq i,k$ if $\{t,s\} \notin E$ then $c_s = c_s - 1$.
If all $c_i = 0$ then stop, otherwise go to (*).

Comment: At the end the set E is the set of edges of G_m. To gain the whole process one should keep the track of the order in which edges are included to E. Vertices i with $d_i < d$ are unsaturated.

Such simulation has been performed by K. Balińska and the data will be published in [2].

2. The structure of d-graphs

Let us denote by $U = U_G$ the set of unsaturated vertices of a d-graph G. Clearly, U induces a complete subgraph of G and therefore $u = |U| \leq d$. The unsaturated vertices have degrees between $u-1$ and $d-1$. Thus the number of edges in an n-vertex d-graph with $|U| = u$ varies from $\binom{u}{2} + \frac{1}{2}d(n-u)$ to $\frac{1}{2}(nd-u)$.
It is not quite obvious that all theoretically possible cases are realisable. A trivial necessary condition is that $d \leq n-u-1$. It is satisfied in the whole range $1 \leq u \leq d$ only if $n \geq 2d+1$.
A more refined necessary condition for the existence of an n-vertex d-graph with $|U| = u$ and $2|E(G)| = L$, relevant when $n \leq 2d$, is

$$d-k \leq n-u-1, \text{ where } k = \lfloor (L - (n-u)d - u(u-1))/(n-u) \rfloor. \quad (1)$$

As it happens (1) is sufficient as well.

Proposition 1: For all $d \geq 2$, $n \geq d+1$, $1 \leq u \leq d$, and $u(u-1) + (n-u)d \leq L \leq nd-u$, L even, there exists an n-vertex d-graph G with $|U_G| = u$ and $2|E(G)| = L$ if and only if condition (1) is satisfied.

Proof: The difference $L - (n-u)d - u(u-1)$ is equal to the number of edges going from U to $V-U$. There must be a vertex in $V-U$ incident to at most k of them. Hence the necessity follows. Let x_o, y_o be the unique integer solution to the system of equations

$$\begin{cases} x + y = n-u, \\ kx + (k+1)y = L - (n-u)d - u(u-1). \end{cases}$$

It can be easily shown that there exists a graph H on $n-u$ vertices with x_o blue vertices of degree $d-k$ and y_o red vertices of degree $d-k-1$. Let K be a complete graph on u vertices disjoint from H. It is possible to join each blue vertex of H to k vertices of K and each red vertex of H to $k+1$ vertices of K and not produce a vertex in K of degree larger than $d-1$. ■

As a consequence of the above result the doubled number of edges in an n-vertex d-graph may be as small as the smallest even integer not smaller than $nd - \frac{1}{4}(d^2+2d)$. The minimum is achieved when $u = \lfloor\frac{1}{2}(d+1)\rfloor$.

To avoid the parity problem we define a d-regular gaph as one with at most one vertex of degree $d-1$ and at least $n-1$ vertices of degree d. As we already know d-graphs are almost d-regular, especially when $n = |V|$ is large compared to d. In order to increase the number of edges in a non-d-regular d-graph one has to remove an edge and then add two new ones if possible. An edge of a d-graph which has the above property is called normal, the name is justified by the fact that every edge with neither endpoint joint to an unsaturated vertex is such and typically there are many such edges in a d-graph. Does every d-graph has a normal edge? It is not immediately seen that the answer is yes.

Proposition 2: Every d-graph is either d-regular or it contains a normal edge.

Proof: The assertion is trivial for $u = |U| = 1$. Assume, therefore, that $u \geq 2$ and set f for the number of edges with exactly one endpoint in U. Clearly $f \leq u(d-u)$.
Suppose, to the contrary, that there is no normal edge. We claim the existence of a pair of vertices $x \in U$, $y \notin U$ such that $N(y) \cap U = U-\{x\}$, where $N(y)$ is the set of neighbours of y. It follows from two facts:

(i) not all vertices of degree d are joined to all ones in U,
(ii) each vertex of degree d has at least $u-1$ unsaturated neighbours.

To prove (ii) suppose that for some $z \notin U$ $|N(z) \cap U| = l \leq u-2$. Then all its neighbours of degree d must be joined to all vertices in U and therefore $f \geq l + (d-1)u > u(d-u)$, a contradiction.
Each saturated (= of degree k) neighbour of y is joined either to all vertices in U or to all except x. Let l of them be of the second kind. If $l = 0$ then

$$f \geq (u-1) + u(d-u+1) > u(d-u).$$

If $l > 0$ then the degree of x is $d-1$ (otherwise there would be a normal edge from y to any of its neighbours not joined to x) and so x is joined to

$$(d-1) - ((u-1) + (d-u+1-l)) = l-1$$

saturated vertices not in $N(y) \cup \{y\}$. But each of these vertices is, in turn, joined to at least $u-1$ vertices in U. Altogether, we get

$$f \geq (u-1) + u(d-u+1-l) + l(u-1) + (l-1)(u-1) > u(d-u),$$

again a contradiction. ■

To make a d-graph d-regular one has to repeat the above operation $\lfloor \frac{1}{2}nd \rfloor - e(G)$ times, where $e(G) = |E(G)|$. This means that

$$f(G) = \min \{|E(G) \Delta E(F)|: d\text{-regular } F \text{ on vertex set } V\}$$

$$\leq 3\ (\lfloor \tfrac{1}{2}nd \rfloor - e(G)).$$

In fact, the equality holds and so the repetitive replacement of a normal edge by two new edges is the fastest way from a d-graph to a d-regular graph.

Proposition 3: For every d-graph on the vertex set V it holds

$$f(G) = 3\,(\lfloor \tfrac{1}{2}nd \rfloor - e(G)).$$

Proof: The sum of degrees of unsaturated vertices bears the whole deficit

$$\varepsilon = 2\lfloor \tfrac{1}{2}nd \rfloor - 2e(G).$$

Let F be a d-regular graph which minimizes $f(G)$. There has to be at least ε edges between U and $V-U$ in $E(F)-E(G)$. This, however, forces us to remove at least $\frac{1}{2}\varepsilon$ edges joining saturated vertices of G. ■

3. Asymptotic enumeration of d-graphs

Let $S = S_G$ be the set of vertices of degree k with at least one unsaturated neighbour. In this section we asymptotically enumerate n-vertex d-graphs ($n \to \infty$) with respect to the size of U and S. We consider separately the cases of dn odd and even. In the former cases almost all d-graphs are d-regular (under the broader definition of Section 2). For dn even, it turns out that for almost all d-graphs $0 \leq |U| \leq 2$ and $|S| = |U|(d-2)$, regardless the value of d. In both cases S is typically an independent set. For convenience, we present our results in the probabilistic form, associating to each n-vertex d-graph the same probability.

Proposition 4: Let P be the uniform probability measure on the set of all d-graphs on the vertex set $V_n = \{ 1,\dots,n \}$. Then

a) $\lim\limits_{\substack{n\to\infty \\ nd \text{ odd}}} P(|U| = 1,\ |S| = d-1) = 1,$

b) $\lim\limits_{\substack{n\to\infty \\ nd \text{ even}}} P(|U| = u,\ |S| = u(d-2)) = \dfrac{(2-\binom{u}{2})(d-1)^u}{d+1},\qquad u = 0,1,2,$

c) $\lim\limits_{n\to\infty} P(S \text{ is an independent set}) = 1.$

Proof: Throughout the proof we use the notations:

$a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$,

$a_n = o(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$,

$a_n = O(b_n)$ if $a_n \leqslant cb_n$ for some $c>0$ and n large enough,

$a_n \asymp b_n$ if $a_n = O(b_n)$ and $b_n = O(a_n)$.

Let us set $u = |U|$, $s = |S|$, and let f be the number of edges from U to S. Denote by $\mathcal{A}(u,s,f)$ the set of all d-graphs on the vertex set V_n and with parameters u, s and f as above. Note that $0 \leqslant u \leqslant d$, $0 \leqslant s \leqslant u(d-u)$, $s \leqslant f \leqslant \min(sd, u(d-u))$. Bender and Canfield [3] proved that the number of graphs on V_n with the degrees $d_1,\ldots,d_n$, $\max d_i \leqslant D$, is asymptotically equal to

$$\exp(-\alpha^2 - \alpha)\ (2q)!/(q!2^q \prod_{i=1}^{n} d_i!)$$

as $n \to \infty$, where $2q = \sum_{i=1}^{n} d_i$ and $\alpha = \frac{1}{4q} \sum_{i=1}^{n} d_i(d_i-1)$.

(Notice that $0 \leqslant \alpha \leqslant (D-1)/2$.)
Applying that result we get

$$|\mathcal{A}(u,s,f)| \asymp n^{u+s} \frac{(dn-du-f)!}{(\frac{1}{2}(dn-du-f)!2^{dn/2}(d!)^n}$$

and, for dn odd,

$$|\mathcal{A}(u,s,f)| \asymp n^{u+s-\frac{1}{2}(du+f+1)}|\mathcal{A}(1,d-1,d-1)|$$

$$= o(|\mathcal{A}(1,d-1,d-1)|)$$

unless $u = 1$ and $s = f = d-1$. This is because $s \leqslant f \leqslant u(d-2)$ for $u \geqslant 2$ and $s = f$ for $u = 1$.
Hence part a) is proved.
Now assumme that dn is even and denote $\mathcal{A}_0 = \mathcal{A}(0,0,0)$, $\mathcal{A}_1 = \mathcal{A}(1,d-2,d-2)$, $\mathcal{A}_2 = \mathcal{A}(2,2d-4,2d-4)$.

Then

$$|\mathcal{A}(u,s,f)|\cdot|\mathcal{A}_0|^{-1} \asymp n^{u+s-\frac{1}{2}(du+f)} = o(1)$$

unless $u \leqslant 2$ and $s = t = u(d-2)$. The reason is that for $u \geqslant 3$ we have $t \leqslant u(d-3)$ and therefore

$$2s \leqslant 2f \leqslant f + u(d-3) < f + u(d-2).$$

Moreover, if $u = 2$ then either of the inequalities $s < f$ and $f < u(d-2)$ implies that $2s < f + u(d-2)$.
Careful calculations show that

$$|\mathcal{A}_1|/|\mathcal{A}_o| \sim d-1 \quad \text{and} \quad |\mathcal{A}_2|/|\mathcal{A}_o| \sim (d-1)^2/2$$

completing the proof of b).
We are left with the proof of c). Let $\mathcal{A}'(u,s,f)$ be the set of all d-graphs belonging to $\mathcal{A}(u,s,f)$ and such that S is an independent set. Denote further by $\mathcal{H}$ the family of all graphs with the vertex set $\{1,2,\ldots,s\}$, with at least one edge and maximum degree at most $d-1$. Given $H \in \mathcal{H}$, let R_H be the number of graphs G on vertices $\{1,\ldots,n-u\}$ with $d_G(i) = d$ for $i = s+1,\ldots,n-u$ and $d_G(i) = d-1-d_H(i)$ for $i = 1,\ldots,s$, where $d_F(v)$ stands for the degree of vertex v in graph F.
With the above notation

$$|\mathcal{A}'(u,s,s)| = O(n^{u+s} \sum_{H \in \mathcal{H}} R_H) = O(\tfrac{1}{n}|\mathcal{A}(u,s,s)|). \quad \blacksquare$$

A weaker version of the above result was proved for $d = 3$ in [1] using a recursion formula of Wormald.

4. 2-processes

Let us recall the question of Erdös:

For a random d-process $(G_o,G_1,\ldots,G_m)$, what is the limit distribution of the number of unsaturated vertices in G_m as $n \to \infty$?

In this section we investigate the case $d = 2$. A connected component of a graph with the maximum degree at most 2 must be either a cycle or a path. We associate to each such graph a triple (a,b,c), called the type of a graph, where a, b, c are the number of isolated vertices, isolated edges, and components being paths of length at least 2, respectively.
Let $(G_o,\ldots,G_m)$ be a 2-process and assume that G_i is of type (a,b,c). Then $a+b+c = n-i$ and G_{i+1} may be one of the following types:

$(a-2,\ b+1,\ c)$, $(a-1,\ b-1,\ c+1)$, $(a,\ b-2,\ c+1)$, $(a-1,\ b,\ c)$, $(a,\ b-1,\ c)$, $(a,\ b,\ c-1)$.

We write $(a,b,c) \rightarrow (a',b',c')$ if (a',b',c') is any of the six triples above. The transition probabilities multiplied by $\binom{a+2b+2c}{2} - b$ are $\binom{a}{2}$, $2ab$, $4\binom{b}{2}$, $2ac$, $4bc$, and $c + 4\binom{c}{2}$, respectively. Thus we have just defined a Markov process whose states are types of graphs and not graphs. Let us denote by $P(a,b,c)$ the probability that in the i-th step, $i = n-(a+b+c)$, the process is in state (a,b,c), equivalently that G_i is of the type (a,b,c).

In particular, $P(0,0,0)$, $P(1,0,0)$, and $P(0,1,0)$ are the probabilities we are interested in, i.e. they are equal to $P(|U_{G_m}| = k)$, $k = 0,1,2$.

There is a simple procedure of computing all $P(a,b,c)$ whose complexity is $O(n^3)$.

Procedure 2:

Set $P(n,0,0) = 1$. For $i = 1,\dots,n$ generate all triples (a',b',c'), $a'+b'+c' = n-i$ (with some further restrictions) and for each triple (a,b,c), $a+b+c = n-i+1$ check if $(a,b,c) \rightarrow (a',b',c')$.

If this is the case multiply $P(a,b,c)$ by the transition probability and add the outcome to the current value of $P(a',b',c')$.

Let us demonstrate how the procedure works in the case $n = 5$ (see Fig. 1).

A sample of the data obtained by the author using his ATARI 130 XE is given in Table 1 below. (The cases of $n = 30,\ 40,\ 48$ were supplied by K. Balinska.) The numbers are rounded to the fourth decimal position.

n	P(0,0,0)	P(1,0,0)
4	.7333	.2667
5	.6296	.2037
10	.7474	.1683
15	.7724	.1586
20	.7875	.1519
25	.7980	.1470
30	.8060	.1432
40	.8173	.1375
48	.8238	.1341

Table 1

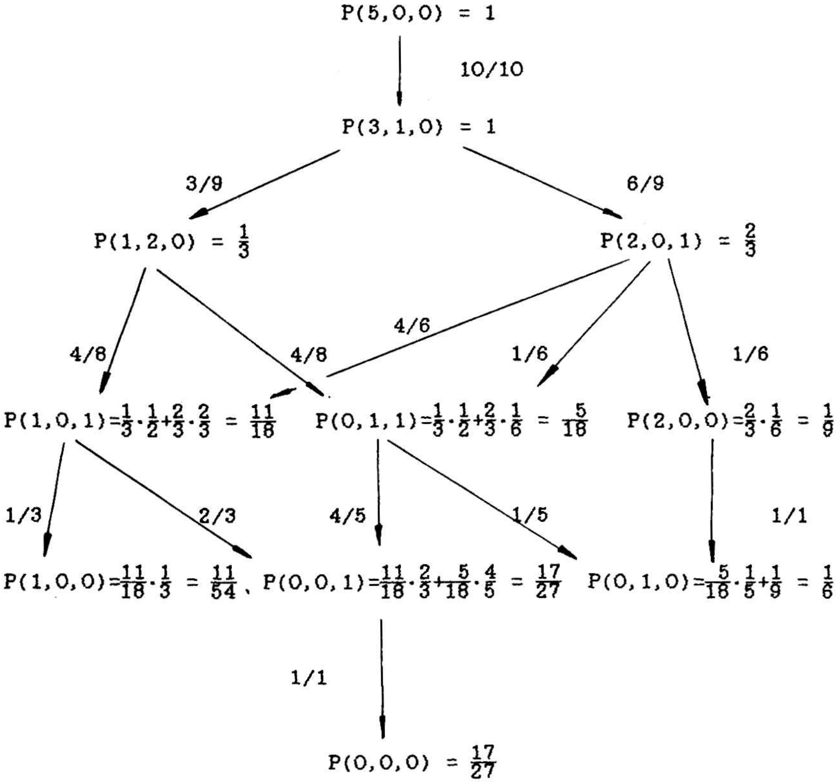
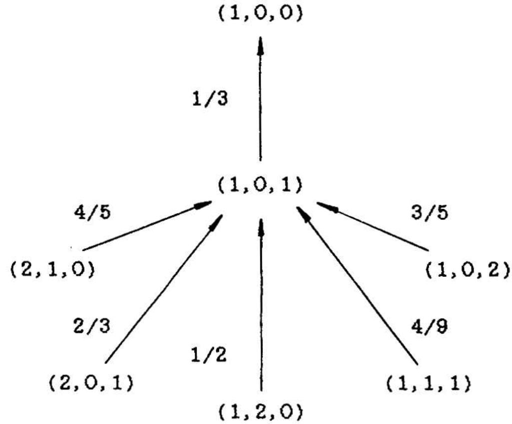


Fig. 1

But what is the limit distribution remains an open question By essentially the same approach but applied backward we get for all n

$$P(1,0,0) \leqslant \frac{4}{15} \quad \text{and} \quad P(0,1,0) \leqslant \frac{1}{5}.$$

The self-explanatory calculations leading to the first inequality together with the corresponding diagram are presented in Fig. 2. The second inequality can be derived similarly.



$$P(1,0,0) = \frac{1}{3} P(1,0,1) = \frac{1}{3} \left(\frac{4}{5} P(2,1,0) + \frac{2}{3} P(2,0,1) + \frac{1}{2} P(1,2,0) + \frac{4}{9} P(1,1,1) + \frac{3}{5} P(1,0,2) \right) \leq \frac{1}{3} \cdot \frac{4}{5}.$$

Fig. 2

If one allows parallel edges, the number of unsaturated vertices in G_m has two-point distribution (0 or 1). In such a case the process can be identified with a 2-dimensional random walk along a special lattice. A simple algorithm with complexity n^2 calculates the distribution. For instance, the probability of no unsaturated vertex is 2/3 for $n = 3$, 7/9 for $n = 4$, 116/150 for $n = 5$, and then it increases but rather slowly to reach, approximately, .835 for $n = 20$, .8597 for $n = 50$, .8973 for $n = 500$, .9087 for $n = 1500$ (the last took 30 hours on ATARI 130 XE).

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Karl Weber

Problems in random communication

We summarize problems and results concerning stochastic communication systems in two directions:

- Either the underlying communication network is fixed but the information transmission is at random
- or the information transmission is deterministic but the communication network is at random.

1. The classical telephone problem (Boyd, A.V.: Problem 3.5. Math. Spectrum 3, 68-69 (1970/71)) asks for a minimum number $g(n)$ of telephone calls for a complete information exchange between n people. (Initially each person has a different piece of information. The people make a sequence of telephone calls in which the two participants exchange all the information they have at the time of the call. If everyone knows everyone else's information then we speak of a complete information exchange.) The solution is $g(n) = 2n - 4$ for $n \geq 4$ (see [6] for a short proof) and was found independently by many authors at the beginning of the 70th. In 1972 J.W. Moon [10] considered the following probabilistic version of this problem. Suppose at every stage two people (among the n) are chosen at random to exchange all the information they know at this time. The choices are made independently of each other and at each stage every pair has the same probability $1/\binom{n}{2}$ to be chosen. Let C_n denote the random number of stages until each person knows all n items of information. (C_n is called the hitting time of complete information exchange.) Moon showed that for any $\varepsilon > 0$ the expectation of C_n satisfies the inequalities

$$(1 - \varepsilon)\, n \ln n < E\, C_n < (2 + \varepsilon)\, n\, (\ln n)^2,$$

and in [2] this was improved to

$$E\, C_n \sim \frac{3}{2}\, n \ln n$$

as $n \longrightarrow \infty$. Independently from Boyd and Steele I proved that

$$(\tfrac{3}{2} - \varepsilon)\, n \ln n < C_n < (\tfrac{3}{2} + \varepsilon)\, n \ln n$$

with probability tending to one (cf. [12]). In the sequel we say C_n is asymptotic to $\frac{3}{2} n \ln n$ almost surely and write $C_n \sim \frac{3}{2} n \ln n$ a.s.

Define C'_n as the hitting time of complete information exchange if we forbid the repeated choice of the same pair, i.e. at each stage all pairs <u>which have not yet been chosen</u> are equiprobable.

<u>Problem 1:</u> Do we get the same asymptotic for C'_n as for C_n?

It is clear that $C'_n \geq \frac{n}{2} \ln n$ a.s. since so many edges are necessary for the connectedness of the random graph formed by considering the calls as edges (cf. [1]).

2. Soon after the solution of the classical telephone problem a lot of variations and generalizations of this problem were considered: variation of the underlying communication network (i.e. not all calls are possible), one-way communication (such as writing a letter), k-party calls (at every stage a complete information exchange between k people takes place) or parallel calls are possible and one asks for the minimum number of stages in these cases. Many of these problems could be solved completely. But there is known almost nothing about probabilistic versions of these problems.

Let be given a connected graph $G = (V,E)$. Now at each stage only those pairs of people are chosen with probability $1/|E|$ which correspond to edges of G. The hitting time for complete information exchange over G is denoted by $C(G)$. Then C_n is just $C(K_n)$ for the complete graph K_n. For the n-path P_n we have

$$C(P_n) \sim n^2 \quad \text{a.s.}$$

For a vertex $x \in V$ denote by $C_x(G)$ the number of calls until everyone knows the information of person x. Define the graph $G_{2n} = (V,E)$ with $V = \{x_1, x_2, \ldots, x_{2n}\}$ and $E = \{x_i x_{i+1}\colon 1 \leq i \leq n\} \cup \{x_i x_j\colon n+1 \leq i < j \leq 2n\}$. Then

$C_{x_1}(G_{2n}) \sim n^3/3$ a.s. (cf. [12]).

It follows that

$C\ (G_{2n}) \gtrsim n^3/3$ a.s.

Graham Brightwell [3] wrote me that he has the following results:

$C_x(G) \geq \frac{n}{2} \ln n$ a.s.

for any x in any connected n-graph G, and this bound is best possible.

$C\ (G) \gtrsim n \ln n$ a.s. and $C\ (G) = O(n^3)$ a.s.

for any connected n-graph G, and these bounds are also best possible.

<u>Problem 2:</u> Determine $C(Q_n)$ for the n-cube Q_n.

The deterministic version of the problem is completely solved. Define $\gamma(G)$ as the minimum number of calls for complete information exchange along the edges of G. Then $\gamma(G) \leq 2n - 3$ for every connected n-graph G and that $\gamma(G) \leq 2n - 4$ if G contains additionally a 4-cycle is easy to show. Considering that $g(n) = \gamma(K_n) = 2n - 4$ it follows that $\gamma(G) = 2n - 4$ for almost all n-graphs G by the well-known fact that almost all n-graphs contain a 4-cycle.

3. The directed case may easily be settled. Let at each stage the communication be only one-way, such as writing a letter, and assume that for each chosen pair of people both directions of communication are equiprobable. The underlying communication network is the complete directed graph $\vec{K}_n$, and at each stage every arc of $\vec{K}_n$ has the same probability $1/n(n-1)$ to be chosen. For the number D_n of stages until each person knows all n items of information I proved (cf. [12]) that

$D_n \sim 3\ n \ln n$ a.s.

Note that the minimum number of letters which have to be written

for the complete information exchange is $2n - 2$.
In the case of k-party calls the minimum number $f(n,k)$ of k-party calls for complete information exchange is

$$f(n,k) = \begin{cases} \lceil \frac{n-k}{k-1} \rceil + \lceil \frac{n}{k} \rceil, & 1 \leq n \leq k^2, \\ 2 \lceil \frac{n-k}{k-1} \rceil, & n \geq k^2 \end{cases}$$

(cf. [6]). Define $C_{n,k}$ as the hitting time for complete information exchange if the k-party calls are randomly chosen and at each stage every k-subset has the same probability $1/\binom{n}{k}$ to be chosen.

<u>Problem 3:</u> Determine $C_{n,k}$ for $k \geq 3$.

We return to the calls between two people. Denote by $t(n)$ the minimum number of stages for complete information exchange when parallel calls are possible in such a way that at each stage each person calls at most one other person. Knödel [7] showed that

$$t(n) = \begin{cases} \lceil \log_2 n \rceil, & \text{if } n \text{ is even,} \\ \lceil \log_2 n \rceil + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Let n be even. Then K_n contains exactly $\binom{n}{2}\binom{n-2}{2} \ldots \binom{4}{2}/\frac{n}{2}!$ differrent perfect matchings. Now at every stage we chose at random one of these perfect matchings (all with equal probability) and the corresponding calls are made.

<u>Problem 4:</u> Determine the number of perfect matchings for complete information exchange.

Varying the underlying network also in the case of parallel calls let $G = (V,E)$ be a connected graph. Define $\tau(G)$ as the minimum number of stages for complete information exchange if at every stage parallel calls are possible (but only along the edges of G and so that two edges corresponding to parallel calls have no common vertex). Then $t(n) = \tau(K_n)$ and the problem of characterization of those n-graphs G with $\tau(G) = t(n)$ is not yet solved.

Problem 5: Is it true that $\tau(G) = t(n)$ for almost all n-graphs G?

I only know that $\tau(G) < 2\lceil \log_2 n\rceil - 1$ for almost all n-graphs G (cf. the result of J. Wierman mentioned below).

4. Suppose at the beginning among n people only one person knows a certain secret. At the first stage that person chooses someone at random to reveal him the secret. At each stage that person who was told the secret last chooses someone randomly from the n people (each - including himself - with the same probability $1/n$), calls that person and reveals him the secret. The choices are made independently of each other, and the number of calls until everyone knows the secret is denoted by A_n.
I proved

$$A_n \sim n \ln n \quad \text{a.s.},$$

and this result remains also true if we exclude that a person chooses himself. The result remains even true if we require more restrictions, e.g. a person cannot call that person who called him at the stage before ([12]).
Now again let be given an arbitrary connected n-graph $G = (V,E)$ as the underlying communication network, i.e. calls are possible only along the edges of G. Assume further that at each stage that person $v \in V$ who was told the secret last chooses at random one of the $d(v)$ neighbours of v in G - each with the same probability $1/d(v)$ - calls that person and tells him the secret. If at the beginning only $x \in V$ knows the secret then $A_x(G)$ denotes the number of independent stages for complete information transmission from x, i.e. until everyone knows the secret. We write $A(G)$ if $A_x(G)$ is the same for all $x \in V$. Then $A_n = A(K_n)$.

Problem 6: Determine $A(Q_n)$ for the n-cube Q_n.

In our last model the secret follows a random walk through the connected graph G starting at the vertex x. In a recent paper G. F. Lawler [9] investigated the expected length $e_G(x,y)$ of a random walk on G with starting point x until it meets the vertex $y \neq x$ for the first time. He showed that

$$e_G(x,y) \leq \Delta\, n(n-1)$$

for every connected n-graph $G = (V,E)$ with maximum degree Δ and all $x, y \in V$, $x \neq y$. Moreover this bound is sharp in order since, for example

$$e_{P_{n+1}}(x_1,\ x_{n+1}) = n^2$$

for the n+1-path P_{n+1} with end vertices x_1 and x_{n+1} and

$$e_{G_{2n}}(x_i,\ x_1) = n^3 + O(n^2)$$

where G_{2n} is defined as above, and $i > n$.

Suppose again that initially among the n people only one person knows a certain secret. But now at each stage <u>all</u> people who know the secret choose independently of each other someone at random from the n people (each - including himself - with the same probability $1/n$) to tell him the secret. Denote by B_n the number of stages for complete information transmission. A. Frieze and G. Grimmett [5] showed that

$$B_n \sim (1 + \ln 2) \log_2 n = 1.69\ldots \log_2 n \quad \text{a.s.}$$

Such a communication process (from one to all, parallel calls are possible) is known as broadcasting (see, for example, [4]). Since the communication network is represented by the complete graph K_n the minimum number of stages for complete information transmission (complete broadcasting) from an arbitrary vertex x (the broadcast time of the vertex x) is $\lceil \log_2 n \rceil$. For a connected graph $G = (V,E)$ and $x \in V$ the broadcast time of x is denoted by $b_x(G)$ and the broadcast time $b(G)$ of G is defined as $\max\limits_{x \in V} b_x(G)$.
Clearly, $\lceil \log_2 n \rceil \leq b_x(G) \leq n - 1$ for all connected n-graphs $G = (V,E)$ and all vertices $x \in V$.
Answering to a question of me ([13]) John Wierman showed that

$$b(G) = \lceil \log_2 n \rceil$$

for almost all n-graphs G ([11]). The bounds $\lceil \log_2 n \rceil \leq b_x(T_n) \leq n - 1$ are also sharp for trees T_n with n vertices.

Obviously, $n - 1$ is the maximum broadcast time for an n-tree.

Roger Labahn [8] constructed trees T_n with minimum broadcast time. He proved that

$$\min_{T_n} b(T_n) \sim \frac{1}{\log_2(1+\sqrt{5}) - 1} \log_2 n = 1.44\ldots \log_2 n$$

where the minimum is taken over all n-trees T_n.

Problem 7: Determine $b(T_n)$ for almost all n-trees T_n.

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Strongly essential variables of functions

In this paper we investigate the properties of functions with a limited number of strongly essential variables.

1. Introduction

In 1962, treating a problem connected to a class of schemes of functional elements, O.B. Lupanov [2] proved that every function from the algebra of logic depending strongly on at least two variables has at least one strongly essential variable.

N.A. Solovev [3] proved that if a function from algebra of logic depends essentially on at least two variables, then every variable is at least of first order towards separable pairs.

The terms "separable pair" and "strongly essential variable" for the functions from the algebra of logic were used firstly by Breithbart (1967). Almost at the same time - as a general conclusion - K.N. Čimev introduced the term "separable m-tuples" for the functions of k-valued logic and then for random functions.

In this paper we use the symbols and terminology of [1-6].

Definition 1: We say that the function $f(x)$ depends essentially on x, if it has at least two values.

Definition 2: We say that the function $f(x_1,x_2,\ldots,x_n)$ depends essentially on x_i, $1 \leq i \leq n$, if there exist such values of the rest of the variables that after replacing them in f we obtain a function which depends essentially on x_i.

The set of all variables essential for the function f we will denote by R_f.
By F_n we will denote the set of all functions $f(x_1,\ldots,x_n)$, $n \geq 1$, which depend essentially on n-variables.

Definition 3: If $f \in F_n$, $n \geq 2$, we will call the variable x_i strongly essential for the function f, if there exist such values c_i of it, that the function $f(x_i=c_i)$ depends strongly essential on every variable which belongs to the set $R_f \setminus \{x_i\}$.

With R_f^* we will denote the set of all strongly essential variables of the function f.

Definition 4: If $f \in F_n$, $n \geq 2$, and $M \subseteq R_f$, then we say that the set M is separable for the function f, if there exist such values for the variables of $R_f \setminus M$ that after replacing them f is a function which depends essentially on every one of the variables from M.

If the set $\{x_1, x_2, \dots, x_m\}$ is separable for te function f, then we will say that the m-tuple $(x_1, x_2, \dots, x_m)$ is separable for f.

With $S_{f,m}$ we will denote the set of separable m-tuples for the function f.
We obtain from Definition 3 and 4 that a variable x is strongly essential for a certain function, when the set $R_f \setminus \{x\}$ is separable for it.

Example 1: For the function

$$f(x_1, x_2, x_3, x_4) = x_1\bar{x}_2 + x_2x_3x_4 \pmod 2 \quad \text{one obtains}$$

$$R_f = \{x_1, x_2, x_3, x_4\} \quad \text{and}$$

$$R_f^* = \{x_1, x_3, x_4\},$$

$$S_{f,2} = \{(x_1, x_2), (x_2, x_3), (x_2, x_4), (x_3, x_4)\}.$$

Breithbart proved in [4] that every function from the algebra of logic which essentially depends on at least two variables has at least two strongly essential variables. Čimev proved in [5] Breithbart's statement firstly for k-valued functions and then for random functions.

2. Some properties of the functions of P_k

Definition 5: Let $P_k(Q)$ be the set of all k-valued functions which have the quality Q. We say, that "almost all" functions of P_k have the quality Q, if

$$\lim_{n \to \infty} \frac{|P_k(Q)|}{k^{k^n}} = 1.$$

As every function $f \in F_n$, $n > 2$, has at least two strongly essential variables, so do exist such functions for which every variable is strongly essential?

The following function proves the positive answer to that question:

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3 \pmod 2.$$

For the function of the k-valued logic there the following theorem holds:

Theorem 1 (Denev, Gjudženov): For "almost all" functions of P_k each set of their essential arguments is separable.

As we say that for "almost all" functions they will be separable of P_k there is a separable set of essential arguments, so for every (n-1)-set of elements. This proves the following

Theorem 2: To "almost all" functions of P_k every essential variable is strongly essential.

Definition 6: If $f \in F_n$, $n > 2$, then we will denote the variable $x_i \in R_f$ c-strongly essential for the function f, if for every one of its values c_i the function $f(x_i = c_i)$ depends essentially on every variable belonging to the set $R_f \setminus \{x_i\}$.

Theorem 3: To "almost all" functions of P_k every essential variable is c-strongly essential.

The proof is cited in [11].

3. Separable pairs of functions with a definite number of strongly essential variables

In despite of the fact that to "almost all" functions of P_k every essential variable is strongly essential, we can give examples that there are functions which have exactly 2, exactly 3, and so on ... exactly k strongly essential variables, where $2 \leq k \leq n-1$.

The properties of functions with two strongly essential variables are studied by K. Čimev and published in [6]. The properties of functions with 3 strongly essential variables are studied in [8]. As we see in [12], Corollary of Theorem 9.14, there is a certain number of functions with 4 strongly essential variables except two cases which can be solved due to the following propositions.

Let the function $f \in F_n$, $n \geq 5$, have 4 strongly essential variables which we denote by x_1, x_2, x_3, x_4.

Assertion 4: If x_1, x_2, x_3, x_4 build up exactly separable pairs and $(x_1, x_2) \in S_{f,2}$, $(x_2, x_3) \in S_{f,2}$, so every of the variables of $R_f \setminus R_f^*$ builds up a separable pair with x_2 and at least one with x_1, x_2, x_4.

Assertion 5: If x_1, x_2, x_3, x_4 build up exactly 3 separable pairs and exactly one of them doesn't take part at a separable pair, so every variable of $R_f \setminus R_f^*$ builds up separable pairs with at least 3 of R_f^*.

The proof is given in Theorems 4.3 and 1.6 from [6].

In [12] some properties of the functions with k strongly essential variables are given. We cite some of the properties of functions which we add to the studies in [12].

Lemma 6: If for the function $f \in F_n$, $n \geq 3$, every variable of $R_f \setminus R_f^*$ builds up a separable pair with every one of R_f^*, so the subgraph of f, where the vertices are the elements of $R_f \setminus R_f^*$, is full.

Proof: Let us assume the opposite. Without loss of generality let us assume that $R_f^* = \{x_1, x_2, \ldots, x_k\}$, $k \geq 2$, and the subgraph of f, where the vertices are the elements of $R_f \setminus R_f^*$, isn't full. Besides let us assume $(x_{k+1}, x_{k+2}) \notin S_{f,2}$ too.

From Theorem 4.3 of [6] it follows that for every value c_{k+1} of x_{k+1} the variable x_{k+2} keeps its order. But x_{k+2} builds up separable pairs with every element of R_f^*. Hence

$$\{x_1, x_2, \ldots, x_k\} \subset R_{f(x_{k+1} = c_{k+1})},$$

and thereby it follows that at least one of the variables of $R_f \backslash R_f^*$ is strongly essential for f, what contradicts the condition. Hence $(x_{k+1}, x_{k+2}) \in S_{f,2}$.

Definition 7: With "graph" of the function $f \in F_n$, $n \geqslant 2$, we will denote an unoriented graph whose vertices are the essential variables of the function f and edges - the separable pairs.

Theorem 7: The graph of a function $f \in F_n$, $n \geqslant 4$, is full if and only if its subgraph whose vertices are build up by the strongly essentials is full.

The proof is given in Theorem 9.13 from [12] and Lemma 6.

From Theorem 7 it follows that the graph of a function $f \in F_n$, $n \geqslant 4$, isn't full if and only if its subgraph, whose vertices are the strongly essential variables of f, isn't full.

Theorem 8: If $f \in F_n$, $n \geqslant 4$, $|R_f^*| = k$ and every variable of R_f^* build up separable pairs with at least $(k-2)$ of the elements of R_f^* then:

a) every variable of $R_f \backslash R_f^*$ builds up a separable pair with every strongly essential variable;
b) the subgraph of f, whose vertices are the elements of $R_f \backslash R_f^*$, is full.

Proof: a) Let be assumed by the conditions given, $f \in F_n$, $n \geqslant 4$, and without loss of generality $R_f^* = \{x_1, x_2, \ldots, x_k\}$. Thus every one of the variables $x_1, x_2, \ldots, x_k$ doesn't build up a separable pair with one of them. We will prove that every one of the variable of $R_f \backslash R_f^*$ builds up a separable pair with every one of the variables $x_1, x_2, \ldots, x_k$.
Let x_p be one of the variables $x_{k+1}, x_{k+2}, \ldots, x_n$. Let us suppose that it doesn't build up a separable pair with at least

one of R_f^* and this is the variable x_1.
From Theorem 4.3 of [6] it follows that for every value c_p of x_p the variable x_1 keeps its order. But x_1 doesn't build up a separable pair with at least one of R_f^*.

1. If x_1 builds up separable pairs with every one of $R_f^*\setminus\{x_1\}$, then from Theorem 4.3 of [6] it follows that $\{x_1, x_2, \dots, x_k\} \subset R_{f(x_p=c_p)}$, i.e. at least one of the variables $x_{k+1}, x_{k+2}, \dots, x_n$ is strongly essential for f what contradicts the conditions mentioned. Thus $(x_p, x_2) \in S_{f,2}$.
2. Let x_1 doesn't build up a separable pair with exactly one of $R_f^*\setminus\{x_1\}$ and that is x_2. We already said that the variable x_1 keeps its value c_p' of x_p, for which the function $f(x_p=c_p')$ depends essentially on x_2. Then it follows that $\{x_1, x_2, \dots, x_k\} \subset R_{f(x_p=c_p')}$ and hence at least one of the variables $x_{k+1}, x_{k+2}, \dots, x_n$ is strongly essential for the function f. This contradicts the conditions considered. Hence $(x_p, x_1) \in S_{f,2}$.

b) Lemma 6 proves the Theorem.

<u>Corollary 1:</u> Let the function $f \in F_n$, $n \geqslant 6$, have 5 strongly essential variables, and the subgraph of f, whose vertices are the elements of $R_f\setminus R_f^*$, have the picture on Fig. 1, Fig. 2, and Fig. 3. Then the number of the separable pairs is equal to $c_n^2 - 2$, $c_n^2 - 1$, c_n^2.

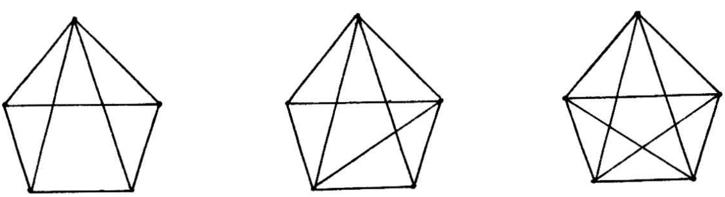

Fig. 1 Fig. 2 Fig. 3

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Hyperidentities in multiple-valued logics

1. Introduction

Clones of functions defined on a set A are collections of functions closed by superposition and containing all projections. Clones are investigated in two- and multiple-valued logics and its applications in the synthesis of networks. Moreover, clones play an essential role in universal algebra.
The expectation in practice for multiple-valued logic in contrast to two-valued logic exists in its higher information density without increasing the size or complexity of the VLSI. Recently, first results were obtained for developing optical three-valued chips ([4]). The major problems in the synthesis of networks are the following: The first one is to find an efficient completeness criterion to decide whether a given set of functions is complete or not, i.e. whether all functions defined on the set A can be constructed from the functions of the given set. The second one is to find criterions to determine whether a subset of a clone is a base for this clone or not. The third one is to investigate an optimum construction of a network from a given base.

Since most properties of a universal algebra depend on the clone of its term functions rather than the choice of its basic operations, in the last fifteen years the theory of clones has become an integral part of universal algebra. The attempt to solve the main problems in the synthesis of networks using concepts and methods of universal algebra led to hyperidentities and its application in multiple-valued logic.

Consider an equation $\sigma = \varrho$ where σ and ϱ are terms constructed from individual variables and operation symbols of a given type, for example

$$F(F(x,y),y) = F(x,y). \qquad (*)$$

Let A be a set, then the function $f: A^2 \longrightarrow A$ fulfils the equation (*) if $f(f(x,y),y) = f(x,y)$ holds for all $x,y \in A$.

Let C be a clone of functions (defined on A). Then we say that $\sigma = \varrho$ is a hyperidentity of C, in sign $C \models \sigma = \varrho$, if $\sigma = \varrho$ holds identically in C for every choice of functions of C with appropriate varieties to represent the operation symbols appearing in σ and ϱ.
If $\underline{A} = (A;G)$ is an algebra of type τ then $\sigma = \varrho$ is a hyperidentity of $\underline{A}$ ($\underline{A} \models \sigma = \varrho$), if $\sigma = \varrho$ is a hyperidentity of the clone of term functions $T(\underline{A})$ of $\underline{A}$. For instance, for any finite set A we consider the set of all unary functions H_A defined on A. Then for $|A| = n > 1$ we have the hyperidentity $H_A \models \varphi^{n-1}(x) = \varphi^{n-1+\varkappa(n)}(x)$ where $\varkappa(n)$ denotes the least common multiple of the numbers $1, \ldots, n$. That means that for all $f \in H_A$ $f^{n-1}(x) = f^{n-1+\varkappa(n)}(x)$ is identically fulfilled.

In this paper we will give a survey on the application of hyperidentities in multiple-valued logic. Our investigation is connected with the first and the second main problem in the synthesis of networks.

2. Clone identities and hyperidentities

Let C be a clone of functions defined on the set A. Then C can be regarded as a universal algebra $\underline{C} = (C;*,\xi,\tau,\Delta,e_1^2)$ of type $(2,1,1,1,0)$ where the operations are defined in the following manner ([3]):

(i) $(f*g)(x_1,\ldots,x_{m+1},\ldots,x_{m+n-1}) = f(g(x_1,\ldots,x_m),x_{m+1},\ldots,x_{m+n-1})$,
f n-ary, g m-ary, $f,g \in C$,

(ii) $(\xi f)(x_1,\ldots,x_n) = f(x_2,\ldots,x_n,x_1)$,

(iii) $(\tau f)(x_1,\ldots,x_n) = f(x_2,x_1,x_3,\ldots,x_n)$,

(iv) $(\Delta f)(x_1,\ldots,x_{n-1}) = f(x_1,x_1,x_2,\ldots,x_{n-1})$, $\quad n > 1$,
$(\xi f)(x_1) = (\tau f)(x_1) = (\Delta f)(x_1) = f(x_1)$,

(v) $e_1^2(x_1,x_2) = x_1$ (projection).

Let $\underline{O}_A$ denote the clone of all functions f defined on A and let $O_A^{(n)}$ be the set of all n-ary functions of O_A
$(O_A = \bigcup_{n\geq 1} O_A^{(n)})$.

For $C \subseteq O_A$ by $\langle\underline{C}\rangle$ we denote the clone generated by C. The crucial point in clone theory is the investigation of the lattice $S(\underline{O}_A)$ of all subclones of $\underline{O}_A$. For $|A| = 2$ this problem was solved by E.L. Post [5]. There are countably many clones on a two-element set, each of them is finitely generated. Post described all these subclones and gave generating systems. For $|A| > 2$ the cardinality of $S(O_A)$ is continuum and not all subclones of O_A are finitely generated. For every finite set A $S(O_A)$ is an atomic and dually atomic lattice with finitely many atoms and dual atoms. The dual atoms of $S(O_A)$ are called maximal clones (maximal classes). For a finite set A ($|A| > 2$) all maximal clones were determined by I.G. Rosenberg [8].

Since a clone is a universal algebra of a certain type we can consider identities of $\underline{C}$ (clone identities) and the variety (equational class) $V(\underline{C})$ generated by $\underline{C}$. By a result of W. Taylor [9] every clone identity of the clone $\underline{C}$ corresponds to a hyperidentity of an algebra $\underline{A} = (A;F)$ with $T(\underline{A}) = \langle \underline{F} \rangle = C$.

The following questions naturally arise:

I. Let $\underline{C}_1$ and $\underline{C}_2$ be clones on A regarded as algebras of type (2,1,1,1,0) such that $\underline{C}_1 \subseteq \underline{C}_2$. What can we say about the inclusion of the varieties generated by $\underline{C}_1$, resp. by $\underline{C}_2$: $V(\underline{C}_1)$ and $V(\underline{C}_2)$?

Is $V(\underline{C}_1) \subsetneq V(\underline{C}_2)$ (i.e. $Id(\underline{C}_1) \supsetneq id(\underline{C}_2)$)? Using the correspondence between clone identities and hyperidentities we have to find a hyperidentity in $\underline{C}_1$ which is no hyperidentity in $\underline{C}_2$. This problem is closely connected with the following one.

II. Characterize the subclones of O_A by hyperidentities: For this purpose construct a set Ω of hyperidentities such that $F \subseteq C$ generates the clone $\underline{C}$ iff $\forall_{\varepsilon \in \Omega} \langle \underline{F} \rangle \not\models \varepsilon$. This is the completeness problem for the clone $\underline{C}$.

To solve these problems we introduce the concept of a characterizing set of hyperidentities of the clone $\underline{C}$. Let $HI(\underline{C})$ be the set of all hyperidentities of the clone $\underline{C}$.

<u>Definition 2.1</u>: $\Sigma \subseteq HI(\underline{C})$ is called a characterizing set of hyperidentities of $\underline{C}$, in sign $CHI(\underline{C})$, if for every clone $\underline{C}'$ with $\underline{C} \subset \underline{C}'$ there is a hyperidentity in Σ which is not contained in $HI(\underline{C}')$ and if Σ is minimal with respect to this property.

(We write $\underline{C}' \not\models \Sigma$ if there exists one hyperidentity in Σ which is not fulfilled in $\underline{C}'$). If a characterizing set of hyperidentities of a clone $\underline{C}$ consists only of one hyperidentity ε then ε is called a characterizing hyperidentity of $\underline{C}$. Let $\underline{C}_1$ and $\underline{C}_2$ be clones of functions defined on a set A and assume that $C_1 \subset C_2$. If $\underline{C}_1 \not\models 6 = e$ then $\underline{C}_2 \not\models 6 = e$.

Therefore, to determine a characterizing set of hyperidentities of a clone $\underline{C}$ for every clone $\underline{C}'$ in which $\underline{C}$ is a maximal subclone (suppose that such clones do exist) we have to find a hyperidentity ε with $C \models \varepsilon$, but $C' \not\models \varepsilon$. Clearly, we have

Proposition 2.2: A clone $\underline{C}$ ($\subseteq \underline{O}_A$) has a characterizing set of hyperidentities iff for every clone $\underline{C}' \subseteq O_A$ with $\underline{C} \subseteq \underline{C}'$ we have $V(\underline{C}) \subsetneq V(\underline{C}')$.

To solve the second problem we remember the following proposition:

Lemma 2.3 ([5]): Let $\underline{C}$ be a finitely generated clone. Then $F \subseteq C$ is a generating system of $\underline{C}$ iff $\langle\underline{F}\rangle$ is not contained in one of the maximal subclones of $\underline{C}$.

Then we obtain:

Theorem 2.4: Let $\underline{C}$ be a finitely generated clone of functions on a set A, let $F \subseteq C$ and let M be the set of all maximal subclones of $\underline{C}$. Then $\langle\underline{F}\rangle = \underline{C}$ iff $\forall_{C \in M} \langle\underline{F}\rangle \not\models CHI(C_m)$.

Proof: 1. If $\langle\underline{F}\rangle = \underline{C}$ then $\langle\underline{F}\rangle \not\models CHI(C_m)$ for every maximal subclone C_m of C by definition of a characterizing set of hyperidentities.

2. If $\langle\underline{F}\rangle \not\models CHI(C_m)$ then $\langle\underline{F}\rangle \nsubseteq \underline{C}_m$ for every $\underline{C}_m \in M$ thus $\langle\underline{F}\rangle = \underline{C}$.

Clearly, every set of characterizing hyperidentities of a maximal subclone of O_A is one-element, i.e. a characterizing hyperidentity.

3. Characterization of all clones on $\{0,1\}$ by hyperidentities

Given a k-ary relation ϱ on a set A (i.e. $\varrho \subseteq A^k$, $k \geq 1$, $k \in \mathbb{N}$). Then we say that a function $f : A^n \longrightarrow A$ preserves $\varrho \subseteq A^k$ iff $(x_1,\dots,x_k), (y_1,\dots,y_k), \dots, (z_1,\dots,z_k) \in \varrho$ implies $(f(x_1,y_1,\dots,z_1),\dots,f(x_k,y_k,\dots,z_k)) \in \varrho$ for all $x_i,y_i,z_i \in A$, $i = 1,\dots,k$. Pol ϱ denotes the set of all functions from O_A preserving ϱ, i.e. Pol $\varrho = \{f \in O_A | \ f$ preserves $\varrho\}$ ("polymorphisms" of ϱ, [6]).

Sets of the form Pol ϱ are always clones.

To characterize all clones on $\mathbf{2} := \{0,1\}$ by hyperidentities at first we introduce some notations for Boolean functions and clones of Boolean functions ([2]):

c_a^n for the n-ary constant Boolean function with value $a \in \mathbf{2}$,

e_i^n for the n-ary projection with $e_i^n(x_1,\dots,x_n) = x_i$, $n \geq 1$, $n \in \mathbb{N}$, $i \in \{1,\dots,n\}$,

vel for the disjunction: $\mathrm{vel}(x,y) := x \vee y$,

et for the conjunction: $\mathrm{et}(x,y) := x \wedge y$,

aut for the addition mod 2: $\mathrm{aut}(x,y) := x + y$,

non for the negation: $\mathrm{non}(x) := \bar{x}$,

$M :=$ Pol$\leq$, where $\leq$ is the ordinary order of the integers 0,1,

$S :=$ Pol $\{(0,1),(1,0)\}$,

$L := \bigcup_{n\geq 1} \{f \in O_2^{(n)} | \ \underset{a_0,\dots,a_n\in 2}{\exists} f(x_1,\dots,x_n) = a_0 + \sum_{i=1}^{n} a_i x_i\}$,

$T_{a,m} :=$ Pol $2^m \setminus \{(\bar{a},\dots,\bar{a}) \mid a \in \mathbf{2}\}$, $m \geq 1$, $m \in \mathbb{N}$,

$T_a := T_{a,1}$,

$T_{a,\infty} := \bigcap_{m\geq 1} T_{a,m}$,

$K := \langle\{\mathrm{et}\}\rangle$,

$D := \langle\{\mathrm{vel}\}\rangle$,

$I := \langle\{e_1^1\}\rangle$, $\bar{I} := \langle\{\bar{e}_1^1\}\rangle$, $C_a := \langle\{e_1^1, c_a^1\}\rangle$, $a \in \mathbf{2}$,

$C := \langle\{\bar{e}_1^1, c_0^1\}\rangle$.

In case of existing characterizing hyperidentities the following table gives a survey which characterzing hyperidentity is fulfilled in which Boolean clone.

Boolean clone	characterizing hyperidentity
T_a	$G(F(x,x),F(x,x)) = F(G(G(x,x),G(x,x)),$ $G(G(x,x),G(x,x)))$
$T_o \cap T_1$	$G(x,y,z) = F(G(x,y,z),\ G(x,y,z),\ G(x,y,z))$
M	$F(x,x) = F(F(x,y),\ F(x,y))$
$M \cap T_a$	$F(G(x,y),\ G(x,y)) = G(F(x,F(x,x)),\ F(F(y,y),y))$
$M \cap T_o \cap T_1$	$F(x,y,z) = F(F(x,G(y,y,y),z),y,z)$
S	$F(F(x,x,x),\ F(x,x,x),\ F(x,x,x)) =$ $F(F(x,x,y),\ F(F(x,x,y),x,x),\ F(x,F(z,z,x),x))$
$S \cap T_a$	$F(x,x,x) = F(F(x,x,y),\ F(F(x,x,y),x,x),$ $F(x,F(z,z,x),x))$
$S \cap M$	$F(x,x,x) = F(F(x,F(x,z,x),y),\ F(z,x,F(x,x,y)),$ $F(F(y,x,x),z,z))$
L	$F(x,x) = F(F(F(x,y),y),\ F(y,F(y,x)))$
$L \cap T_a$	$F(G(x,x),\ G(x,x)) = G(F(G(x,F(y,y)),\ G(x,F(x,y))),$ $F(G(F(y,x),x),\ G(F(y,y),x)))$
$L \cap S$	$F(x,x,x) = F(x,F(y,F(z,x,y),F(y,z,z)),$ $F(z,F(z,z,y),F(y,z,x)))$
$L \cap S \cap T_a$	$F(x,x,G(G(x,y,y,),\ G(y,x,y),\ G(y,y,x))) =$ $G(x,G(y,G(z,x,y),\ G(y,zy)),\ G(z,G(z,z,y),G(y,z,x))$
$K\ (D)$	$F(F(x,G(x,y)),\ F(G(y,x),x)) = F(x,F(G(y,y),x))$
$K \cup C_o(D \cup C_1)$	$G(F(x,y,x),\ F(x,x,x),\ F(x,y,x)) = F(G(x,x,x),$ $G(G(y,x,x),x,x)\ G(x,x,x))$
$K \cup C_1(D \cup C_o)$	$F(F(x,x),x) = F(x,F(G(x,x),x))$
$K \cup C\ (D \cup C)$	$F(F(x,y,x)\ F(x,x,y)\ F(y,x,x)) =$ $F(F(x,G(y,y,y),G(y,y,y)),F(x,x,G(y,y,y)),F(x,x,x))$
C	$F(F(x,x),\ F(y,x)) = F(F(x,y),\ F(y,x))$
$C_o \cup C_1$	$F(x,x) = F(F(x,y),\ F(y,x))$
C_a	$G(F(x,x),\ F(x,x)) = F(G(x,x),\ G(G(x,y),x))$
I	$F(G(x,x,x),\ G(y,y,y),\ G(z,z,z)) =$ $G(F(x,y,z),\ F(x,y,z),\ F(x,y,F(x,y,F(z,z,z)))$
I	$F(x,x,x) =$ $G(F(x,F(y,x,z),\ F(y,y,x)),\ F(F(x,y,z),x,F(y,z,x)),$ $F(F(x,y,z),\ F(y,x,z),x))$

Boolean clone	characterizing hyperidentity
$T_{a,2}$	$G(F^{\oplus},F^{\oplus},F^{\oplus}) = G(F^{+},F^{+},F^{+})$ with $F^{\oplus} := F(F(x,x,G^{+}),\ F(G^{+},x,x),\ F(x,G^{+},x))$ and $F^{+} := F(F(x,x,y),\ F(y,x,x),\ F(x,y,x))$ $G^{+} := G(G(y,y,x),\ G(x,y,y),\ G(y,x,y))$
$T_{a,m}$, $m>2$, $m\in\mathbb{N}$	$G(Q_1,Q_1) = G(Q_2,Q_2)$ with $Q_1 := F(F(x,R^{+},y,\dots,y),\ F(y,x,R^{+},y,\dots,y),\dots,$ $F(y,\dots,y,x,R^{+}),\ F(R^{+},y,\dots,y,x)),$ $Q_2 := F(F(x,y,\dots,y),\ F(y,x,y,\dots,y),\dots,$ $F(y,\dots,y,x,x\wedge y),\ F(x\wedge y,y,\dots,y,x)),$ $R^{+} = R(R(y,x),\ R(x,y))$
$T_{a,2} \wedge M$	$G(F^{\oplus},F^{\oplus},F^{\oplus}) = F(F_1^{+},F_2^{+},F_3^{+})$ with $F_1^{+} := F(F(x,x,x),\ F(x,y,x),\ F(x,x,x)),$ $F_2^{+} := F(F(x,x,x),\ F(x,x,x),\ F(x,x,y)),$ $F_3^{+} := F(F(y,x,x),\ F(x,x,x),\ F(x,x,x))$
$T_{a,2} \wedge T_{\bar{a}}$	$F(F(x,x,G^{+}),\ F(G^{+},x,x),\ F(x,G^{+},x)) =$ $F(F(x,x,y),\ F(y,x,x),\ F(x,y,x))$
$T_{a,2} \wedge T_{\bar{a}} \wedge M$	$G(F^{\oplus},F^{\oplus},F^{\oplus}) = G(F',F',F')$ with $F' := F(F_1^{+},F_2^{+},F_3^{+})$
$T_{a,m} \wedge M$, $m>2$, $m\in\mathbb{N}$	$h_1 = h_2$ $h_1 := F(F(x,y,x,y,\dots,y),\ F(y,x,x,y,\dots,y),$ $F(y,y,x,y,\dots,y),\dots,F(y,\dots,y,x)),$ $h_2 := F(F(x,y,\dots,y,x),\ F(y,x,y,\dots,y,x),$ $F(y,y,x,y,\dots,y),\dots,F(y,\dots,y,x))$
$T_{a,m} \wedge T_{\bar{a}}$, $m>2$, $m\in\mathbb{N}$	$H(Q_1) = H'(Q_2)$
$T_{a,m} \wedge M \wedge T_{\bar{a}}$, $m>2$, $m\in\mathbb{N}$	$G(h_1) = G'(h_2)$

For the Boolean clones $T_{a,\infty}$, $T_{a,\infty} \wedge M$, $T_{a,\infty} \wedge T_{\bar{a}}$, $T_{a,\infty} \wedge T_{\bar{a}} \wedge M$ we obtain the following characterizing sets of hyperidentities:

$\{CHI(T_{a,m}) \mid m\geq 3\}$, $\{CHI(T_{a,m} \wedge M) \mid m\geq 3\}$,

$\{CHI(T_{a,m} \wedge T_{\bar{a}}) \mid m\geq 3\}$, $\{CHI(T_{a,m} \wedge T_{a} \wedge M) \mid m\geq 3\}$.

(We remark that for isomorphic clones $\underline{C}$ and $\underline{C}'$ (such as K and D or $T_{o,m}$ and $T_{1,m}$) we have $HI(\underline{C}) = HI(\underline{C}')$ and $\Sigma = CHI(\underline{C})$ iff $\Sigma = CHI(\underline{C}')$) The proofs of all propositions in the table can be found in [7]. By the table and by Proposition 2.2 we can answer our first question for Boolean clones:

<u>Theorem 3.1:</u> Let $\underline{C}_1$ and $\underline{C}_2$ be Boolean clones with $\underline{C}_1 \subset \underline{C}_2$. Then we have $V(\underline{C}_1) \subset V(\underline{C}_2)$ for the varieties generated by $\underline{C}_1$ and $\underline{C}_2$.

By Theorem 2.4 for every Boolean clone C we can derive a criterion for a subset of C to be a generating system of C.

4. Completeness criterions by hyperidentities

Applying Theorem 2.4 one can solve the completeness problem for the clone O_A with $|A| = n > 1$. At first we give a survey on characterizing hyperidentities of the maximal subclones of $\underline{O}_A$. Each of these clones can be represented in the form Pol ϱ for some relation ϱ on A. Here are all classes of relations ϱ over A which one needs ([8]):

<u>Class (1):</u> The class of prime permutations. A permutation is regarded as a binary relation on A, $s = \{(x,s(x)) \mid x \in A\}$. A prime permutation is one having all its cycles of equal length, the cycle length being a prime integer.

<u>Class (2):</u> The class of non-linear bounded (with a least and a greatest element) partial orders ϱ.

<u>Class (3):</u> The class of linear orders ϱ_L.

<u>Class (4):</u> The class of non-trivial equivalence relations Θ on A with more than one non-trivial block.

<u>Class (5):</u> The class of all non-trivial equivalence relations Θ_B on A with exactly one non-trivial block B.

<u>Class (6):</u> The class of prime affine relations. These are relations $\varrho \subseteq A^4$ such that $\varrho = \{(x,y,u,v) \mid x+y = u+v\}$ for some operation + under which (A,+) is an Abelian group of prime exponent p. This class is empty unless n is a prime power.

<u>Class (7):</u> The class of central relations. These are the relations $\varrho \subseteq A^k$ for some $k \geq 1$ which satisfy

(i) ϱ is totally symmetric,
(ii) ϱ is totally reflexive,
(iii) the center of ϱ, defined as $\{a \mid (a,x_1,\ldots,x_{n-1}) \in \varrho$ for all $x_1,\ldots,x_{n-1}\}$, is a non-empty proper subset of A.

Class (8): Let $E_t = \{1,2,\dots,t\}$ $(3\le t\le n)$ and let $\iota_t = \{(c_1,\dots,c_t) \in E_t^t \mid c_i = c_j$ for some $1\le i<j\le t\}$. A t-ary relation $\varrho \subseteq A^t$ $(t\ge3)$ is called t-universal if there are an $m\ge1$ and a surjective mapping $\mu\colon A \longrightarrow E_t^m$ such that $\varrho = \varrho(\mu) := \{(a_1,\dots,a_t) \in A^t \mid (\mu(a_1),\dots,\mu(a_t)) \in \iota_t^{\otimes m}\}$, where $\iota_t^{\otimes m} = \iota_t \otimes \dots \otimes \iota_t = \{((c_{11},\dots,c_{1m}),\dots,(c_{t1},\dots c_{tm})) \in (E_t^m)^t \mid (c_{1i},\dots,c_{ti}) \in \iota_t$ for $i = 1,\dots,m\}$. Suppose that μ is not injective and $\ker \mu$ has blocks of equal length.

Class (9): Let $\varrho = \varrho(\mu)$ be a t-universal relation where μ is not injective and $\ker \mu$ has blocks of different length.

Class (10): Let $\varrho = \varrho(\mu)$ be a t-universal relation where μ is bijective, $m>2$.

Class (11): Let ϱ be an n-universal relation $(t = n,\ m = 1)$, i.e. $\varrho = \iota_n = (a_1,\dots,a_n) \in A^n \mid a_i = a_j$ for some $1\le i<j\le n\}$. Clearly, $\mathrm{Pol}\ \iota_n$ is the set of all functions of O_A being non-surjective or essentially depending on at most one variable.

Consider the following hyperidentities:

$\varphi^{n-2}(x) = \varphi^{n-2+\varkappa(n)}(x)$, φ unary operation symbol, (A)

$\varphi^i(x) = \varphi(\dots(\varphi(x))\dots)$, $\varkappa(n)$ = least common multiple of the numbers $1,\dots,n$,

$\varphi^{n-1}(x) = \varphi^{n-1+\varkappa(n-1)}(x)$, (B)

$\psi^{n^2-2}\psi\ \psi^{n^2-1}(x_1,x_2) = \psi^{n^2-2+\varkappa(n^2)}\psi'\ \psi^{n^2-1}(x_1,x_2)$, where (C)

$\psi(x_1,x_2) = (\varphi_1(x_1,x_2),\ \varphi_2(x_1,x_2))$, $\psi'(x_1,x_2) = (\varphi_2(x_1,x_2),$
$\varphi_1(x_1,x_2))$.

The composition $\psi\psi(x_1,x_2)$ is defined by

$\psi(\psi(x_1,x_2)) = \psi(\varphi_1(x_1,x_2),\ \varphi_2(x_1,x_2))$
$= (\varphi_1(\varphi_1(x_1,x_2),\ \varphi_2(x_1,x_2)),\ \varphi_2(\varphi_1(x_1,x_2),\ \varphi_2(x_1,x_2)))$ and the expression $\psi^m(x_1,x_2)$ stands for $\underbrace{\psi \dots \psi}_{m\text{-times}}(x_1,x_2)$. Each such product is a pair. Hyperidentities in terms of such ψ are interpreted as hyperidentities of the first component.

(C) is a characterizing hyperidentity of every maximal subclone of O_A. (A) is a characterizing hyperidentity of every maximal clone $\mathrm{Pol}\ \varrho$ where ϱ is a relation from one of the classes (1), (2), (4), (6), (8) or (10). (B) is a characterizing hyperidentity of every maximal clone $\mathrm{Pol}\ \varrho$ where ϱ is a relation from one of the classes (3), (5), (7), (9) if $\varkappa(n) \neq \varkappa(n-1)$.

Note that $\varkappa(n) \neq \varkappa(n-1)$ iff n is a prime power. Since $O_A^{(1)} \subseteq \mathrm{Pol}\,\iota_n$ the maximal clone $\mathrm{Pol}\,\iota_n$ has no characterizing hyperidentity with only unary operation symbols.
Proposition 2.2 shows that for all maximal subclones $\underline{C}_m$ of $\underline{O}_A$ the variety $V(\underline{C}_m)$ is properly contained in the variety $V(\underline{O}_A)$.
From Theorem 2.4 we obtain

Proposition 4.1 ([1]):

(i) Let $F \subseteq O_A$, then $\langle \underline{F} \rangle = \underline{O}_A$ iff $\langle \underline{F} \rangle \models\!\!\!\!/\;\; (C)$.

(ii) Let F be a subset of O_A containing a surjective and essentially at least binary function and let $n = p^m > 2$ be a prime power, then $\langle \underline{F} \rangle = O_A$ iff $\langle \underline{F} \rangle \models\!\!\!\!/\;\; (A)$ and $\langle \underline{F} \rangle \models\!\!\!\!/\;\; (B)$.

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Andreas Straßburg

Optimale lineare Erweiterungen Boolescher Verbände

1. Einführung

Vorliegende Arbeit befaßt sich mit optimalen linearen Erweiterungen Boolescher Verbände B_n. Hierzu wird zunächst der Zusammenhang zwischen optimalen und greedy-linearen Erweiterungen des B_n untersucht. Im weiteren wird eine algorithmisch günstige Vorschrift zur Erzeugung gewisser optimaler linearer Erweiterungen des B_n vorgestellt.

Eine teilweise geordnete Menge $P = (X, \leq_P)$, im weiteren kurz Ordnung P genannt, besteht aus einer endlichen nichtleeren Trägermenge X und einer binären Relation $\leq_P$ auf X. Die Relation $\leq_P$ ist reflexiv, transitiv und antisymmetrisch.
Es werden folgende Bezeichnungen vereinbart:

$x \lessdot_P y$ für x ist unmittelbarer Vorgänger von y in P,
$x \|_P y$ für x und y sind in P unvergleichbar.

Eine lineare Erweiterung $L(P) = (X, \leq_L)$ ist eine totale Ordnung mit der Eigenschaft $\leq_P \subseteq \leq_L$. Die Menge aller linearen Erweiterungen von P sei mit $\mathfrak{L}(P)$ bezeichnet.
Zu den am häufigsten betrachteten Parametern von Ordnungen P gehört die Sprungzahl s(P).
Sei $L = L(P)$ eine lineare Erweiterung von P. Dann heißt

$$s(P,L) := |\{(a,b): a,b \in P,\ a \lessdot_L b \text{ und } a \|_P b\}|$$

die Sprungzahl von L bezüglich P.
Die Sprungzahl einer Ordnung P ist nun erklärt durch

$$s(P) := \min \{s(P,L)\} \quad \text{(Chaty, Chein 1979, [2])}.$$

Gilt $s(P,L) = s(P)$, so heißt L optimale lineare Erweiterung von P. Die Menge aller optimalen linearen Erweiterungen von P sei mit $\mathfrak{L}_o(P)$ bezeichnet.

Vom algorithmischen Standpunkt sind die erstmals von Cogis und

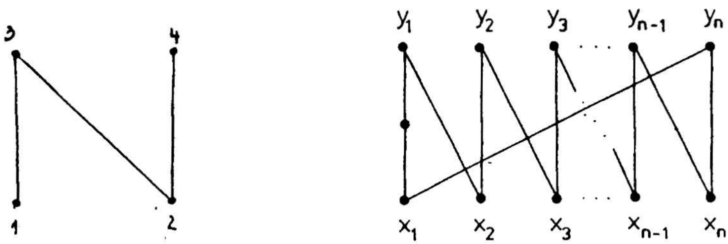
Habib ([2]) beim Studium der Sprungzahl betrachteten <u>greedy-linearen Erweiterungen</u> von besonderer Bedeutung. El-Zahar und Rival [3] beschrieben den Algorithmus zur Erzeugung greedy-linearer Erweiterungen wie folgt:

<u>Greedy-Algorithmus</u> (1)

$P = (X,<)$, $|X| = n$
$U_1 := X$
$x_1 \in \min U_1$
  $i = 1,2,\dots,n-1$
  $U_{i+1} := U - \{x_i\}$
   $x_{i+1} \in \{x\colon x \in \min U_{i+1},\ x_i <_P x\}$, falls x existiert, (1a)
   sonst
   $x_{i+1} \in \min U_{i+1}$. (1b)

Die so konstruierte lineare Erweiterung $x_1 < x_2 < \dots < x_n$ heißt greedy linear. Die Menge aller greedy-linearen Erweiterungen von P sei mit $\mathfrak{X}_g(P)$ bezeichnet.

Der Zusammenhang zwischen optimalen und greedy-linearen Erweiterungen von Ordnungen ist in verschiedenen Arbeiten untersucht worden. Zwei Resultate seien nachfolgend genannt. Zu deren besserem Verständnis dienen die in Abbildung 1 a) und b) dargestellten Ordnungen N und C_n.



a) b) Abbildung 1

<u>Satz A (El-Zahar, Rival [3]):</u> Sei P eine endliche Ordnung, die keine zu C_n, $n>2$ (s. Abbildung 1b)), isomorphe Teilordnung enthält. Dann ist

$\mathfrak{X}_o(P) \subseteq \mathfrak{X}_g(P)$. ∎

Satz B (Rival [4]): Sei P eine endliche, N-freie Ordnung. Dann gilt für alle $L \in \mathcal{L}_g(P)$

$$s(P,L) = s(P), \text{ d.h. } \mathcal{L}_g(P) \subseteq \mathcal{L}_o(P). \qquad \blacksquare$$

Bemerkung 1: Eine Ordnung $P = (X,<)$ heißt N-frei, wenn keine Teilmenge der Überdeckungsrelation R von P ($R := \{(x,y): x,y \in X, x \lessdot_P y\}$) isomorph zur in Abbildung 1a) dargestellten Ordnung N ist.

2. Optimale lineare Erweiterungen Boolescher Verbände

Unter dem Booleschen Verband $B_n = (2^X, <_B)$, $|X| = n$, versteht man die nach Inklusion geordnete Menge aller Teilmengen einer n-elementigen Menge.

Es ist bekannt, daß $s(B_n) = 2^{n-1} - 1$ gilt und daß es in $L \in \mathcal{L}(B_n)$ keine 3-elementigen Teilketten zwischen zwei aufeinanderfolgenden Sprüngen gibt (vgl. [1, 5]).

Bemerkung 2: Jede optimale lineare Erweiterung des B_n ist darstellbar als lineare Summe von 2^{n-1} 2-elementigen Teilketten in B_n.

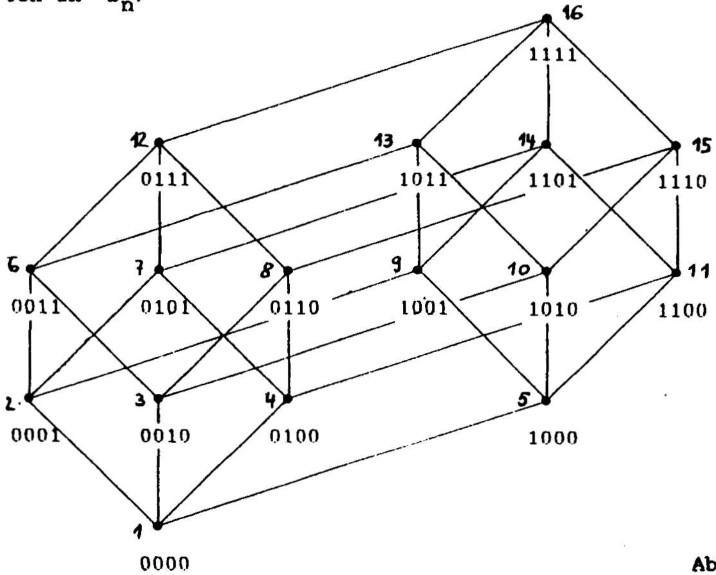


Abbildung 2

Für $n = 4$ ist $L_o(B_4)$ (vgl. Tabelle 1) ein Beispiel für eine optimale lineare Erweiterung des B_4.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}
$L_g(B_4)$	1	2	4	7	5	11	3	10	6	8	12	9	14	13	15	16
$L_o(B_4)$	1	2	3	6	4	8	5	9	7	12	10	13	11	14	15	16
$L_A(B_4)$	1	2	3	6	4	7	5	9	8	12	10	13	11	14	15	16

Tabelle 1

<u>Bemerkung 3:</u> Für Boolesche Verbände können die Sätze A und B nicht zur Anwendung kommen. Hierzu betrachte man in Abbildung 2 die N und C_2 darstellenden Teilordnungen über den Punktmengen $\{2,3,6,8\}$ bzw. $\{2,4,6,12,14\}$ des B_4.

Nachfolgender Satz 1 beschreibt den Zusammenhang zwischen optimalen und greedy-linearen Erweiterungen Boolescher Verbände. Dabei werden folgende Fragen untersucht:

- Ist jede optimale lineare Erweiterung des B_n durch den greedy-Algorithmus konstruierbar?
- Gibt es greedy-lineare Erweiterungen des B_n, die nicht optimal sind?

<u>Satz 1:</u> $\mathcal{L}_o(B_n) \subset \mathcal{L}_g(B_n)$.

<u>Beweis:</u> Sei $L_o = \{x_1 < x_2 < \ldots < x_m\}$ eine beliebige optimale lineare Erweiterung des B_n. Es wird zunächst

$$L_o \in \mathcal{L}_g(B_n) \tag{2}$$

gezeigt.

Angenommen, es ist $L_o \notin \mathcal{L}_g(B_n)$. Dann gibt es einen kleinsten Index i, so daß zwar $x_1, x_2, \ldots, x_i$, aber nicht $x_1, x_2, \ldots, x_i, x_{i+1}$ nach dem greedy-Algorithmus (1) gebildet werden kann. Offenbar ist $i \geqslant 1$.
Dann ergeben sich für x_{i+1} zwei Fälle.

a) $x_i <_B x_{i+1}$: Sei x_{i+1}^- die Menge aller Vorgänger von x_{i+1} in B_n. Dann ist $x_{i+1}^- \subseteq \{x_1, x_2, \ldots, x_i\}$. Folglich muß x_{k+1} minimal in $B_n - \{x_1, x_2, \ldots, x_i\}$ sein, d.h. $x_{i+1} \in \min U_{i+1}$; offenbar wurde x_{i+1} nach (1a) konstruiert, und man hat einen Widerspruch.

b) $x_i \| _B x_{i+1}$: Es gilt $x_{i-1} <_B x_i$ (vgl. Bemerkung 2) und $\bar{x}_{i+1} \subset \{x_1, x_2, \ldots, x_i\}$, d.h., x_{i+1} ist minimal in $B_n - \{x_1, x_2, \ldots, x_i\}$, d.h. $x_{i+1} \in \min U_{i+1}$. Damit wird der Schritt (1a) aus Algorithmus (1) nicht anwendbar und x_{i+1} kann nur nach (1b) konstruiert werden. Damit ergibt sich erneut ein Widerspruch.

Folglich ist jede optimale lineare Erweiterung des B_n durch den greedy-Algorithmus (1) konstruierbar.

Die Existenz nicht-optimaler greedy-linearer Erweiterungen des B_n wird durch Angabe des folgenden Beispiels gezeigt.

Für die greedy-lineare Erweiterung $L_g(B_4) \in \mathfrak{L}_g(B_4)$ aus Tabelle 1 gilt

$$s(B_4, L_g(B_4)) = 2^3 \neq 2^3-1 = s(B_4).$$

Offenbar ist $L_g(B_4)$ nicht optimal. Also gilt allgemein

$$\mathfrak{L}_o(B_n) \neq \mathfrak{L}_g(B_n). \qquad (3)$$

Mit (2) und (3) ist der Satz bewiesen. ∎

3. Gerichtet-greedy-lineare Erweiterungen

Gibt es eine nicht-triviale Konstruktion, die genau alle optimalen linearen Erweiterungen des B_n liefert? (4)

Die Suche nach der Antwort auf diese Frage führte über den greedy-Algorithmus zu einer hieraus modifizierten Konstruktion A.

Konstruktion A (5)

$B_n = (2^X, <_B)$, $|X| = n$

$U_1 := 2^X$

$x_1 := \min B_n$

$U_2 := U_1 - \{x_1\}$

$x_2 \in \{x: x \in U_2\}$ (5a)

$i = 2, 3, \ldots, 2^n-1$

$U_{i+1} := U_i - \{x_i\}$

$x_{i+1} := \{x: x \in \min U_{i+1}, x_i <_B x, x_2 <_B x\}$, falls x existiert, sonst

$x_{i+1} \in \min U_i+1.$

Die durch die Konstruktion A gebildeten linearen Erweiterungen

$x_1, x_2, \ldots, x_{|X|}$ heißen gerichtet-greedy. Die Menge aller so gebildeten linearen Erweiterungen sei mit $\mathcal{L}_A(B_n)$ bezeichnet (Man betrachte $L_A(B_4)$ aus Tabelle 1 in Abbildung 2).
In der Konstruktion wird durch (5a), d.h. durch die Wahl von $x_2 = \{j\}$ (derjenige n-elementige 0-1-Vektor, der an j-ter Stelle eine 1 hat und sonst Nullen), $1 \le j \le n$, die Richtung festgelegt.
Es gilt

Satz 2: $\mathcal{L}_A(B_n) \subset \mathcal{L}_o(B_n)$.

Beweisidee: Sei $L \in \mathcal{L}_A(B_n)$ und $L = C_1 \oplus C_2 \oplus \ldots \oplus C_m$, $|C_i| \le 2$, $1 \le i \le m$, eine Zerlegung von L in Teilketten von B_n, die jeweils nicht verlängerbar sind. Es genügt zu zeigen, daß

$$L = C'_1 \oplus C'_2 \oplus \ldots \oplus C'_t, \quad C'_i = \{a_{2i-1}, a_{2i}\}, \quad i = 1, \ldots, t, \qquad (6)$$

gilt, wobei a_{2i} der Nachfolger von a_{2i-1} in Richtung $\{j\}$ ist.

Angenommen, i sei der kleinste Index, mit $|C_i| < 2$. Sei $C_i = \{a_{2i-1}\}$ und b der a_{2i-1}-Nachfolger in $\{j\}$-Richtung. Offenbar ist $i>1$ und $a_{2i-1} \|_B a_{2i-2}$. Dann existiert in b^-, der Menge aller Vorgänger von b in B_n, ein c mit $c \notin C_1 \cup \ldots \cup C_i$. Sei a' der Vorgänger von c in $\{j\}$-Richtung. Dann gilt $a' <_B a_{2i-1}$, und es gibt ein r, $1 \le r < i$, so daß $a' \in C_r$ gilt. Mit a' muß aber auch c in der Teilkette C_r liegen im Widerspuch zur Annahme.
Folglich gilt (6), und wegen Bemerkung 2 ist jedes L aus $\mathcal{L}_A(B_n)$ in $\mathcal{L}_o(B_n)$ enthalten.
Aus der Betrachtung von $L_o(B_4)$ (vgl. Tabelle 1) folgt $L_o(B_4) \notin \mathcal{L}_A(B_4)$. Damit ist $\mathcal{L}_A(B_n) \neq \mathcal{L}_o(B_n)$ und die Behauptung gezeigt (siehe auch [6]).

Die unter (4) formulierte Frage bleibt weiterhin offen.

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Problem session

(This problem session was held on September 17, 1988 during the 7th Fischland-Colloquium. Prof. G. Burosch summarized the problems for the proceedings.)

Problem 1 (P. Erdös)

Let Q_n be the n-dimensional cube. Is it true that every subgraph $G' \subseteq Q_n$ having $(\frac{1}{2}+\varepsilon)\cdot n\cdot 2^{n-1}$ edges contains a C_4?
Perhaps $n\cdot 2^{n-1} + c2^n$ edges will suffice for sufficiently large c.
Fan Chung proved that for $|E(G')| > 0.6\cdot n\cdot 2^{n-1}$ G' contains a C_4.

Is it true that for $n > n_o(\varepsilon)$ from $|E(G')| > \varepsilon\cdot n\cdot 2^{n-1}$ it follows that G' contains a C_6? Here perhaps $n^c\cdot 2^n$ edges will suffice if c is close enough to 1.

Problem 2 (P. Erdös, L. Pyber, H. Tura)

Let n be even. Colour the edges of the complete graph K_{6n+1} so that every colour has degree n in every vertex. Is it then true that there is a totally multicoloured C_6 (i.e. a cycle of length 6 each edge of it gets different colour)? Is there a multicoloured K(4)?

Problem 3 (P. Erdös)

Let $G(n)$ be a graph of n vertices each edge of which is contained in a triangle. Denote by e the number of edges of our graph. Gallai defined e(min) as the smallest integer for which there is a set of e(min) edges which represent every triangle (i.e. the omission of these edges makes the graph triangle free). e(max) is the largest integer for which there are e(max) edges no two of them occur in a triangle. Gallai asked for the investgation of these quantities. Erdös and Tura conjectured that

$$e(\min) + e(\max) \leqslant \frac{n^2}{4}. \qquad (1)$$

(1) is still open. One difficulty seems to be that there are many graphs for which there is equality in (1). Lehel suggested in 1988 the following strengthening of (1). Denote by $e_b(\min)$ the smallest integer for which there are $e_b(\min)$ edges whose omission makes the graph bipartite. Clearly $e_b(\min) \geqslant e(\min)$. Now Lehel suggested

$$e_b(\min) + e(\max) \leqslant \frac{n^2}{4}. \qquad (2)$$

What is the maximum possible value of

$$e_b(\min) - e(\min),$$

where the maximum is to be taken over all graphs of n vertices. Perhaps

$$e_b(\min) - e(\min) \leqslant \frac{n^2}{25}.$$

Is it true that for every ε there is a graph $G(n)$, $n > n_o(\varepsilon)$, for which

$$e(\min) > (\tfrac{1}{2}-\varepsilon)\cdot n \quad \text{and} \quad e(\max) > (\tfrac{1}{2}-\varepsilon)\cdot n?$$

Problem 4 (P. Erdös)

Here is an old problem of Gallai: Let $G(n)$ be a graph of n vertices which contains no wheel. Is it then true that it contains at most $\frac{n^2}{8}$ triangles? The Turan-graph $K(\frac{n}{2},\frac{n}{2})$ with a matching on one side shows that if this is true then it is best possible. Erdös was able to prove only $(\frac{1}{6} - c)\cdot n^2$ for some fairly small c.

Problem 5 (V. Turán-Sos)

Let

$$R(G) := \max \{r\colon G \text{ has a regular subgraph of } r \text{ edges}\}$$

and let $\mathcal{G}_n^e$ denote the class of graphs on n vertices and e edges. Determine

$$f(n,e) := \min_{G \in \mathcal{G}_n^e} R(G).$$

Remark: P. Erdös and Z. Füredi proved that

$$f(n, kn) = k^2 \quad \text{for} \quad k < c\sqrt{\log n}.$$

More generally:

Let $\mathcal{H}(S,\mathcal{A})$ be a hypergraph with vertex set $S = \{x_1,\dots,x_n\}$ and edge set $\mathcal{A} = \{A_1,\dots,A_m\}$. For $i \neq j$ let

$$d_{ij} = |\{A : A \in \mathcal{A};\ x_i, x_j \in A\}|$$

(the "degree" of the pair (x_i,x_j)).

How large regular or pair-regular (etc) subgraphs must be contained in $\mathcal{H}$ (in terms of $|S|$ and $|\mathcal{A}|$)?

Problem 6 (A. Burr, P. Erdös, R. Graham, V. Turán-Sos)

Let S_n be an arbitrary Steiner-system on n points and $S^* \subseteq S$ be a subsystem with the property that every 6 points span at most 2 triangles. Put

$$f(S,n) := \max_{S^* \subseteq S_n} |S^*| \quad \text{and}$$

$$t(n) := \min_{S} f(S,n).$$

a) Define $t(n)$ or give upper and lower bounds.

Remark: From a result of Ruzsa-Szemeredi we have $t(n) = o(n^2)$.

b) Let the Steiner system $\bar{S}$ on the vertex set Z_2^n defined by

$$(\underline{a},\underline{b},\underline{c}) \in S \iff \underline{a} \oplus \underline{b} \oplus \underline{c} = \underline{0}$$

(where $\underline{a},\underline{b},\underline{c} \in Z_2^n$ and $\oplus$ denotes the Boolean operation). Determine $f(\bar{S},n)$! Is it true that for every $\varepsilon > 0$

$$f(\bar{S},n) > n^{2-\varepsilon} \quad \text{if} \quad n > n_0(\varepsilon).$$

Problem 7 (M. Fiedler)

Find a direct proof of the following assertion (Fiedler, M.: Modified algebraic connectivity of trees. Math. Inst. Acad. Praha 1988, preprint 36, p.27):

Let $T = (V,E)$ be a finite directed rooted tree directed toward the root r. Let $\mathcal{F}$ be the class of all strictly decreasing functions $x = (x_i)$ on V with the value zero at r (i.e. $x_r = 0$, $x_i > x_j$ if $(i,j) \in E$).
For $(i,j) \in E$, let $B(i,j)$ denote the set of vertices in V from which i can be reached by a path in T. Then the following sharp inequality holds for $x \in \mathcal{F}$:

$$\sum_{(i,j)\in E} \frac{1}{x_i - x_j} \sum_{k\in B(i,j)} x_k \geqslant \sum_{k\in V} d^2(k,r)$$

where $d(p,q)$ means the distance from p to q. Equality is attained for the distance function from r:

$$x_k = d(k,r), \quad k \in V.$$

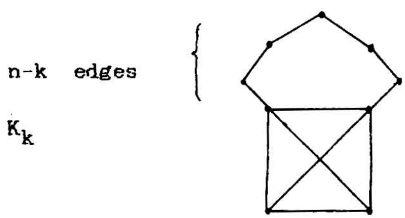
Example: For the simplest nontrivial case of T with $V = \{1,2,3\}$, $E = \{(2,1),(3,2)\}$, thus $r = 1$, the conditions for $\mathcal{F}$ read $0 = x_1 < x_2 < x_3$, and the assertion is that

$$\frac{x_2 + x_3}{x_2 - x_1} + \frac{x_3}{x_3 - x_2} \geqslant 5 \ (= 1 + 4).$$

Equality is attained for $x_1 = 0$, $x_2 = 1$, $x_3 = 2$.

Problem 8 (I. Tomescu)

Find all extremal graphs relative to the number of k-colourings (partitions) in the class of graphs G of order n and chromatic number k, which are 2-connected. I conjecture that for $k \geqslant 4$ the graph G which reaches a maximal number of k-colourings is unique:



Note that for minimal number of k-colourings there is also a unique minimal graph, namely the Turan graph $T(n,k)$.

Problem 9 (Z. Füredi, P. Seymour)

Let $\mathcal{F}$ be an intersecting family over the n-element set X. Are there n pairs $\{x_i, x_{i+1}\} \subseteq X$ such that all $F \in \mathcal{F}$ contain one of them?

Remarks: If this is true, then it implies Lennon's conjecture: Any intersecting family $\mathcal{F}$ on X can be decomposed into $<n$ 2-intersecting families.
We have proved the following much weaker version: There is a pair $\{x,y\} \subseteq X$, such that

$$|\{F \in \mathcal{F} : F \supseteq \{x,y\}\}| \geq \frac{1}{n} \cdot |\mathcal{F}|.$$

This is a generalization of the de-Bruijn-Erdös Theorem. If the above conjecture is not true, then still it seems to be interesting to determine the minimum number of pairs which cover every intersecting family. It is trivially $\leq n^{3/2}$.

Problem 10 (W. Mader)

I conjecture that every minimal n-connected digraph has two vertices x such that $d^+(x) = d^-(x) = n$.

Problem 11 (W. Mader)

Every critically n-connected digraph D has a vertex of small outdegree.
I conjecture that there exist at least 4 vertices $x_1, \ldots, x_4$ such that $d^{\varepsilon_i}(x_j) < 2n$, $\varepsilon_i \in \{0,1\}$.

Problem 12 (B. Voigt)

Let $\mathcal{L}(n,q)$ be the linear lattice and denote by N_k the level of k-dimensional subvectorspaces. For $k < l$ cover the intervall between N_k and N_l by linear lattices. Take the number of those lattices so small as possible. Is this number equal to

$$\max\left(\binom{n}{l}_q, \binom{n}{k}_q\right) ?$$

Problem 13 (I, Havel, J.M. Laborde, P. Liebl)

For $n \geq 1$ and $S \subseteq \{1,2,\ldots,n\}$ let $Q_n(S)$ denote the graph derived from the hypercube Q_n in the following way:

- $V(Q_n(S)) := V(Q_n)$,
- $(u,v) \in E(Q_n(S)) :\Longleftrightarrow d_{Q_n}(u,v) \in S$

(i.e. iff their Hamming distance in Q_n is an element of S).
In that sense $Q_n(1) = Q_n$. (They have been called somewhere "cube-like" by L. Lovasz).

1. Describe some nontrivial chromatic properties of $Q_n(S)$!
   Is it true that the chromatic number $\chi(Q_n(S))$ is always equal to some power of 2?

2. Determine particular $\chi(Q_8(1,2))$ and $\chi(Q_5(1,4,5))$!

n	1	2	3	4	5	6	7	8	9
$\chi_n(1,2)$	2	4	4	8	8	8	8	?	?

It is known that

$$13 \leq \chi(Q_8(1,2)) \leq 16.$$

3. Determine $\chi(Q_5(1,4,5))$!

It is known that

$$7 \leq \chi(Q_5(1,4,5)) \leq 8.$$

Problem 14 (A. Rucinski)

In a random graph $K(n,p)$, for what $p = p(n)$ it is true that the probability that $K(n,p)$ contains $\lfloor \frac{n}{3} \rfloor$ vertex-disjoint triangles tends to 1 as $n \to \infty$?

Problem 15 (G. Burosch)

For any poset $P = (X, <)$ and its Hasse Diagram $G = (X, E)$, $E = \{(x, y) : x \lessdot y\}$ consider the poset $P' = (E, <')$, where for $k = (x, y)$, $k' = (x', y')$ $k <' k' :\Longleftrightarrow k = k'$ or $y < x'$. Let $\dim Q$ denote the dimension of the poset Q. It is known that for P with 0 and 1 it holds $\dim P' \geq \dim P$ and there exist examples showing the cases $\dim P' = \dim P + 1$, $\dim P' = \dim P + 2$. Prove that $\dim P' - \dim P$ can take arbitrary great values.

Hinweise für Autoren

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