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Harry Poppe

Applications of the Bartsch-Poppe duality approach

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1 Introduction

In the papers [1], [2], [3] by R. Bartsch and H. Poppe a general duality system was defined and studied:

$$(X, Y, X^d, X^{dd}, J :\to X^{dd}).$$

Here X, Y are spaces, X^d is the first dual space of X with respect to Y, X^{dd} denotes the second dual space of X w. r. t. Y and J is the canonical map as is known, from classical examples.

The map J we define by the evaluation map ω: let X, Y be nonempty sets,

$$\omega : X \times Y^X \to Y, \quad \forall (x, h) \in X \times Y^X : \omega(x, h) := h(x).$$

Hence we find:

$$\forall x \in X : Jx = \omega(x, \cdot), \quad \omega(x, \cdot) : Y^X \to Y : \forall h \in Y^X : \omega(x, \cdot)(h) = \omega(x, h) = h(x).$$

In short we call it the B/P duality approach.

In the papers [1], [2], [3] this general duality approach was applied to well known examples of representation theorems.

Let for instance X be a unital commutative Banachalgebra, or let X be a Boolean ring.

We used suitable spaces Y and defined then the dual spaces X^d and X^{dd} and proved the Gelfand and the Stone representation theorem respectively using the general B/P duality approach.

We also obtained new results. For example, [2], theorem 5.4 shows the embedding of a vector lattice X into X^{dd}, in [3], theorem 4.5 one finds the representation of an unital, noncommutative C^*-algebra.

What is the aim of this paper?

1. We want to improve the definitions of the first dual space X^d and the second dual space X^{dd} of a given space, X as were defined in [1]. For this purpose we will repeat in short the very basic definitions and some results of the B/P duality approach.
2. We apply the B/P duality approach to get new, well arranged proofs of
   (a) the representation of a nonunital commutative C^*-algebra (Gelfand-Naimark theorem)
   (b) the embedding theorem of Kadison.

2 The duality approach

2.1 Abstract definition

Let X, Y be sets or spaces. Y^X means of course the set of all functions from X to Y.

Now we will define an abstract scheme of duality.

Definition 2.1 1. *Let be $A \subseteq Y^X$, $A \neq \emptyset$.*

We call A to be the first dual space of X with respect to Y.

2. *We use here the definition of the map J. Let $B \subseteq Y^A$, $B \neq \emptyset$; let further be: $J : X \to Y^A$, hence as we know:*

$$\forall x \in X : Jx = \omega(x, \cdot), \omega(x, \cdot) : A \to Y : \forall h \in A : \omega(x, \cdot)(h) = \omega(x, h) = h(x)\,.$$

If $J(X) \subseteq B$, i. e. $\forall x \in X : \omega(x, \cdot) \in B$ then we call B to be the second (abstract) dual space of X w. r. t. Y.

Remarks 2.2 (a) If we in definition 2.1 only consider sets X, Y we cannot formulate nice properties or prove useful theorems concerning the abstract dual spaces A, B. But of course this is possible for spaces X, Y, where we can use the special properties of these spaces to give A, B concrete forms.

(b) We will consider spaces with algebraic, order, and topological structures, where topologies can be derived from metrics, norms or inner products. We also use measurable spaces.

We put emphasis on spaces with algebraic and topological structures.

2.2 Concrete definition of the first and of the second dual space

2.2.1 The first dual space X^d of a space X with respect to a space Y At first glance we can say:

X^d consists of homomorphism from X to Y. But this can only work if we state an assumption.

Basic Assumption 2.3 X and Y belong to the same class of spaces.

We consider three simple examples:

(a) X and Y are vector spaces over $\mathbb{R}$ If necessary we add: $\dim X = \dim Y$.

Then X and Y are in the same class of spaces.

(b) Let be X and Y C^*-algebras over $\mathbb{C}$; X and Y are commutative. If X has no unit element and Y has an unit then X and Y do not belong to the same class of spaces.

(c) Let X, Y be lattices. Then of course X and Y fullfill (2.3).

In case (a) the first dual space is well known. Let

$$Y = \mathbb{R}, X^d = \{h : X \to \mathbb{R} | h \text{ is linear } \}\,.$$

If X is a topological vector space, then we get:

$$X^d = \{h : X \to \mathbb{R} | h \text{ is linear and } h \text{ is continuous}\}.$$

Here we consider a continuous map as a topological homomorphism. In case (b) we cannot set $Y = \mathbb{C}$ since the C^*-algebra $\mathbb{C}$ has an unit and, hence $Y = \mathbb{C}$ contradicts assumption (2.3). We will later come back to this example.

In case (c) we at once can write:

$$X^d = \{h : X \to Y | h \text{ is a lattice-homomorphism}\}.$$

Let X, Y be spaces with algebraic or order operations. By the basic assumption (2.3) we find for each operation in X a corresponding operation in Y.

By $A(X, Y)$ we denote the set of all such pairs of operation in X and Y respectively.

We assume $\emptyset \neq A(X, Y)$ and $A(X, Y)$ is a finite set.

Definition 2.4 (a) *$H(X, Y) = \{h : X \to Y | h$ is a homomorphism for each pair of operations from $A(X, Y)\}$*

(b) *If both spaces X, Y have also a topology then we consider $H(X, Y) \cap C(X, Y)$, where $C(X, Y)$ is the space of all continuous functions from X to Y. $X^d = H(X, Y)$ or $X^d = H(X, Y) \cap C(X, Y)$ and we also find:*

$$X^d \subseteq H(X, Y) \text{ or } X^d \subseteq H(X, Y) \cap C(X, Y)\,.$$

If this is possible and useful we provide X^d with a topology η, $\tau_p \leq \eta$ where τ_p denotes the pointwise topology.

We call X^d to be the first dual space of X with respect to Y. To define the pointwise topology τ_p for X^d we must have a topology for Y. As we soon will see, in some cases we indeed will use τ_p. Hence we come to:

Basic Assumption 2.5 Y always has a topology. If for Y no topology is given we will define: Y is provided with

$$\begin{cases} \text{the discrete topology, if } X \text{ has no topology} \\ \text{the trivial topology, if } X \text{ has a topology} \end{cases}$$

If we want that all $h \in H(X, Y)$ are continuous too and X has no topology we provide X also with the discrete topology. The elements of X^d are functions or maps. Using the operations in Y and in X we want to define corresponding operations in X^d too. In most cases we define these operations pointwise. For instance let be in X and in Y an addition is defined:

$$X = (X, +),\ Y = (Y, +)\,.$$

If now $h_1, h_2 \in X^d$:

$$h_1 + h_2 : \forall x \in X : (h_1 + h_2)(x) := h_1(x) + h_2(x) \in Y\,.$$

If for example we have $X = Y$ than we can h_1, h_2 also compose: $h_1 \circ h_2$.

Definition 2.6 *If X, Y are spaces and we have defined X^d then for X^d there exists two possibilities:*

1. *X and X^d belong to the same class of spaces*
2. *X and X^d do not belong to the same class of spaces.*

Now let us consider some examples to clear up the situation.

Examples 2.7 1. Let X be a normed vector space over $\mathbb{R}$ and let be $Y = \mathbb{R}$.

$$X^d = \{h : X \to \mathbb{R} \,|\, h \text{ is linear and } h \text{is continuous}\}\,.$$

With pointwise defined vector operations and the sup-norm (on bounded sets) X^d is a normed vector space over $\mathbb{R}$ too. Hence X and X^d belong to the same class of spaces.

2. Let $X = (X, \|\cdot\|)$ a $\mathbb{R}$-normed space again, $Y = \mathbb{R}$ and $X^d = \{h : X \to \mathbb{R} \,|\, h$ is linear and continuous and $\|h\| = 1\}$. But here X^d is no vector space:

   we assume that X^d is a vector space, hence

   $$h \in X^d \Rightarrow 2h \in X^d\,,$$

   but $\|2h\| = 2\|h\| = 2 \neq 1$, a contradiction.

   Thus X and X^d do not belong to the same class of spaces.

3. Let X be a vector lattice (a Riesz space), $\mathbb{R}$ with natural order also is a vector lattice. It is known that $X^d = \{h : X \to \mathbb{R} |\ h$ is linear and order bounded$\}$ is a vector lattice too, showing that X and X^d belong to the same class of spaces.

4. In the paper [2], definition 5.1 we find for a vector lattice

$$X : X^d = \{h : X \to \mathbb{R} \,|\, h \text{ is a linear lattice homomorphism}\} .$$

the following example 5.3 shows that (in general) X^d is no vector lattice.

Hence X and X^d do not belong to the same class of spaces.

Remarks 2.8 (a) If X and X^d belong to the same class of spaces we can define the second dual space by: $X^{dd} := (X^d)^d$. But otherwise we must find a suitable definition of X^{dd}.

(b) As special cases of definition (2.4) we get:

X and Y respectively have only:

(b.a) topologies

(b.b) algebraic operations

(b.c) lattice operations

Case (b.a) was treated in our paper [2], concerning (b.b) in [1], 5. Some examples and applications, [1], page 290 we considered two communicative rings X, Y with units.

(c) Let $(X, \underline{A}, \mu)$ be a measure space, where X is a set, $\underline{A}$ is a σ-algebra of subsets of X and $\mu : \underline{A} \to [0, +\infty]$ is a measure.

Let $p \in \mathbb{R}$, $1 \le p < \infty$, let $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ be measurable. Then the L^p-norm of f is given by

$$\|f\|_p = \left(\int_X |f|^p\right)^{\frac{1}{p}} .$$

$f : X \to \mathbb{R}$ is called p-integrable or a L^p-function if f is measureable and $\|f\|_p < \infty$.

$$L^p(\mu) = \{f : X \to \mathbb{R} \,|\, f \text{ is } \underline{A}\text{-measurable and } \|f\|_p < \infty\} .$$

$L^p(\mu) = (L^p(\mu), \|\cdot\|_p)$ is a normed space, even a Banach space. Hence

$$(L^p(\mu))^d = \{h : L^p(\mu) \to \mathbb{R} \,|\, h \text{ is linear and continuous}\}.$$

Now let be $p \in \mathbb{R}$ and $1 < p < \infty$ and let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$.

There exists a isomorphic and isometric map from $L^q(\mu)$ onto $(L^p(\mu))^d$.

This result has the advantage that we can much better work with $L^q(\mu)$ than with $(L^p(\mu))^d$. This situation we find by many dual spaces X^d, especially if the space X is a normed space. This procedure, where the dual space will be replaced by a *better space* we also will apply to the two following examples in this paper, where we will use the B/P duality approach. But here the starting spaces X are not only normed spaces.

Precise definitions and proofs of the above statements about $L^p(\mu)$-spaces one finds in modern books on measure and integration theory, for instance in [7].

Now we come back to 2.6. Following [1], definition 4.1, page 282 we define:

Definition 2.9 *We say that X^d has the defect D, D, if X and X^d do not belong to the same class of spaces; not the defect D, non D, otherwise.*

Now we can define the second dual space.

2.2.2 The second dual space X^{dd} with respect to a space Y.

Definition 2.10 *Let X, Y be spaces in the sense of 2.2, (b). X, Y fulfill basic assumption 2.3. According to basis assumption 2.5 X^d has a topology η with $\tau_p \leq \eta$, since τ_p is defined.*

Part 1

$$X^{dd} = \begin{cases} \big((X^d,\eta)^d, \mu\big) & \text{if non } D \\ \big(\mathcal{C}((X^d,\tau_p),(Y,\sigma)), \mu\big) & \text{if } D \end{cases}$$

where $\mathcal{C}(\cdot,\cdot)$ means the space of continuous maps.

Here we also assume:

$$\tau_p \leq \mu\,.$$

X^{dd} is called the second dual space of X w. r. t. Y, σ, η, μ.

By [1], lemma 4.1, page 283 and corollary 4.1, page 284 we know that $J(X) \subseteq X^{dd}$ holds.

Basic Assumption 2.11 X and X^{dd} are in the same class of spaces.

Part 2

$$X^{dd} = \begin{cases} X^{dd} \text{ as defined in part 1, if (2.11) holds} \\ J(X) \text{ otherwise} \end{cases}$$

Remark 2.12 The operations in X^{dd} we define pointwise using the operations in X^d and in Y. See also [1], page 283.

3 The Gelfand-Naimark theorem for nonunital commutative C^*-algebras

At first we will repeat some well-known definitions and results:

Let X be a commutative nonunital C^*-algebra.

Then $X_1 = X \times \mathbb{C}$ is a commutative C^*-algebra with unit, if we provide X_1 with the defined algebraic operations and the C^*-norm for X_1.

The unit for X_1 is then $(0,1) \in X \times \mathbb{C}$.

The map: $x \to (x,0)$ from X to X_1 is a *-isomorphic, isometric homomorphism onto $X \times \{0\}$ with $\|(x,0)\| = \|x\|$.

Thus we can identify X with $X \times \{0\} \subseteq X_1$ and by this way X can be considered as a subspace of X_1.

$x \to (x,0)$ is also an uniform bijective map implying that $X \times \{0\}$ is complete since X is complete hence $X \times \{0\}$ is a closed subspace of X_1; this set is even a maximal ideal in X_1.

We state:

Proposition 3.1 *$X \times \{0\} = \{(x,0) | x \in X\}$ is a nonunital C^*-subalgebra of $X_1 = X \times \mathbb{C}$.*

3.1 The first dual spaces of X, X_1 and the second dual space of X

According to definition 2.4 we can define:

$$\begin{aligned} X^d &= \{h : X \to \mathbb{C} \,|\, h \text{ is a } ^*\text{-homomorphism and } h \text{ is continuous}\} \\ &= \{h : X \to \mathbb{C} \,|\, h \text{ is a } ^*\text{-homomorphism}\}\,, \\ X_1^d &= \{g : X_1 \to \mathbb{C} \,|\, g \text{ is } ^*\text{-homomorphismus}\}\,. \end{aligned}$$

If 0 is the zero-homomorphism, by definition 3.2 of [1], page 281, $0 \in X^d$, but by lemma 4.2 of [1], page 288, $0 \notin X_1^d$, hence $X_1^d \backslash \{0\}$ is the new dual space.

For X_1 we know the second dual space

$$X_1^{dd} = \left(C((X_1^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_{\|\cdot\|} \right)\,,$$

where τ_p is the pontwise topology and $\tau_{\|\cdot\|} = \tau_u$ is the uniform topology generated by the sup-norm, [1], [3], and the Gelfand-Naimark theorem for unital algebras. We can use the sup-norm here because $((X_1^d \backslash \{0\}), \tau_p)$ is compact and Hausdorff and thus X^{dd} consists of bounded functions:

$$X_1^{dd} = \left(C_b((X_1^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_u \right)\,.$$

Remark 3.2 Concerning the Gelfand-Naimark theorem for unital C^*-algebras look at [8] and relevant books and papers.

3.2 Preliminaries

At first we show a result, which is important for our purposes: the dual spaces (X^d, τ_p) and $(X_1^d \backslash \{0\}, \tau_p)$ are homeomorphic. Moreover we consider a simple criterion that a continuous function already belongs to the space of continuous functions vanishing at infinity.

And we need the Stone-Weierstrass theorem for the case that the basic space is not compact but only locally compact.

It is nearby that there exists a connection between the first dual space X^d of X and the first dual space X_1^d of the unital extension, $X_1 = X \times \mathbb{C}$, of X.

Indeed:

$$\forall (h, (x, \alpha)) \in X^d \times X_1 : \tilde{h} : \tilde{h}(x, \alpha) = h(x) + \alpha$$

By the following proposition we show that holds:

$$\forall h \in X^d : \tilde{h} \in X_1^d .$$

This proposition is well known (see for instance [4]). We will not prove the proposition.

Proposition 3.3 1. $\tilde{h}(0, 1) = 1$

2. $\tilde{h}$ *is uniquely determined by* h
3. $\tilde{h}$ *is a* $*$*-homomorphism and thus* $\tilde{h}$ *is continuous*
4. $\tilde{h}$ *is an extension of* h
5. *If* $0 \in X^d$ *is the zero-element then* $\tilde{0}$ *is not, the zero-element of*

$$X_1^d : \forall (x, \alpha) \in X_1 : \tilde{0}(x, \alpha) = \hat{0}(x) + \alpha = 0 + \alpha = \alpha .$$

6. *If* $g \in X_1^d \backslash \{0\}$ *then* $g|(X \times \{0\}) \in X^d$

Now we can define the map

$$G : X^d \to X_1^d \backslash \{0\} : \forall h \in X^d : G(h) = \tilde{h} .$$

By proposition 3.3 we know that $\tilde{h} \in X_1^d \backslash \{0\}$

Theorem 3.4 (a) *The map* G *is bijective*

(b) G *is neither linear nor multiplicative*

(c) $G : (X^d, \tau_p) \to (X_1^d \backslash \{0\}, \tau_p)$ *is continuous*

(d) $G : (X^d, \tau_p) \to (X_1^d \backslash \{0\}, \tau)$ *is open*

Proof. (a) G is injective:

$$\forall (h_1, h_2) \in X^d \times X^d,\ h_1 \neq h_2$$

and we assume

$$\begin{aligned} G(h_1) = G(h_2);\ h_1 \neq h_2 \Rightarrow \exists x_0 \in X : h_1(x_0) \neq h_2(x_0),\ \tilde{h}_1 = \tilde{h}_2 \\ \tilde{h}_1(x_0, 0) = \tilde{h}_2(x_0, 0) \Rightarrow h_1(x_0) + 0 = h_2(x_0) + 0 \\ \Rightarrow h_1(x_0) = h_2(x_0)\,, \end{aligned}$$

a contradiction.

G is surjective too:

$$\forall\ f \in X_1^d \backslash \{0\},\ f \neq 0\,,$$

(a.a) $f = \tilde{0}$: we know:

$0 \in X^d$ and hence $G(0) = \tilde{0} = f$;

(a.b) $f \neq \tilde{0}$, by 3.3, 6.:

$$\begin{aligned} f|X \times \{0\} \in X^d, & G(f|X \times \{0\}) \\ &= (f|\widetilde{X \times \{0\}}) :\ \forall (x, \alpha) \in X_1 : (f|X \times \{0\})(x, \alpha) \\ &= (f|X \times \{0\})(x) + \alpha = f(x, 0) + \alpha = f(x, 0) + 1\alpha \\ &= f(x, 0) + \alpha f(0, 1) = f(x, 0) + f(0, \alpha) = f(x, \alpha)\,. \end{aligned}$$

Thus G is bijective

(b) Let be $f, g \in X^d$ and $f + g \in X^d$, $f \neq 0$, $g \neq 0$;

$$G(f + g) = \widetilde{f + g}\,;$$

let be $(x, \alpha) \in X_1$, $\alpha \neq 0$,

$$\begin{aligned} (\widetilde{f + g})(x, \alpha) &= (f + g)(x) + \alpha = f(x) + g(x) + \alpha \\ &\neq \tilde{f}(x, \alpha) + \tilde{g}(x, \alpha) = (f(x) + \alpha) + (g(x) + \alpha)\,. \end{aligned}$$

Analogously one shows that G is not multiplicative too.

(c) Let (h_i) be a net from X^d, $h \in X^d$ and $h_i \overset{\tau_p}{\to} h$,

$$\forall (x, \alpha) \in X_1;\ h_i(x) \to h(x) \Rightarrow h_i(x) + \alpha \to h(x) + \alpha \text{ in } \mathbb{C}\,; \Rightarrow \tilde{h}_i(x, \alpha) \to \tilde{h}(x, \alpha)\,,$$

hence $G(h_i) \overset{\tau_p}{\to} G(h)$.

(d) Let be $H \subseteq X^d$ be τ_p-open, we will show that $G(H)$ is τ_p-open in

$$X_1^d \backslash \{0\} : \forall f \in G(H) \exists h \in H : f = \tilde{h} = G(h)\,;$$

now let be (f_i) a net from $G(H)$, $f_i \xrightarrow{\tau_p} f$;

$$f_i = \tilde{h}_i, h_i \in H.\ \forall \times \in X : f_i(x,0) \to f_i(x,0)\,,$$

hence $\tilde{h}_i(x,0) \to \tilde{h}(x,0) \Rightarrow h_i(x) \to h(x)$, hence $h_i \xrightarrow{\tau_p} h$; but then there exists i_o:

$$\forall_i \geq i_o : h_i \in H \Rightarrow \forall i \geq i_o : f_i = G(h_i) \in G(H)\,.$$

Thus $G(H)$ is τ_p-open in $X_1^d \backslash \{0\}$.

Corollary 3.5 *The map G is a topological map from (X^d, τ_p) onto $((X_1^d \backslash \{0\}), \tau_p)$.*

Remark 3.6 The two dual spaces (X^d, τ_p) and $(X_1^d \backslash \{0\}, \tau_p)$ respectively are topologically equivalent, but (in general) not algebraically. We see here once more that in our duality approach the essential space is the second dual space X^{dd} of X and not the first dual space X^d of X. Of course X^d is necessary to construct X^{dd}, but in some sense X^d is not so important.

When does a continuous function already vanish at infinity?

It is not hard to find an answer to this question.

Let X be a locally compact, non-compact Hausdorff space, and let $\alpha X = X \cup \{\infty\}$, $\infty \notin X$, be the one-point – compactificativen of X. If $f \in C(X, \mathbb{K})$, we define:

$$f_\infty : \alpha X \to \mathbb{K} :$$
$$f_\infty(x) = \begin{cases} f(x), & x \in X \\ 0, & x = \infty\,. \end{cases}$$

By the definition of a continuous function vanishing at infinity and by the definitions of the topology for αX we see at once:

Proposition 3.7 (a) *$f \in C_0(X, \mathbb{K}) \Leftrightarrow f_\infty$ is continuous in $x = \infty \Leftrightarrow$*

(b) *For each net (x_i) from αX, $\forall i : x_i \neq \infty$, $x_i \to \infty$ in $\alpha X \Rightarrow f(x_i) \to 0$ in $\mathbb{K}$.*

A Stone-Weierstrass theorem

Theorem 3.8 *Let X be a locally compact noncompact Hausdorff space. Suppose A is a closed, selfadjoint subalgebra of $C_0(X, \mathbb{C})$. If A separates the points of X and for every $x \in X$ there exists $f \in A$ with $f(x) \neq 0$ then $A = C_0(X, \mathbb{C})$.*

The homorphic image units

Theorem 3.9 *Let X, Y be rings and $h : X \to Y$ a ring-homomorphism*

(a) *If h is surjective and e is a (multiplicative) unit in X then $h(e)$ is an unit in Y.*

(b) *Let h be bijective and let e_y be a unit in Y.*

Then $h^{-1}(e_y)$ is a unit in X.

We do not prove this proposition.

3.3 The second dual space of X

X is our starting space: X is a nonunital C^*-algebra. As in the case of an unital Banachalgebra or an unital C^*-algebra here also X^d has by Definition 2.9 the defect D and hence by definition 2.10 we get:

$$X^{dd} = \big(C((X^d, \tau_p), \mathbb{C}), \mu\big) \ ,$$

where the topology μ still must be determined. And we have the canonical map

$$J : \ X \to X^{dd} .$$

The constant function

$$1 : \ \forall h \in X^d : 1(h) = 1$$

is a multiplicative unit in X^{dd} . But this means that X and X^{dd} do not belong to the same class of spaces. Hence according to definition 2.10 we must look at $J(X) \subseteq X^{dd}$ and show that X and $J(X)$ belong to the same class of spaces.

X_1 is an unital C^*-algebra and hence $(X_1^d \backslash \{0\}, \tau_p)$ is compact and Hausdorff yielding by corollary 3.5 that (X^d, τ_p) is compact and Hausdorff too. This implies that hold

$$X^{dd} = \big(C_b((X^d, \tau_p), \mathbb{C}), \mu\big)$$

But now we can choose $\mu = \tau_{\|\cdot\|_{\text{sup}}}$: the uniform topology generated by the sup-norm.

Proposition 3.10

$$J(X) \subseteq \big(C_b((X^d, \tau_p), \mathbb{C}), \tau_{\|\cdot\|_{sup}}\big)$$
$$\textit{and} \ \ J : X \to J(X)$$

is an isomorphy and an isometry.

Proof.

$$J_1 : \ X_1 \to X_1^{dd}, \quad J_1(x, \alpha) = \omega((x, \alpha), \cdot) .$$
$$J_1 : \ X_1 \to \big(C_b((X_1^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_{\|\cdot\|_{\text{sup}}}\big) = X_1^{dd}$$

is a bijective, isomorphic and isometric map. $X \times \{0\}$ is a C^*-subalgebra of X_1.

$$(J_1|(X \times \{0\}))(x, \alpha) = J_1(x, 0) = \omega((x, 0), \cdot) = J(x, 0) \in J(X) .$$

Hence J maps X isomorphically and isometrically to $J(X) = J(X \times \{0\})$.

We consider $0 \in X^d$; 0 is either a τ_p-isolated point or a τ_p-accumulation point (clusterpoint) of (X^d, τ_p).

If 0 is isolated then $(X^d \backslash \{0\}, \tau_p)$ is still a compact Hausdorff space implying that

$$X^{dd} = \left(C_b((X^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_{\|\cdot\|_{\text{sup}}}\right) .$$

Since $1 \in X^{dd}$, X and X^{dd} do not belong to the same class of spaces. Hence $0 \in X^d$ must be a τ_p accumulation point.

3.4 Proof of the Gelfand-Naimark theorem

Theorem 3.11

1. *X^d has enough elements*
2. *$(X^d \backslash \{0\}, \tau_p)$ is a Hausdorff, locally compact, noncompact topological space and $(X \backslash \{0\}) \cup \{0\}$ is the onepoint-compactification of $(X^d \backslash \{0\}, \tau_p)$*
3. *$J :\ X \to \left(C_b((X^d \backslash \{0\}, \tau_p), \mathbb{C}, \tau_{\|\cdot\|_{sup}}\right)$ and $J(X) = \left(C_0((X^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_{\|\cdot\|_{sup}}\right)$*
4. *J is an isomorphic and isometric map from X onto $C_0(X^d \backslash \{0\}, \tau_p), \mathbb{C})$*
5. *X and $J(X) = C_0((X^d \backslash \{0\}, \tau_p), \mathbb{C})$ belong to the same class of spaces.*

Proof.

1. X_1 is a commutative, unital C^*-algebra, hence we know that $X_1^d \backslash \{0\}$ has enough elements. But by the theorem 3.4 we get for the cardinal numbers:
$$|X^d| = |X_1^d \backslash \{0\}| .$$
2. $0 \in X^d$ is a τ_p-accumulation point and hence it is well-known that 2. holds, since (X^d, τ_p) is compact and Hausdorff.
3. At first we show that
$$J(X) \subseteq C_0((X^d \backslash \{0\}, \tau_p), \mathbb{C}) :$$
$$J(X) = \{\omega(x, \cdot) | x \in X\} ,$$
$$\omega(x, \cdot) : X^d \to \mathbb{C} :\ \forall h \in X^d :\ \omega(x, \cdot)(h) = \omega(x, h) = h(x) .$$
We consider $\omega(x, \cdot)$ for some $x \in X$; the zerohomomorphism fromX^d is the point at infinity of $(X^d \backslash \{0\}, \tau_p)$. Let (h_i) be an arbitrary net from $X^d \backslash \{0\}$ and $h_i \stackrel{\tau_p}{\to} 0$, then
$$h_i(x) \mapsto 0(x) = 0 \in \mathbb{C} \Rightarrow \omega(x_i, \cdot)(h_i) \to 0$$

showing by proposition 3.7 that

$$\omega(x,\cdot) = Jx \in C_0((X^d \backslash \{0\}, \tau_p), \mathbb{C})$$

holds.

If we can show that the assumptions of the Stone-Weierstrass theorem 3.8 are fullfilled for $J(X)$ then

$$J(X) = \left(C_0((X^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_{\|\cdot\|_{\sup}}\right) .$$

Now, by proposition 3.10 J is an isometry from X into $C_0((X^d \backslash \{0\}, \tau_p), \mathbb{C})$ and thus $J(X)$ is closed in

$$\left(C_0((X^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_{\|\cdot\|_{\sup}}\right)$$

Corollary 3.7 of [3] shows that $J(X)$ is selfadjoined too. J is injective and hence by [1], proposition 4.5, page 290, $J(X)$ separates the points of $X^d \backslash \{0\}$; now finally:

$$\forall h \in X^d \backslash \{0\} \Rightarrow h \neq 0 \Rightarrow \exists x \in X :\ h(x) \neq 0 \in \mathbb{C}\,;$$

then $x \neq 0$ holds too; now, $\omega(x,\cdot) \in J(X)$ and $\omega(x,\cdot)(h) = h(x) \neq 0$. Thus the assumptions of the Stone-Weierstrass theorem are fulfilled.

4. This follows from 3. and from proposition 3.10.

5. X and $\left(C_0((X^d \backslash \{0\}, \tau_p), \mathbb{C}), \tau_{\|\cdot\|_{\sup}}\right)$ are commutative, nonunital C^*-algebras and hence both spaces belong to the same class of spaces.

Corollary 3.12 *Equivalent are:*

(1) *X has the unit e*

(2) *$0 \in X^d$ is an isolated point of (X^d, τ_p)*

(3) *(X^d, τ_p) is compact (and Hausdorff)*

Proof. (1) $\Rightarrow$ (2): this assertion follows from [1], lemma 4.3, page 389

(2) $\Rightarrow$ (3): Since 0 is an isolated point then $(X^d \backslash \{0\}) \cup \{0\}$ cannot be the one-point compactification of $(X^d \backslash \{0\}, \tau_p)$ and thus $(X^d \backslash \{0\}, \tau_p)$ is compact implying that (X^d, τ_p) is compact.

(3) $\Rightarrow$ (1): We have

$$X^{dd} = \left(C_b((X^d, \tau_p), \mathbb{C}), \tau_{\|\cdot\|_{\sup}}\right)$$

because (X^d, τ_p) is compact and Hausdorff.

Hence the constant function 1 is unit in X^{dd} implying by proposition 3.10 that $e := J^{-1}(1)$ is unit in X.

4 The embedding theorem of Kadison

4.1 The spaces X_{sa} and $S(X)$

Let X be an unital C^*-algebra. By X_{sa} we denote the set of all selfadjoined elements of X and by $S(X)$ we mean the set of states of X.

$X_{sa} \subseteq X$ is a real vector space and $(X_{sa}, \|\cdot\|)$ is real Banach subspace of X. The unit $e \in X$ belongs to $X_{sa} : e^* = e$. For instance:

$$x \in X \Rightarrow x^* \in X \Rightarrow x^*x \in X ,$$

but $x^*x \in X_{sa}$ too: $(x^*x)^* = x^*x^{**} = x^*x$.

If X is commutative then of course X_{sa} is closed under multiplication.

4.2 The first and the second dual space of X_{sa}

According to our duality theory we define now the first dual space of X_{sa}.

e is the multiplicative unit in X and $e \in X_{sa}$. Hence we define:

Definition 4.1 $(X_{sa})^d = \{h: \ X_{sa} \to \mathbb{R} | h \textit{ is linear, continuous and } h(e) = 1\}$

Remark 4.2 1. $(X_{sa})^d$ is not identical with the the Banachspace – dual

$$X'_{sa} = \{h: \ X_{sa} \to \mathbb{R} \,|\, h \text{ is linear and continuous}\} .$$

2. For $(X_{sa})^d$ does not hold:

$$h_1, h_2 \in (X_{sa})^d \Rightarrow h_1 + h_2 \in (X_{sa})^d : \text{ if } h_1 + h_2 \in (X_{sa})^d \text{ then } (h_1 + h_2)(e) = 1 ,$$

but otherwise:

$$(h_1 + h_2)(e) = h_1(e) + h_2(e) = 2 ,$$

a contradiction.

Hence $(X_{sa})^d$ is no vectorspace.

From remark 4.2, 2. we get: $(X_{sa})^d$ has the defect D according to 2.9. Hence by 2.10 the second dual space of X_{sa} reads:

Remark 4.3

$$(X_{sa})^{dd} = \left(C((X_{sa})^d, \tau_p), (\mathbb{R}, \tau_{|\cdot|}), \mu\right) ,$$

where are: $\tau_{|\cdot|}$ the Euclidian topology and μ a topology for the space of continuous functions. μ still must be specified.

Remark 4.4 We don't know the properties of $((X_{sa})^d, \tau_p)$, especially we don't know wether or not, $((X_{sa}^d), \tau_p)$ is compact Hausdorff or not. But in exchange we find a Hausdorff and compact space as the next proposition will show.

Proposition 4.5 *The topological spaces (X_{sa}^d, τ_p) and $(S(X), \tau_p)$ are homeomorphic.*

For the proof we need a result, which we provide by the following proposition.

For the $\mathbb{C}^*$-algebra $\mathbb{C}$ we easily can prove the characterization of the convergence of a sequence (z_n), $\forall n \in \mathbb{N} : z_n \in \mathbb{C}$, $z \in \mathbb{C}$: let be $\forall n \in \mathbb{N} :\ z_n = x_n + iy_n,\ z = x + iy$.

Then holds:

$$z_n \to z \Leftrightarrow x_n \to x \text{ and } y_n \to y\,.$$

Somewhat more difficult to prove is the corresponding characterization in an arbitrary C^*-algebra.

Proposition 4.6 *Let X an unital $C*$-algebra, X_{sa} denotes the set of all selfadjoint elements of X. Let (x_n) be a sequence in X, $x \in X$. Convergence means norm-convergence. We write:*

$$x_n = a_n + ib_n,\ \ x = a + ib\,;\ \forall n :\ \ a_n, b_n \in X_{sa},\ \ a, b \in X_{sa}\,.$$

Then holds: Equivalent are:

(1) $x_n \to x$

(2) $a_n \to a$ *and* $b_n \to b$

Proof. (2) $\to$ (1):

$$\begin{aligned}\|x_n - x\| &= \|(a_n - a) + i(b_n - b)\| \\ &\leq \|a_n - a\| + |i|\|b_n - b\| \\ &= \|a_n - a\| + \|b_n - b\| \to 0\,,\end{aligned}$$

hence $\|x_n - x\| \to 0$ too.

(1) $\to$ (2):

$$\forall n :\ \ a_n - a,\ \ b_n - b \in X_{sa}\,;$$

but then

$$(a_n - a)^2,\ (b_n - b)^2 \in X_{sa} \text{ and } (a_n - a)^2,\ (b_n - b)2$$

are positive.

Now, for instance

$$(b_n - b)^2 \leq (a_n - a)^2 + (b_n - b)^2\,,$$

since

$$\left[(a_n - a)^2 + (b_n - b)^2\right] - (b_n - b)^2 = (a_n - a)^2 \geq 0 \, .$$

But

$$0 \leq (b_n - b)^2 \leq (a_n - a)^2 + (b_n - b)^2 \Rightarrow \|(b_n - b)^2\| \leq \|(a_n - a)^2 + (b_n - b)^2\| \, .$$

Otherwise: $x_n - x = (a_n - a) + i(b_n - b)$ yielding

$$\begin{aligned} \|x_n - x\|^2 &= \|[(a_n - a) + i(b_n - b)]^*[(a_n - a) + i(b_n - b)]\| \\ &= \|[(a_n - a) - i(b_n - b)][(a_n - a) + i(b_n - b)]\| \\ &= \|(a_n - a)^2 + (b_n - b)^2\| \end{aligned}$$

Hence we get:

$$\|(b_n - b)^2\| \leq \|x_n - x\|^2 \, ;$$

$b_n - b \in X_{sa}$ and hence $b_n - b$ is normal $\forall n$, which gives us:

$$\|(b_n - b)^2\| = \|(b_n - b\|^2 \, ;$$

thus

$$\|b_n - b\|^2 \leq \|x_n - x\|^2 \Rightarrow \|b_n - b\| \leq \|x_n - x\| \text{ and } \|x_n - x\| \to 0 \Rightarrow \|b_n - b\| \to 0 \, .$$

By this way we show

$$\|a_n - a\| \to 0$$

too.

Thus (1) $\Rightarrow$ (2) is proved too.

Proof of proposition 4.5

We define a map φ:

$$\varphi : S(X) \to (X_{sa})^d : \; \forall h \in S(X) : \; \varphi(h) = h|X_{sa}$$

Lemma 4.7 *φ is an injective and surjective map from $S(X)$ to $(X_{sa})^d$.*

Proof of the lemma.

At first we show:

$$\varphi(S(X)) \subseteq (X_{sa})^d : \; \forall h \in S(X) :$$

1. $\varphi(h) = h|X_{sa}$ is linear:

$$\forall x_1, x_2 \in X_{sa};\ \forall \alpha_1, \alpha_2 \in \mathbb{R} \Rightarrow \alpha_1 x_1 + \alpha_2 x_2 \in X_{sa}\,,$$

   but this linear combination is also an element of X, hence

$$h(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 h(x_1) + \alpha_2 h(x_2)\,,$$

   yielding:

$$(h|X_{sa})(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 (h|X_{sa})(x_1) + \alpha_2 (h|X_{sa})(x_2)$$

2. h continuous $\Rightarrow \varphi(h) = h|X_{sa}$ is continuous
3. $h \in S(X) \Rightarrow h(e) = 1$;

$$e \in X_{sa}:\ 1 = h(e) = (h|X_{sa})(e) \Rightarrow \varphi(h)(e) = (h|X_{sa})(e) = 1\,.$$

By 1, 2 and 3 we get:

$$h|X_{sa} \in (X_{sa})^d\,,$$

hence $\varphi(S(X)) \subseteq (X_{sa})^d$.

4. φ is injective:

   $\forall f, g \in S(X)$: let be

$$\varphi(f) = f|X_{sa} = g|X_{sa} = \varphi(g)$$

   We want to show: $\forall x \in X : f(x) = g(x)$, hence $f = g$:

   (a) $x \in X_{sa} : f(x) = (f|X_{sa})(x) = (g|X_{sa})(x) = g(x)$

   (b) $x \in X \backslash X_{sa} : x = x_1 + ix_2$, $x_1, x_2 \in X_{sa}$; f, g are linear on X:

$$f(x) = f(x_1) + if(x_2),\ g(x) = g(x_1) + ig(x_2)\,,$$

   but:

$$x_1, x_2 \in X_{sa} \Rightarrow f(x_1) = g(x_1),\ f(x_2) = g(x_2)$$

   showing $f(x) = g(x)$ and hence, finally $f = g$.

5. We show that φ is surjective too.

   $\forall h \in (X_{sa})^d$: we define the function $\tilde{h}$:

$$\forall x \in X:\ x = x_1 + ix_2,\ x_1, x_2 \in x_{sa}\,;$$
$$\tilde{h} : X \to \mathbb{C}:\ \tilde{h}(x) = h(x_1) + ih(x_2)$$

Lemma 4.8 *$\tilde{h} \in S(X)$ and $\varphi(\tilde{h}) = \tilde{h}|X_{sa} = h$*

Proof. (a) $\tilde{h}$ is linear.

We know that h is linear.

$$\forall x, y \in X,\ \forall \alpha, \beta \in \mathbb{C}:$$

We can write:

$$\begin{aligned} x &= x_1 + ix_2, & y &= y_1 + iy_2 \\ \alpha &= \alpha_1 + i\alpha_2, & \beta &= \beta_1 + i\beta_2 \end{aligned}$$

Now we can compute $\alpha x + \beta y$ in X:

$$\begin{aligned} \alpha x + \beta y &= (\alpha_1 + i\alpha_2)(x_1 + ix_2) + (\beta_1 + i\beta_2)(y_1 + iy_2) \\ &= \alpha_1 x_1 + i\alpha_2 x_1 + i\alpha_1 x_2 - \alpha_2 x_2 + \beta_1 y_1 + i\beta_2 y_1 + i\beta_1 y_2 - \beta_2 y_2 \\ &= \alpha_1 x_1 - \alpha_2 x_2 + \beta_1 y_1 - \beta_2 y_2 + i(\alpha_2 x_1 + \alpha_1 x_2 + \beta_2 y_1 + \beta_1 y_2)\,. \end{aligned}$$

Then follows:

$$\begin{aligned} \tilde{h}(\alpha x + \beta y) &= h(\alpha_1 x_1 - \alpha_2 x_2 + \beta_1 y_1 - \beta_2 y_2) + ih(\alpha_2 x_1 + \alpha_1 x_2 + \beta_2 y_1 + \beta_1 y_2) \\ &= \alpha_1 h(x_1) - \alpha_2 h(x_2) + \beta_1 h(y_1) - \beta_2 h(y) \\ &\quad + i\alpha_2 h(x_1) + i\alpha_1 h(x_2) + i\beta_2 h(y_1) + i\beta_1 h(y_2) \\ &= (\alpha_1 + i\alpha_2)h(x_1) + (i\alpha_1 + i^2\alpha_2)h(x_2) + \ldots \\ &= (\alpha_1 + i\alpha_2)h(x_1) + i(\alpha_1 + i\alpha_2)h(x_2) + \ldots \\ &= (\alpha_1 + i\alpha_2)(h(x_1) + ih(x_2) + \ldots \\ &= \alpha\tilde{h}(x) + \ldots\,; \end{aligned}$$

hence h is linear:

$$\tilde{h}(\alpha x + \beta y) = \alpha\tilde{h}(x) + \beta\tilde{h}(y)\,,$$

(b) $\tilde{h}$ is continuous on X: let be (x_n) a sequence from X, $x \in X$ and $\|x_n - x\| \to 0$ for $n \to +\infty$; let further be:

$$\forall n:\ x_n = x_n^1 + ix_n^2,\ x = x_1 + ix_2$$

we want to show:

$$\tilde{h}(x_n) \to \tilde{h}(x):$$

by proposition 4.5 we get: $x_n \to x \Leftrightarrow x_n^1 \to x_1$ and $x_n^2 \to x_2$ yielding:

$$\tilde{h}(x_n) = h(x_n^1) + ih(x_n^2) \to h(x_1) + ih(x_2) = \tilde{h}(x)\,,$$

since h is continuous on $(X_{sa}, \|\cdot\|)$.

(c) $\tilde{h}|X_{sa} = h:\ \forall x \in X_{sa}:\ x = x + i \cdot 0$, hence:

$$(\tilde{h}|X_{sa})(x + i \cdot 0) = \tilde{h}(x + i \cdot 0) = h(x) + ih(0) = h(x)\,,$$

since $0 \in X_{sa}$ and h is linear yields: $h(0) = 0$

(d) $\tilde{h}(e) = 1$: $e \in X_{sa} \Rightarrow \tilde{h}(e) = (\tilde{h}|X_{sa})(e) = h(e) = 1$ by (c).

By a well-known theorem of the C^*-algebra-theory follows by (a), (b) and (d) that $\tilde{h}$ is positive, yielding by another theorem:

$$\|\tilde{h}\| = \tilde{h}(e)\,,$$

and hence we have $\|\tilde{h}\| = 1$ too.

Thus we have shown:

$$\tilde{h} \in S(X) \text{ and } \varphi(\tilde{h}) = h\,.$$

Hence indeed we got: $\varphi : S(X) \to (X_{sa})^d$ is injective and surjective.

Lemma 4.9 *$\varphi:\ (S(X), \tau_p) \to ((X_{sa})^d, \tau_p)$ is continuous.*

Proof. Let be (h_i) a net from $S(X)$, $h \in S(X)$ and $h_i \xrightarrow{\tau_p} h$; we want to show that $\varphi(h_i) \xrightarrow{\tau_p} \varphi(h)$ holds: $\forall x \in X_{sa}$, then $x \in X$ too and thus $h_i(x) \to h(x)$ in $\mathbb{R}$. Now,

$$\varphi(h_i)(x) = (h_i|X_{sa})(x) \to (h|X_{sa})(x)\,,$$

since $x \in X_{sa}$. Hence

$$\varphi(h_i) \xrightarrow{\tau_p} \varphi(h) \text{ in } (X_{sa})^d\,.$$

Finally we must still show:

Lemma 4.10 *$\varphi : (S(X), \tau_p) \to ((X_{sa})^d, \tau_p)$ is open:*

Proof. Let $G \subseteq S(X)$ be τ_p-open, we show: $\varphi(G)$ is τ_p-open in $(X_{sa})^d$: let be $h \in \varphi(G)$ and (h_k) a net from $(X_{sa})^d$ such that $h_k \xrightarrow{\tau_p} h$.

φ is bijective, hence there exists $g \in G$,

$$\forall k : g_k \in S(X) : \varphi(g) = h = g|X_{sa},\ \forall k : \varphi(g_k) = g_k|X_{sa} = h_k\,.$$

Now we want to show:

$$g_k \xrightarrow{\tau_p} g \text{ in } S(X) :$$

(a) $\forall x \in X_{sa}:\ g(x) = (g|X_{sa})(x) = \varphi(g)(x) = h(x)$; $\forall k:\ g_k(x) = h_k(x)$. Hence $h_k(x) \to h(x)$ meaning that holds:

$$g_k(x) \xrightarrow{\tau_p} g \text{ on } X_{sa}\,.$$

(b) $\forall x \in X \backslash X_{sa}: \ x = x_1 + ix_2,\ x_1, x_2 \in X_{sa}$; by (a) we get:

$$\begin{aligned} g_k(x_1) &= h_k(x_1) \to h(x_1) = g(x_1)\,, \\ g_k(x_2) &= h_k(x_2) \to h(x:2) = g(x_2) \\ &\Rightarrow g_k(x_1) + ig_k(x_2) \to g(x_1) + ig(x_2)\,. \end{aligned}$$

Now $g \in S(X)$ and $\forall k: g_k \in S(X)$ showing that these functions are linear:

$$\begin{aligned} x = x_1 + ix_2 \Rightarrow g(x) &= g(x_1) + ig(x_2), \\ \forall k: g_k(x) &= g_k(x_1) + ig_k(x_2) \end{aligned}$$

But then follows:

$g_k(x) \to g(x)$ on $X \backslash X_{sa}$, and thus from (a), (b) we get:

$$g_k(x) \to g(x),\ \forall x \in X,\ g_k \overset{\tau_p}{\to} g\,.$$

Since $g \in G$ and G is τ_p-open there exists k_o:

$\forall k \geq k_o: \ g_k \in G$ showing that holds:

$$\forall k \geq k_o: \varphi(g_k) = h_k \in \varphi(G)\,,$$

hence $\varphi(G)$ is τ-open in $(X_{sa})^d$.

Final *proof* of proposition 4.5. By lemma 4.7, 4.8, 4.9 and 4.10

$$\varphi: (S(X), \tau_p) \to ((X_{sa})^d, \tau_p)$$

is bijective, continuous and open yielding that φ is a topological map onto $(X_{sa})^d$ and thus $(S(X), \tau_p)$ and $((X_{sa})^d, \tau_p)$ are homeomorphic.

Corollary 4.11 *The first dual space of X_{sa} is a Hausdorff and compact topological space w. r. t. the pointwise topology τ_p.*

Proof. We know that the state space $(S(X), \tau_p)$ is a compact and Hausdorff space.

We come now back to the second dual space 4.3 of X_{sa}:

$$\begin{aligned} (X_{sa})^{dd} &= \left(C(((X_{sa})^d, \tau_p), (\mathbb{R}, \tau_{|\cdot|})), \mu\right) \\ &= \left(C_b(((X_{sa})^d, \tau_p), (\mathbb{R}, \tau_{|\cdot|})), \mu\right) \\ &= \left(C_b((S(X), \tau_p), (\mathbb{R}, \tau_{|\cdot|})), \mu\right)\ , \end{aligned}$$

where $C_b(\cdot, \cdot)$ of course means the space of bounded and continuous real functions.

Then for μ we can choose the sup-norm and hence the uniform topology.

We state now:

1. $(X_{sa}, \|\cdot\|)$ (and $S(X), \|\cdot\|$) and $(\mathbb{R}, |\cdot|)$ both are real Banach spaces.

   X_{sa} and $\mathbb{R}$ both have a multiplicative unit. Hence both spaces belong to the same class of spaces.

2. $(X_{sa}, \|\cdot\|)$ and $((X_{sa})^{dd}, \|\cdot\|_{\sup})$ are real Banach spaces; both have a multiplicative unit. $X_{sa}, (X_{sa})^{dd}$ belong to the same class of spaces.

We still need a lemma.

Lemma 4.12

$$\forall x \in X_{sa} : \|x\| \in \sigma(x)\,.$$

Proof. 1. $0 \in X_{sa}$, but 0^{-1} does not exist and hence $\|0\| = 0 \in \sigma(0)$.

2. $x \in X_{sa}$ and $x \neq 0$; $x \in X_{sa} \Rightarrow \sigma(x) \subseteq \mathbb{R}$ and x is normal and thus:

$$r(x) = s = \sup\left\{|x|\middle|\lambda \in \mathbb{R} \text{ and } \lambda \in \sigma(x)\right\} = \|x\|\,.$$

   $\|x\| > 0 \Rightarrow \exists$ sequence $(\lambda_n):\ \forall n:\ \lambda_n \in (\sigma(x), \|x\|)$ such that $\lambda_n \to s$.

   $\sigma(x)$ is Hausdorff and compact and thus $\sigma(x)$ is sequentially compact and Hausdorff too since $(X, \|\cdot\|)$ is a metric space. Thus we find a subsequence (λ_{n_k}) of (λ_n) and $\lambda \in \sigma(x)$, $\lambda > 0 : \lambda_{n_k} \to \lambda$, but also $\lambda_{n_k} \to s = \|x\|$, implying $\lambda = \|x\| \in \sigma(x)$.

4.3 Proof of the Kadison embedding theorem

Theorem 4.13 *Let X be an unital C^*-algebra. Then holds:*

1. $J:\ (X_{sa}, \|\cdot\|) \to ((X_{sa})^{dd}, \|\cdot\|_{sup}) = \big(C_b((X_{sa})^d, \tau_p), (\mathbb{R}, \tau_{|\cdot|}), \tau_{\|\cdot\|_{sup}}\big)$ *is an isometric and isomorphic map onto* $(X_{sa})^{dd}$
2. $J(X_{sa})$ *separates the points of* $(X_{sa})^d$
3. $J(X_{sa})$ *is a closed subspace of* $(X_{sa})^{dd}$

Proof. By corollary 4.1 of [1], p. 284 we get: $J(X_{sa}) \subseteq (X_{sa})^{dd}$. Now

$$(X_{sa})^d \subseteq X'_{sa} = \{h:\ X_{sa} \to \mathbb{R}|\ \text{his linear and } h \text{ is continuous}\}$$
$$\forall h \in X_{sa}^d\ \exists g \in S(X):\ \varphi(g) = g|X_{sa} = h\,;$$

hence

$$\|h\| = \|g|X_{sa}\| = \sup\{|g(x)|\big|x \in x_{sa} \text{ and}$$
$$\|x\| \leq 1\} \leq \sup\{|g(x)|\big|\ x \in X \text{ and } \|x\| \leq 1\} = \|g\| = 1$$

and thus $\|h\| \leq 1$.

But then we can apply proposition 4.3, p. 287 of [1]. At first we get:

$$\forall x \in X_{sa}: \ \|J(x)\|_{\sup} \leq \|x\| \, .$$

Moreover we have:

$\forall x \in X_{sa}$, $x \neq 0$, by lemma 4.12 we know: either $\|x\| \in \sigma(x)$ or $-\|x\| \in \sigma(x)$. Let us consider $-\|x\|$: there exists $h \in S(X) : h(x) = -\|x\|$; but $x \in X_{sa} \Longrightarrow h(x) = h\big|X_{sa}(x)$ showing that $h|X_{sa} \in X_{sa}^d$ and $h|X_{sa}(x) = -\|x\|$, implying $|h|X_{sa}(x)| = |-\|x\|| = \|x\|$ and hence $\|x\| \leq |h|X_{sa}|$. Of course this last result we get also if $\|x\| \in \sigma(x)$. This implies by the above mentioned proposition that holds $\|x\| \leq \|J(x)\|_{\sup}$. Hence we have:

$$\forall x \in X_{sa}: \ \|J(x)\|_{\sup} = \|x\| \, ,$$

yielding that $J : X_{sa} \to J(X_{sa})$ is an isometric map.

Now J is then an injective map onto $J(X_{sa})$ and thus the homomorphy theorem 4.4, p. 284 of [1] shows that J is an isomorphic map for real Banach spaces too, meaning that point 1. of our theorem is proved, but only for $J : X_{sa} \to J(X_{sa})$.

Proposition 4.3 of [1] shows also 2.. $J(X_{sa})$ separates the points of $(X_{sa})^d$.

Since J is an isometric map J is an uniform isomorphy too, yielding that $J(X_{sa})$ is a complete subspace of $(X_{sa}^{dd}, \|\cdot\|)$ since X_{sa} is complete.

Thus we proved 3.:

$J(X_{sa})$ is a closed subspace of $(X_{sa})^{dd}$.

Concluding we find:

$$e \in X_{sa} \Rightarrow \omega(e, \cdot) \in (X_{sa})^{dd}, \ \text{but: } \forall h \in (X_{sa})^d : \ \omega(e, \cdot)(h) = h(e) = 1 \, ,$$

showing that the constant function $\omega(e, \cdot) \equiv 1$ belongs to $J(X_{sa})$.

But this result together with assertions 2., 3. shows that $J(X_{sa}) = (X_{sa})^{dd}$ by the theorem of Stone-Weierstrass.

Now our proof is complete.

Concluding remarks We consider our basic assumptions 2.3, 2.5 and 2.11:

(1) X and Y belong to the same class of spaces

(2) Y always has a topology

(3) X and X^{dd} are in the same class of spaces

As we have shown in our text the general procedure runs as follows:

We start with the space X and want to define the second dual space of X and to embedd X into X^{dd} using the canonical map J. To do so we must choose a suitable space Y such that (1) is fulfilled. Then we can define the first dual space X^d of X with respect to Y, where (2) holds. According to the properties of X^d we are able to define the second dual space X^{dd} of X w. r. t. Y such that (3) is fulfilled and $J: \ X \to X^{dd}$ embedds X into or onto X^{dd}.

References

[1] **Bartsch, R.**, and **Poppe, H. :** *An abstract algebraic-topological approach to the notions of a first and a second dual space, I.* In: Theory and applications of proximity, nearness and uniformity. Caserta: Dipartimento di Matematica, Seconda Università di Napoli, Quaderni di Matematica 22; Rome: Aracne 2009, 275 – 297

[2] **Bartsch, R.**, and **Poppe, H. :** *An abstract algebraic-topological approach to the notions of a first and a second dual space, II.* Int. J. Pure, Appl. Math. **84**, 2013, 651 – 667

[3] **Bartsch, R.**, and **Poppe, H. :** *An abstract algebraic-topological approach to the notions of a first and a second dual space, III.* N. Z. J. **46**, 2016, 1 – 8

[4] **Folland, G. B. :** *A course in Abstarct Harmonic Analysis.* CRC Press 1985

[5] **Kadison, R. V.**, and **Ringrose, J. R. :** *Fundamentals of the Theory of Operator Algebras, Volumes I, II, III, IV.* Amer. Math. Soc.1997

[6] **Poppe, H. :** *Compactness in General Function Spaces.* Deutscher Verlag der Wissenschaften, Berlin 1974

[7] **Salamon, D. A. :** *Measure and Integration.* EMS 2016

[8] **Zhu, K. H. :** *An Introduction to Operator Algebras.* CRC Press 1993

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An inverse Problem for a parabolic System in an unbounded Guide

ABSTRACT. In this article we consider a two-by-two parabolic system defined on an unbounded guide with coefficients depending both on the space variable and on the time variable. The main aim of this paper is to obtain a stability result for the coefficients depending on the space variable. Using Carleman inequalities adapted for the guide, we obtain Hölder estimates of these coefficients in any finite portion of the guide with boundary measurements, given two sets of initial conditions.

KEY WORDS. inverse problems, Carleman inequalities, heat operator, system, unbounded guide

1 Introduction

Let ω be a bounded connex domain in $\mathbb{R}^{n-1}$, $n \geq 2$ with C^2 boundary. Denote $\Omega = \mathbb{R} \times \omega$ and $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$. We consider the following problem

$$\begin{cases} \partial_t u = \Delta u + \alpha\phi_1 u + \beta\phi_2 w + g_1 \text{ in } Q, \\ \partial_t w = \Delta w + \gamma\phi_3 u + \delta\phi_4 w + g_2 \text{ in } Q, \\ u(.,0) = a_1, \; w(.,0) = a_2 \text{ in } \Omega, \\ u = a_3, \; w = a_4 \text{ in } \Sigma, \end{cases} \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta$ are bounded coefficients defined on Ω such that

$$\alpha, \beta, \gamma, \delta \in \Lambda_1(M_0) = \{f \in L^\infty(\Omega), \|f\|_{L^\infty(\Omega)} \leq M_0\} \text{ for some } M_0 > 0,$$

and $\phi_1, \phi_2, \phi_3, \phi_4$ are bounded coefficients defined on $[0,T]$ such that for $i = 1, \cdots, 4$

$$\phi_i \in \Lambda_2(M_0) = \{f \in C^1([0,T]), f(\frac{T}{2}) \neq 0 \text{ and } \|f\|_{C^1([0,T])} \leq M_0\}.$$

The main problem is to estimate the coefficients $(\alpha, \beta, \gamma, \delta)$ from boundary observations of (u, w).

We will consider two sets of Cauchy and Dirichlet conditions A and B and denote

$$G=(g_1,g_2),\ A=(a_1,a_2,a_3,a_4),\ B=(b_1,b_2,b_3,b_4),\ \rho=(\alpha,\beta,\gamma,\delta,\phi_1,\phi_2,\phi_3,\phi_4),$$

$$\tilde{\rho_1}=(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma},\widetilde{\delta},\phi_1,\phi_2,\phi_3,\phi_4),\ \tilde{\rho_2}=(\alpha,\widetilde{\beta},\widetilde{\gamma},\widetilde{\delta},\tilde{\phi_1},\phi_2,\phi_3,\phi_4),\ \tilde{\rho_3}=(\alpha,\tilde{\beta},\tilde{\gamma},\tilde{\delta},\phi_1,\phi_2,\phi_3,\phi_4). \tag{1.2}$$

Let two positive reals l, L be such that $l<L$. Denote

$$\Omega_L=(-L,L)\times\omega \text{ and } \Omega_l=(-l,l)\times\omega.$$

The first result of this paper gives a Hölder stability result (3.4) for the coefficients $\alpha,\beta,\gamma,\delta$ and is the following (see Theorem 3.1)

$$\|\alpha-\tilde{\alpha}\|^2_{L^2(\Omega_l)}+\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)}$$
$$\leq K\left(\int_{\gamma_L\times(0,T)}\sum_{k=0}^{1}(|\partial_\nu(\partial_t^k(u_A-\tilde{u}_A))|^2+|\partial_\nu(\partial_t^k(w_A-\tilde{w}_A))|^2)\ d\sigma\ dt\right.$$
$$\left.+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{1}(|\partial_\nu(\partial_t^k(u_B-\tilde{u}_B))|^2+|\partial_\nu(\partial_t^k(w_B-\tilde{w}_B))|^2)\ d\sigma\ dt\right)^{\kappa}$$

where K is a positive constant, $\kappa\in(0,1)$, γ_L is a part of the boundary (see (2.2)), and assuming that the hypothesis (3.3) is satisfied. We consider in the above result $V_A=(u_A,w_A)$ (resp. $\tilde{V}_A=(\widetilde{u}_A,\widetilde{w}_A)$) a solution of (1.1) associated with the coefficients (ρ,G,A) (resp. $(\tilde{\rho_1},G,A)$) and $V_B=(u_B,w_B)$ (resp. $\tilde{V}_B=(\widetilde{u}_B,\widetilde{w}_B)$) a solution of (1.1) associated with the coefficients (ρ,G,B) (resp. $(\tilde{\rho_1},G,B)$) where A is a set of Cauchy and Dirichlet conditions and B is a suitable change of initial and boundary conditions. The above result is an improvement of results obtained in [5] with different and less restrictive hypotheses but with two choices of Cauchy and Dirichlet conditions A and B. In abbreviated form we will call A and B the two sets of initial conditions. It is an improvement because on one hand the hypotheses, though quite differents, are easier to satisfy than in [5] and on the other hand there are no observation terms of the solutions (u,w) at a fixed time on the right-hand side of the estimate, such as $\|(u_A-\tilde{u}_A)(.,\frac{T}{2})\|^2_{H^2(\Omega_L)}$ (see [5]). The idea of choosing two different sets of initial conditions can be found in [2] for a hyperbolic equation in a bounded domain (see also [6] for a hyperbolic system).

A consequence of the above result is given in Theorem 3.2 where the measurements are given for only one component (for example u) and is the following (see (3.6))

$$\|\alpha-\tilde{\alpha}\|^2_{L^2(\Omega_l)}+\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)}$$
$$\leq K\Big(\|u_A-\tilde{u}_A\|^2_{H^2([0,T],H^2(\omega'\cap\Omega_L))}+\|u_A-\tilde{u}_A\|^2_{H^1([0,T],H^4(\omega'\cap\Omega_L))}$$

$$+\|u_B-\tilde{u}_B\|^2_{H^2([0,T],H^2(\omega'\cap\Omega_L))}+\|u_B-\tilde{u}_B\|^2_{H^1([0,T],H^4(\omega'\cap\Omega_L))}$$
$$\left.+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{1}(|\partial_\nu(\partial_t^k(u_A-\tilde{u}_A))|^2+|\partial_\nu(\partial_t^k(u_B-\tilde{u}_B))|^2)\,d\sigma\,dt\right)^\kappa$$

where $K>0$, $\kappa\in(0,1)$ and ω' is a neighborhood of γ_L, ω' being a subdomain of Ω such that $\gamma_L\subset\partial\omega'$, and assuming that $\alpha=\tilde{\alpha}$ and $\beta=\tilde{\beta}$ in ω'. We can relax the hypothesis that the coefficients α and β are supposed known in ω' when these coefficients are in $H^2(\Omega)$ and we obtain a similar result with the L^2-norms replaced by the H^2-norms for the coefficients α and β on the left-hand side of the above estimate and additional terms such as $\|(u_A-\tilde{u}_A)(.,\frac{T}{2})\|^2_{H^4(\Omega_L)}$ on the right-hand side of this estimate (see (3.7)).
The third result gives a Hölder result (3.10) for the coefficients $\phi_1,\beta,\gamma,\ \delta$ (assuming also that $\phi_i\in C^2([0,T])$) and is the following (see Theorem 3.3)

$$\sum_{i=0}^{2}\|\partial_t^i(\phi_1-\tilde{\phi}_1)\|^2_{L^2((0,T))}+\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)}$$
$$\leq K\left(\sum_{k=0}^{1}(\|\partial_t^k(u_A-\tilde{u}_A)(\cdot,\frac{T}{2})\|^2_{H^2(\Omega_L)}+\|\partial_t^k(u_B-\tilde{u}_B)(\cdot,\frac{T}{2})\|^2_{H^2(\Omega_L)})\right.$$
$$+\|\partial_t^2(u_A-\tilde{u}_A)(\cdot,\frac{T}{2})\|^2_{L^2(\Omega_L)}+\|\partial_t^2(u_B-\tilde{u}_B)(\cdot,\frac{T}{2})\|^2_{L^2(\Omega_L)})+\|(w_A-\tilde{w}_A)(\cdot,\frac{T}{2})\|^2_{H^2(\Omega_L)}$$
$$+\|(w_B-\tilde{w}_B)(\cdot,\frac{T}{2})\|^2_{H^2(\Omega_L)}+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u_A-\tilde{u}_A))|^2+|\partial_\nu(\partial_t^k(w_A-\tilde{w}_A))|^2)\,d\sigma\,dt$$
$$\left.+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u_B-\tilde{u}_B))|^2+|\partial_\nu(\partial_t^k(w_B-\tilde{w}_B))|^2)\,d\sigma\,dt)\right)^\kappa$$

where K is still a positive constant, $\kappa\in(0,1)$, and $\tilde{\phi}_1$ belongs to a set of admissible coefficients (namely $\Lambda_3(M_3)$, see (3.8)). In the above case we denote $V_A=(u_A,w_A)$ (resp. $\tilde{V}_A=(\widetilde{u}_A,\widetilde{w}_A)$) a solution of (1.1) associated with (ρ,G,A) (resp. $(\tilde{\rho}_2,G,A)$) and $V_B=(u_B,w_B)$ (resp. $\tilde{V}_B=(\widetilde{u}_B,\widetilde{w}_B)$) a solution of (1.1) associated with (ρ,G,B) (resp. $(\tilde{\rho}_2,G,B)$). So this third result gives a determination of one coefficient depending on the time variable. Be careful that the meanings of $\tilde{V}_A$ and $\tilde{V}_B$ are not the same in Theorems 3.1 and 3.2 on one hand and Theorem 3.3 on the other hand.
Finally the fourth theorem gives a Hölder result (3.11) for the following reaction-diffusion system

$$\begin{cases}\partial_t u=\Delta u+\alpha\phi_1 u+\beta\phi_2 w+\Theta_1\cdot\nabla u+\Theta_2\cdot\nabla w+g_1 \text{ in } Q,\\ \partial_t w=\Delta w+\gamma\phi_3 u+\delta\phi_4 w+\Theta_3\cdot\nabla u+\Theta_4\cdot\nabla w+g_2 \text{ in } Q,\\ u(.,0)=a_1,\ w(.,0)=a_2 \text{ in } \Omega,\\ u=a_3,\ w=a_4 \text{ in } \Sigma,\end{cases}\tag{1.3}$$

where all the coefficients $\alpha, \beta, \gamma, \delta, \phi_1, \phi_2, \phi_3, \phi_4, \Theta_1, \Theta_2, \Theta_3, \Theta_4$ are bounded. We present here a result for the four coefficients $\beta, \gamma, \delta, \Theta_1$ (and assuming that Θ_1 has the form $\Theta_1 = \nabla \xi_1$). So denote now

$$\Theta = (\Theta_1, \cdots, \Theta_4), \quad \tilde{\Theta} = (\tilde{\Theta}_1, \Theta_2, \Theta_3, \Theta_4). \tag{1.4}$$

We get the following result

$$\begin{aligned}
&\|\beta - \tilde{\beta}\|^2_{L^2(\Omega_l)} + \|\gamma - \tilde{\gamma}\|^2_{L^2(\Omega_l)} + \|\delta - \tilde{\delta}\|^2_{L^2(\Omega_l)} + \|\Theta_1 - \tilde{\Theta}_1\|_{(L^2(\Omega_l))^n} \\
&\leq K \Bigg(\|(u_A - \tilde{u}_A)(., \frac{T}{2})\|^2_{H^3(\Omega_L)} + \|(u_B - \tilde{u}_B)(., \frac{T}{2})\|^2_{H^3(\Omega_L)} \\
&+ \int_{\gamma_L \times (0,T)} \sum_{k=0}^{1} (|\partial_\nu(\partial_t^k(u_A - \tilde{u}_A))|^2 + |\partial_\nu(\partial_t^k(w_A - \tilde{w}_A))|^2) \, d\sigma \, dt \\
&+ \int_{\gamma_L \times (0,T)} \sum_{k=0}^{1} (|\partial_\nu(\partial_t^k(u_B - \tilde{u}_B))|^2 + |\partial_\nu(\partial_t^k(w_B - \tilde{w}_B))|^2) \, d\sigma \, dt \Bigg)^\kappa
\end{aligned}$$

where K is a positive constant, $\kappa \in (0, 1)$. This time we denote $V_A = (u_A, w_A)$ (resp. $\tilde{V}_A = (\tilde{u}_A, \tilde{w}_A)$) a solution of (1.3) associated with (ρ, G, A, Θ) (resp. $(\tilde{\rho}_3, G, A, \tilde{\Theta})$) and $V_B = (u_B, w_B)$ (resp. $\tilde{V}_B = (\tilde{u}_B, \tilde{w}_B)$) a solution of (1.3) associated with (ρ, G, B, Θ) (resp. $(\tilde{\rho}_3, G, B, \tilde{\Theta})$).

Note that all our results imply uniqueness results. Up to our knowledge, there are few results concerning the simultaneous identification of more than one coefficient in each equation (see for examples [1, 2, 5, 6, 9, 10]) and note that in these papers the coefficients only depend on the space variable. Also notice that there are very few results where the measurements are given with only one component. Here the first and fourth theorems (Theorems 3.1 and 3.4) extend some results obtained in [5, Theorem 3.2] but with hypotheses (see (3.2) and (3.3)) less restrictive than in [5]. The second result (Theorem 3.2) gives a result for four coefficients depending on the space variable and with measurements of only one component. The third theorem (Theorem 3.3) also gives a result for four coefficients but one of each depending on the time variable. Furthermore, usually the papers investigate the case of bounded domains and give results with observations on a subdomain of the domain (see for example [1, 2, 10]). Here we present results with observations on a part of the boundary (see Theorems 3.1, 3.3, 3.4). Besides, because of our unbounded domain and our choice of weight functions (2.3), we will use cut-off functions in time and in the direction x_1 (see for example [12] where cut-off functions are removed but in a bounded domain). Finally, usually the results have observations terms with data of the solution at a fixed time (such as $\|(u_A - \tilde{u}_A)(., \frac{T}{2})\|^2_{H^2(\Omega_L)}$, see for example [5, 7, 8]). We have been able to remove them in Theorems 3.1, 3.2i) thanks to the properties of the weight functions. So the theorems presented here give stability results for four coefficients for a system defined on an unbounded

domain, with boundary measurements in Theorems 3.1, 3.3 and 3.4, measurements for only one component in Theorem 3.2, with a time variable coefficient in Theorem 3.3. These results extend previous results for one equation [7, 8] or for a system [5] defined on an unbounded guide. Last we recall that the method of Carleman estimates used for solving inverse problems has been initiated by [3].
This Paper is organized as folows: in Section 2, we recall the weight functions adapted for our unbounded domain and the Carleman estimate (2.6) as well as the crucial inequality (2.4) for our Hölder estimates. Then in Section 3 we state and prove our results.

2 Carleman estimate

Denote $Q_L = \Omega_L \times (0,T) = (-L,L) \times \omega \times (0,T)$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $x' = (x_2, \cdots, x_n)$ and define the operator

$$A_0 u = \partial_t u - \Delta u.$$

Let $l > 0$, following [7] we are going to carry out special weight functions allowing us to avoid observations on the cross section of the wave guide in our inverse problem. For this we consider some positive real $L > l$ and we choose $\hat{a} = (a_1, a') \in \mathbb{R}^n \setminus \Omega$ such that if $\hat{d}(x) = |x' - a'|^2 - x_1^2$ for $x \in \Omega_L$, then

$$\hat{d} > 0 \text{ in } \Omega_L, \quad |\nabla \hat{d}| > 0 \text{ in } \overline{\Omega_L}. \tag{2.1}$$

Moreover we define

$$\Gamma_L = \{x \in \partial\Omega_L, < x - \hat{a}, \nu(x) > \geq 0\} \text{ and } \gamma_L = \Gamma_L \cap \partial\Omega. \tag{2.2}$$

Here $< .,. >$ denotes the usual scalar product in $\mathbb{R}^n$ and $\nu(x)$ is the outwards unit normal vector to $\partial\Omega_L$ at x. Notice that γ_L does not contain any cross section of the guide. From [14]-[15] we consider weight functions as follows: for $t \in (0,T)$, if $M_1 > \sup_{0<t<T}(t - T/2)^2 = (T/2)^2$,

$$\psi(x,t) = \hat{d}(x) - \left(t - \frac{T}{2}\right)^2 + M_1 \text{ and } \phi(x,t) = e^{\lambda\psi(x,t)}. \tag{2.3}$$

The constant $\lambda > 0$ will be set in Proposition 2.2 and is usually used as a large parameter in Carleman inequalities. Since we will not use it, we will consider λ fixed in the article. We recall from [7] and [8] the following result.

Proposition 2.1 *There exist $T > 0$, $L > l$, $\hat{a} \in \mathbb{R}^n \setminus \Omega_L$ and $\epsilon > 0$ such that (2.1) holds and, setting*

$$O_{L,\epsilon} = (\Omega_L \times ((0, 2\epsilon) \cup (T - 2\epsilon, T))) \cup (((-L, -L + 2\epsilon) \cup (L - 2\epsilon, L)) \times \omega \times (0,T)),$$

we have

$$d_1 < d_0 < d_2 \tag{2.4}$$

where

$$d_0 = \inf_{\Omega_l} \phi\left(\cdot, \theta\right), \qquad d_1 = \sup_{O_{L,\epsilon}} \phi, \qquad d_2 = \sup_{\overline{\Omega_L}} \phi\left(\cdot, \theta\right) \ \textit{and}\ \theta = \frac{T}{2}.$$

From now on and from simplicity we denote $\theta = \frac{T}{2}$ throughout the paper. These two above estimates (2.4) will be fruitful in Section 3 to solve our inverse problem. In the sequel C will be a generic positive constant. When needed, we will specify its dependency with respect to the different parameters. We will use the following notations: Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index with $\alpha_i \in \mathbb{N} \cup \{0\}$. We set $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and define

$$H^{2,1}(Q_L) = \{u \in L^2(Q_L), \partial_x^\alpha \partial_t^{\alpha_{n+1}} u \in L^2(Q_L), |\alpha| + 2\alpha_{n+1} \leq 2\}$$

endowed with its norm

$$\|u\|^2_{H^{2,1}(Q_L)} = \sum_{|\alpha|+2\alpha_{n+1} \leq 2} \|\partial_x^\alpha \partial_t^{\alpha_{n+1}} u\|^2_{L^2(Q_L)}.$$

We recall now a global Carleman-type estimate proved in [7, Proposition 4.2] or in [8, Proposition 3], based on a classical Carleman estimate (see Yamamoto [14, Theorem 7.3]). The key difference with the classical Carleman inequality in [14, Theorem 7.3] is to remove, on the cross-sections of Ω_L, the boundary condition and the observation. For that we need cut-off functions in time. On the other hand, to manage our infinite wave guide we also need to consider cut-off functions in space but only in the infinite direction x_1. These cut-off functions will induce additive terms coming from the commutator between the evolution operator and these cut-off functions. Let χ, η be C^∞ cut-off functions such that $\chi, \nabla\chi, \Delta\chi \in \Lambda_1(M_0)$, $0 \leq \chi \leq 1$, $0 \leq \eta \leq 1$,

$$\chi(x) = 0 \text{ if } x \in ((-\infty, -L+\epsilon) \cup (L-\epsilon, +\infty)) \times \omega),$$

$$\chi(x) = 1 \text{ if } x \in (-L+2\epsilon, L-2\epsilon) \times \omega,$$

$$\eta(t) = 0 \text{ if } t \in (0,\epsilon) \cup (T-\epsilon, T),\ \ \eta(t) = 1 \text{ if } t \in \times(2\epsilon, T-2\epsilon). \tag{2.5}$$

with ϵ defined in Proposition 2.1.

Proposition 2.2 [7, Proposition 4.2] *There exist a value of $\lambda > 0$ and positive constants s_0 and $C = C(\lambda, s_0)$ such that*

$$I(u) = \int_{Q_L} \left(\frac{1}{s\phi} (|\partial_t u|^2 + |\Delta u|^2) + s\phi\ |\nabla u|^2 + s^3\phi^3 |u|^2 \right) e^{2s\phi} dx\ dt$$

$$\leq C\|e^{s\phi} A_0 u\|^2_{L^2(Q_L)} + Cs^3 e^{2sd_1} \|u\|^2_{H^{2,1}(Q_L)} + Cs \int_{\gamma_L \times (0,T)} |\partial_\nu u|^2 e^{2s\phi} d\sigma\ dt, \tag{2.6}$$

for all $s > s_0$ and all $u \in H^{2,1}(Q_L)$ satisfying $u(.,0) = u(.,T) = 0$ in Ω_L, $u = 0$ on $(\partial\Omega \cap \partial\Omega_L) \times (0,T)$. We denote $\partial_\nu u = \nu \cdot \nabla u$ and recall that $A_0 u = \partial_t u - \Delta u$.

Since the method of Carleman estimates requires several time differentiations, we assume in the following that u, w (solution of (1.1) or (1.3)) belong to $\mathcal{H} = H^2([0,T], H^2(\Omega)) \cap W^{2,\infty}(\Omega \times (0,T))$ for Theorems 3.1, $\mathcal{H} = H^3([0,T], H^4(\Omega)) \cap W^{4,\infty}(\Omega \times (0,T))$ for Theorem 3.2, $\mathcal{H} = H^3([0,T], H^2(\Omega)) \cap W^{3,\infty}(\Omega \times (0,T))$ for Theorem 3.3, $\mathcal{H} = H^2([0,T], H^3(\Omega)) \cap W^{3,\infty}(\Omega \times (0,T))$ for Theorem 3.4, satisfying the a-priori bound

$$\|u\|_{\mathcal{H}} < M_2 \text{ and } \|w\|_{\mathcal{H}} < M_2 \text{ for given } M_2 > 0.$$

From now on, we use the notation $f(\theta) = f(.,\theta)$ for any function f defined on Q.

3 Inverse problem

3.1 Preliminary lemmas

From [11, Lemma 4.2], we derive the following result, also used in [7] or [5, Lemma 3.1].

Lemma 3.1 *There exist positive constants s_1 and C such that*

$$\int_{\Omega_L} e^{2s\phi(\theta)}(f(\theta))^2\, dx \le Cs \int_{Q_L} e^{2s\phi} f^2\, dx\, dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}(\partial_t f)^2\, dx\, dt$$

for all $s \ge s_1$ and $f \in H^1(0,T; L^2(\Omega_L))$.

For the sake of completeness, we recall its proof.

Proof. Consider η defined by (2.5) and any $w \in H^1(0,T; L^2(\Omega_L))$. Since $\eta(\theta) = 1$ and $\eta(0) = 0$, we have

$$\int_{\Omega_L} w(x,\theta)^2 dx = \int_{\Omega_L} (\eta(\theta) w(x,\theta))^2 dx = \int_{\Omega_L} \int_0^\theta \partial_t(\eta^2(t)|w(x,t)|^2) dt\, dx$$

$$= 2\int_0^\theta \int_{\Omega_L} \eta^2(t) w(x,t)\partial_t w(x,t) dx\, dt + 2\int_0^\theta \int_{\Omega_L} \eta(t)\partial_t\eta(t)|w(x,t)|^2 dx\, dt.$$

As $0 \le \eta \le 1$, using Young's inequality, it comes that for any $s > 0$,

$$\int_{\Omega_L} w(x,\theta)^2\, dx \le Cs \int_{Q_L} |w|^2 dx\, dt + \frac{C}{s}\int_{Q_L} |\partial_t w|^2 dx\, dt. \tag{3.1}$$

Then we can conclude replacing w by $e^{s\phi} f$ in (3.1). □

The following lemma will be only used for Theorem 3.4. It is a classical lemma for a first order partial differential operator but which necessites a strong positivity condition (3.2). This condition is nevertheless weaker than the one used in [8] or [5] (which was

$|\nabla\hat{d}\cdot\nabla\tilde{u}(\theta)| \geq R > 0$ in Ω_L). So we follow an idea developed in [13] for Lamé system in bounded domains, also used for example in [8] or in [5]. The lemma below will be used in the proof of Theorem 3.4 with $(v_1, \cdots, v_4) = (\tilde{w}_B(\theta), \tilde{u}_A(\theta), \tilde{w}_A(\theta), \tilde{u}_B(\theta))$. Recall that $\hat{d}$ is defined by (2.1).

Lemma 3.2 *Assume that the following assumption*

$$|v_1\nabla\hat{d}\cdot\nabla v_2 - v_3\nabla\hat{d}\cdot\nabla v_4| \geq R \text{ in } \Omega_L \text{ for some } R > 0 \tag{3.2}$$

holds. Consider the first order partial differential operator $Pf = v_1\nabla f\cdot\nabla v_2 - v_3\nabla f\cdot\nabla v_4$. Then there exist positive constants $s_1' > 0$ and $C > 0$ such that for all $s \geq s_1'$,

$$s^2\int_{\Omega_L} e^{2s\phi(\theta)}f^2\,dx \leq C\int_{\Omega_L} e^{2s\phi(\theta)}|Pf|^2\,dx,$$

for all $f \in H_0^1(\Omega_L)$.

Proof. The proof follows [8] or [5]. Let $f \in H_0^1(\Omega_L)$. Denote $w = e^{s\phi(\theta)}f$ and $Qw = e^{s\phi(\theta)}P(e^{-s\phi(\theta)}w)$. So we get $Qw = Pw - s\lambda\phi(\theta)w(P\hat{d})$. Therefore we have

$$\int_{\Omega_L}|Qw|^2\,dx \geq s^2\lambda^2\int_{\Omega_L}(\phi(\theta))^2w^2(P\hat{d})^2\,dx - 2s\lambda\int_{\Omega_L}\phi(\theta)(Pw)w(P\hat{d})\,dx.$$

So

$$\int_{\Omega_L}|Qw|^2\,dx \geq s^2\lambda^2\int_{\Omega_L}(\phi(\theta))^2w^2(P\hat{d})^2\,dx - s\lambda\int_{\Omega_L}\phi(\theta)(Pw^2)(P\hat{d})\,dx.$$

Thus integrating by parts

$$\int_{\Omega_L}|Qw|^2\,dx \geq s^2\lambda^2\int_{\Omega_L}(\phi(\theta))^2w^2(P\hat{d})^2\,dx + s\lambda\int_{\Omega_L}w^2\nabla\cdot(\phi(\theta)(P\hat{d})(v_1\nabla v_2 - v_3\nabla v_4))\,dx.$$

And we can conclude for s sufficiently large. □

3.2 Statements of results

3.2.1 First result Consider $V_A = (u_A, w_A)$ (resp. $\tilde{V}_A = (\tilde{u}_A, \tilde{w}_A)$) a strong solution of (1.1) associated with (ρ, G, A) defined by (1.2) (resp. $(\tilde{\rho}_1, G, A)$) where A is a set of initial and boundary conditions. Consider also $V_B = (u_B, w_B)$ (resp. $\tilde{V}_B = (\tilde{u}_B, \tilde{w}_B)$) a strong solution of (1.1) associated with (ρ, G, B) (resp. $(\tilde{\rho}_1, G, B)$) and where B is another set of initial and boundary conditions. Assume that all the coefficients $\alpha, \beta, \gamma, \delta, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, belong to $\Lambda_1(M_0)$ and all the coefficients ϕ_i to $\Lambda_2(M_0)$ (for $i = 1, \cdots, 4$).
Our main result is the following

Theorem 3.1 *Let $l > 0$. Let $T > 0$, $L > l$ and $\hat{a} \in \mathbb{R}^n \setminus \Omega$ satisfying the conditions of Proposition 2.1. Assume that*

$$|\tilde{u}_A(\cdot,\theta)\tilde{w}_B(\cdot,\theta) - \tilde{u}_B(\cdot,\theta)\tilde{w}_A(\cdot,\theta)| \geq R \text{ in } \Omega_L \text{ for some } R > 0. \tag{3.3}$$

Then there exists a sufficiently small number $\tau_0 > 0$ such that if $\tau \in (0,\tau_0)$,

$$\sum_{k=0}^{1} \int_{\gamma_L \times (0,T)} (|\partial_\nu(\partial_t^k(u_A - \tilde{u}_A))|^2 + |\partial_\nu(\partial_t^k(w_A - \tilde{w}_A))|^2$$

$$+|\partial_\nu(\partial_t^k(u_B - \tilde{u}_B))|^2 + |\partial_\nu(\partial_t^k(w_B - \tilde{w}_B))|^2) d\sigma\ dt \leq \tau$$

then the following Hölder stability estimate holds

$$\|\alpha - \tilde{\alpha}\|^2_{L^2(\Omega_l)} + \|\beta - \tilde{\beta}\|^2_{L^2(\Omega_l)} + \|\gamma - \tilde{\gamma}\|^2_{L^2(\Omega_l)} + \|\delta - \tilde{\delta}\|^2_{L^2(\Omega_l)} \leq K\tau^\kappa \text{ for all } \tau \in (0,\tau_0). \tag{3.4}$$

Here, $K > 0$ and $\kappa \in (0,1)$ are two constants depending on R, L, l, M_0, M_1, M_2, T and $\hat{a}$.

3.2.2 Second result As a consequence of Theorem 3.1, we can give a stability result with measurements of only one component. Theorem 3.2i) gives an estimate of the four coefficients $\alpha, \beta, \gamma, \delta \in L^2(\Omega)$ when $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$ in a neighborhood ω' of the boundary of interest γ_L. That means that these two coefficients α and β are supposed known in ω'. We relax this last hypothesis in Theorem 3.2ii) where an estimate of these four coefficients is given for $\alpha, \beta \in H^2(\Omega)$. Consider $V_A = (u_A, w_A)$ (resp. $\tilde{V}_A = (\tilde{u}_A, \tilde{w}_A)$) a strong solution of (1.1) associated with (ρ, G, A) defined by (1.2) (resp. $(\tilde{\rho}_1, G, A)$). Consider also $V_B = (u_B, w_B)$ (resp. $\tilde{V}_B = (\tilde{u}_B, \tilde{w}_B)$) a strong solution of (1.1) associated with (ρ, G, B) (resp. $(\tilde{\rho}_1, G, B)$). Assume that all the coefficients $\alpha, \beta, \gamma, \delta, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, belong to $\Lambda_1(M_0)$ and all the coefficients ϕ_i to $\Lambda_2(M_0)$ (for $i = 1, \cdots, 4$). For Theorem 3.2ii) we also suppose that $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} \in \Lambda'(M_0) = \{f \in H^2(\Omega),\ \|f\|_{H^2(\Omega)}\| \leq M_0\}$ and $\phi_i \in C^2([0,T])$.

Theorem 3.2 *Let $l > 0$. Let $T > 0$, $L > l$ and $\hat{a} \in \mathbb{R}^n \setminus \Omega$ satisfying the conditions of Proposition 2.1. Let ω' be a neighborhood of γ_L, $\omega' \subset \Omega_{L+\epsilon}$ such that $\gamma_L \subset \partial\omega'$, $\partial\omega'$ being C^2. Assume that the hypothesis(3.3) holds and that we also have*

$$|\beta\phi_2| \geq R > 0 \text{ in } Q_L. \tag{3.5}$$

i) *We suppose that $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$ in ω'.*

Then there exists a sufficiently small number $\tau_0 > 0$ such that if $\tau \in (0,\tau_0)$,

$$\|u_A - \tilde{u}_A\|^2_{H^2([0,T],H^2(\omega'\cap\Omega_L))} + \|u_A - \tilde{u}_A\|^2_{H^1([0,T],H^4(\omega'\cap\Omega_L))}$$

$$+\|u_B - \tilde{u}_B\|^2_{H^2([0,T],H^2(\omega'\cap\Omega_L))} + \|u_B - \tilde{u}_B\|^2_{H^1([0,T],H^4(\omega'\cap\Omega_L))}$$

$$+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{1}(|\partial_\nu(\partial_t^k(u_A-\tilde u_A))|^2+|\partial_\nu(\partial_t^k(u_B-\tilde u_B))|^2)\ d\sigma\ dt\le\tau$$

then the following Hölder stability estimate holds

$$\|\alpha-\tilde\alpha\|^2_{L^2(\Omega_l)}+\|\beta-\tilde\beta\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde\gamma\|^2_{L^2(\Omega_l)}+\|\delta-\tilde\delta\|^2_{L^2(\Omega_l)}\le K\tau^\kappa \text{ for all } \tau\in(0,\tau_0). \tag{3.6}$$

ii) *We suppose that $\alpha,\beta,\tilde\alpha,\tilde\beta\in H^2(\Omega)$.*
Then there exists a sufficiently small number $\tau_0>0$ such that if $\tau\in(0,\tau_0)$,

$$\|(u_A-\tilde u_A)(\cdot,\theta)\|^2_{H^4(\Omega_L)}+\|(u_B-\tilde u_B)(\cdot,\theta)\|^2_{H^4(\Omega_L)}+\|u_A-\tilde u_A\|^2_{H^3([0,T],H^2(\omega'\cap\Omega_L))}$$

$$+\|u_A-\tilde u_A\|^2_{H^2([0,T],H^4(\omega'\cap\Omega_L))}+\|u_B-\tilde u_B\|^2_{H^3([0,T],H^2(\omega'\cap\Omega_L))}+\|u_B-\tilde u_B\|^2_{H^2([0,T],H^4(\omega'\cap\Omega_L))}$$

$$+\int_{\gamma_L\times(0,T)}\sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u_A-\tilde u_A))|^2+|\partial_\nu(\partial_t^k(u_B-\tilde u_B))|^2)\ d\sigma\ dt\le\tau$$

then the following Hölder stability estimate holds

$$\|\alpha-\tilde\alpha\|^2_{H^2(\Omega_l)}+\|\beta-\tilde\beta\|^2_{H^2(\Omega_l)}+\|\gamma-\tilde\gamma\|^2_{L^2(\Omega_l)}+\|\delta-\tilde\delta\|^2_{L^2(\Omega_l)}\le K\tau^\kappa \text{ for all } \tau\in(0,\tau_0). \tag{3.7}$$

Here, $K>0$ and $\kappa\in(0,1)$ are two constants depending on R, L, l, M_0, M_1, M_2, T, $\|g_0\|_{(C^1(\omega'))^n}$ and $\hat a$.

3.2.3 Third result Now we present a result for the four coefficients $(\phi_1,\beta,\gamma,\delta)$. We consider here $V_A=(u_A,w_A)$ (resp. $\tilde V_A=(\tilde u_A,\tilde w_A)$) a strong solution of (1.1) associated with (ρ,G,A) defined by (1.2) (resp. $(\tilde\rho_2,G,A)$). Consider also $V_B=(u_B,w_B)$ (resp. $\tilde V_B=(\tilde u_B,\tilde w_B)$) a strong solution of (1.1) associated with (ρ,G,B) (resp. $(\tilde\rho_2,G,B)$). Assume that all the coefficients $\alpha,\beta,\gamma,\delta,\tilde\beta,\tilde\gamma,\tilde\delta$, belong to $\Lambda_1(M_0)$ and all the coefficients $\phi_i,\tilde\phi_1$ to $\Lambda_2(M_0)$ (for $i=1,\cdots,4$). Let the set of admissible coefficients

$$\Lambda_3(M_3)=\{f\in C^2([0,T]),|\partial_t^2(f-\phi_1)(t)|\le M_3|(f-\phi_1)(\theta)| \text{ for all } t\in[0,T]\} \tag{3.8}$$

with M_3 a positive constant.
Our result is the following.

Theorem 3.3 *Let $l>0$. Let $T>0$, $L>l$ and $\hat a\in\mathbb{R}^n\setminus\Omega$ satisfying the conditions of Proposition 2.1. We suppose that $\tilde\phi_1\in\Lambda_3(M_3)$. Assume that Assumption (3.3) holds and that*

$$|\alpha|\ge R>0 \text{ in } \Omega_L. \tag{3.9}$$

Then there exists a sufficiently small number $\tau_0 > 0$ such that if $\tau \in (0, \tau_0)$,

$$\sum_{k=0}^{1}(\|\partial_t^k(u_A - \tilde{u}_A)(\cdot,\theta)\|^2_{H^2(\Omega_L)} + \|\partial_t^k(u_B - \tilde{u}_B)(\cdot,\theta)\|^2_{H^2(\Omega_L)}) + \|\partial_t^2(u_A - \tilde{u}_A)(\cdot,\theta)\|^2_{L^2(\Omega_L)}$$

$$+\|\partial_t^2(u_B - \tilde{u}_B)(\cdot,\theta)\|^2_{L^2(\Omega_L)} + \|(w_A - \tilde{w}_A)(\cdot,\theta)\|^2_{H^2(\Omega_L)} + \|(w_B - \tilde{w}_B)(\cdot,\theta)\|^2_{H^2(\Omega_L)}$$

$$+ \int_{\gamma_L\times(0,T)} \sum_{k=0}^{2}(|\partial_\nu(\partial_t^k(u_A - \tilde{u}_A))|^2 + |\partial_\nu(\partial_t^k(w_A - \tilde{w}_A))|^2)\; d\sigma\; dt \le \tau,$$

then the following Hölder stability estimate holds

$$\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)}+\sum_{i=0}^{2}\|\partial_t^i(\phi_1-\tilde{\phi}_1)\|^2_{L^2(0,T)} \le K\tau^\kappa \text{ for all } \tau \in (0,\tau_0). \tag{3.10}$$

Here, $K > 0$ and $\kappa \in (0, 1)$ are two constants depending on R, L, l, M_0, M_1, M_2, M_3, T, $\hat{a}$.

Remark 1 • Notice that the hypothesis $\tilde{\phi}_1 \in \Lambda_3(M_3)$ is satisfied when $\tilde{\phi}_1 \in C^2([0,T])$ is such that $\phi_1(\theta) \neq \tilde{\phi}_1(\theta)$ and $\frac{\sup_{t\in[0,T]}|\partial_t(\phi_1-\tilde{\phi}_1)(t)|}{|\phi_1(\theta)-\tilde{\phi}_1(\theta)|} \le M_3$. Moreover note also that if $\tilde{\phi}_1 \in C^2([0,T])$ is such that $\phi_1(\theta) \neq \tilde{\phi}_1(\theta)$, then if we denote $f_1 = \phi_1 - \tilde{\phi}_1$, we have $f_1(\theta) \neq 0$. Therefore $t \mapsto |\frac{f_1(t)}{f_1(\theta)}|$ is bounded on $[0,T]$ so there exists a positive constant C_0 such that for all $t \in [0,T]$, $|f_1(t)| \le C_0|f_1(\theta)|$. Similarly there exists a positive constant C_1 such that $|\partial_t f_1(t)| \le C_1|f_1(\theta)|$ and there exists a positive constant C_2 such that $|\partial_t^2 f_1(t)| \le C_2|f_1(\theta)|$. Note also that if $\tilde{\phi}_1 \in \Lambda_3(M_3)$ and $\tilde{\phi}_1(\theta) = \phi_1(\theta)$, then $\partial_t^2(\tilde{\phi}_1 - \phi_1) = 0$ in $[0,T]$. Therefore $\tilde{\phi}_1$ has the form $\tilde{\phi}_1(t) = \phi_1(t) + k(t-\theta)$ with k any real.

• Moreover if the function ϕ_1 is more regular, for example if $\phi_1 \in C^p([0,T])$ with $p \ge 2$, then Theorem 3.3 is still valid with a more generalized admissible set of coefficients $\Lambda'_3(M_3) = \{f \in C^p([0,T]), |\partial_t^p(f-\phi_1)(t)| \le M_3|(f-\phi_1)(\theta)| \text{ for all } t \in [0,T]\}$. But in this case, because of our method, the observations terms at the fixed time θ on the right-hand side of the estimate (3.10) would demand more regularity.

• On the contrary, we can relax some of the observations terms on u (u_A and $\tilde{u}_A$) at θ on the right-hand side of (3.10) and only have $\|(u-\tilde{u})(\cdot,\theta)\|^2_{H^2(\Omega_L)}$ but for a more restrictive admissible set of coefficients $\Lambda''_3(M_3) = \{f \in C^2([0,T]), |\partial_t^i(f-\phi_1)(t)| \le M_3|(f-\phi_1)(\theta)| \text{ for all } i = 0,1,2 \text{ and } t \in [0,T]\}$.

3.2.4 Fourth result

Finally, we consider the system (1.3). Consider $V_A = (u_A, w_A)$ (resp. $\tilde{V}_A = (\tilde{u}_A, \tilde{w}_A)$) a strong solution of (1.3) associated with (ρ, G, A, Θ) defined by (1.2) and (1.4) (resp. $(\tilde{\rho}_3, G, A, \tilde{\Theta})$). Consider also $V_B = (u_B, w_B)$ (resp. $\tilde{V}_B = (\tilde{u}_B, \tilde{w}_B)$) a strong solution of (1.3) associated with (ρ, G, B, Θ) (resp. $(\tilde{\rho}_3, G, B, \tilde{\Theta})$). Assume that all the coefficients $\alpha, \beta, \gamma, \delta, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, belong to $\Lambda_1(M_0)$ and all the coefficients ϕ_i to $\Lambda_2(M_0)$

(for $i = 1, \cdots, 4$). Moreover we suppose that $\Theta_i, \tilde{\Theta}_1$ belong to $(\Lambda_1(M_0))^n \cap (L^2(\Omega))^n$ (for $i = 1, \cdots, 4$) and there exist functions $\xi_1, \tilde{\xi_1}$ such that

$$\Theta_1 = \nabla \xi_1, \ \tilde{\Theta}_1 = \nabla \tilde{\xi_1} \text{ in } \Omega.$$

Theorem 3.4 *Let $l > 0$. Let $T > 0$, $L > l$ and $\hat{a} \in \mathbb{R}^n \setminus \Omega$ satisfying the conditions of Proposition 2.1. Assume that Assumptions (3.2) and (3.3) are satisfied with $(v_1, \cdots, v_4) = (\tilde{w}_B(\cdot,\theta), \tilde{u}_A(\cdot,\theta), \tilde{w}_A(\cdot,\theta), \tilde{u}_B(\cdot,\theta))$.*
If $\xi_1 = \tilde{\xi_1}$ and $\Theta_1 = \tilde{\Theta}_1$ on $\partial\Omega \cap \partial\Omega_L$, then there exists a sufficiently small number $\tau_0 > 0$ such that if $\tau \in (0, \tau_0)$,

$$\sum_{k=0}^{1} \int_{\gamma_L \times (0,T)} (|\partial_\nu(\partial_t^k(u_A - \tilde{u}_A))|^2 + |\partial_\nu(\partial_t^k(w_A - \tilde{w}_A))|^2 + |\partial_\nu(\partial_t^k(u_B - \tilde{u}_B))|^2$$

$$+|\partial_\nu(\partial_t^k(w_B - \tilde{w}_B))|^2) d\sigma\, dt + \|(u_A - \tilde{u}_A)(\cdot,\theta)\|^2_{H^3(\Omega_L)} + \|(u_B - \tilde{u}_B)(\cdot,\theta)\|^2_{H^3(\Omega_L)} \leq \tau$$

then the following Hölder stability estimate holds

$$\|\beta - \tilde{\beta}\|^2_{L^2(\Omega_l)} + \|\gamma - \tilde{\gamma}\|^2_{L^2(\Omega_l)} + \|\delta - \tilde{\delta}\|^2_{L^2(\Omega_l)} + \|\Theta_1 - \tilde{\Theta}_1\|_{(L^2(\Omega_l))^n} \leq K\tau^\kappa \tag{3.11}$$

for all $\tau \in (0, \tau_0)$.
Here, $K > 0$ and $\kappa \in (0,1)$ are two constants depending on R, L, l, M_0, M_1, M_2, T and $\hat{a}$.

3.3 Proofs of theorems

3.3.1 Proof of Theorem 3.1 Let $V_A = (u_A, w_A)$ (resp. $\tilde{V}_A = (\tilde{u}_A, \tilde{w}_A)$) be a solution of (1.1) associated with (ρ, G, A) (resp. $(\tilde{\rho_1}, G, A)$) and $V_B = (u_B, w_B)$ (resp. $\tilde{V}_B = (\tilde{u}_B, \tilde{w}_B)$) be a solution of (1.1) associated with (ρ, G, B) (resp. $(\tilde{\rho_1}, G, B)$). We decompose the proof in several steps.

- First step:

Denote $V = (u, w) = V_A$, $\tilde{V} = (\tilde{u}, \tilde{w}) = \tilde{V}_A$ and

$$U = u - \tilde{u},\ W = w - \tilde{w},\ a = \alpha - \tilde{\alpha}.\ b = \beta - \tilde{\beta},\ c = \gamma - \tilde{\gamma},\ d = \delta - \tilde{\delta}. \tag{3.12}$$

Then (U, W) satisfy the following system

$$\begin{cases} \partial_t U = \Delta U + \alpha\phi_1 U + \beta\phi_2 W + a\phi_1\tilde{u} + b\phi_2\tilde{w} \text{ in } Q, \\ \partial_t W = \Delta W + \gamma\phi_3 U + \delta\phi_4 W + c\phi_3\tilde{u} + d\phi_4\tilde{w} \text{ in } Q, \\ U = W = 0 \text{ on } \Sigma. \end{cases} \tag{3.13}$$

Define

$$y_0 = \eta\chi U,\ z_0 = \eta\chi W,\ y_1 = \partial_t y_0,\ z_1 = \partial_t z_0 \tag{3.14}$$

We deduce that (y_i, z_i) for $i = 0, 1$ satisfy the following systems

$$\begin{cases} \partial_t y_0 = \Delta y_0 + \alpha\phi_1 y_0 + \beta\phi_2 z_0 + a\eta\chi\phi_1\tilde{u} + b\eta\chi\phi_2\tilde{w} + R_1 \ \text{ in } \ Q_L, \\ \partial_t z_0 = \Delta z_0 + \gamma\phi_3 y_0 + \delta\phi_4 z_0 + c\eta\chi\phi_3\tilde{u} + d\eta\chi\phi_4\tilde{w} + R_2 \text{ in } \ Q_L, \\ y_0 = z_0 = 0 \text{ on } \partial\Omega_L \times (0, T) \end{cases} \tag{3.15}$$

with

$$R_1 = -(\Delta\chi)\eta U - 2\eta\nabla\chi \cdot \nabla U + \chi\partial_t\eta U, \ R_2 = -(\Delta\chi)\eta W - 2\eta\nabla\chi \cdot \nabla W + \chi\partial_t\eta W.$$

We have

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \alpha\phi_1 y_1 + \beta\phi_2 z_1 + R_3 \ \text{ in } \ Q_L, \\ \partial_t z_1 = \Delta z_1 + \gamma\phi_3 y_1 + \delta\phi_4 z_1 + R_4 \ \text{ in } \ Q_L, \\ y_1 = z_1 = 0 \text{ on } \partial\Omega_L \times (0, T), \end{cases} \tag{3.16}$$

with

$$R_3 = a\chi\partial_t(\eta\phi_1\tilde{u}) + b\chi\partial_t(\eta\phi_2\tilde{w}) + \partial_t R_1 + \alpha y_0\partial_t\phi_1 + \beta z_0\partial_t\phi_2,$$

$$R_4 = c\chi\partial_t(\eta\phi_3\tilde{u}) + d\chi\partial_t(\eta\phi_4\tilde{w}) + \partial_t R_2 + \gamma y_0\partial_t\phi_3 + \delta z_0\partial_t\phi_4.$$

• Second step: we estimate $\sum_{i=0}^{1}(I(y_i) + I(z_i))$ by the Carleman inequalities (2.6). Note that all the terms in $A_0 y_i$ or $A_0 z_i$ with derivatives of χ or η will be bounded above by Ce^{2sd_1} with C a positive constant (see Proposition 2.1 for the definitions of d_1 and d_2). Moreover all the terms such as $\int_{Q_L} e^{2s\phi} y_i^2 \, dx \, dt$ on the right-and side of the estimates (2.6) will be absorbed by $I(y_i)$ for s sufficiently large. So we have for s sufficiently large,

$$\sum_{i=0}^{1}(I(y_i) + I(z_i)) \leq C \int_{Q_L} e^{2s\phi}(a^2 + b^2 + c^2 + d^2)\chi^2 \, dx \, dt + Cs^3 e^{2sd_1}$$

$$+Cs \int_{\gamma_L \times (0,T)} e^{2s\phi} \sum_{i=0}^{1}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2) \, d\sigma \, dt.$$

Since $e^{2s\phi} \leq e^{2s\phi(\theta)} \leq e^{2sd_2}$ we get

$$\sum_{i=0}^{1}(I(y_i) + I(z_i)) \leq C \int_{Q_L} e^{2s\phi}(a^2 + b^2 + c^2 + d^2)\chi^2 \, dx \, dt + Cs^3 e^{2sd_1} + Cse^{2sd_2} F_0(\gamma_L) \tag{3.17}$$

with $F_0(\gamma_L) = \int_{\gamma_L \times (0,T)} \sum_{i=0}^{1}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2) \, d\sigma \, dt$.

• Third step: now we estimate $\int_{\Omega_L} e^{2s\phi(\theta)} |\partial_t^i f(\theta)|^2 \, dx$ and $\int_{\Omega_L} e^{2s\phi(\theta)} |\Delta f(\theta)|^2 \, dx$ for $f = y_0$ or $f = z_0$ and $i = 0, 1$. By Lemma 3.1, we have (since $\phi \geq 1$ and $\frac{1}{\phi} \geq \frac{1}{d_2}$)

$$\int_{\Omega_L} e^{2s\phi(\theta)} |y_0(\theta)|^2 \, dx \leq Cs \int_{Q_L} e^{2s\phi} y_0^2 \, dx \, dt + \frac{C}{s} \int_{Q_L} e^{2s\phi} y_1^2 \, dx \, dt \leq \frac{C}{s^2}(I(y_0) + I(y_1)),$$

$$\int_{\Omega_L} e^{2s\phi(\theta)} |\partial_t y_0(\theta)|^2 \, dx \leq Cs \int_{Q_L} e^{2s\phi} y_1^2 \, dx \, dt + \frac{C}{s} \int_{Q_L} e^{2s\phi} |\partial_t y_1|^2 \, dx \, dt \leq CI(y_1),$$

$$\int_{\Omega_L} e^{2s\phi(\theta)}|\Delta y_0(\theta)|^2\,dx \le Cs\int_{Q_L} e^{2s\phi}|\Delta y_0|^2\,dx\,dt+\frac{C}{s}\int_{Q_L} e^{2s\phi}|\Delta y_1|^2\,dx\,dt \le Cs^2(I(y_0)+I(y_1)).$$

Notice that the three above inequalities are satisfied replacing (y_0, y_1, y_2) by (z_0, z_1, z_2). Therefore

$$\int_{\Omega_L} e^{2s\phi(\theta)}(|y_0(\theta)|^2 + |\partial_t y_0(\theta)|^2 + |\Delta y_0(\theta)|^2 + |z_0(\theta)|^2 + |\partial_t z_0(\theta)|^2 + |\Delta z_0(\theta)|^2)\,dx$$

$$\le Cs^2\sum_{i=0}^{1}(I(y_i) + I(z_i)).$$

So using (3.17) we deduce that

$$\int_{\Omega_L} e^{2s\phi(\theta)}(|y_0(\theta)|^2 + |\partial_t y_0(\theta)|^2 + |\Delta y_0(\theta)|^2 + |z_0(\theta)|^2 + |\partial_t z_0(\theta)|^2 + |\Delta z_0(\theta)|^2)\,dx$$

$$\le Cs^2\int_{Q_L} e^{2s\phi}(a^2 + b^2 + c^2 + d^2)\chi^2\,dx\,dt + Cs^5e^{2sd_1} + Cs^3e^{2sd_2}F_0(\gamma_L). \tag{3.18}$$

At last in this step, denote

$$R = (R_1, R_2, R_3, R_4). \tag{3.19}$$

• Fourth step: here we estimate $\int_{\Omega_L} e^{2s\phi(\theta)}(a^2 + b^2 + c^2 + d^2)\chi^2\,dx$.
We choose now the two sets of conditions A and B and consider V_A, $\tilde{V}_A$, V_B and $\tilde{V}_B$. From now on, each function f defined in the precedent steps is denoted either f_A or f_B when it is related either by the conditions A or B. Denote now $F_{0A}(\gamma_L) = F_0(\gamma_L)$ associated with $(V_A, \tilde{V}_A)$, and $F_{0B}(\gamma_L) = F_0(\gamma_L)$ associated with $(V_B, \tilde{V}_B)$ (see (3.17) in the second step):

$$F_{0A}(\gamma_L) = \int_{\gamma_L\times(0,T)}\sum_{i=0}^{1}(|\partial_\nu y_{iA}|^2 + |\partial_\nu z_{iA}|^2)\,d\sigma\,dt,\ F_{0B}(\gamma_L) =$$

$$\int_{\gamma_L\times(0,T)}\sum_{i=0}^{1}(|\partial_\nu y_{iB}|^2 + |\partial_\nu z_{iB}|^2)\,d\sigma\,dt.$$

Let R_A be defined by (3.19) for $(V_A, \tilde{V}_A)$ (resp. R_B for $(V_B, \tilde{V}_B)$). Multiplying the first equation of (3.15) written for y_{0A} by $\tilde{w}_B$ and the first equation of (3.15) written for y_{0B} by $\tilde{w}_A$ and subtracting, we eliminate the term in b and we get

$$a\eta\chi\phi_1(\tilde{u}_A\tilde{w}_B - \tilde{u}_B\tilde{w}_A) = \tilde{w}_B(\partial_t y_{0A} - \Delta y_{0A} - \alpha\phi_1 y_{0A} - \beta\phi_2 z_{0A} - R_{1A})$$

$$-\tilde{w}_A(\partial_t y_{0B} - \Delta y_{0B} - \alpha\phi_1 y_{0B} - \beta\phi_2 z_{0B} - R_{1B}). \tag{3.20}$$

By hypothesis (3.3), applying (3.20) for $t = \theta$, since $\eta = 1$ in a neighborhood of θ we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}a^2\chi^2(\phi_1(\theta))^2\,dx \le C\int_{\Omega_L} e^{2s\phi(\theta)}\left(|\partial_t y_{0A}(\theta)|^2 + |\partial_t y_{0B}(\theta)|^2 + |\Delta y_{0A}(\theta)|^2 + |\Delta y_{0B}(\theta)|^2\right.$$

$$+|y_{0A}(\theta)|^2+|z_{0A}(\theta)|^2+|y_{0B}(\theta)|^2+|z_{0B}(\theta)|^2\big)\ dx+Ce^{2sd_1}.$$

But $\phi_1 \in \Lambda_2(M_0)$. So from (3.18) applied for $y_{0A}, y_{0B}, z_{0A}, z_{0B}$ we obtain

$$\int_{\Omega_L} e^{2s\phi(\theta)}a^2\chi^2\ dx \le Cs^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt+Cs^5e^{2sd_1}+Cs^3e^{2sd_2}F_1(\gamma_L) \quad (3.21)$$

with $F_1(\gamma_L)=F_{0A}(\gamma_L)+F_{0B}(\gamma_L)$. Similarly we can replace a by b on the left-hand side of (3.21), still using (3.15) for y_{0A} and y_{0B}. Indeed

$$-b\eta\chi\phi_2(\tilde u_A\tilde w_B-\tilde u_B\tilde w_A)=\tilde u_B(\partial_t y_{0A}-\Delta y_{0A}-\alpha\phi_1 y_{0A}-\beta\phi_2 z_{0A}-R_{1A})$$

$$-\tilde u_A(\partial_t y_{0B}-\Delta y_{0B}-\alpha\phi_1 y_{0B}-\beta\phi_2 z_{0B}-R_{1B}).$$

So we have

$$\int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2)\chi^2\ dx \le Cs^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt+Cs^5e^{2sd_1}+Cs^3e^{2sd_2}F_1(\gamma_L). \quad (3.22)$$

We do the same to obtain c and d using this time (3.15) for z_{0A} and z_{0B} and the hypothesis (3.3). Therefore

$$\int_{\Omega_L} e^{2s\phi(\theta)}(c^2+d^2)\chi^2\ dx \le Cs^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt+Cs^5e^{2sd_1}+Cs^3e^{2sd_2}F_1(\gamma_L). \quad (3.23)$$

Adding (3.22) and (3.23), we have

$$\int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt \le$$
$$Cs^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt+Cs^5e^{2sd_1}+Cs^3e^{2sd_2}F_1(\gamma_L).$$

Now we proceed as in [2, 11, 12] in order to prove that $s^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt$ can be absorbed by the left-hand side of the above estimate for s sufficiently large ($s \ge s_2$). Indeed

$$s^2\int_{Q_L} e^{2s\phi}(a^2+b^2+c^2+d^2)\chi^2\ dx\ dt=\int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2+c^2+d^2)\chi^2\Big(\int_0^T s^2e^{2s(\phi-\phi(\theta))}\ dt\Big)\ dx.$$

But $\phi-\phi(\theta)=-e^{\lambda(\hat d+M_1)}(1-e^{-\lambda(t-\theta)^2})$ and there exists a positive constant C such that $\phi-\phi(\theta)\le -C(1-e^{-\lambda(t-\theta)^2})$. Therefore $\int_0^T s^2e^{2s(\phi-\phi(\theta))}\ dt \le \int_0^T s^2e^{-2sC(1-e^{-\lambda(t-\theta)^2})}\ dt$ uniformly in x. Moreover by the Lebesgue convergence theorem, we have

$$\int_0^T s^2e^{-2sC(1-e^{-\lambda(t-\theta)^2})}\ dt \to 0 \text{ as } s\to\infty.$$

Thus for s sufficiently large, we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2+c^2+d^2)\chi^2\ dx \leq Cs^5e^{2sd_1}+Cs^3e^{2sd_2}F_1(\gamma_L).$$

Since $e^{2sd_0} \leq e^{2s\phi(\theta)}$ in Ω_l and $\chi=1$ in Ω_l, we deduce that

$$e^{2sd_0}(\|\alpha-\tilde{\alpha}\|^2_{L^2(\Omega_l)}+\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)}) \leq Cs^3(e^{2sd_2}F_1(\gamma_L)+s^2e^{2sd_1})$$

which can be rewritten

$$\|\alpha-\tilde{\alpha}\|^2_{L^2(\Omega_l)}+\|\beta-\tilde{\beta}\|^2_{L^2(\Omega_l)}+\|\gamma-\tilde{\gamma}\|^2_{L^2(\Omega_l)}+\|\delta-\tilde{\delta}\|^2_{L^2(\Omega_l)} \leq Cs^3(e^{2s(d_2-d_0)}F_1(\gamma_L)+s^2e^{2s(d_1-d_0)}). \tag{3.24}$$

As $d_1-d_0<0$ and $d_2-d_0>0$, we can optimize the above inequality with respect to s (see for example [5, 7, 8]). Indeed, note that if $F_1(\gamma_L)=0$, since (3.24) holds for any $s \geq s_2$ and $d_1-d_0<0$ we get (3.4). Now if $F_1(\gamma_L) \neq 0$ is sufficiently small ($F_1(\gamma_L) < \frac{d_0-d_1}{d_2-d_0}$), we optimize (3.24) with respect to s. Indeed denote

$$f(s)=e^{2s(d_2-d_0)}F_1(\gamma_L)+e^{2s(d_1-d_0)} \text{ and } g(s)=e^{2s(d_2-d_0)}F_1(\gamma_L)+s^2e^{2s(d_1-d_0)}.$$

We have $f(s) \sim g(s)$ at infinity. Moreover the function f has a minimum in

$$s_3=\frac{1}{2(d_2-d_1)}\ln\left(\frac{d_0-d_1}{(d_2-d_0)F_1(\gamma_L)}\right) \text{ and } f(s_3)=K'F_1(\gamma_L)^{\kappa}$$

with $\kappa=\frac{d_0-d_1}{d_2-d_1}$ and $K'=\left(\frac{d_0-d_1}{d_2-d_0}\right)^{\frac{d_2-d_0}{d_2-d_1}}+\left(\frac{d_0-d_1}{d_2-d_0}\right)^{\frac{d_1-d_0}{d_2-d_0}}$. Finally the minimum s_3 is sufficiently large ($s_3 \geq s_2$) if the following condition $F_1(\gamma_L) \leq \tau_0$, with $\tau_0=\frac{d_0-d_1}{(d_2-d_0)e^{2s_2(d_2-d_1)}}$, is satisfied. So we conclude for Theorem 3.1.

3.3.2 Proof of Theorem 3.2 We keep the notations of the proof of Theorem 3.1. In this theorem, we want to remove all the observation terms on w obtained in Theorem 3.1 and express them in terms of u. So we look at the terms $\int_{\gamma_L\times(0,T)} e^{2s\phi}|\partial_\nu z_i|^2\ d\sigma\ dt$ for $i=0,1$ appearing in step 2 of Theorem 3.1. Recall that $z_i=0$ outside $\Omega_{L-\epsilon}$ and $\gamma_L \subset \partial\omega'$.
As in [4, Lemma 2] we choose $g_0 \in C^2(\overline{\omega'},\mathbb{R}^n)$ such that $g_0=\nu$ on the C^2-boundary $\partial\omega'$ where ν is the normal vector to $\partial\omega'$. We have by integration by parts for any integer $i=0,1$,

$$\int_{\omega'\times(0,T)} e^{2s\phi}\Delta z_i\ g_0\cdot\nabla z_i\ dx\ dt = -\int_{\omega'\times(0,T)} \nabla(e^{2s\phi}g_0\cdot\nabla z_i)\cdot\nabla z_i\ dx\ dt$$
$$+\int_{\partial\omega'\times(0,T)} e^{2s\phi}g_0\cdot\nabla z_i\ \partial_\nu z_i\ d\sigma\ dt.$$

So

$$\int_{\omega'\times(0,T)} e^{2s\phi}\Delta z_i\ g_0\cdot\nabla z_i\ dx\ dt = -\int_{\omega'\times(0,T)} \nabla(e^{2s\phi}g_0\cdot\nabla z_i)\cdot\nabla z_i\ dx\ dt$$

$$+\int_{\partial\omega'\times(0,T)} e^{2s\phi}\ |\partial_\nu z_i|^2\ d\sigma\ dt.$$

and we get

$$\int_{\gamma_L\times(0,T)} e^{2s\phi}|\partial_\nu z_i|^2\ d\sigma\ dt \leq Cs\int_{(\omega'\cap\Omega_L)\times(0,T)} e^{2s\phi}(|\nabla z_i|^2+|\Delta z_i|^2)\ dx\ dt. \tag{3.25}$$

From the first equation in (3.15) we have

$$\beta\phi_2 z_0 = \partial_t y_0 - \Delta y_0 - \alpha\phi_1 y_0 - a\eta\chi\phi_1\tilde{u} - b\eta\chi\phi_2\tilde{w} - R_1 \text{ in } Q_L. \tag{3.26}$$

By the same way, from (3.16) we have

$$\beta\phi_2 z_1 = \partial_t y_1 - \Delta y_1 - \alpha\phi_1 y_1 - R_3 \text{ in } Q_L. \tag{3.27}$$

i) First assume that $a=b=0$ in ω'. From hypothesis (3.5), (3.25)-(3.27) we get

$$\sum_{i=0}^{1}\int_{\gamma_L\times(0,T)} e^{2s\phi}|\partial_\nu z_i|^2\ d\sigma\ dt \leq Cs\sum_{i=0}^{1}\int_{(\omega'\cap\Omega_L)\times(0,T)} e^{2s\phi}(|\nabla\partial_t y_i|^2+|\nabla(\Delta y_i)|^2+|\nabla y_i|^2+|y_i|^2$$

$$+|\Delta\partial_t y_i|^2+|\Delta(\Delta y_i)|^2+|\Delta y_i|^2)\ dx\ dt + Cse^{2sd_1}.$$

So

$$\sum_{i=0}^{1}\int_{\gamma_L\times(0,T)} e^{2s\phi}|\partial_\nu z_i|^2\ d\sigma\ dt \leq Cse^{2sd_1}+Cse^{2sd_2}G_0(\omega')$$

with $G_0(\omega') = \|y_0\|^2_{H^1(0,T,H^4(\omega'\cap\Omega_L))} + \|y_0\|^2_{H^2(0,T,H^2(\omega'\cap\Omega_L))}$.
Therefore (3.17) is still valid with $sF_0(\gamma_L)$ replaced by $s^2G_1(\gamma_L) = s^2\int_{\gamma_L\times(0,T)}$ $\sum_{i=0}^{1}|\partial_\nu y_i|^2\ d\sigma\ dt + s^2G_0(\omega')$. Thus we follow the proof of Theorem 3.1 substituting $F_0(\gamma_L)$ by $G_1(\gamma_L)$. The rest of the proof (steps 3 and 4) remains unchanged.

ii) Here we suppose that $\alpha,\beta,\tilde{\alpha},\tilde{\beta}\in H^2(\Omega)$. We will need to differentiate y_0 and z_0 twice with respect to t (in order to get (3.35)) and we have

$$\begin{cases} \partial_t y_2 = \Delta y_2 + \alpha\phi_1 y_2 + \beta\phi_2 z_2 + \partial_t R_3 + \alpha\partial_t\phi_1 y_1 + \beta\partial_t\phi_2 z_1 \ \text{ in } \ Q_L, \\ \partial_t z_2 = \Delta z_2 + \gamma\phi_3 y_2 + \delta\phi_4 z_2 + \partial_t R_4 + \gamma\partial_t\phi_3 y_1 + \delta\partial_t\phi_4 z_1 \ \text{ in } \ Q_L, \\ y_2 = z_2 = 0 \text{ on } \partial\Omega_L\times(0,T). \end{cases} \tag{3.28}$$

Therefore

$$\beta\phi_2 z_2 = \partial_t y_2 - \Delta y_2 - \alpha\phi_1 y_2 - \partial_t R_3 - \alpha\partial_t\phi_1 y_1 - \beta\partial_t\phi_2 z_1 \text{ in } Q_L. \tag{3.29}$$

Notice that we can take $\sum_{k=0}^{2}\int_{\gamma_L\times(0,T)}(|\partial_\nu(\partial_t^k(u_A-\tilde{u}_A))|^2+|\partial_\nu(\partial_t^k(w_A-\tilde{w}_A))|^2+\partial_\nu(\partial_t^k(u_B-\tilde{u}_B))|^2+|\partial_\nu(\partial_t^k(w_B-\tilde{w}_B))|^2)d\sigma\ dt$ as observation terms in (3.4). So we apply (3.25) for

$i = 0, 1, 2$.
From (3.25)-(3.29) we get

$$\sum_{i=0}^{2} \int_{\gamma_L \times (0,T)} e^{2s\phi} |\partial_\nu z_i|^2 \, d\sigma \, dt \leq Cs \sum_{i=0}^{2} \int_{(\omega' \cap \Omega_L) \times (0,T)} e^{2s\phi} (|\nabla \partial_t y_i|^2 + |\nabla(\Delta y_i)|^2 + |\nabla y_i|^2 + |y_i|^2$$
$$+ |\Delta \partial_t y_i|^2 + |\Delta(\Delta y_i)|^2 + |\Delta y_i|^2$$
$$+ (a^2 + b^2)\chi^2 + |\nabla(a\chi)|^2 + |\nabla(b\chi)|^2 + |\Delta(a\chi)|^2 + |\Delta(b\chi)|^2) \, dx \, dt + Cse^{2sd_1}.$$

So

$$\sum_{i=0}^{2} \int_{\gamma_L \times (0,T)} e^{2s\phi} |\partial_\nu z_i|^2 \, d\sigma \, dt \leq Cse^{2sd_2} \tilde{G}_0(\omega') + Cse^{2sd_1}$$

$$+ Cs \int_{Q_L} e^{2s\phi} ((a^2 + b^2)\chi^2 + |\nabla(a\chi)|^2 + |\nabla(b\chi)|^2 + |\Delta(a\chi)|^2 + |\Delta(b\chi)|^2) \, dx \, dt$$

with $\tilde{G}_0(\omega') = \|y_0\|^2_{H^2(0,T,H^4(\omega' \cap \Omega_L))} + \|y_0\|^2_{H^3(0,T,H^2(\omega' \cap \Omega_L))}$.
Thus the estimate (3.17) becomes

$$\sum_{i=0}^{2} (I(y_i) + I(z_i)) \leq Cs^3 e^{2sd_1} + Cs^2 e^{2sd_2} \tilde{G}_1(\gamma_L)$$

$$+ Cs^2 \int_{Q_L} e^{2s\phi} ((a^2 + b^2 + c^2 + d^2)\chi^2 + |\nabla(a\chi)|^2 + |\Delta(a\chi)|^2 + |\nabla(b\chi)|^2 + |\Delta(b\chi)|^2) \, dx \, dt \quad (3.30)$$

with $\tilde{G}_1(\gamma_L) = \int_{\gamma_L \times (0,T)} \sum_{i=0}^{2} |\partial_\nu y_i|^2 \, d\sigma \, dt + \tilde{G}_0(\omega')$.
As in the third step of Theorem 3.1 when we get (3.18), by Lemma 3.1 we have

$$\sum_{i=0}^{1} \int_{\Omega_L} e^{2s\phi(\theta)} (|y_i(\theta)|^2 + |\nabla y_i(\theta)|^2 + |\Delta y_i(\theta)|^2 + |z_i(\theta)|^2 + |\nabla z_i(\theta)|^2 + |\Delta z_i(\theta)|^2) \, dx$$

$$\leq Cs^2 \sum_{i=0}^{2} (I(y_i) + I(z_i)).$$

So from (3.30)

$$\sum_{i=0}^{1} \int_{\Omega_L} e^{2s\phi(\theta)} (|y_i(\theta)|^2 + |\nabla y_i(\theta)|^2 + |\Delta y_i(\theta)|^2 + |z_i(\theta)|^2 + |\nabla z_i(\theta)|^2 + |\Delta z_i(\theta)|^2) \, dx$$

$$\leq Cs^4 \int_{Q_L} e^{2s\phi} ((a^2 + b^2 + c^2 + d^2)\chi^2 + |\nabla(a\chi)|^2 + |\Delta(a\chi)|^2 + |\nabla(b\chi)|^2 + |\Delta(b\chi)|^2) \, dx \, dt$$

$$+ Cs^5 e^{2sd_1} + Cs^4 e^{2sd_2} \tilde{G}_1(\gamma_L). \quad (3.31)$$

Now we estimate $\int_{\Omega_L} e^{2s\phi(\theta)}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx$ as in the fourth step of Theorem 3.1. We consider two sets of initial conditions A and B and the corresponding solutions $V_A, \tilde{V}_A, V_B, \tilde{V}_B$ of (1.1). As in (3.20)-(3.23) we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2+c^2+d^2)\chi^2\,dx \le C\int_{\Omega_L} e^{2s\phi(\theta)}(|\partial_t y_{0A}(\theta)|^2+|\partial_t y_{0B}(\theta)|^2+|\Delta y_{0A}(\theta)|^2$$
$$+|\Delta y_{0B}(\theta)|^2+|y_{0A}(\theta)|^2+|y_{0B}(\theta)|^2+|\partial_t z_{0A}(\theta)|^2+|\partial_t z_{0B}(\theta)|^2+|\Delta z_{0A}(\theta)|^2$$
$$+|\Delta z_{0B}(\theta)|^2+|z_{0A}(\theta)|^2+|z_{0B}(\theta)|^2)\,dx+Ce^{2sd_1}.$$

So from (3.31) we obtain

$$\int_{\Omega_L} e^{2s\phi(\theta)}(a^2+b^2+c^2+d^2)\chi^2\,dx \le Cs^5e^{2sd_1}+Cs^4e^{2sd_2}G_2(\gamma_L)$$
$$+Cs^4\int_{Q_L} e^{2s\phi}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2))\,dx\,dt \quad (3.32)$$

with $G_2(\gamma_L)=\tilde{G}_{1A}(\gamma_L)+\tilde{G}_{1B}(\gamma_L)$.
We apply the same ideas for $\nabla(a\chi), \nabla(b\chi), \Delta(a\chi), \Delta(b\chi)$.
For any integer $1\le i\le n$, taking the space derivative with respect to x_i in (3.20), we obtain

$$\partial_{x_i}(a\chi)\eta\phi_1(\tilde{u}_A\tilde{w}_B-\tilde{u}_B\tilde{w}_A)+a\eta\chi\phi_1\partial_{x_i}(\tilde{u}_A\tilde{w}_B-\tilde{u}_B\tilde{w}_A)$$
$$=\partial_{x_i}\left(\tilde{w}_B(\partial_t y_{0A}-\Delta y_{0A}-\alpha\phi_1 y_{0A}-\beta\phi_2 z_{0A}-R_{1A})\right)$$
$$-\partial_{x_i}\left(\tilde{w}_A(\partial_t y_{0B}-\Delta y_{0B}-\alpha\phi_1 y_{0B}-\beta\phi_2 z_{0B}-R_{1B}\right). \quad (3.33)$$

Therefore by hypothesis (3.3) we deduce that

$$\int_{\Omega_L} e^{2s\phi(\theta)}|\nabla(a\chi)|^2\,dx \le C\int_{\Omega_L} e^{2s\phi(\theta)}(a\chi)^2\,dx+Ce^{2sd_1}$$
$$+\int_{\Omega_L} e^{2s\phi(\theta)}(|\nabla\partial_t y_{0A}(\theta)|^2+|\nabla\Delta y_{0A}(\theta)|^2+|\nabla y_{0A}(\theta)|^2+|\nabla z_{0A}(\theta)|^2$$
$$+|\nabla\partial_t y_{0B}(\theta)|^2+|\nabla\Delta y_{0B}(\theta)|^2+|\nabla y_{0B}(\theta)|^2+|\nabla z_{0B}(\theta)|^2)\,dx.$$

From (3.31)-(3.32) we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}|\nabla(a\chi)|^2\,dx \le Cs^5e^{2sd_1}+Cs^4e^{2sd_2}G_2(\gamma_L)+Ce^{2sd_2}(\|y_{0A}(\theta)\|^2_{H^3(\Omega_L)}+\|y_{0B}(\theta)\|^2_{H^3(\Omega_L)})$$
$$+Cs^4\int_{Q_L} e^{2s\phi}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx\,dt. \quad (3.34)$$

Taking again the space derivative with respect to x_i in (3.33) we obtain

$$\int_{\Omega_L} e^{2s\phi(\theta)}|\Delta(a\chi)|^2\,dx \le Cs^5e^{2sd_1}+Cs^4e^{2sd_2}G_2(\gamma_L)+Ce^{2sd_2}(\|y_{0A}(\theta)\|_{H^4(\Omega_L)}+\|y_{0B}(\theta)\|_{H^4(\Omega_L)})$$

$$+Cs^4\int_{Q_L} e^{2s\phi}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx\,dt. \quad (3.35)$$

Similarly for b, so from (3.32),(3.34),(3.35) we have

$$\int_{\Omega_L} e^{2s\phi(\theta)}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx$$
$$\leq Cs^5e^{2sd_1}+Cs^4e^{2sd_2}G_2(\gamma_L)+Ce^{2sd_2}(\|y_{0A}(\theta)\|_{H^4(\Omega_L)}+\|y_{0B}(\theta)\|_{H^4(\Omega_L)})$$
$$+Cs^4\int_{Q_L} e^{2s\phi}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx\,dt.$$

As in the proof of Theorem 3.1 (see the fourth step) we can absorb the last term of the above estimate by the left-hand side so we deduce that for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(\theta)}((a^2+b^2+c^2+d^2)\chi^2+|\nabla(a\chi)|^2+|\Delta(a\chi)|^2+|\nabla(b\chi)|^2+|\Delta(b\chi)|^2)\,dx$$
$$\leq Cs^5e^{2sd_1}+Cs^4e^{2sd_2}G_3(\gamma_L)$$

with $G_3(\gamma_L)=G_2(\gamma_L)+\|y_{0A}(\theta)\|_{H^4(\Omega_L)}+\|y_{0B}(\theta)\|_{H^4(\Omega_L)}$ and we conclude as for Theorem 3.1.

3.3.3 Proof of Theorem 3.3

Let $V_A=(u_A,w_A)$ (resp. $\tilde V_A=(\widetilde u_A,\widetilde w_A)$) be a solution of (1.1) associated with (ρ,G,A) (resp. $(\tilde\rho_2,G,A)$) and $V_B=(u_B,w_B)$ (resp. $\tilde V_B=(\widetilde u_B,\widetilde w_B)$) be a solution of (1.1) associated with (ρ,G,B) (resp. $(\tilde\rho_2,G,B)$). As for Theorems 3.1 and 3.2 we decompose the proof in several steps.

• First step: We keep the notations of (3.12)

$$V=(u,w)=V_A,\ \tilde V=(\tilde u,\tilde w)=\tilde V_A,\ U=u-\tilde u,\ W=w-\tilde w, b=\beta-\tilde\beta,\ c=\gamma-\tilde\gamma,\ d=\delta-\tilde\delta.$$

and now define

$$f_1=\phi_1-\tilde\phi_1.$$

We still define (see (3.14)) (for $i=0,1,2$)

$$y_0=\eta\chi U,\ z_0=\eta\chi W,\ y_i=\partial_t^i y_0,\ z_i=\partial_t^i z_0.$$

The systems (3.13), (3.15), (3.16) become

$$\begin{cases}\partial_t U=\Delta U+\alpha\phi_1 U+\beta\phi_2 W+\alpha f_1\tilde u+b\phi_2\tilde w \text{ in } Q,\\ \partial_t W=\Delta W+\gamma\phi_3 U+\delta\phi_4 W+c\phi_3\tilde u+d\phi_4\tilde w \text{ in } Q,\\ U=W=0 \text{ in } \Sigma,\end{cases}$$

and (y_i,z_i) for $i=0,1$ satisfy the following systems

$$\begin{cases}\partial_t y_0=\Delta y_0+\alpha\phi_1 y_0+\beta\phi_2 z_0+\alpha f_1\eta\chi\tilde u+b\phi_2\eta\chi\tilde w+S_1 \text{ in } Q_L,\\ \partial_t z_0=\Delta z_0+\gamma\phi_3 y_0+\delta\phi_4 z_0+c\phi_3\eta\chi\tilde u+d\phi_4\eta\chi\tilde w+S_2 \text{ in } Q_L,\\ y_0=z_0=0 \text{ on } \partial\Omega_L\times(0,T)\end{cases} \quad (3.36)$$

with

$$S_1 = R_1 = -(\Delta\chi)\eta U - 2\eta\nabla\chi\cdot\nabla U + \chi\partial_t\eta U,\ S_2 = R_2 = -(\Delta\chi)\eta W - 2\eta\nabla\chi\cdot\nabla W + \chi\partial_t\eta W.$$

We have

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \alpha\phi_1 y_1 + \beta\phi_2 z_1 + S_3 \ \text{ in } \ Q_L, \\ \partial_t z_1 = \Delta z_1 + \gamma\phi_3 y_1 + \delta\phi_4 z_1 + S_4 \ \text{ in } \ Q_L, \\ y_1 = z_1 = 0 \text{ on } \partial\Omega_L \times (0,T), \end{cases}$$

with

$$S_3 = \partial_t\left(\alpha f_1\eta\chi\tilde{u} + b\phi_2\eta\chi\tilde{w}\right) + \partial_t S_1 + \alpha y_0\partial_t\phi_1 + \beta z_0\partial_t\phi_2,$$

$$S_4 = R_4 = \partial_t\left(c\phi_3\eta\chi\tilde{u} + d\phi_4\eta\chi\tilde{w}\right) + \partial_t S_2 + \gamma y_0\partial_t\phi_3 + \delta z_0\partial_t\phi_4.$$

We also have

$$\begin{cases} \partial_t y_2 = \Delta y_2 + \alpha\phi_1 y_2 + \beta\phi_2 z_2 + \partial_t S_3 + \alpha\partial_t\phi_1 y_1 + \beta\partial_t\phi_2 z_1 \ \text{ in } \ Q_L, \\ \partial_t z_2 = \Delta z_2 + \gamma\phi_3 y_2 + \delta\phi_4 z_2 + \partial_t S_4 + \gamma\partial_t\phi_3 y_1 + \delta\partial_t\phi_4 z_1 \ \text{ in } \ Q_L, \\ y_2 = z_2 = 0 \text{ on } \partial\Omega_L \times (0,T). \end{cases}$$

- In the second step we estimate $\sum_{i=0}^{2}(I(y_i) + I(z_i))$ as in Theorem 3.1 and we get

$$\sum_{i=0}^{2}(I(y_i) + I(z_i)) \le C\int_{Q_L} e^{2s\phi}(b^2 + c^2 + d^2)\chi^2\ dx\ dt + C\int_{Q_L} e^{2s\phi}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\ dx\ dt$$
$$+ Cs^3 e^{2sd_1} + Cse^{2sd_2}\tilde{F}_0(\gamma_L) \tag{3.37}$$

with $\tilde{F}_0(\gamma_L) = \int_{\gamma_L\times(0,T)}\sum_{i=0}^{2}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\ d\sigma\ dt$ (nearly same definition as before since (3.17)).

Now following the proof of Theorem 3.1 we look at (3.18) in this context. First note that because of the fourth step of this proof, we can no longer use the estimates of the Laplacian terms in (3.18) and contrary to Theorems 3.1, 3.2, 3.4, we have to take care of the powers of s on the right-hand sides of our estimates. In fact we could only look at the estimate of $\int_{\Omega_L} e^{2s\phi(\theta)}|\partial_t z_0(\theta)|^2\ dx$ but because of the remarks given just after the proof of this theorem, we will keep more terms. So we will not estimate $\int_{\Omega_L} e^{2s\phi(\theta)}|\partial_t z_0(\theta)|^2\ dx$ as in Theorems 3.1, 3.2, 3.4 (see the third step of Theorem 3.1) and for that, we need to differentiate twice y_0 and z_0 with respect to t. Thus

$$\int_{\Omega_L} e^{2s\phi(\theta)}|\partial_t z_0(\theta)|^2\ dx \le Cs\int_{Q_L} e^{2s\phi}|z_1|^2\ dx\ dt + \frac{C}{s}\int_{Q_L} e^{2s\phi}|z_2|^2 \le \frac{C}{s^2}(I(z_1) + I(z_2)).$$

So we have (coming from Lemma 3.1 as in (3.18)) and by (3.37)

$$\int_{\Omega_L} e^{2s\phi(\theta)}(y_0(\theta)|^2 + |\partial_t y_0(\theta)|^2 + |z_0(\theta)|^2 + |\partial_t z_0(\theta)|^2)\ dx \le \frac{C}{s^2}\sum_{i=0}^{2}(I(y_i) + I(z_i))$$

$$\leq \frac{C}{s^2}\int_{Q_L} e^{2s\phi}(b^2+c^2+d^2)\chi^2\ dx\ dt + \frac{C}{s^2}\int_{Q_L} e^{2s\phi}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\ dx\ dt + Cse^{2sd_1} + \frac{C}{s}e^{2sd_2}\tilde{F}_0(\gamma_L).$$

Since $\phi \leq \phi(\theta)$ we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}(|y_0(\theta)|^2 + |\partial_t y_0(\theta)|^2 + |z_0(\theta)|^2 + |\partial_t z_0(\theta)|^2)\ dx \leq \frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(\theta)}(b^2+c^2+d^2)\chi^2\ dx$$

$$+\frac{C}{s^2}\int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\ dx\ dt + Cse^{2sd_1} + \frac{C}{s}e^{2sd_2}\tilde{F}_0(\gamma_L). \tag{3.38}$$

• Third step: here we estimate $\int_{\Omega_L} e^{2s\phi(\theta)}\chi^2(b^2+c^2+d^2)\ dx$ as in Theorem 3.1 with two different sets of conditions A and B. We recall that each function f precendently defined is denoted either f_A or f_B when it is related either by the conditions A or B.
For the coefficient b we can write from the first equation of (3.36)

$$-b\eta\chi\phi_2(\tilde{u}_A\tilde{w}_B - \tilde{u}_B\tilde{w}_A) = \tilde{u}_B(\partial_t y_{0A} - \Delta y_{0A} - \alpha\phi_1 y_{0A} - \beta\phi_2 z_{0A} - \alpha f_1\eta\chi\tilde{u}_A - S_{1A})$$

$$-\tilde{u}_A(\partial_t y_{0B} - \Delta y_{0B} - \alpha\phi_1 y_{0B} - \beta\phi_2 z_{0B} - \alpha f_1\eta\chi\tilde{u}_B - S_{1B}).$$

Note that the terms in f_1 disappear in the above equality. For the coefficients c and d we use the second equation of (3.36) and proceed as in Theorem 3.1. Indeed, for example for c, we have

$$c\eta\chi\phi_3(\tilde{u}_A\tilde{w}_B - \tilde{u}_B\tilde{w}_A) = \tilde{w}_B(\partial_t z_{0A} - \Delta z_{0A} - \gamma\phi_3 y_{0A} - \delta\phi_4 z_{0A} - S_{2A})$$

$$-\tilde{w}_A(\partial_t z_{0B} - \Delta z_{0B} - \gamma\phi_3 y_{0B} - \delta\phi_4 z_{0B} - S_{2B}).$$

Therefore by hypothesis (3.3) and (3.38) we obtain for s sufficiently large

$$\int_{\Omega_L} e^{2s\phi(\theta)}(b^2+c^2+d^2)\chi^2\ dx \leq \frac{C}{s^2}\int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\ dx\ dt + Cse^{2sd_1} + Ce^{2sd_2}F_2(\theta) \tag{3.39}$$

with $F_2(\theta) = \tilde{F}_{0A}(\gamma_L) + \tilde{F}_{0B}(\gamma_L) + \|\Delta y_{0A}(\theta)\|_{L^2(\Omega_L)} + \|\Delta y_{0B}(\theta)\|_{L^2(\Omega_L)} + \|\Delta z_{0A}(\theta)\|_{L^2(\Omega_L)} + \|\Delta z_{0B}(\theta)\|_{L^2(\Omega_L)}$.
• Fourth step: we estimate now $\int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\ dx\ dt$. Here again we use the two different sets of coefficients A and B. From (3.36) for y_{0A} and y_{0B}, we get

$$\alpha\eta\chi f_1(\tilde{u}_A\tilde{w}_B - \tilde{u}_B\tilde{w}_A) = \tilde{w}_B(\partial_t y_{0A} - \Delta y_{0A} - \alpha\phi_1 y_{0A} - \beta\phi_2 z_{0A} - S_{1A})$$

$$-\tilde{w}_A(\partial_t y_{0B} - \Delta y_{0B} - \alpha\phi_1 y_{0B} - \beta\phi_2 z_{0B} - S_{1B}). \tag{3.40}$$

Applying (3.40) for $t = \theta$, by hypotheses (3.3) and (3.9), using again (3.38) we obtain

$$\int_{\Omega_L} e^{2s\phi(\theta)}\chi^2(f_1(\theta))^2\ dx \leq \frac{C}{s^2}\int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\ dx\ dt$$

$$+\frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(\theta)}(b^2+c^2+d^2)\chi^2\ dx + Cse^{2sd_1} + Ce^{2sd_2}F_2(\theta). \tag{3.41}$$

Deriving now (3.40) with respect to t, we have

$$(\partial_t f_1)\alpha\eta(\tilde{u}_A\tilde{w}_B - \tilde{u}_B\tilde{w}_A) + f_1\partial_t(\alpha\eta\chi(\tilde{u}_A\tilde{w}_B - \tilde{u}_B\tilde{w}_A)) =$$
$$\partial_t(\tilde{w}_B(\partial_t y_{0A} - \Delta y_{0A} - \alpha\phi_1 y_{0A} - \beta\phi_2 z_{0A} - S_{1A}) - \tilde{w}_A(\partial_t y_{0B} - \Delta y_{0B} - \alpha\phi_1 y_{0B} - \beta\phi_2 z_{0B} - S_{1B}))$$

and evaluating this last equation at $t=\theta$, still by hypotheses (3.3) and (3.9), we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}\chi^2(\partial_t f_1(\theta))^2\ dx \le C\int_{\Omega_L} e^{2s\phi(\theta)}\chi^2(f_1(\theta))^2\ dx$$
$$+C\int_{\Omega_L} e^{2s\phi(\theta)}\sum_{i=0}^{1}(|\partial_t^i z_{0A}(\theta)|^2 + |\partial_t^i z_{0B}(\theta)|^2) + Ce^{2sd_2}F_3(\theta) \tag{3.42}$$

with

$$F_3(\theta) = \sum_{k=0}^{2}(\|\partial_t^k y_{0A}(\theta)\|^2_{L^2(\Omega_L)} + \|\partial_t^k y_{0B}(\theta)\|^2_{L^2(\Omega_L)})$$
$$+\sum_{k=0}^{1}(\|\partial_t^k \Delta y_{0A}(\theta)\|^2_{L^2(\Omega_L)} + \|\partial_t^k \Delta y_{0B}(\theta)\|^2_{L^2(\Omega_L)}).$$

From (3.38), (3.41) and (3.42) we have

$$\int_{\Omega_L} e^{2s\phi(\theta)}\chi^2((f_1(\theta))^2 + (\partial_t f_1(\theta))^2)\ dx \le \frac{C}{s^2}\int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\ dx\ dt$$
$$+\frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(\theta)}(b^2+c^2+d^2)\chi^2\ dx + Cse^{2sd_1} + Ce^{2sd_2}F_4(\theta) \tag{3.43}$$

with $F_4(\theta) = F_2(\theta) + F_3(\theta)$.
Moreover by Taylor's formula, we have

$$f_1(t) = f_1(\theta) + \partial_t f_1(\theta)(t-\theta) + \partial_t^2 f_1(c_\theta)\frac{(t-\theta)^2}{2} \text{ and } \partial_t f_1(t) = \partial_t f_1(\theta) + \partial_t^2 f_1(c'_\theta)(t-\theta)$$

with $c_\theta, c'_\theta \in [0,T]$. Therefore, since $\tilde{\phi_1} \in \Lambda_3(M_3)$ the admissible set of coefficients, we get

$$\sum_{i=0}^{2}(\partial_t^i f_1)^2 \le C(f_1(\theta))^2 + (\partial_t f_1(\theta))^2),$$

so from (3.43) we deduce that for s sufficiently large

$$\int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\ dx\ dt \le \frac{C}{s^2}\int_{\Omega_L} e^{2s\phi(\theta)}(b^2+c^2+d^2)\chi^2\ dx + Cse^{2sd_1} + Ce^{2sd_2}F_4(\theta). \tag{3.44}$$

• Fifth and last step: now addding (3.39) and (3.44) we obtain

$$\int_{\Omega_L} e^{2s\phi(\theta)}(b^2+c^2+d^2)\chi^2\, dx + \int_{Q_L} e^{2s\phi(\theta)}\chi^2(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\, dx\, dt \le Cse^{2sd_1} + Ce^{2sd_2}F_4(\theta).$$

So

$$\int_{\Omega_l} e^{2s\phi(\theta)}(b^2+c^2+d^2)\, dx + \int_{\Omega_l\times(0,T)} e^{2s\phi(\theta)}(\sum_{i=0}^{2}(\partial_t^i f_1)^2)\, dx\, dt \le Cse^{2sd_1} + Ce^{2sd_2}F_4(\theta)$$

and we conclude as for Theorem 3.1 by optimizing the above inequality with respect to s.

Remark 2 • If the admissible set of coefficients is $\Lambda_3'(M_3)$ (thus less restrictive than $\Lambda_3(M_3)$), then we would have to derive $p-1$ times (3.40) with respect to t and that would demand more regularity for the observation terms on u.
• On the contrary if the admissible set of coefficients is $\Lambda_3''(M_3)$, so more restrictive than $\Lambda_3(M_3)$ (or if $\tilde{\phi_1} \in C^2([0,T])$ is such that $\tilde{\phi_1}(\theta) \neq \phi_1(\theta)$ and $\frac{\sup_{t\in[0,T]}|\partial_t^i(\phi_1-\tilde{\phi_1})(t)|}{|\phi_1(\theta)-\tilde{\phi_1}(\theta)|} \le M_3$ for $i=0,1,2$), then we can drop (3.42) and (3.43) in the above proof. Therefore the result remains valid without $F_3(\theta)$ and so $F_4(\theta)=F_2(\theta)$. Thus the observations terms on u are only $\|(u_A-\tilde{u_A})(\cdot,\theta)\|^2_{H^2(\Omega_L)}$ and $\|(u_B-\tilde{u_B})(\cdot,\theta)\|^2_{H^2(\Omega_L)}$.

3.3.4 Proof of Theorem 3.4

Here again we follow the method described before. Let $V_A=(u_A,w_A)$ (resp. $\tilde{V}_A=(\tilde{u}_A,\tilde{w}_A)$) be a strong solution of (1.3) associated with (ρ,G,A,Θ) defined by (1.2) and (1.4) (resp. $(\tilde{\rho_3},G,A,\tilde{\Theta})$). Consider also $V_B=(u_B,w_B)$ (resp. $\tilde{V}_B=(\tilde{u}_B,\tilde{w}_B)$) a strong solution of (1.3) associated with (ρ,G,B,Θ) (resp. $(\tilde{\rho_3},G,B,\tilde{\Theta})$).
• As before, in a first step we define

$$V=(u,w)=V_A,\ \tilde{V}=(\tilde{u},\tilde{w})=\tilde{V}_A,\ U=u-\tilde{u},\ W=w-\tilde{w}, b=\beta-\tilde{\beta},\ c=\gamma-\tilde{\gamma},\ d=\delta-\tilde{\delta}$$

and also

$$H=\Theta_1-\tilde{\Theta_1}=\nabla h \text{ with } h=\xi_1-\tilde{\xi_1}.$$

Recall that for $i=0,1$,

$$y_0=\eta\chi U,\ z_0=\eta\chi W,\ y_1=\partial_i y_0,\ z_1=\partial_t z_0.$$

Then

$$\begin{cases} \partial_t y_0 = \Delta y_0 + \alpha\phi_1 y_0 + \beta\phi_2 z_0 + \Theta_1\cdot\nabla y_0 + \Theta_2\cdot\nabla z_0 + b\eta\chi\phi_2\tilde{w} + \eta\nabla(\chi h)\cdot\nabla\tilde{u} + T_1 \text{ in } Q_L,\\ \partial_t z_0 = \Delta z_0 + \gamma\phi_3 y_0 + \delta\phi_4 z_0 + \Theta_3\cdot\nabla y_0 + \Theta_4\cdot\nabla z_0 + c\eta\chi\phi_3\tilde{u} + d\eta\chi\phi_4\tilde{w} + T_2 \text{ in } Q_L,\\ y_0=z_0=0 \text{ on } \partial\Omega_L\times(0,T) \end{cases} \tag{3.45}$$

with

$$T_1 = (\partial_t \eta)\chi U - (\Delta\chi)\eta U - 2\nabla\chi\cdot\nabla(\eta U) - \eta U\Theta_1\cdot\nabla\chi - \eta W\Theta_2\cdot\nabla\chi - \eta h\nabla\tilde{u}\cdot\nabla\chi$$

$$T_2 = (\partial_t \eta)\chi W - (\Delta\chi)\eta W - 2\nabla\chi\cdot\nabla(\eta W) - \eta U\Theta_3\cdot\nabla\chi - \eta W\Theta_4\cdot\nabla\chi.$$

And

$$\begin{cases} \partial_t y_1 & = \Delta y_1 + \alpha\phi_1 y_1 + \beta\phi_2 z_1 + \Theta_1\cdot\nabla y_1 + \Theta_2\cdot\nabla z_1 + b\eta\chi\partial_t(\phi_2\tilde{w}) + \eta\nabla(\chi h)\cdot\nabla\partial_t\tilde{u} + T_3 \\ & \quad \text{in } Q_L, \\ \partial_t z_1 & = \Delta z_1 + \gamma\phi_3 y_1 + \delta\phi_4 z_1 + \Theta_3\cdot\nabla y_1 + \Theta_4\cdot\nabla z_1 + c\eta\chi\partial_t(\phi_3\tilde{u}) + d\eta\chi\partial_t(\phi_4\tilde{w}) + T_4 \\ & \quad \text{in } Q_L, \\ y_1 & = z_1 = 0 \text{ on } \partial\Omega_L\times(0,T) \end{cases}$$

with

$$T_3 = \alpha y_0\partial_t\phi_1 + \beta z_0\partial_t\phi_2 + \partial_t\eta(b\chi\phi_2\tilde{w} + \nabla(\chi h)\cdot\nabla\tilde{u}) + \partial_t T_1,$$

$$T_4 = \gamma y_0\partial_t\phi_3 + \delta z_0\partial_t\phi_4 + \partial_t\eta(c\chi\phi_3\tilde{u} + d\chi\phi_4\tilde{w}) + \partial_t T_2.$$

Thus we obtain

$$\sum_{i=0}^{1}(I(y_i)+I(z_i)) \le C\int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2 + |\nabla(\chi h)|^2)\,dx\,dt + Cs^3e^{2sd_1}$$
$$+Cs\sum_{i=0}^{1}\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\partial_\nu y_i|^2 + |\partial_\nu z_i|^2)\,d\sigma\,dt.$$

We deduce that (see the third step of Theorem 3.1)

$$\sum_{i=0}^{1}\int_{\Omega_L} e^{2s\phi(\theta)}(|y_i(\theta)|^2+|\nabla y_i(\theta)|^2+|z_i(\theta)|^2+|\nabla z_i(\theta)|^2)\,dx + \int_{\Omega_L} e^{2s\phi(\theta)}(|\Delta y_0(\theta)|^2+|\Delta z_0(\theta)|^2)\,dx$$
$$\le Cs^2\sum_{i=0}^{1}(I(y_i)+I(z_i))$$
$$\le Cs^2\int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2+|\nabla(\chi h)|^2)\,dx\,dt + Cs^5e^{2sd_1} + Cs^3e^{2sd_2}F_0(\gamma_L) \qquad (3.46)$$

with $F_0(\gamma_L)$ defined by (3.17).

• In a second step we consider the solutions of (1.3) associated with two different sets of initial conditions A and B and we recall that each function f precendently defined is denoted either f_A or f_B when it is related either by the conditions A or B. As in the fourth step of Theorem 3.1 we have a similar estimate to (3.23) for the coefficients c and d. Indeed, writing (3.45) for z_{0A} and z_{0B}, by the hypothesis (3.3) and from (3.46) we have

$$\int_{\Omega_L} e^{2s\phi(\theta)}(c^2+d^2)\chi^2\,dx \le$$
$$Cs^2\int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2+|\nabla(\chi h)|^2)\,dx\,dt + Cs^5e^{2sd_1} + Cs^3e^{2sd_2}F_1(\gamma_L) \quad (3.47)$$

with $F_1(\gamma_L)$ defined by (3.21). Now we eliminate b in (3.45) in order to estimate the coefficient h and we evaluate at $t=\theta$. We use here the partial differential operator P defined in Lemma 3.2.

$$P(\chi h)=\tilde{w}_B(\theta)\nabla(\chi h)\cdot\nabla\tilde{u}_A(\theta)-\tilde{w}_A(\theta)\nabla(\chi h)\cdot\nabla\tilde{u}_B(\theta)$$

$$\begin{aligned}P(\chi h)=\tilde{w}_B(\theta)[\partial_t y_{0A}(\theta)-\Delta y_{0A}(\theta)-\alpha\phi_1 y_{0A}(\theta)-\beta\phi_2 z_{0A}(\theta)\\ -\Theta_1\cdot\nabla y_{0A}(\theta)-\Theta_2\cdot\nabla z_{0A}(\theta)-T_{1A}(\theta)]\\ -\tilde{w}_A(\theta)[\partial_t y_{0B}(\theta)-\Delta y_{0B}(\theta)-\alpha\phi_1 y_{0B}(\theta)-\beta\phi_2 z_{0B}(\theta)-\Theta_1\cdot\nabla y_{0B}(\theta)-\Theta_2\cdot\nabla z_{0B}(\theta)-T_{1B}(\theta)].\end{aligned} \tag{3.48}$$

From Lemma 3.2 we have

$$s^2\int_{\Omega_L} e^{2s\phi(\theta)}(\partial_{x_i}(h\chi))^2\,dx\le C\int_{\Omega_L} e^{2s\phi(\theta)}|P(\partial_{x_i}(\chi h))|^2\,dx.$$

So taking the space derivative with respect to x_i (for $i=1,\cdots,n$) in (3.48), from (3.46) we get that

$$\begin{aligned}s^2\int_{\Omega_L} e^{2s\phi(\theta)}|\nabla(\chi h)|^2\,dx\le C\int_{\Omega_L} e^{2s\phi(\theta)}|\nabla(\chi h)|^2\,dx+\\ Cs^2\int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2+|\nabla(\chi h)|^2)\,dx\,dt\\ +Ce^{2sd_2}(\|y_{0A}(\theta)\|^2_{H^3(\Omega_L)}+\|y_{0B}(\theta)\|^2_{H^3(\Omega_L)})+Cs^5e^{2sd_1}+Cs^3e^{2sd_2}F_1(\gamma_L)\end{aligned}$$

and for s sufficiently large,

$$\begin{aligned}s^2\int_{\Omega_L} e^{2s\phi(\theta)}|\nabla(\chi h)|^2\,dx\le Cs^2\int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2+|\nabla(\chi h)|^2)\,dx\,dt\\ +Cs^5e^{2sd_1}+Cs^3e^{2sd_2}F_5(\theta)\end{aligned} \tag{3.49}$$

with $F_5(\theta)=F_1(\gamma_L)+\|y_{0A}(\theta)\|^2_{H^3(\Omega_L)}+\|y_{0B}(\theta)\|^2_{H^3(\Omega_L)}$. Now we look at the coefficient b. We also use (3.45) for y_{0A} and y_{0B}

$$\begin{aligned}-b\eta\chi\phi_2(\tilde{u}_A\tilde{w}_B-\tilde{u}_B\tilde{w}_A)=\tilde{u}_B(\partial_t y_{0A}-\Delta y_{0A}-\alpha\phi_1 y_{0A}-\beta\phi_2 z_{0A}-\Theta_1\cdot\nabla y_{0A}-\Theta_2\cdot\nabla z_{0A}\\ -\eta\nabla(\chi h)\cdot\nabla\tilde{u}_A-T_{1A})-\tilde{u}_A(\partial_t y_{0B}-\Delta y_{0B}-\alpha\phi_1 y_{0B}-\beta\phi_2 z_{0B}-\Theta_1\cdot\nabla y_{0B}\\ -\Theta_2\cdot\nabla z_{0B}-\eta\nabla(\chi h)\cdot\nabla\tilde{u}_B-T_{1B}).\end{aligned} \tag{3.50}$$

Therefore, evaluating (3.50) at $t=\theta$, still using hypothesis (3.3), from (3.46) we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}b^2\chi^2\,dx\le C\int_{\Omega_L} e^{2s\phi(\theta)}|\nabla(\chi h)|^2\,dx$$

$$+Cs^2 \int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2 + |\nabla(\chi h)|^2)\ dx\ dt + Cs^5 e^{2sd_1} + Cs^3 e^{2sd_2} F_1(\gamma_L). \qquad (3.51)$$

Thus from (3.49)-(3.51) we obtain

$$\int_{\Omega_L} e^{2s\phi(\theta)}(b\chi)^2\ dx \le Cs^2 \int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2 + |\nabla(\chi h)|^2)\ dx\ dt$$
$$+Cs^5 e^{2sd_1} + Cs^3 e^{2sd_2} F_5(\theta). \qquad (3.52)$$

Finally adding (3.47), (3.49), (3.52), as in the proof of Theorem 3.1 we can neglect $s^2 \int_{Q_L} e^{2s\phi}((b^2+c^2+d^2)\chi^2 + |\nabla(\chi h)|^2)\ dx\ dt$ by the left-hand side so we get

$$\int_{\Omega_L} e^{2s\phi(\theta)}((b^2+c^2+d^2)\chi^2 + |\nabla(\chi h)|^2) \le Cs^5 e^{2sd_1} + Cs^3 e^{2sd_2} F_5(\theta)$$

and we conclude as in Theorem 3.1.

References

[1] **Benabdallah, A., Cristofol, M., Gaitan, P.**, and **Yamamoto, M. :** *Inverse problem for a parabolic system with two components by measurements of one component.* Applicable Analysis Vol. 88, No. 5 (2009), 683 – 709

[2] **Beilina, L., Cristofol, M., Li, S.**, and **Yamamoto, M. :** *Lipschitz stability for an inverse hyperbolic problem of determining two coefficients by a finite number of observations.* Inverse Problems 34 (2018) 015001 (27pp)

[3] **Bukhgeim, A. L.**, and **Klibanov, M. V. :** *Uniqueness in the Large of a Class of Multidimensional Inverse Problems.* Soviet Math. Dokl. 17, (1981), 244 – 247

[4] **Baudouin, L.**, and **Puel, J.-P. :** *Uniqueness and stability in an inverse problem for the Schrödinger equation.* Inverse Problems 18, (2002), 1537 – 1554

[5] **Cardoulis, L. :** *Applications of Carleman inequalities for a two-by-two parabolic system in an unbounded guide.* Rostock. Math. Kolloq. 72, (2019/2020), 49 – 71

[6] **Cardoulis, L. :** *An inverse problem for a hyperbolic system in a bounded domain.* Comptes Rendus Mathématique Vol. 361, (2023), 653 – 665

[7] **Cardoulis, L.**, and **Cristofol, M. :** *An inverse problem for the heat equation in an unbounded guide.* Applied Mathematics Letters 62, (2016), 6317

[8] **Cardoulis, L., Cristofol, M.**, and **Morancey, M. :** *A stability result for the diffusion coefficient of the heat operator defined on an unbounded guide.* Mathematical Control and Related Fields, doi:10.3934/mcrf.2020054 (2021)

[9] **Cardoulis, L.**, and **Gaitan, P. :** *Simultaneous identification of the diffusion coefficient and the potential for the Schrödinger operator with only one observation.* Inverse Problems 26 (2010) 035012 (10pp)

[10] **Cristofol, M., Gaitan, P.**, and **Ramoul, H. :** *Inverse problems for a 2x2 reaction diffusion system using a Carleman estimate with one observation.* Inverse Problems 22 (2006), 1561 – 1573

[11] **Cristofol, M., Li, S.**, and **Soccorsi, E. :** *Determining the waveguide conductivity in a hyperbolic equation from a single measurement on the lateral boundary.* Mathematical Control and Related Fields, Volume 6, Number 3 (2016), 407 – 427

[12] **Huang, X., Imanuvilov, O. Yu.**, and **Yamamoto, M. :** *Stability for inverse source problems by Carleman estimates.* arXiv:1912.10484 (2019) (20pp)

[13] **Immanuvilov, O. Yu., Isakov, V.**, and **Yamamoto, M. :** *An inverse problem for the dynamical Lamé system with two set of boundary data.* CPAM 56 , 1366 – 1382 (2003)

[14] **Yamamoto, M. :** *Carleman estimates for parabolic equations and applications.* Inverse Problems 25, (2009), 123013

[15] **Yuan, G.**, and **Yamamoto, M. :** *Lipschitz stability in the determination of the principal part of a parabolic equation.* ESAIM : Control Optim. Calc. Var. 15, (2009), 525 – 554

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[8] **Gnedenko, B.W.:** *Über die Arbeiten von C.F. Gauß zur Wahrscheinlichkeitsrechnung.* In: Reichardt, H. (Ed.): C.F. Gauß, Gedenkband anläßlich des 100. Todestages. S. 193-204, Leipzig 1957

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