

Partial inner products

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Abstract

A 'partial' inner product with respect to a given subset of $\text{Irr}(G)$ is introduced and its basic properties are studied. As applications, a weak second orthogonality is shown in a Clifford setting and a version of the Burnside-Brauer theorem is given.

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Throughout this paper, G is a finite group. Characters and class functions are over $\mathbb{C}$ and $\chi \mapsto \bar{\chi}$ denotes complex conjugation (of numbers or class functions).

Definition *Partial inner product*

Let $S \subseteq \text{Irr}(G)$ be any subset.

- (1) For class functions α and β of G , call

$$(\alpha, \beta)_G^S := \sum_{\sigma \in S} (\alpha, \sigma)_G (\sigma, \beta)_G$$

the S -partial inner product of α and β .

Closely related is the projection

$$\alpha_S := \sum_{\sigma \in S} (\alpha, \sigma)_G \sigma,$$

because clearly $(\alpha, \beta)_G^S = (\alpha_S, \beta)_G = (\alpha, \beta_S)_G = (\alpha_S, \beta_S)_G$ for any α and β .

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(2) For $g \in G$, denote

$$\rho_S^g = \sum_{\sigma \in S} \overline{\sigma(g)} \sigma \quad .$$

Remark 1

- (i) Clearly, ρ_S^g depends only on the conjugacy class of g .
- (ii) There is a symmetry: $\rho_S^g(h) = \overline{\rho_S^h(g)}$ for all $g, h \in G$.
- (iii) By its very definition, $\rho_S^g \in \langle S \rangle$, the subspace of all class functions spanned by S . In fact, $\langle S \rangle$ is generated by the ρ_S^g 's, because for $\tau \in S$, we have

$$\sum_{g \in G} \frac{\tau(g)}{|G|} \rho_S^g = \sum_{g \in G} \frac{\tau(g)}{|G|} \sum_{\sigma \in S} \overline{\sigma(g)} \sigma = \sum_{\sigma \in S} \left(\frac{1}{|G|} \sum_{g \in G} \tau(g) \overline{\sigma(g)} \right) \sigma = \sum_{\sigma \in S} (\tau, \sigma)_G \sigma = \tau \quad .$$

- (iv) The most prominent member of the family $\{\rho_S^g\}$ is of course the regular character $\rho = \rho_{\text{Irr}(G)}^1$, and as with the regular character, one can calculate inner products as values: $(\alpha, \rho_S^g)_G = \alpha_S(g)$ for every class function α .

Example

If $S = \text{Irr}(G/N)$ for some $N \triangleleft G$, then $\alpha_S(g) = \frac{1}{|N|} \sum_{n \in N} \alpha(gn)$ for any $g \in G$.

Proposition 1 *Associativity for partial inner products*

Let $S \subseteq \text{Irr}(G)$ and a class function η of G be given and denote $T = \text{Irr}(G) \setminus S$. For $c \in \mathbb{C}$, let $G(c) = G(c, \eta) = \{g \in G \mid \eta(g) = c\}$. Then the following are equivalent:

- (i) For all class functions α and β of G , one has $(\alpha\eta, \beta)_G^S = (\alpha, \bar{\eta}\beta)_G^S$.
- (ii) If $0 \neq \sum_{\sigma \in S} \sigma(g) \overline{\sigma(h)}$ for elements $g, h \in G$, then $\eta(g) = \eta(h)$.
- (iii) Every $0 \neq \rho_S^g$ is an eigenvector for η .
- (iv) $\langle S \rangle$ is closed under multiplication with η .
- (v) $\sum_{g \in G(c)} \sigma(g) \overline{\tau(g)} = 0$ for every $c \in \mathbb{C}$, $\sigma \in S$, $\tau \in T$.
- (vi) $\langle S \rangle$ is closed under multiplication with $\bar{\eta}$.
- (vii) $\langle S \rangle$ and $\langle T \rangle$ are closed under multiplication with η .
- (viii) $(\alpha\eta)_S = \alpha_S \eta$ for every α .

Proof:

(i) $\Rightarrow$ (ii) Let $K = g^G$ and $L = h^G$, and let χ_K and χ_L be the characteristic functions for these conjugacy classes. Then it is easily seen that

$$(\chi_K \eta, \chi_L)_G^S = \eta(g) \frac{|K||L|}{|G|^2} \sum_{\sigma \in S} \overline{\sigma(g)} \sigma(h) \quad ,$$

while

$$(\chi_K, \bar{\eta} \chi_L)_G^S = \eta(h) \frac{|K||L|}{|G|^2} \sum_{\sigma \in S} \overline{\sigma(g)} \sigma(h) \quad .$$

Since these two values are equal by assumption, (ii) follows.

(ii) $\Rightarrow$ (iii) If $0 \neq \rho_S^g(h) = \sum_{\sigma \in S} \overline{\sigma(g)}\sigma(h)$, then $\eta(g) = \eta(h)$, so $\eta(h)\rho_S^g(h) = \eta(g)\rho_S^g(h)$ for every h . This means $\eta\rho_S^g = \eta(g)\rho_S^g$, so ρ_S^g is an eigenvector for η with eigenvalue $\eta(g)$.

(iii) $\Rightarrow$ (iv) This is clear since $\langle S \rangle$ is spanned by the ρ_S^g 's.

(iv) $\Rightarrow$ (v) Take $\sigma \in S$ and $\tau \in T$; let $c_1, \dots, c_m \in \mathbb{C}$ be all the different values of η . Define $a_i(\sigma, \tau) = \sum_{g \in G(c_i)} \sigma(g)\overline{\tau(g)}$. For fixed i , let $\pi = \pi(i) = \prod_{j \neq i} (\eta - c_j \mathbf{1})$. Then π vanishes on $G \setminus G(c_i)$ and is a non-zero constant d , say, on $G(c_i)$. Since π is a polynomial in η , it leaves $\langle S \rangle$ invariant, so

$$0 = (\pi\sigma, \tau)_G = \frac{d}{|G|} a_i(\sigma, \tau),$$

which implies $a_i(\sigma, \tau) = 0$. This proves the statement if c is one of the c_i 's. If not, $G(c)$ is empty and the assertion holds trivially.

(v) $\Rightarrow$ (vi) Keep the notation and let γ be any class function which is constant, say equal to γ_i , on every $G(c_i)$; then

$$(\gamma\sigma, \tau)_G = \frac{1}{|G|} \sum_{i=1}^m \gamma_i a_i(\sigma, \tau) = 0$$

for every $\sigma \in S$, $\tau \in T$. This implies $\gamma\sigma \in \langle S \rangle$. The argument applies in particular to $\gamma = \bar{\eta}$, proving (vi).

(vi) $\Rightarrow$ (vii) Using (iv) $\Rightarrow$ (vi) with $\bar{\eta}$ instead of η , one finds that $\langle S \rangle$ is closed under multiplication with η . Let $\tau \in T$. If $\eta\tau \notin \langle T \rangle$, there is some $\sigma \in S$ with $0 \neq (\sigma, \eta\tau)_G = (\sigma\bar{\eta}, \tau)_G$, a contradiction, since $\sigma\bar{\eta} \in \langle S \rangle$ by assumption.

(vii) $\Rightarrow$ (viii) Let $\alpha = \alpha_S + \alpha_T$ be given. Then in $\alpha\eta = \alpha_S\eta + \alpha_T\eta$, the first summand belongs to $\langle S \rangle$, the second to $\langle T \rangle$, by hypothesis. Therefore $(\alpha\eta)_S = \alpha_S\eta$.

(viii) $\Rightarrow$ (i)

$$(\alpha\eta, \beta)_G^S = ((\alpha\eta)_S, \beta)_G = (\alpha_S\eta, \beta)_G = (\alpha_S, \bar{\eta}\beta)_G = (\alpha, \bar{\eta}\beta)_G^S,$$

where the second equality holds by assumption.

Remark 2

- (i) The values $c_1, \dots, c_m$ of η come in no particular order. However, it is reasonable to agree that $c_1 = \eta(1)$. Note that this means $G(c_1) = \text{Ker}(\eta)$ if η is a character; also, all the $G(c_i)$'s are then unions of cosets of $G(c_1)$.
- (ii) The class function $\pi(i)$ constructed in the above proof will in fact be a generalized character if η is and if c_i is rational (for instance $i = 1$). This follows by observing that then the c_j 's are algebraic integers permuted by the Galois group.
- (iii) The functions constant on all $G(c_i)$ are precisely the polynomials in η (of degree $< m$), as the usual interpolation argument shows.
- (iv) Statements (ii) and (v) are versions of the orthogonality relations: Condition (iv) is trivially satisfied for every S if one takes $\eta = \mathbf{1}$. Then $G(1) = G$, so (v) reduces to $(\sigma, \tau)_G = 0$ for $\sigma \neq \tau \in \text{Irr}(G)$ (taking $S = \{\sigma\}$).

Again, condition (iv) is trivially satisfied for every η if one takes $S = \text{Irr}(G)$. Since we may choose η such that $\eta(g) \neq \eta(h)$ if g, h are not conjugate, we conclude from (ii) that then $0 = \sum_{\sigma \in \text{Irr}(G)} \sigma(g)\overline{\sigma(h)}$. This can be generalized:

Corollary 1

Let φ be an irreducible character of some normal subgroup N of G and let $g, h \in G$. If $\bar{g}$ and $\bar{h}$ are not conjugate in $\bar{G} = G/N$ then

$$0 = \sum_{\chi \in \text{Irr}(G|\varphi)} \chi(g)\overline{\chi(h)} .$$

Proof: It is clear that $\langle \text{Irr}(G|\varphi) \rangle$ is invariant under multiplication with any class function of $\bar{G}$. Since there is such a class function η with $\eta(g) \neq \eta(h)$, the statement follows from (iv) $\Rightarrow$ (ii) above.

Remark 3

- (i) Trivially, there are class functions which take different values on all conjugacy classes of G . It is not hard to prove that there is a *character* (in general reducible) with that property.
- (ii) The question naturally arises what the value of $\sum_{\chi \in \text{Irr}(G|\varphi)} \chi(g)\overline{\chi(h)}$ is if $\bar{g}$ and $\bar{h}$ are conjugate in $\bar{G}$. This will be investigated in a separate paper [2].

Corollary 2

Assume that η takes only non-negative real values. Then the conditions of the proposition are equivalent to $(\alpha\eta, \alpha)_G^S \geq 0$ for every α .

Proof: Assume that condition (viii) holds. Then

$$(\alpha\eta, \alpha)_G^S = ((\alpha\eta)_S, \alpha_S)_G = (\alpha_S\eta, \alpha_S)_G \geq 0 .$$

Conversely, assume that $(\alpha\eta, \alpha)_G^S \geq 0$ for every α . It is enough to show that $\langle S \rangle$ is closed under multiplication with $\bar{\eta}$ ($= \eta$). If not, there is $\sigma \in S$ and $\tau \in T = \text{Irr}(G) \setminus S$ such that $(\tau, \bar{\eta}\sigma)_G \neq 0$. Choose $c \in \mathbb{C}$ such that $c \cdot (\tau, \bar{\eta}\sigma)_G = -(\sigma, \bar{\eta}\sigma)_G - 1$ and let $\alpha = \sigma + c\tau$, so $\alpha_S = \sigma$. It follows that

$$0 \leq (\alpha\eta, \alpha)_G^S = (\alpha\eta, \alpha_S)_G = (\alpha\eta, \sigma)_G = (\sigma + c\tau, \bar{\eta}\sigma)_G = (\sigma, \bar{\eta}\sigma)_G + c \cdot (\tau, \bar{\eta}\sigma)_G = -1 ,$$

a contradiction.

Corollary 3

Let $S \subseteq \text{Irr}(G)$. Define a graph $\Delta = \Delta(S)$ with vertices all the conjugacy classes of G and an edge between two classes K and L if and only if $0 \neq \sum_{\sigma \in S} \sigma(g)\overline{\sigma(h)}$, where $g \in K$ and $h \in L$. The class functions η which are constant on the connected components of Δ are then precisely the ones satisfying the conditions of the proposition.

Proof: Clear from condition (ii) of the proposition.

Corollary 4

Let η be a class function of G taking exactly m values $c_1, \dots, c_m$. Define a graph $\Gamma = \Gamma(\eta)$ with vertices all the irreducible characters of G and an edge between σ and τ if and only if $0 \neq \sum_{x \in G(c_i)} \sigma(x)\overline{\tau(x)}$ for some i ; here again $G(c_i) = \{g \in G \mid \eta(g) = c_i\}$. The unions of connected components of Γ are then precisely the sets S satisfying the conditions of the proposition.

Proof: Clear from condition (v) of the proposition.

Corollary 5

Let η be a faithful class function of G , i.e. $\eta(g) \neq \eta(1)$ for $g \neq 1$. Then $S = \text{Irr}(G)$ and $S = \emptyset$ are the only subsets of $\text{Irr}(G)$ such that the conditions of the proposition are satisfied.

Proof: Taking $c_1 = \eta(1)$, we have $G(c_1) = \{1\}$, so $\sum_{x \in G(c_1)} \sigma(x)\overline{\tau(x)} = \sigma(1)\overline{\tau(1)} \neq 0$ for every $\sigma, \tau \in \text{Irr}(G)$. Therefore the graph Γ of the last corollary is connected (even complete).

Corollary 5 can be read as a version of the Burnside-Brauer theorem (compare [1], Theorem 4.3), because $\eta < S > \leq < S >$ is certainly true if η is a character and $S = \{\sigma \in \text{Irr}(G) \mid (\sigma, \eta^i)_G \neq 0 \text{ for some } i\}$.

Even if η is not faithful, an idea in the proof of Proposition 1 can be used to predict, for a given character γ , which irreducibles of G will show up as constituents of some $\gamma\eta^i$:

Proposition 2 Burnside-Brauer

Let γ and η be characters of G and let $K = \text{Ker}(\eta)$; further let m be the number of different values of η . Then for any $\chi \in \text{Irr}(G)$, the following are equivalent:

- (i) $(\chi, \gamma)_K \neq 0$
- (ii) $(\chi, \gamma\eta^i)_G \neq 0$ for some $m > i \geq 0$
- (iii) $(\chi, \gamma\eta^i)_G \neq 0$ for some $i \geq 0$.

Proof:

(i) $\Rightarrow$ (ii) Let $\eta(1) = c_1, \dots, c_m$ be the values of η and let $\pi = \prod_{i>1} (\eta - c_i \mathbf{1})$. Then π vanishes off K and is a non-zero constant on K , so $\pi = c \cdot \mathbf{1}_K^G$ for some $c \neq 0$. Therefore $(\chi, \gamma\pi)_G = \bar{c} \cdot (\chi, \gamma \mathbf{1}_K^G)_G = \bar{c} \cdot (\chi, \gamma)_K \neq 0$. Since π is a polynomial of degree $m - 1$ in η , it follows that χ is a constituent of $\gamma\eta^i$ for some $i < m$.

(ii) $\Rightarrow$ (iii) Trivial.

(iii) $\Rightarrow$ (i) Since $(\chi, \gamma\eta^i)_G \neq 0$ for some i , a fortiori $0 \neq (\chi, \gamma\eta^i)_K = \eta^i(1)(\chi, \gamma)_K$.

Remark 4

- (i) Note that the first condition depends on η only via its kernel.
- (ii) Clearly, one gets the classical result for $\gamma = \mathbf{1}$ and η faithful, since then $K = 1$.

- (iii) As the proof of (i) $\Rightarrow$ (ii) shows, the assumption that γ and η are characters can be weakened for this implication: γ may be any class function; for η , we need only assume that $K = \{g \in G \mid \eta(g) = \eta(1)\}$ is a (normal) subgroup; there are many class functions besides the characters satisfying this condition.
- (iv) However, here are two simple examples showing that the condition that γ and η are characters is essential for the implication (iii) $\Rightarrow$ (i). In both examples, one of γ, η is a character, the other a generalized character. The group is C_4 and λ is a faithful linear character.
- First, take $\gamma = \mathbf{1} - \lambda^2$ and $\eta = \mathbf{1} + \lambda^2$, so $K = C_2$. Then $\mathbf{1}$ is a constituent of $\gamma\eta^0$ but $(\mathbf{1}, \gamma)_K = 0$.
- Second, take $\gamma = \mathbf{1}$ and $\eta = \lambda(\mathbf{1} - \lambda^2)$, so again $K = C_2$ (even though η is not a character). Then λ is a constituent of $\gamma\eta^1$ but $(\lambda, \gamma)_K = 0$.

References

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- [2] R. Knörr, Isaacs' bilinear form and the second orthogonality relation. In preparation.